Lowig, H. F. J.
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On the Composition of Some Representations of Lattices of Law Relations

By H. F. J. LOWIG

By (IV) (see the bibliography at the end of the paper) Theorems 1.4 and 3.9,

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \supset r \quad \text{for} \quad r \in L\mathfrak{C}_1,$$

and

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) = r \quad \text{for} \quad r \in L\mathfrak{C}_1$$

if and only if $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$. In the present paper, I wish to prove that, more generally,

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \supset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad \text{for} \quad r \in L\mathfrak{C}_1,$$

and that

$$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\}$$

if and only if $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$ or $\mathfrak{C}_3 \sqsubset \mathfrak{C}_2$. Besides I am going to prove some further theorems of this sort. It is understood that all conventions on terminology and notation introduced in (IV) hold in this paper.

Theorem 1. *Let* $r \in L\mathfrak{C}_1$. *Then*

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \supset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r.$$

Proof. By (IV), Definition 1.1 and Theorem 1.3,

$$\begin{aligned}(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) &= (\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Phi\mathfrak{C}_2)(\mathfrak{C}_1/r)) \supset (\Phi\mathfrak{C}_3)(\mathfrak{C}_1/r) \\ &= (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r.\end{aligned}$$

Theorem 2. *Let* $r \in L\mathfrak{C}_1$. *Then*

$$(\psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r.$$

Proof. By (IV), Theorem 1.5,

$$r \subset (\Psi\{\mathfrak{C}_3, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r).$$

By Theorem 1,

$$(\Psi\{\mathfrak{C}_3, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r) \subset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r)).$$

Hence

$$r \subset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r)).$$

By (IV), Theorem 1.8,

$$(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r \subset (\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r), \tag{1}$$

and

$$(\psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r.$$

If, in (1), we interchange $\mathfrak{C}_2$ and $\mathfrak{C}_3$, we obtain the following theorem:

Theorem 3.

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \supset (\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad \textit{for} \quad r \in L\mathfrak{C}_1.$$

Theorem 4. *Let* $r \in L\mathfrak{C}_1$. *Then*

$$(\psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r.$$

Proof. By Theorem 1,

$$(\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r) \supset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r.$$

By (IV), Theorem 1.8,

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \supset (\psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r).$$

Theorem 5.

$$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\} \tag{2}$$

if and only if

$$\mathfrak{C}_1 \sqsubset \mathfrak{C}_2 \quad \textit{or} \quad \mathfrak{C}_3 \sqsubset \mathfrak{C}_2. \tag{3}$$

Proof. (i) Let $r \in L\mathfrak{C}_1$. Then, by Theorem 1 and by (IV), Theorem 1.2,

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)), \tag{4}$$

and

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r)). \tag{5}$$

If (3) holds then, by (IV), Theorem 3.3, $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ or $\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_3, \mathfrak{C}_2\}$ is the identical representation of $L\mathfrak{C}_1$ or $L\mathfrak{C}_3$,

respectively, (4) or (5) implies that

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r ,$$

Theorem 1 implies that

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) = (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r ,$$

and (2) holds.

(ii) Let (2) hold, and let $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$ not hold. Then, by (IV), Theorem 3.9, $\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$ is not simple. By (2), $\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\}$ is not simple. By (IV), Theorem 3.9, $\mathfrak{C}_1 \sqsubset \mathfrak{C}_3$ does not hold. Hence $\mathfrak{C}_3 \sqsubset \mathfrak{C}_1$. By the result obtained under (i),

$$(6) \qquad \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} \cdot \Psi\{\mathfrak{C}_3, \mathfrak{C}_1\} = \Psi\{\mathfrak{C}_3, \mathfrak{C}_2\} .$$

Let $r \in L\mathfrak{C}_3$. Then, by (2) and (6),

$$\begin{aligned}(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\})r) &= (\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Psi\{\mathfrak{C}_3, \mathfrak{C}_1\})r)) \\ &= (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_3, \mathfrak{C}_1\})r) .\end{aligned}$$

By (IV), Theorem 3.9,

$$\begin{aligned}(\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_3, \mathfrak{C}_1\})r) &= r , \\ (\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\})r) &= r ,\end{aligned}$$

$\mathfrak{C}_3 \sqsubset \mathfrak{C}_2$, and (3) holds, completing the proof.

