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BP is the Brown-Peterson spectrum (for some prime $p$) and $BP_*X = \pi_*(BP \wedge X)$ is the Brown-Peterson homology of the CW spectrum (or complex) $X$. $BP_*X$ is a left module over the coefficient ring $BP_* \cong \mathbb{Z}_p[v_1, v_2, \ldots]$ and a left comodule over the coalgebra $BP_*BP$. A now classical result is that the stable Hurewicz homomorphism $\pi_*X \to H_*(X; \mathbb{Z})$ is an isomorphism modulo torsion. In our context, we restate this as: the Hurewicz homomorphism $h_0(X): \pi_*(BP \wedge X) \to H_*(BP \wedge X; \mathbb{Q})$ has as its kernel the $p$-torsion subgroup of $BP_*X$. This is a prototype of our results.

Instead of restricting our attention to $BP_*X$, it is convenient to study abstract $BP_*BP$-comodules $(M, \psi)$, $\psi: M \to BP_*BP \otimes_{BP_*BP} M$. A priori, $M$ is a left $BP_*$-module. As such, it has a richer potential for torsion than mere $p$-torsion. For any polynomial generator $v_n$ of $BP_*$ (by convention $v_0 = p$), we say that an element $y \in M$ is $v_n$-torsion if $v_n^s y = 0$ for some exponent $s$. If all elements of $M$ are $v_n$-torsion ones, we say that $M$ is a $v_n$-torsion module. If no non-zero element of $M$ is $v_n$-torsion, we say that $M$ is $v_n$-torsion free. Being a $BP_*BP$-comodule severely constrains the $BP_*$-module structure of $M$.

**Theorem 0.1.** Let $M$ be a $BP_*BP$-comodule. If $y \in M$ is a $v_n$-torsion element, then it is a $v_{n-1}$-torsion element. Consequently, if $M$ is a $v_n$-torsion module, then it is a $v_{n-1}$-torsion module. Or: if $M$ is $v_n$-torsion free, it is $v_{n-1}$-torsion free (Lemma 2.3 and Proposition 2.5).

The primitive elements of a $BP_*BP$-comodule $M$ are those elements $a$ for which $\psi(a) = 1 \otimes a$ under $M$'s coproduct $\psi: M \to BP_*BP \otimes_{BP_*BP} M$. We find that some qualitative properties of $BP_*BP$-comodules are determined by these primitives.

**Theorem 0.2.** Let $M$ be an associative $BP_*BP$-comodule. If all the primitives of $M$ are $v_n$-torsion, then $M$ itself is a $v_n$-torsion module. Or: if none of the

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non-zero primitives of \( M \) is \( v_n \)-torsion, then \( M \) is \( v_n \)-torsion free (Proposition 2.7).

We may localize a \( BP_*BP \)-comodule \( M \) with respect to \( v_n \) to form \( v_n^{-1}M \). Generally, the resulting \( BP_*BP \)-module is not a \( BP_*BP \)-comodule; we characterize when it is.

**Theorem 0.3.** Let \( M \) be an associative \( BP_*BP \)-comodule. \( M \) is a \( v_{n-1} \)-torsion module if and only if \( v_n^{-1}M \) is an associative \( BP_*BP \)-comodule. (Proposition 2.9) (The “only if” part is due to Miller and Ravenel [11].)

There is no dearth of homology theories associated to \( BP \), but some of the most interesting are the periodic homology theories \( E(n)_* \). The coefficients of \( E(n)_*( ) \) are \( E(n)_* \cong \mathbb{Z}_p[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}] \); the representing spectrum is \( E(n) \). \( E(0)_*X \) is the familiar rational homology of \( X \). \( E(1)_*X \) is a summand of localized (at \( p \)) complex \( K \)-homology of \( X \). There is a Boardman map \( BP \to E(n) \wedge BP \) which induces a Hurewicz homomorphism \( h_n(X) : \pi_*(BP \wedge X) \to E(n)_*(BP \wedge X) \). When \( n=1 \), this is properly called the Hattori-Stong homomorphism. We prove:

