Osaka Mathematical Journal,
Vol. 2, No. 1, March, 1950

A Topological Characterization of Affine Transformations in E^2

By Hidetaka TERASAKA

The problem of the topological characterization of the group of congruent transformations in the Euclidean space E^n is solved completely for the plane by D. HILBERT [6] and S. S. CAIRNS [4] and for the 3-dimensional space by B. VON KERÉKJÁRTÓ [7] and D. MONTGOMERY and L. ZIPPIN [9]. For the general case of n dimension the problem was also approached by the author [11]. On the other hand no attempt seems to have been made for the affine transformations. The object of the present paper is to characterize topologically the group of affine transformations in the plane, leaning upon the notion of linear dependency introduced by H. WHITNEY [12] G. BIRKHOFF [1], T. NAKASAWA [10] and O. HAUPT, G. NÖBELING, C. PAUC [5].

§ 1. For every p-ple of points $x_1, x_2, \dots, x_p$ of E^n we assume that either $A[x_1, x_2, \dots, x_p]$ — $x_1, x_2, \dots, x_p$ are dependent — or its negative $U[x_1, x_2, \dots, x_p]$ — $x_1, x_2, \dots, x_p$ are independent — holds, A and U satisfying the following axioms:

I. AXIOMS FOR LINEAR DEPENDENCY [5].

(i) AXIOM OF COINCIDENCE. *For every x, $A[x, x]$.*

(ii) AXIOM OF INDUCTION.

If $A[x_1, x_2, \dots, x_p]$, then $A[x_1, x_2, \dots, x_p, y]$ for all y.

(iii) AXIOM OF EXCHANGE.

If $U[x_1, x_2, \dots, x_p]$, $A[x_1, x_2, \dots, x_p, y]$

and $A[x_1, x_2, \dots, x_p, z]$, then $A[x_2, \dots, x_p, y, z]$.

To these we add further

(iv) *$U[x]$ for all x, and $U[x, y]$ for all x, y with $x \neq y$.*

II. *If $U[a_1, a_2, \dots, a_p]$, provided that $1 \leq p \leq n$, there exists for any point a and any neighborhood $U(a)$ of a at least one point $x \in U(a)$ such that $U[a_1, a_2, \dots, a_p, x]$.*

III. *If $U[a_1, a_2, \dots, a_{n+1}]$ there exists some neighborhood $U(a_1)$ of a_1, such that for any point $x \in U(a_1)$ the relation $U[x, a_2, \dots, a_{n+1}]$ holds.*

Now let $\mathfrak{H}$ be a *continuous group of homeomorphisms* of E^n on itself, which satisfies the following basic axiom:

IV. AXIOM FOR TRANSFORMATIONS. *If $U[a_1, a_2, \ldots, a_{n+1}]$ and $U[b_1, b_2, \ldots, b_{n+1}]$ there exists one and only one transformation $T \in \mathfrak{H}$ such that $T(a_1, a_2, \ldots, a_{n+1}) = (b_1, b_2, \ldots, b_{n+1})$* [1].

It is most probable that the group $\mathfrak{H}$ is topologically equivalent to the group of affine transformations of E^n onto itself. In the following we shall show that it is indeed the case when $n=2$, viz., when E^n is a plane.

To this end we prove first that there passes through each pair of distinct points a and b one and only one "L-line" $L(a, b)$, namely an open line, which is by definition a closed, topological image of a straight line, and which is transformed by the transformation of the group under consideration into another L-line. These L-lines are then proved to form a family of curves which can be transformed by a suitable homeomorphism of the plane into the family of all straight lines on it, and thus our theorem follows.

§ 2. If a triple of points (a, b, c) can not by any $T \in \mathfrak{H}$ be transformed into another triple of points (a', b', c') with $U[a', b', c']$, the points (a, b, c) are said to be *collinear* or in the *L-relation,* written $L[a, b, c]$; otherwise, (a, b, c) is said to be *non-collinear,* written *non-L* $[a, b, c]$.