Theorem 6.

$$(2') \qquad \psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \psi\{\mathfrak{C}_1, \mathfrak{C}_3\}$$

if and only if (3) *holds.*

A proof of Theorem 6 is obtained from the proof of Theorem 5 by replacing Ψ, $\subset$, (2), Theorem 1, and Theorem 1.2 by ψ, $\supset$, (2′), Theorem 2, and Theorem 1.12, respectively.

Theorem 7. *Let B be a relation on C_1, let $\mathfrak{c}_2'$ and $\mathfrak{c}_2''$ be elements of C_2 with*

$$\mathfrak{c}_2'((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B))\mathfrak{c}_2'' ,$$

and let $|D| \geqq 2m_\sigma$. (See (III), Definition 2.2, and (II), Definition 3.4.) *Then there exist elements, $\mathfrak{c}'$ and $\mathfrak{c}''$, of C such that*

$$\mathfrak{c}'((\psi\{\mathfrak{C}_1, \mathfrak{C}\})((\Theta\mathfrak{C}_1)B))\mathfrak{c}''$$

and

$$\{\mathfrak{c}', \mathfrak{c}''\}(\{\mathfrak{C}, \mathfrak{C}_2\}\text{-conf})\{\mathfrak{c}_2', \mathfrak{c}_2''\} .$$

(See (III), Definition 1.1.)

Proof. By (IV), Definition 2.2, $\mathfrak{C}_1 \sqsubset \mathfrak{C}$. By (IV), Theorem 1.20, and by Theorem 6,

$$(\psi_0\{\mathfrak{C}, \mathfrak{C}_2\})((\psi\{\mathfrak{C}_1, \mathfrak{C}\})((\Theta\mathfrak{C}_1)B)) = (\psi\{\mathfrak{C}, \mathfrak{C}_2\})((\psi\{\mathfrak{C}_1, \mathfrak{C}\})((\Theta\mathfrak{C}_1)B)) = (\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B) .$$

The assertion is now obvious from (IV), Definition 1.3.

Let B be a relation on C_1, let $\mathfrak{C}$ be fixed, and let $|D| \geqq 2m_\sigma$. Theorem 7 shows that then our $(\psi\{\mathfrak{C}_1, \mathfrak{C}\})((\Theta\mathfrak{C}_1)B)$ may be regarded as an analogue of Birkhoff's $\Phi(B)$. (See (I), p. 440, Definition 5.)

Theorem 8. *Let B be a relation on C_1. Let $\mathfrak{c}'$ and $\mathfrak{c}''$ be elements of C such that*

$$(7) \qquad \begin{cases} h\mathfrak{c}' = h\mathfrak{c}'' & \text{for every homomorphism } h \text{ of } \mathfrak{C} \text{ into} \\ \mathfrak{C}_2/(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B) \end{cases}$$

and such that

$$|\mathfrak{S}[(P\mathfrak{C})\mathfrak{c}', (P\mathfrak{C})\mathfrak{c}'']| \leqq |D_2| .$$

Then

$$\mathfrak{c}'((\psi\{\mathfrak{C}_1, \mathfrak{C}\})((\Theta\mathfrak{C}_1)B))\mathfrak{c}'' .$$

Proof. By (III), Definition 3.1, (7) is a $\mathfrak{C}$-law of $\mathfrak{C}_2/(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B)$, and

$$\mathfrak{c}'((\Phi\mathfrak{C})(\mathfrak{C}_2/(\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B)))\mathfrak{c}'' .$$

By (IV), Definition 1.1,

$$\mathfrak{c}'((\Psi\{\mathfrak{C}_2, \mathfrak{C}\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B)))\mathfrak{c}'' .$$

By (IV), Theorems 1.17 and 1.18,

$$\mathfrak{c}'((\psi\{\mathfrak{C}_2, \mathfrak{C}\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\Theta\mathfrak{C}_1)B)))\mathfrak{c}'' .$$

By Theorem 2,

$$\mathfrak{c}'((\psi\{\mathfrak{C}_1, \mathfrak{C}\})((\Theta\mathfrak{C}_1)B))\mathfrak{c}'' .$$

Compare Theorem 8 with the following statement occurring in (I), p. 441, lines 4 and 5: "Every law of $F(B, m)$ involving m primitive symbols is an equation of $\Phi(B)$."