**Theorem 0.4.** Let \( X \) be a CW spectrum. The kernel of the Hurewicz homomorphism \( h_n(X) : \pi_*(BP \wedge X) \to E(n)_*(BP \wedge X) \) is the \( v_n \)-torsion subgroup of \( BP_*X \). (Theorem 4.10)

We can localize \( BP_*X \) to form \( v_n^{-1}BP_*X \). We prove:

**Theorem 0.5.** Let \( X \) be a CW spectrum. \( v_n^{-1}BP_*X = 0 \) if and only if \( E(n)_*X = 0 \). Hence \( v_n^{-1}BP_*X \) and \( E(n)_*X \) have the same acyclic spaces. (Corollary 4.11)

During a provocative talk at the Northwestern conference of March 1977, Douglas Ravenel shared his insight that Theorem 0.5 should hold. Our attempts to substantiate his intuition led to this paper. We thank Ravenel for making the manuscript [12] of his Northwestern talk available to us, for his stimulating correspondence, and for his kind hospitality.

An obvious generalization presents itself. Let \( J = \{ q_0, q_1, \ldots, q_{n-1} \} \) be an invariant regular sequence of elements of \( BP_* \). There is a left \( BP \)-module spectrum \( BPJ \) whose homotopy is \( BPJ_* \cong BP_*/(q_0, \ldots, q_{n-1}) \). When \( J \) is empty, \( BPJ \) is just \( BP \). As we do prove our results for \( BPJ_*BPJ \)-comodules, we must list properties of such comodules (§1), prove some simple change-of-ring (\( BPJ_* \) to \( BP_* \)) lemmas in §3, and sketch some proofs of the properties of \( BPJ \) (§5). A reader who is interested only in \( BP_*BP \)-comodules may neglect the “\( J \)” in the \( BPJ \) notation and read only the even-numbered sections: §2, “\( v_n \)-Torsion Properties,” and §4, “Hurewicz Homomorphisms.”
1. \( BPJ_*BPJ \)-comodules

Let \( J=\{q_0, \ldots, q_{n-1}\} \) be an invariant regular sequence of elements of \( BP_*=\pi_*BP \). \( J \) and \( n \) will remain fixed throughout this section. There is an associative left \( BP \)-module spectrum \( BPJ \) which has homotopy \( \pi_*BPJ=BPJ_*\cong BP_*/\langle q_0, \ldots, q_{n-1} \rangle \). A map of spectra \( j: BP \to BPJ \) induces an epimorphism in homotopy. Let \( \varphi_0: BP \wedge BP \to BP \) give \( BP \) its ring spectrum structure and let \( \varphi=\varphi_0: BP \wedge BPJ \to BPJ \) give \( BPJ \) its \( BP \)-module structure. Then \( \varphi(1 \wedge j)=jo\varphi_0 \) and \( \varphi(\varphi_0 \wedge 1)=\varphi(1 \wedge \varphi) \).

When \( J \) is empty, \( BPJ \) is just \( BP \). For \( J=\{p, v_1, \ldots, v_{n-1}\} \), \( BPJ \) is known as \( P(n) \) [6; 16; 17]. The following properties have become classical for \( BP_*BP \)-comodules [7; 8; 9]. Würgler has established these properties for \( P(n)_*P(n) \)-comodules (\( p \) odd) [16]. We defer some of our exposition and our proof sketches until §5.

Let \( i: S^0 \to BP \) be the unit map for the Brown-Peterson spectrum. There are pairings \( \mu: BPJ \wedge BPJ \to BPJ \) which make \( BPJ \) into a quasi-associative ring spectrum with unit \( t_*=jo: S^0 \to BPJ \). Here \( \mu(j \wedge 1)=\varphi \) and \( \mu(j \wedge j)=jo\varphi_0 \). (See Proposition 5.5.) These pairings are not generally unique; the (co) multiplicative structures which follow can depend on the particular (fixed) choice of \( \mu \).