1. *If non-$L[a, b, c]$ and non-$L[a', b', c']$, there exists one and only one transformation T of $\mathfrak{H}$ such that $T(a, b, c) = (a', b', c')$.*

For on account of the definition of non-L and Axiom IV (AXIOM FOR TRANSFORMATION) there exist transformations T_1 and T_2 of $\mathfrak{H}$ and a triple of points (p, q, r) with $U[p, q, r]$ such that $T_1(a, b, c) = (p, q, r) = T_2(a', b', c')$, whence the existence of the transformation $T_2^{-1}T_1\ (a, b, c) = (a', b', c')$ follows. Conversely if $T(a, b, c) = (a', b', c')$, then $T_2 T T_1^{-1}(p, q, r) = (p, q, r)$, hence by Axiom IV $T = T_2^{-1} T_1$, proving the unicity of T in question.

2. *If $a \neq b$, then x's with $L[a, b, x]$ are by threes collinear.*

Proof. Let

$$L[a, b, x] \quad \text{and} \quad L[a, b, y].$$

First let a, b, x and y be all distinct. If non-$L[b, x, y]$, there would exist some $T \in \mathfrak{H}$ with $T(b, x, y) = (b', x', y')$, where b', x', y' are some points with $U[b', x', y']$. Then it follows from Axiom of Exchange that for four points $T(a)$, b', x', y', either $U[T(a), b', x']$ or $U[T(a),$

1) Instead of $T(a_1) = b_1$, $T(a_2) = b_2$, ... we write briefly $T(a_1, a_2, \ldots) = (b_1, b_2, \ldots)$.

$b', y']$ or both must hold, whence we have either non-L $[a, b, x]$ or non-$L[a, b, y]$ or both, contrary to hypothesis. Hence we have $L[b, x, y]$.

If a, b, x, and y are not all distinct, $L[b, x, y]$ is a direct consequence of the hypothesis. In any case we have therefore the following implication:

$$a \neq b,\ L[a, b, x],\ L[a, b, y] \rightarrow L[b, x, y].$$

The required implication

$$L[a, b, x],\ L[a, b, y],\ L[a, b, z] \rightarrow L[x, y, z]$$

is then immediate. —

If $a \neq b$, the set of all points x with $L[a, b, x]$ will be called an *L-line* and denoted by $L(a, b)$. From Proposition 2 we have then

3. *If a', b' are two distinct points of an L-line $L(a, b)$, then* $L(a', b') = L(a, b)$.

The following proposition is also clear.

4. *An L-line is transformed by a transformation $T \in \mathfrak{H}$ again into an L-line.*

From Axiom III we see easily

5. *An L-line is a closed set.*

6. *Every point of an L-line L is accessible from each of the components of E^2—L.*

Proof. Let x be a given point of L and let a be any one of the points of L which is rectilinearly accessible from a given point p of E^2—L, so that the segment pa with end-points p and a has no point in common with L except for p. We may suppose $x \neq a$. Then, since non-$L[x, a, p]$, there exists by Proposition 1 a transformation $T \in \mathfrak{H}$ such that $T(a, x, p) = (x, a, p)$. $T(pa)$ is then the arc from p to x in $E^2 - L$ except for x, which proves the proposition.

7. *An L-line contains no bounded component.*

Proof. Suppose on the contrary that the L-line L contains a bounded component A. Construct a polygon Π having no point in common with L and containing A in its interior. Let a and c be points of A and Π respectively and let γ be an arc from c to some point b of L distinct from a having no point in common with L except for b.

Since for every point x of Π non-$L[a, b, x]$ holds, there exists by Proposition 1 one and only one T of $\mathfrak{H}$ such that $T(a, b, c) = (a, b, x)$. Such T form therefore a continuous family of transformations depending upon the parameter x, and $T(\gamma)$ form in their turn a continuous family of arcs joining b to all points x of Π. On account of a well-known theorem we see then that $T(\gamma)$ pass through every point of the interior of Π, in particular through a, which is evidently a contradiction. Thus the proposition was proved.