Theorem 9. *The following propositions* (3), (2), (2′), *and* (8) *to* (14)

are equivalent:

(3) $$\mathfrak{C}_1 \sqsubset \mathfrak{C}_2 \quad or \quad \mathfrak{C}_3 \sqsubset \mathfrak{C}_2 .$$

(2) $$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\} .$$

(2′) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \psi\{\mathfrak{C}_1, \mathfrak{C}_3\} .$$

(8) $$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r) \supset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})r \quad for \quad r \in L\mathfrak{C}_2 .$$

(9) $$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad for \quad r \in L\mathfrak{C}_1 .$$

(10) $$(\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r) \subset (\psi\{\mathfrak{C}_2, \mathfrak{C}_3\})r \quad for \quad r \in L\mathfrak{C}_2 .$$

(11) $$(\psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \supset (\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad for \quad r \in L\mathfrak{C}_1 .$$

(12) $$\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\} \cdot \psi\{\mathfrak{C}_2, \mathfrak{C}_1\} .$$

(13) $$\psi\{\mathfrak{C}_1, \mathfrak{C}_3\} \cdot \psi\{\mathfrak{C}_2, \mathfrak{C}_1\} = \psi\{\mathfrak{C}_1, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} .$$

(14) $$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r) \supset (\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r) \quad for \quad r \in L\mathfrak{C}_2 .$$

Corollary. Each of the propositions (2), (2′), and (8) to (14) is equivalent to the proposition arising from it by interchanging the subscripts 1 and 3.

Proof. That (3), (2) and (2′) are equivalent follows from Theorems 5 and 6. If (2) holds, then

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r) = (\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r)) \quad \text{for} \quad r \in L\mathfrak{C}_2 ,$$

and (8) holds by (IV), Theorems 1.2 and 1.5. If (8) holds, then

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)) \quad \text{for} \quad r \in L\mathfrak{C}_1 ,$$

and (9) holds by (IV), Theorems 1.2, 1.9 and 3.4. If (9) holds, and $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$, then (3) holds, hence (2) holds. If (9) holds, and $\mathfrak{C}_2 \sqsubset \mathfrak{C}_1$, then

$$\psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\}$$

by (IV), Theorem 2.5,

$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad \text{for} \quad r \in L\mathfrak{C}_1 ,$$

and (2) holds by Theorem 1. Hence (2), (8) and (9) are equivalent.

By a similar argument it can be shown that (2′), (10) and (11) are equivalent.

By (IV), Theorem 3.4,

$$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot (\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} \cdot \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}) = \Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot (\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} \cdot \psi\{\mathfrak{C}_2, \mathfrak{C}_1\}) .$$

Hence (2) implies (12). Let (12) hold. Then

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)) = (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r))$$
$$\text{for} \quad r \in L\mathfrak{C}_1 .$$

By (IV), Theorems 1.2, 1.4 and 1.9,

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r)) = (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad \text{for} \quad r \in L\mathfrak{C}_1 .$$

If $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$, (2) holds. If $\mathfrak{C}_2 \sqsubset \mathfrak{C}_1$ then, for the same reason,

$$\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} ,$$
$$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) = (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad \text{for} \quad r \in L\mathfrak{C}_1 ,$$

and (2) holds again. Hence (2) is equivalent to (12).

In a similar way, it can be shown that (2′) is equivalent to (13).