Let \( c: BPJ \wedge BPJ \to BPJ \wedge BPJ \) be the conjugation. \( BPJ \wedge BPJ \) is a free left \( BP/J \)-module with basis given by symbols \( z^{E,A} \) of dimension \( \sum_i e_i(2^i-2)+\sum_j a_j(\dim(q_j)+1) \). Here \( E=(e_1, e_2, \ldots) \) is a finite sequence of non-negative integers and \( A=(a_0, \ldots, a_{n-1}) \) is an \( n \)-tuple of zeros and ones. \( BPJ \wedge BPJ \) is an associative left \( BP_*BP \approx BP_*[t_1, t_2, \ldots] \)-module with structure given by the formula:

\[
(1.1) \quad t^E z^{F,A} = z^{E+F,A} \quad \text{for} \quad t^E = t_1^{e_1} t_2^{e_2} \cdots \in BP_*BP.
\]

In particular, \( (j \wedge j)_* (t^E) = z^{E,0} \). The \( c(z^{E,A}) \) give a basis for \( BPJ_*BPJ \) as a free right \( BPJ_* \)-module. Because of this right freeness, there is a natural isomorphism \( BPJ_* (BPJ \wedge X) \cong BPJ_*BPJ \otimes_{BPJ_*BPJ} BPJ_*X \) for any \( CW \) spectrum \( X \). The map \( 1 \wedge t_* \wedge 1: BPJ \wedge S^0 \wedge X \to BPJ \wedge BPJ \wedge X \) induces a coproduct:

\[
\psi_X: BPJ_*X \to BPJ_* (BPJ \wedge X) \cong BPJ_*BPJ \otimes_{BPJ_*BPJ} BPJ_*X.
\]

We define natural homomorphisms \( s_{E,A}: BPJ_*X \to BPJ_*X \) by the following recipe

\[
(1.2) \quad \psi_X(x) = \sum_{E,A} c(z^{E,A}) \otimes s_{E,A}(x).
\]

We call these \( s_{E,A} \) elementary \( BPJ \) operations. When \( J \) is empty and \( BPJ \) is \( BP \), the \( s_{E,0} \) coincide with \( BP \) operations \( r_E \) [2]. The elementary \( BPJ \) operations satisfy the following properties.
(1.3) Under the natural map $j_*: BP_*X \rightarrow BPJ_*X$, $s_{E,A}j_*(x) = j_*r_E(x)$.

(1.4) The elementary $BPJ$ operations generate all the $BPJ$ operations in that any $BPJ$ operation $\theta$ can be written uniquely as a (possibly infinite) sum

$$\theta = \sum_{E,A} q_{E,A} s_{E,A}$$

(See 5.12).

(1.5) The dimension of $s_{E,A}$ is $d = \sum_i e_i(2p^i - 2) + \sum_j a_j (\dim (g_j) + 1)$ where $E = (e_1, e_2, \cdots)$ and $A = (a_0, a_1, \cdots)$. That is: if $x \in BP_*X$, then $s_{E,A}(x) \in BP_{n-d}X$. (This follows from (1.2).)

(1.6) For any element $x \in BPJ_*X$, $s_{E,A}(x)$ is zero except for finitely many indices $E$ and $A$. (The proof is trivial.)

(1.7) There is a Cartan formula. If $y \in BP_*X$ and $x \in BPJ_*X$, then

$$s_{E,A}(yx) = \sum_{F,G,B} s_F(y) s_{G,A}(x).$$

(This follows from (1.1).)

(1.8) There are coefficients $q_A \in BPJ_*$ such that

$$s_{0,0}(x) = x + \sum_{A \neq 0} q_A s_{0,A}(x)$$

for any $x \in BPJ_*X$ and for any $X$. (See Remark 5.13.)