8. *If L is an L-line, every component of E^2-L is unbounded.*

Proof. Suppose on the contrary that E^2-L contains a bounded component D. Since L is unbounded by the foregoing proposition, there exists a point a on L lying wholly without D. If p is an arbitrary point in D and if b is a point on the boundary of D, such that the segment pb has no point in common with the latter except for b, there exists a transformation $T \in \mathfrak{H}$ such that $T(a, b, p) = (b, a, p)$. Then $T(pb)$ would be an arc from p to a having no point in common with L except for a, which involves a contradiction.

9. *If L is an L-line, there exists for each pair of distinct points a and b of L a bounded subcontinuum K of L containing a and b.*

Proof. L has evidently a bounded subcontinuum. By a suitable transformation T of $\mathfrak{H}$ we can bring this into a continuum containing a and b, which proves the proposition.

If $a \neq b$, we shall denote by $K(a, b)$ an irreducible bounded subcontinuum of $L(a, b)$ which contains a and b.

10. *If $U[a, b, c]$, then the transformation T of $\mathfrak{H}$ with $T(a, b, c) = (b, a, c)$ is involutorial and orientation-reversing.*

Proof. That T is involutorial is obvious from Axiom IV.

To prove that T is orientation-reversing, suppose the contrary. Then by a well-known theorem of v. KERÉKJÁRTÓ [8] T must be topologically equivalent to the half-rotation of the plane with center at c. Denoting $K(a, b)$ by K for brevity, $K + T(K) \subset L$ would then enclose a bounded domain. To show this let h be a homeomorphism of the plane on itself such that the transform $T' = hTh^{-1}$ of T becomes a half rotation about $c' = h(c)$. Then we assert that $C = h(K) + hTh^{-1}.\ h(K) = h(K + T(K))$ would separate c' from the point at infinity (which is a contradiction); for if not, let ρ be a half ray from a point p having no point in common with C, and join p to c' by a polygonal line π which has no point in common with C. $(\pi+\rho) + hTh^{-1}(\pi+\rho)$ has in general double points; however it is a matter of no difficulty to obtain a subray ρ' of ρ beginning from c' such that $\rho' + hTh^{-1}(\rho') = l$ becomes an open line. l divides then the plane into two domains which interchanges themselves by T. Now since $b' = hTh^{-1}(a')$ and since a' and b' do not lie upon l, a' lies in the one domain and b' in the other, while the continuum $h(C)$ contains both a' and b' and has no point in common with l, which is impossible. These contradictions show the absurdity of the original assumption that T should be orientation-preserving, and the proposition was thus proved.

11. *An L-line L is an open line.*

Proof. Let a, b be two distinct points of L and let S be an involutorial and orientation-reversing transformation of $\mathfrak{H}$ leaving both a and b fixed; such an S may be obtained by transforming the transformation T considered in Proposition 10. Again by the theorem of v. KERÉKJÁRTÓ [8], the set of fixed points under S consists of an open line passing through a and b. Call this L^*. We assert, that $L^* = L$. For first, if $L^* - L$ contains a point c, since $S(a, b, c) = (a, b, c)$ and $U[a, b, c]$, S must be equal to the identical mapping, which is absurd. Therefore $L^* \subset L$. If on the other hand $L - L^*$ contains a point c, let p be a point lying on the other side of L^* with respect to c and not on L. Then the point c on L is not accessible from the component of $E^2 - L$ in which p lies, contrary to Proposition 6. Thus we have indeed $L^* - L$.

§ 3. In the foregoing paragraph we have shown the existence and the uniqueness of the open line $L(a, b)$ through every pair of points a and b of E^2. We now proceed to investigate the general behavior of the family of these L-lines and then establish the affine geometry based on them.