If (8) and (10) hold then (14) holds by (IV), Theorem 1.13. Let (14) hold. If $\mathfrak{C}_1 \sqsubset \mathfrak{C}_3$ then, by (IV), Theorems 1.2, 1.13 and 3.9,

$$\begin{aligned}(\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r &= (\Psi\{\mathfrak{C}_3, \mathfrak{C}_1\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r)) \\ &\supset (\Psi\{\mathfrak{C}_3, \mathfrak{C}_1\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r)) \\ &= (\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r \quad \text{for} \quad r \in L\mathfrak{C}_1 , \\ \psi\{\mathfrak{C}_2, \mathfrak{C}_1\} &= \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} ,\end{aligned}$$

$\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$, and (3) holds. If $\mathfrak{C}_3 \sqsubset \mathfrak{C}_1$ then

$$\psi\{\mathfrak{C}_1, \mathfrak{C}_3\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\} ,$$
$$(\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r) \supset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\})r) \quad \text{for} \quad r \in L\mathfrak{C}_2 ,$$
$$(\psi\{\mathfrak{C}_2, \mathfrak{C}_3\})r \supset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})r \quad \text{for} \quad r \in L\mathfrak{C}_2$$

by Theorems 1 and 2,

$$\psi\{\mathfrak{C}_2, \mathfrak{C}_3\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} ,$$

$\mathfrak{C}_3 \sqsubset \mathfrak{C}_2$, and (3) holds. Thus (3) holds in both cases.

This completes the proof that the propositions (3), (2), (2′), and (8) to (14) are equivalent.

Theorem 10. *The following three propositions are equivalent*:

(15) $\mathfrak{C}_1 \sqcap \mathfrak{C}_2$, *or* $\mathfrak{C}_3 \sqsubset \mathfrak{C}_1$ *as well as* $\mathfrak{C}_3 \sqsubset \mathfrak{C}_2$. (See (IV), Definition 2.3.)

(16) $$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\} .$$

(17) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \psi\{\mathfrak{C}_1, \mathfrak{C}_3\} .$$

Corollary. (16) as well as (17) is equivalent to the equation arising from it by interchanging the subscripts 1 and 2.

Proof. By Theorem 9, the following three propositions are equivalent:

(18) $$\mathfrak{C}_2 \sqsubset \mathfrak{C}_1 \quad \text{or} \quad \mathfrak{C}_3 \sqsubset \mathfrak{C}_1 .$$

(19) $$(\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \supset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad \text{for} \quad r \in L\mathfrak{C}_1 .$$

(20) $$(\psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad \text{for} \quad r \in L\mathfrak{C}_1 .$$

By the same theorem, (3), (9) and (11) are equivalent. Hence the statement that both (3) and (18) hold is equivalent to the statement that both (9) and (19) hold and also to the statement that both (11) and (20) hold. This implies that (15), (16) and (17) are equivalent.

Theorem 11. *The following propositions* (21) *to* (32) *are equivalent*:

(21) $$\mathfrak{C}_3 \sqsubset \mathfrak{C}_1 \quad \textit{and} \quad \mathfrak{C}_3 \sqsubset \mathfrak{C}_2 .$$

(22) $$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \psi\{\mathfrak{C}_1, \mathfrak{C}_3\} .$$

(23) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\} .$$

(24) $$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} .$$

(25) $$\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_3, \mathfrak{C}_2\} \quad \textit{is simple.}$$

(26) $$\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \psi\{\mathfrak{C}_3, \mathfrak{C}_2\} \quad \textit{is simple.}$$

(27) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_3, \mathfrak{C}_2\} \quad \textit{is simple.}$$

(28) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \psi\{\mathfrak{C}_3, \mathfrak{C}_2\} \quad \textit{is simple.}$$

(29) $$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} \quad \textit{takes all values of} \quad L\mathfrak{C}_3 .$$

(30) $$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} \quad \textit{takes all values of} \quad L\mathfrak{C}_3 .$$

(31) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} \quad \textit{takes all values of} \quad L\mathfrak{C}_3 .$$

(32) $$\psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_2\} \quad \textit{takes all values of} \quad L\mathfrak{C}_3 .$$

Corollary. Each of the propositions (22) to (32) is equivalent to the proposition arising from it by interchanging the subscripts 1 and 2.