(1.9) For the elementary $BPJ$ operations $s_{E,A}$ and $s_{F,B}$, there are coefficients $q_{G,C} = q_{G,C}(E, A; F, B) \in BPJ_*$ such that

$$s_{E,A}(s_{F,B}(x)) = \sum_{G,C} q_{G,C} s_{G,C}(x)$$

for any $x \in BPJ_*X$ and for any $X$. Furthermore, the dimension of $s_{G,C}$ is not less than the sum of the dimensions of $s_{E,A}$ and $s_{F,B}$. (See Remark 5.14.)

Let $M$ be a left $BPJ_*$-module. $M$ is defined to be a $BPJ_*BPJ$-comodule if the elementary $BPJ$ operations act on $M$ satisfying (1.5) through (1.8). The $BPJ_*BPJ$ coaction of $M$ is given by $\psi_M: M \rightarrow BPJ_*BPJ \otimes_{BPJ_*} M$ with

$$\psi_M(x) = \sum_{A} c(x^{E,A}) \otimes s_{E,A}(x).$$

If (1.9) is also satisfied, we call $(M, \psi_M)$ an associative $BPJ_*BPJ$-comodule. The following remark follows from (1.2).

Remark 1.10. Let $M$ be a $BPJ_*BPJ$-comodule and let $x \in M$. The following are equivalent statements.
(i) $\psi_M(x) = 1 \otimes x$

(ii) $s_{E,A}(x) = 0$ if $(E, A) \neq (0, 0)$ and $s_{E,A}(x) = x$.

If $x$ satisfies these equivalent statements, we call $x$ primitive. Let $PM$ be the subgroup of primitive elements of $M$.

Define the primitive degree $d(x)$ of an element $x$ of a $BPJ_*BPJ$-comodule $M$ as follows. If there is an elementary operation $s_{E,A}$ of dimension $m$ such that $s_{E,A}(x) \neq 0$, then $d(x) \geq m$. Define $d(0) = 0$. By (1.6), $d(x) \geq 0$ is always finite. We record two observations.

(1.11) If $x \in M$, $d(x) = 0$ if and only if $x$ is primitive. (See Remark 1.10.)

(1.12) Let $M$ be an associative $BPJ_*BPJ$-comodule and let $s_{E,A}$ be an elementary $BPJ$ operation of dimension $m$. For $x \in M$, $d(s_{E,A}(x)) \leq \text{maximum} \{d(x) - m, 0\}$.

(See (1.9).)

**Lemma 1.13.** Let $M$ be an associative or a connective $BPJ_*BPJ$-comodule. Then $M$ coincides with the union of all of its finitely-generated subcomodules.

**Proof.** This follows routinely using (1.6) and (1.9) or (1.5).

**Lemma 1.14.** Let $M$ be an associative $BPJ_*BPJ$-comodule. There is an epimorphism of associative $BPJ_*BPJ$-comodules $f : F \to M$ with $F$ $BPJ_*$-free. $F$ may be chosen to be finitely-generated in the case that $M$ is finitely-generated.

**Proof.** Follow the proof of Proposition 2.4 of [9].

**Lemma 1.15.** Every associative $BPJ_*BPJ$-comodule $M$ is a direct limit of finitely-presented associative comodules.

**Proof.** See the proof of Lemma 2.11 of [11] or see [17].

Recall that $I_0 = (p)$, $I_m = (p, v_1, \ldots, v_{m-1})$, and $I_\infty = (p, v_1, v_2, \ldots)$ are the non-trivial prime ideals of $BP_*$ invariant under the $BP_*BP$-coaction [7;5]. By Landweber [10], the ideal-theoretic radical of $(q_0, \ldots, q_{s-1})$ is $I_s$.