We know already that two different L-lines have at most one point in common. From this we infer easily that they "intersect", when they have one point in common. Furthermore, from the assumption of continuity of the group $\mathfrak{H}$ and from the fact that L-line is transformed by a transformation of $\mathfrak{H}$ again into L-line, we can deduce without difficulty that the totality $\mathfrak{L}$ of L-lines forms a uniformly continuous family: that is, if $a_n \to a$, $b_n \to b$ and $a \neq b$, then $K(a_n, b_n) \to K(a, b)$ and $L(a_n, b_n) \to L(a, b)$. We are thus led to the proof of the following proposition:

Given a continuous group $\mathfrak{H}$ of homeomorphisms and a continuous family $\mathfrak{L}$ of open lines on E^2, such that i) through each pair of distinct points passes one and only one line of $\mathfrak{L}$; ii) for each pair of triple of points (a, b, c) and (a', b', c') which do not lie respectively on any single line of $\mathfrak{L}$, there exists one and only one element T of $\mathfrak{H}$ with $T(a, b, c) = (a', b', c')$; and iii) each $T \in \mathfrak{H}$ carries $\mathfrak{L}$ onto $\mathfrak{L}$. Then $\mathfrak{L}$

is topologically equivalent to the family of straight lines and $\mathfrak{H}$ is isomorphic to the group of affine transformations of the plane.

In the first place let α be an L-line and b a point outside of it. We assign on α a positive direction, and let a point x move in the positive direction along α as far as ∞. Then $L(a, x)$ will converge to an L-line β, which we shall call the *asymptote* [3] to α in the positive direction through b. Likewise we can define the asymptote β^* in the negative direction. In reality however β and β^* coincide.

Suppose on the contrary that β and β^* are different. Then for every L-line α' the two asymptotes through a point b' outside of it are also different, since a transformation of $\mathfrak{H}$ can be found such that α and b are transformed respectively into α' and b'. Now let a be a point on α above considered and let a point x move along α toward ∞ in the other side of β with respect to a and denote by γ the asymptote obtained as the limit of $L(a, x)$. Then γ must have a point in common with β, for otherwise γ would be asymptotic to β and coincide with α; if S is the transformation of $\mathfrak{H}$ which leaves b fixed and which interchanges two different points on α, then β and β^* interchange themselves by S and γ would again be asymptotic to β, but now in the opposite direction. Therefore the asymptotes through a to β would coincide, which is absurd. Thus γ and β has a point, say c, in common. Take now on α two points a' and a'' such that a' lies between a and a'', and let T be the transformation of $\mathfrak{H}$ with $T(b, a, a') = (b, a, a'')$. By T the L-lines α, β, β^*, and γ are mapped respectively on themselves, and consequently c must be a fixed point under T, which involves a contradiction, since T would leave three points a, b, c fixed while it is not evidently an identical transformation.

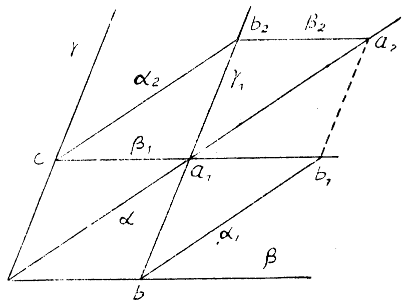
Thus we have shown that the asymptotes to an L-line through a point outside of it in both directions coincide. Hereafter we shall call the asymptotes *parallel L-lines.*

Any triple system of parallel L-lines constitutes a hexagonal texture (Sechseckgewebe) in the sense of W. BLASCHKE [2].