Proof. If (21) holds, (22) holds by Theorem 5 and by (IV), Theorem 2.5. If (22) holds, then

$$(\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \supset (\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad \text{for} \quad r \in L\mathfrak{C}_1$$

by Theorem 1,

(33) $$\psi\{\mathfrak{C}_1, \mathfrak{C}_3\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\}$$

by (IV), Theorem 1.13,

$$\mathfrak{C}_3 \sqsubset \mathfrak{C}_1$$

by (IV), Theorem 2.5,

$$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\},$$

(3) holds by Theorem 5, and (21) holds. Hence (21) is equivalent to (22).

By a similar argument it can be shown that (21) is equivalent to (23).

As (21) implies (22), (23) and (33), (21) implies (24). Conversely, if (24) holds, then

$$(\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \subset (\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\})((\Psi\{\mathfrak{C}_1, \mathfrak{C}_2\})r) \subset (\psi\{\mathfrak{C}_1, \mathfrak{C}_3\})r \quad \text{for} \quad r \in L\mathfrak{C}_1$$

by Theorems 1 and 2, and (22) holds by (IV), Theorem 1.13.

We may now assert that (21), (22), (23) and (24) are equivalent.

If (21) holds, $\Psi\{\mathfrak{C}_3, \mathfrak{C}_1\}$ is simple by (IV), Theorem 3.9,

$$\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_3, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_3, \mathfrak{C}_1\} \tag{34}$$

by Theorem 5, and (25) holds. If (25) holds, then $\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\}$ is simple, $\mathfrak{C}_3 \sqsubset \mathfrak{C}_2$, (34) holds, $\Psi\{\mathfrak{C}_3, \mathfrak{C}_1\}$ is simple, $\mathfrak{C}_3 \sqsubset \mathfrak{C}_1$, and (21) holds. Hence (21) is equivalent to (25). For a similar reason, (21) is equivalent to (28).

Also, if $\mathfrak{C}_1 \sqsubset \mathfrak{C}_2$, then

$$\psi\{\mathfrak{C}_2, \mathfrak{C}_1\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$$

by (IV), Theorem 2.5, (25) is equivalent to (27), and (26) is equivalent to (28). If, on the other hand, $\mathfrak{C}_2 \sqsubset \mathfrak{C}_1$, and one of (25) to (28) holds, then $\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\}$ or $\psi\{\mathfrak{C}_3, \mathfrak{C}_2\}$ is simple; by (IV), Theorem 3.9, $\Psi\{\mathfrak{C}_3, \mathfrak{C}_2\}$, $\psi\{\mathfrak{C}_3, \mathfrak{C}_2\}$, $\Psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$, and $\psi\{\mathfrak{C}_2, \mathfrak{C}_1\}$ are all simple, and (25) to (28) all hold. In both cases, the propositions (25) to (28) are equivalent.

If (21) holds, $\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\}$ takes all values of $L\mathfrak{C}_3$ by (IV), Theorem 3.9,

$$\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_2\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_3\} \tag{2}$$

by Theorem 5, and (29) holds. If (29) holds, then $\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\}$ takes all values of $L\mathfrak{C}_3$, $\mathfrak{C}_3 \sqsubset \mathfrak{C}_2$, (2) holds, $\Psi\{\mathfrak{C}_1, \mathfrak{C}_3\}$ takes all values of $L\mathfrak{C}_3$, $\mathfrak{C}_3 \sqsubset \mathfrak{C}_1$, and (21) holds. Thus (21) is equivalent to (29). For a similar reason, (21) is equivalent to (32).

Finally, if one of (29) to (32) holds, $\Psi\{\mathfrak{C}_2, \mathfrak{C}_3\}$ or $\psi\{\mathfrak{C}_2, \mathfrak{C}_3\}$ takes all values of $L\mathfrak{C}_3$, $\mathfrak{C}_3 \sqsubset \mathfrak{C}_2$, and

$$\psi\{\mathfrak{C}_2, \mathfrak{C}_3\} = \Psi\{\mathfrak{C}_2, \mathfrak{C}_3\}$$

by (IV), Theorem 2.5; hence (29) is equivalent to (31), and (30) is equivalent to (32).

This completes the proof that the propositions (21) to (32) are equivalent.