**Theorem 1.16** (Filtration Theorem). Let $J = \{q_0, \ldots, q_{s-1}\}$ be an invariant regular sequence in $BP_*$ of length $n$. Let $M$ be a finitely-presented, associative $BPJ_*BPJ$-comodule. Then $M$ has a finite filtration

$$M = M_s \supset M_{s-1} \supset \cdots \supset M_1 \supset M_0 = \{0\}$$

by finitely-presented, associative $BPJ_*BPJ$-subcomodules. As a $BPJ_*BPJ$-comodule, each quotient $M_i/M_{i-1}$, $1 \leq i \leq s$, is isomorphic to some suspension of some $BP_*/I_n$, $n \leq k$.

**Proof.** Follow the patterns of the proofs of Theorem 3.3 of [8] and
Theorem 3.4 of [17].

2. \(v_n\)-torsion properties

Again, \(J\) will be a fixed invariant regular sequence and \(BPJ\) will be the resulting spectrum. \(BPJ_*BPJ\)-comodules are \(BP_*\)-modules through the epimorphism \(BP_* \to BPJ_*\). This section studies certain \(BP_*\)-module properties of \(BPJ_*BPJ\)-comodules which are independent of the particular sequence \(J\). (Here, we use the letter "\(n\)" as a variable and not as the length of the fixed sequence \(J\).)

Our study begins with a lemma which descends directly from the "Ballentine Lemma" of Smith (and Stong) [14]. For the exponent sequence \(E = (e_1, e_2, \ldots)\), let \(|E| = \sum; e_i(2p^{i-2})\). Let \(\Delta = (0, \cdots, 0, 1, 0, \cdots)\) with the single "1" in the \(k\)-th position. Exponent sequences are added (or multiplied by positive integers) term-wise.

Lemma 2.1. Let \(E\) be an exponent sequence with \(|E| > 2kp'(p^n - p^m)\), \(n > m\), \(s \geq 0\), and \(k \geq 1\). Then

\[
re(v_n^{kp'}) = \begin{cases} 
 v_m^{kp} \text{ modulo } I_m^{s+1} & \text{if } E = kp^{s+m}\Delta_{n-m} \\
0 \text{ modulo } I_m^{s+1} & \text{otherwise.}
\end{cases}
\]

Proof. The \(s=0\) case is Corollary 1.8 of [5]. The general case follows by induction on \(s\) using the Cartan formula and the fact that \(p \in I_m^s\).

Lemma 2.2. Let \(M\) be a \(BPJ_*BPJ\)-comodule and let \(s_{E,A}\) be any elementary \(BPJ\) operation. If an element \(x \in M\) is \(v_m\)-torsion for all \(m\) satisfying \(0 \leq m \leq n\), then \(s_{E,A}(x)\) is also \(v_m\)-torsion for such \(m\), \(0 \leq m \leq n\).

Proof. Assume inductively that \(s_{F,B}(x)\) is \(v_k\)-torsion for every elementary \(BPJ\) operation \(s_{F,B}\) and for all \(k\) satisfying \(0 \leq k < m\). (The initial \(m=0\) case is the same as the inductive step.) Recalling (1.6), there is a non-negative integer \(s=s(x, m)\) such that \(v_m^{p}\cdot x = 0\) and \(I_m^{s+1} s_{F,B}(x) = 0\) for all elementary \(BPJ\) operations \(s_{F,B}\). By (1.7) and Lemma 2.1,

\[
0 = s_{E,A}(v_m^{p}\cdot x) = v_m^{p}s_{E,A}(x)
\]

and so \(s_{E,A}(x)\) is \(v_m\)-torsion.

Lemma 2.3. Let \(M\) be a \(BPJ_*BPJ\)-comodule. If an element \(x \in M\) is \(v_n\)-torsion, then it is \(v_m\)-torsion for each \(m\) satisfying \(0 \leq m \leq n\).