To show this let α, β, γ be three different L-lines through a point a, and let a_1 be a point on α different from a. Draw parallel L-lines β_1 and γ_1 through a_1 to β and γ respectively and put $b = \beta \frown \gamma_1$, $c = \beta_1 \frown \gamma$ [2]). Draw parallel L-lines α_1 and α_2 to α through b and c respectively and put $b_1 = \alpha_1 \frown \beta_1$, $b_2 = \alpha_2 \frown \gamma_1$. Finally draw parallel L-line β_2 to β through b_2 and put $a_2 = \alpha \frown \beta_2$. It is to show that $L(a_2, b_1)$ is parallel to γ. To this end consider the transformation $T \in \mathfrak{H}$:

---

2) $\alpha \frown \beta$ denotes the intersection of α and β.



$T(a, b, c) = (a, c, b)$. Since b and c; $L(a, b)$ and $L(a, c)$ are respectively interchanged by T, the parallel L-lines $L(c, a_1)$ and $L(b, a_1)$ are interchanged by T. Therefore a_1 is fixed under T, and hence $L(a, a_1)$ is a fixed line of T. Then, since b_1 and b_2 are interchanged, a_2 is fixed, and $L(b_2, a_2)$ is parallel to $L(a_1, b_1) = \beta_1$, $L(a_2, b_1)$ must be parallel to $L(b_2, a_1) = \gamma_1$, thus the hexagonal property of the hexagonal texture was proved.

Starting from a, b, c, a_1, b_1, etc. we can construct after BLASCHKE [2] by virtue of the hexagonal property a net of triangles $\mathfrak{N}$, made up from triple system of parallel L-lines, whose intersections may be denoted by $z_{n,m}$ with the double suffixes in integers n, m, setting in particular:

$$z_{0,0} = x_0 = y_0 = a, \qquad z_{1,0} = x_1 = b,$$

$$z_{n,0} = x_n, \qquad z_{0,m} = y_m.$$

It will be seen that the whole plane is triangulated in this way. We shall prove indeed that:

For any point x and y on $L(a, b)$ and $L(a, c)$ respectively we can find integers n and m such that $x \in K(x_n, x_{n+1})$ and $y \in K(y_m, y_{m+1})$ respectively.

Denote for the sake of brevity the parallel L-lines to $L(x_0, x_1)$ through y_n by ξ_n, the parallel L-lines to $L(x_0, y_1)$ through x_n by η_n and finally the L-lines $L(x_n, y_{-n})$ which are parallel one another, by ζ_n. Suppose now the proposition is false and suppose without loss of generality that x_n converge to a point $\bar{x}$ for $n \to \infty$. Since by the transformation $S \in \mathfrak{H}$: $S(x_0, x_1, y_1) = (x_0, y_1, x_1)$, x_n and y_n are interchanged, we must have $y_n \to S(\bar{x}) = \bar{y}$. Consider $T \in \mathfrak{H}$: $T(x_0, x_1, y_1) = (x_0, x_2, y_2)$. In consequence of the preservation of parallelism under T, we have successively the implications:

From $T(\xi_1) = \xi_2$, $T(\eta_1) = \eta_2$ follows $T(z_{11}) = z_{22}$, whence $T(\zeta_0) = \zeta_0$. From $T(\zeta_1) = \zeta_2$ and $T(\xi_1) = \xi_2$ follows $T(z_{21}) = z_{42}$, whence $T(\xi_2) = \xi_4$. Similarly $T(\eta_2) = \eta_4$.

We have thus in general $T(\xi_n) = \xi_{2n}$ and $T(\eta_n) = \eta_{2n}$. Consequently we must have $T(\bar{x}) = \bar{x}$ and $T(\bar{y}) = \bar{y}$, which in connection with $T(x_0) = x_0$ leads us to a contradiction, and the proposition is proved.

By the continuity property we can subdivide the original triangural net in smaller net of $1/2^n$-th size [2] and thus finally we can assign for every point z of E^2 its coordinates x and y: $z = z_{x,y}$.