For the rest of this paper, n is a non-zero finite cardinal, $\mathfrak{C}_\nu$ ($\nu \in \mathfrak{b}\{1, n\}$) are freely generated algebras (even if $n > 3$),

(35) each of δ_1, Δ_1, λ_1 and Λ_1 is the identical representation of $L\mathfrak{C}_1$,

(36) δ_ν and Δ_ν are representations of $L\mathfrak{C}_1$ into $L\mathfrak{C}_\nu$ for $\nu \in \mathfrak{b}\{2, n\}$,

(37) λ_ν and Λ_ν are representations of $L\mathfrak{C}_\nu$ into $L\mathfrak{C}_1$ for $\nu \in \mathfrak{b}\{2, n\}$,

(38) $$\delta_{\nu+1} = \psi\{\mathfrak{C}_\nu, \mathfrak{C}_{\nu+1}\} \cdot \delta_\nu \quad \text{for} \quad \nu \in \mathfrak{a}\{1, n\},$$

(39) $$\Delta_{\nu+1} = \Psi\{\mathfrak{C}_\nu, \mathfrak{C}_{\nu+1}\} \cdot \Delta_\nu \quad \text{for} \quad \nu \in \mathfrak{a}\{1, n\},$$

(40) $$\lambda_{\nu+1} = \lambda_\nu \cdot \psi\{\mathfrak{C}_{\nu+1}, \mathfrak{C}_\nu\} \quad \text{for} \quad \nu \in \mathfrak{a}\{1, n\},$$

and

(41) $$\Lambda_{\nu+1} = \Lambda_\nu \cdot \Psi\{\mathfrak{C}_{\nu+1}, \mathfrak{C}_\nu\} \quad \text{for} \quad \nu \in \mathfrak{a}\{1, n\}.$$

It is obvious that the functions δ_ν, Δ_ν, λ_ν and Λ_ν ($\nu \in \mathfrak{b}\{1, n\}$) are uniquely determined by (35) to (41) if the algebras $\mathfrak{C}_\nu$ ($\nu \in \mathfrak{b}\{1, n\}$) are given.

Theorem 12. *The following propositions* (42) *to* (46) *are equivalent*:

(42) $$\mathfrak{C}_1 \sqsubset \mathfrak{C}_\nu \quad \textit{or} \quad \mathfrak{C}_{\nu+1} \sqsubset \mathfrak{C}_\nu \quad \textit{for} \quad \nu \in \mathfrak{a}\{1, n\}.$$

(43) $$\delta_\nu = \psi\{\mathfrak{C}_1, \mathfrak{C}_\nu\} \quad \textit{for} \quad \nu \in \mathfrak{b}\{1, n\}.$$

(44) $$\Delta_\nu = \Psi\{\mathfrak{C}_1, \mathfrak{C}_\nu\} \quad \textit{for} \quad \nu \in \mathfrak{b}\{1, n\}.$$

(45) $$\lambda_\nu = \psi\{\mathfrak{C}_\nu, \mathfrak{C}_1\} \quad \textit{for} \quad \nu \in \mathfrak{b}\{1, n\}.$$

(46) $$\Lambda_\nu = \Psi\{\mathfrak{C}_\nu, \mathfrak{C}_1\} \quad \textit{for} \quad \nu \in \mathfrak{b}\{1, n\}.$$

Proof. If $\alpha = 1$, it is obvious that the following five propositions are equivalent:

(42′) $$\mathfrak{C}_1 \sqsubset \mathfrak{C}_\nu \quad \text{or} \quad \mathfrak{C}_{\nu+1} \sqsubset \mathfrak{C}_\nu \quad \text{for} \quad \nu \in \mathfrak{a}\{1, \alpha\}.$$

(43′) $$\delta_\nu = \psi\{\mathfrak{C}_1, \mathfrak{C}_\nu\} \quad \text{for} \quad \nu \in \mathfrak{b}\{1, \alpha\}.$$

(44′) $$\Delta_\nu = \Psi\{\mathfrak{C}_1, \mathfrak{C}_\nu\} \quad \text{for} \quad \nu \in \mathfrak{b}\{1, \alpha\}.$$