Proof. Our proof is by double induction. The first induction (on \(m\)) assumes that if \(x\) is \(v_n\)-torsion, then \(x\) is \(v_k\)-torsion for \(k < m\). For such an \(x\) and for any elementary \(BPJ\) operation \(s_{E,A}\), \(s_{E,A}(x)\) is \(v_k\)-torsion for all \(k < m\) by Lemma
2.2. We may choose an \( s > 0 \) such that \( v_n^s x = 0 \) and \( I_{s+1} s_{E, A}(x) = 0 \) for all \( s_{E, A} \). Suppose \( s_{H, A} \) is an elementary \( BPJ \)-operation of dimension \( d(x) \). (See (1.11) and (1.12).) Let \( G(p^{m+1} \Delta_n) \). By (1.7) and Lemma 2.1,

\[
0 = s_{G+H, A}(v_n^s x) = \sum_{\beta, s_{H, A}} r_{\beta}(v_n^s) s_{E, A}(x) = r_{G}(v_n^s) s_{H, A}(x)
\]

If \( d(x) = 0 \), this computation shows that \( x \) is \( v_m \)-torsion. If \( d(x) > 0 \), it shows that \( d(v_m^s x) < d(x) \). By a second induction on the primitive degree \( d() \), \( v_m^s x \) is assumed to be \( v_m \)-torsion. Hence \( x \) is \( v_m \)-torsion as desired.

**Corollary 2.4.** Let \( M \) be a \( BPJ^*BPJ \)-comodule. If \( x \in M \) is \( v_n \)-torsion, then \( s_{E, A}(x) \) is \( v_m \)-torsion for all \( m \) satisfying \( 0 \leq m \leq n \) and for all elementary \( BPJ \) operations \( s_{E, A} \).

**Proof.** Lemmas 2.2 and 2.3.

Recall that a \( BP^* \)-module \( M \) (e.g. a \( BPJ^*BPJ \)-comodule) is \( v_n \)-torsion if every element \( x \in M \) is \( v_n \)-torsion. \( M \) is \( v_n \)-torsion free if no non-zero element is \( v_n \)-torsion. The following proposition follows immediately from Lemma 2.3.

**Proposition 2.5.** Let \( M \) be a \( BPJ^*BPJ \)-comodule. If \( M \) is \( v_n \)-torsion, then it is \( v_{n-1} \)-torsion. At the other extreme: if \( M \) is \( v_n \)-torsion free, then it is \( v_{n+1} \)-torsion free.

Let \( Y \) be an associative \( BP \)-module spectrum. We can form a new spectrum \( v_n^{-1} Y \) which is defined to be the mapping telescope \( S^{\geq 2(n-1)} Y \) of the map

\[
S^{2(n-1)} Y \xrightarrow{v_n^{-1}} BP \wedge Y \rightarrow Y.
\]

Note that \( v_n^{-1} Y \) is a \( BP \)-module spectrum which is possibly non-associative. We have a canonical isomorphism \( v_n^{-1}(Y_{BP}(X)) \rightarrow (v_n^{-1} Y)_{BP}(X) \).

**Corollary 2.6.** Let \( X \) be a CW spectrum. If \( (v_n^{-1} BPJ)_{BP}(X) = 0 \), then \( (v_n^{-1} BPJ)_{BP}(X) = 0 \).

**Proposition 2.7.** Let \( M \) be a \( BPJ^*BPJ \)-comodule which is either associative or connective.

(i) If all the primitive elements of \( M \) are \( v_n \)-torsion, then \( M \) is a \( v_n \)-torsion module.

(ii) If none of the non-zero primitive elements of \( M \) is \( v_n \)-torsion, then \( M \) is a \( v_n \)-torsion free module.

**Proof.** To prove (i), assume \( M \) is an associative comodule with \( v_n \)-torsion primitives. Assume inductively that \( M \) is a \( v_k \)-torsion module for \( k < m \leq n \). If \( y \in M \) with \( d(y) = 0 \) (see (1.11)), \( y \) is \( v_k \)-torsion for all \( k \leq n \) by our hypothesis and by Lemma 2.3. Let \( x \in M \) with \( d(x) > 0 \). Let \( s_{E, A} \) be any positive dimen-
sional elementary $BPJ$ operation. Since $M$ is associative, $d(s_{E,A}(x))<d(x)$ by (1.12). By a subsidiary induction on $d(y)$, we may assume that such $s_{E,A}(x)$ are $v_n$-torsion. Hence there is an $s \geq 0$ such that $I_{m+k}^s x = 0$ and $I_{m+k}^{s+1} s_{E,A}(x) = 0$. Note that (1.8) implies that $s_{0,0}(x) = x + z$ with $I_{m+1}^{s+1} z = 0$. For any positive dimensional $s_{E,A}$,