It remains now to prove that L-lines can be analytically expressed in linear equations. This will be done by introducing the "translation". Let T be the transformation of $\mathfrak{H}$ with $T(x_0, x_1, z_{11}) = (x_1, x_2, z_{21})$. Again by virtue of the preservation of parallelism we see easily that $T(\xi_1) = \xi_1$, $T(\zeta_1) = \zeta_2$, hence $T(x_1, x_2, z_{21}) = (x_2, x_3, z_{31})$ and furthermore $T(\eta_0) = \eta_1$, $T(\eta_1) = \eta_2$, $T(\zeta_{-1}) = \zeta_0$, whence $T(y_1, z_{11}, z_{12}) = (z_{11}, z_{21}, z_{22})$. From these we see that the triangural net $\mathfrak{N}$ is transformed by T into itself, and we can conclude by the consideration of subdivision of triangles and continuity of transformations again, that every point $z = z_{x,y}$ of E^2 is carried by T into the point $z' = z_{x',y'}$ such that

$$x' = x + 1, \qquad y' = y.$$

Every L-line is carried therefore by T into an L-line having no point in common with itself, so that λ and $T(\lambda)$ are parallel to each other.

The same is also true for the transformation S of $\mathfrak{H}$ with $S(x_0, y_1, z_{11}) = (y_1, y_2, z_{12})$, which maps $\mathfrak{N}$ on itself, and may be expressed analytically by

$$x' = x, \qquad y' = y + 1.$$

Combining T and S we get the general "translation" $S^m T^n$ which carries a point x_0 into the point $z_{n,m}$ with coordonates n, m in integers. On account of the preservation of parallelism $S^m T^n$ maps $L(x_0, z_{n,m})$ on itself, whence we see that $z_{2n,2m}$ is a point of $L(x_0, z_{n,m})$, and generally, p being integers, the points $z_{x,y}$ with $x = pn$, $y = pm$ are all points of $L(x_0, z_{n,m})$. The principle of subdivision of triangles leads us immediately to the conclusion that whenever r are rational numbers, the points $z_{x,y}$ with $x = rn$, $y = rm$ are also points of $L(x_0, z_{n,m})$ and we obtain finally by the consideration of continuity the required linear expressions for L-lines in general.

The isomorphism of the group of affine transformations and $\mathfrak{H}$ results immediately from this.

REFERENCES

[1] G. Birkhoff: Abstract linear dependence and lattices. Amer. Journ. 57 (1935).

[2] W. Blaschke: Topologische Fragen der Differentialgeometrie, I. Math. Zeitschr. 28 (1928).

[3] H. Busemann: On translations in general plane geometries. Amer. Journ. 60 (1938).

[4] S. S. Cairns: An axiomatic basis for plane geometry. Trans. Amer. Math. Soc. 35 (1933).

[5] O. Haupt, G. Nöbeling, C. Pauc: Über Abhängigkeitsräume. Crelles Journal 181 (1940).

[6] D. Hilbert: Über die Grundlagen der Geometrie. Math. Ann. 56 (1902), Grundlagen der Geometrie, 7te Aufl. Anhang IV.

[7] B. von Kerékjártó: On a geometrical theory of continuous groups, II. Euclidean and hyperbolic groups of the three dimensional space. Ann. of Math. 29 (1928).

[8] B. von Kerékjártó: Topologische Charakterisierung der linearen Abbildungen. Acta Litt. ac Sci. Szeged 6 (1934).

[9] D. Montgomery and L. Zippin: Topological group foundations of rigid geometry. Trans. Amer. Math. Soc. 48 (1940).

[10] T. Nakasawa: Zur Axiomatik der linearen Abhängigkeit, I–III. Science Reports of the Tokyo Bunrika Daigaku. Section A. 2 (1935), 3 (1936).

[11] H. Terasaka: Eine topologische Charakterisierung der kongruenten Transformationen in E^n. Osaka Math. Journ. 1 (1949).

[12] H. Whitney: The abstract properties of linear dependence. Amer. Journ. 57 (1935).

(Received November 7, 1949)