(45′) $$\lambda_\nu = \psi\{\mathfrak{C}_\nu, \mathfrak{C}_1\} \quad \text{for} \quad \nu \in \mathfrak{b}\{1, \alpha\}.$$

(46′) $$\Lambda_\nu = \Psi\{\mathfrak{C}_\nu, \mathfrak{C}_1\} \quad \text{for} \quad \nu \in \mathfrak{b}\{1, \alpha\}.$$

Let m be an element of $\mathfrak{a}\{1, n\}$ such that the propositions (42′) to (46′) are equivalent if $\alpha = m$. Let one of (42′) to (46′) hold for $\alpha = m+1$,

Then one of (42′) to (46′) holds for $\alpha=m$, and one of the following propositions (47) to (51) holds:

$$\mathfrak{C}_1 \sqsubset \mathfrak{C}_m \quad \text{or} \quad \mathfrak{C}_{m+1} \sqsubset \mathfrak{C}_m . \tag{47}$$

$$\delta_{m+1} = \psi\{\mathfrak{C}_1, \mathfrak{C}_{m+1}\} . \tag{48}$$

$$\Delta_{m+1} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_{m+1}\} . \tag{49}$$

$$\lambda_{m+1} = \psi\{\mathfrak{C}_{m+1}, \mathfrak{C}_1\} . \tag{50}$$

$$\Lambda_{m+1} = \Psi\{\mathfrak{C}_{m+1}, \mathfrak{C}_1\} . \tag{51}$$

Hence all of (42′) to (46′) hold for $\alpha=m$, and all of (42′) to (46′) hold for $\alpha=m+1$ if and only if the propositions (47) to (51) all hold. Because of (38) to (41), and because (42′) to (46′) hold for $\alpha=m$, the propositions (47) to (51) are, respectively, equivalent to the following propositions (47′) to (51′):

$$\mathfrak{C}_1 \sqsubset \mathfrak{C}_m \quad \text{or} \quad \mathfrak{C}_{m+1} \sqsubset \mathfrak{C}_m . \tag{47′}$$

$$\psi\{\mathfrak{C}_m, \mathfrak{C}_{m+1}\} \cdot \psi\{\mathfrak{C}_1, \mathfrak{C}_m\} = \psi\{\mathfrak{C}_1, \mathfrak{C}_{m+1}\} . \tag{48′}$$

$$\Psi\{\mathfrak{C}_m, \mathfrak{C}_{m+1}\} \cdot \Psi\{\mathfrak{C}_1, \mathfrak{C}_m\} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_{m+1}\} . \tag{49′}$$

$$\psi\{\mathfrak{C}_m, \mathfrak{C}_1\} \cdot \psi\{\mathfrak{C}_{m+1}, \mathfrak{C}_m\} = \psi\{\mathfrak{C}_{m+1}, \mathfrak{C}_1\} . \tag{50′}$$

$$\Psi\{\mathfrak{C}_m, \mathfrak{C}_1\} \cdot \Psi\{\mathfrak{C}_{m+1}, \mathfrak{C}_m\} = \Psi\{\mathfrak{C}_{m+1}, \mathfrak{C}_1\} . \tag{51′}$$

But these five propositions are equivalent by Theorems 5 and 6. Hence the propositions (47) to (51) are equivalent. Hence these propositions all hold. Hence all of (42′) to (46′) hold for $\alpha=m+1$. Dropping the hypothesis that one of (42′) to (46′) holds for $\alpha=m+1$, we have the result that the propositions (42′) to (46′) are equivalent if $\alpha=m+1$. By induction, these propositions are equivalent if $\alpha=n$, or the propositions (42) to (46) are equivalent as asserted.

Theorem 13. *The following propositions* (52) *to* (56) *are equivalent*:

$$\mathfrak{C}_1 \sqsubset \mathfrak{C}_\nu \quad \textit{for} \quad \nu \in \mathfrak{b}\{1, n\} . \tag{52}$$

$$\delta_n \quad \textit{is simple.} \tag{53}$$

$$\Delta_n \quad \textit{is simple.} \tag{54}$$

$$\lambda_n \quad \textit{takes all values of} \quad L\mathfrak{C}_1 . \tag{55}$$

$$\Lambda_n \quad \textit{takes all values of} \quad L\mathfrak{C}_1 . \tag{56}$$

Proof. If $\alpha=1$, it is obvious that the following five propositions are equivalent:

(52′) $\mathfrak{C}_1 \sqsubset \mathfrak{C}_\nu$ for $\nu \in \mathfrak{b}\{1, \alpha\}$.

(53′) δ_α is simple.

(54′) Δ_α is simple.

(55′) λ_α takes all values of $L\mathfrak{C}_1$.

(56′) Λ_α takes all values of $L\mathfrak{C}_1$.

Let m be an element of $\mathfrak{a}\{1, n\}$ such that the propositions (52′) to (56′) are equivalent if $\alpha = m$. If one of (53′) to (56′) holds for $\alpha = m+1$ then, by (38) or (39) or (40) or (41), this proposition holds for $\alpha = m$. Let one of (52′) to (56′) hold for $\alpha = m+1$. Then one of (52′) to (56′) holds for $\alpha = m$. Hence all of (52′) to (56′) hold for $\alpha = m$. Because (52′) holds for $\alpha = m$,

$$\mathfrak{C}_1 \sqsubset \mathfrak{C}_\nu \quad \text{or} \quad \mathfrak{C}_{\nu+1} \sqsubset \mathfrak{C}_\nu \quad \text{for} \quad \nu \in \mathfrak{a}\{1, m+1\} .$$

By Theorem 12, with n replaced by $m+1$,

$$\delta_{m+1} = \psi\{\mathfrak{C}_1, \mathfrak{C}_{m+1}\} ,$$
$$\Delta_{m+1} = \Psi\{\mathfrak{C}_1, \mathfrak{C}_{m+1}\} ,$$
$$\lambda_{m+1} = \psi\{\mathfrak{C}_{m+1}, \mathfrak{C}_1\} ,$$

and

$$\Lambda_{m+1} = \Psi\{\mathfrak{C}_{m+1}, \mathfrak{C}_1\} .$$

Therefore and because of (IV), Theorem 3.9, the following five propositions are equivalent:

(57) $\mathfrak{C}_1 \sqsubset \mathfrak{C}_{m+1}$.

(58) δ_{m+1} is simple.

(59) Δ_{m+1} is simple.

(60) λ_{m+1} takes all values of $L\mathfrak{C}_1$.

(61) Λ_{m+1} takes all values of $L\mathfrak{C}_1$.

Because one of (52′) to (56′) holds for $\alpha = m+1$, one of (57) to (61) holds. Hence all of (57) to (61) hold. Hence all of (52′) to (56′) hold for $\alpha = m+1$. Dropping the hypothesis that one of (52′) to (56′) holds for $\alpha = m+1$, we have the result that the propositions (52′) to (56′) are equivalent if $\alpha = m+1$. By induction, these propositions are equivalent if $\alpha = n$.

Theorem 14. *Let* $\mathfrak{C}_n = \mathfrak{C}_1$. *Then each of the propositions* (52) *to* (56) (see Theorem 13) *is equivalent to each of the following propositions* (62) *to* (65):

(62) δ_n *is the identical representation of* $L\mathfrak{C}_1$.

(63) Δ_n *is the identical representation of* $L\mathfrak{C}_1$.

(64) λ_n *is the identical representation of* $L\mathfrak{C}_1$.

(65) Λ_n *is the identical representation of* $L\mathfrak{C}_1$.

Proof. If (52) holds then, by Theorem 12,

$$\delta_n = \psi\{\mathfrak{C}_1, \mathfrak{C}_n\},$$
$$\Delta_n = \Psi\{\mathfrak{C}_1, \mathfrak{C}_n\},$$
$$\lambda_n = \psi\{\mathfrak{C}_n, \mathfrak{C}_1\},$$
$$\Lambda_n = \Psi\{\mathfrak{C}_n, \mathfrak{C}_1\},$$

and (62) to (65) hold by (IV), Theorems 1.1 and 1.11. Conversely, if (62), (63), (64) or (65) holds then (53), (54), (55) or (56) holds, respectively.

Department of Mathematics,
University of Alberta.

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