$$s_{E,A}(v_m^{s'} x) = \sum_{F + G = H} r_F(v_m^{s'}) s_{G,A}(x) = 0.$$  

So $v_m^{s'} x$ is primitive and hence $v_m$-torsion. Thus $x$ itself is $v_m$-torsion. This completes both the auxiliary and the original inductions.

We turn to (ii). Let $M$ be an associative comodule with no non-zero $v_n$-torsion primitives. We assume inductively that all non-zero elements $y \in M$ with $d(y) < l$ are not $v_n$-torsion. If non-primitive $x \in M$ has $d(x) = l$, there is an elementary $BPJ$ operation $s_{E,A}$ with $s_{E,A}(x) = 0$ and $d(s_{E,A}(x)) < d(x)$ (1.12). So $s_{E,A}(x)$ is not $v_n$-torsion. By Corollary 2.4, $x$ fails to be $v_n$-torsion also. Thus $M$ is $v_n$-torsion free.

Finally, assume $M$ is connective. With a few minor modifications, the above proofs of (i) and (ii) work if we replace the primitive degree $d(x)$ of the element $x$ by $x$'s dimension $|x|$.

A $BP_\ast$-module (e.g. a $BPJ_\ast BPJ$-comodule) $M$ is said to be $v_n$-divisible if multiplication by $v_n$ on $M$ is epic.

**Proposition 2.8.** If an associative $BPJ_\ast BPJ$-comodule $M$ is $v_n$-divisible, then it is $v_{n-1}$-torsion. (Cf. [11, Proposition 3.5].)

Proof. Assume inductively that $M$ is $v_k$-torsion for $k < m < n$. Let $0 \neq x \in M$ be a primitive element. By Proposition 2.7, it will suffice to show that $x$ is $v_m$-torsion. There is an integer $t \geq 0$ such that $I_{m+t}^s x = 0$. Note that this implies that $I_{m+t}^s s_{E,A}(x) = 0$ (1.7). By the divisibility of $M$, there is an element $y \in M$ with $v_m^{s'} y = x$. In preparation for a second induction, we do a curious computation. For any integer $u \geq 0$, our (primary) inductive hypothesis gives us an integer $s \geq t$ such $I_{m+s}^{s+t} s_{E,A}(v_m^{s'} y) = 0$ for all elementary $BPJ$ operations $s_{E,A}$. Suppose $d(v_m^{s'} y) = l$ and let $s_{H,A}$ be any elementary $BPJ$ operation of that maximal dimension $l$. Let $G = p^m + \Delta_{n-m}$. Using (1.7) and Lemma 2.1 repeatedly, we compute:

$$0 = r_{G+H}(v_m^{s'-s'} v_m^{s'} s_{0,A}(x)) = r_{G+H}(v_m^{s'-s'} s_{0,A}(v_m^{s'} x)) = s_{G+H,A}(v_m^{s'-s'} v_m^{s'} v_m^{s'} x) = s_{0,A}(v_m^{s'} y) = r_{0,A}(v_m^{s'} y) = s_{H,A}(v_m^{s'} y) = v_m^{s'} s_{H,A}(v_m^{s'} y) = s_{H,A}(v_m^{s'} y).$$  

If $d(v_m^{s'} y) = 0$, this shows that $v_m^{s'} y$ and hence $y$ and $x$ are $v_m$-torsion. If $d(v_m^{s'} y) > 0$, the computation shows that $d(v_m^{s'} (s'-s+u) y) < d(v_m^{s'} y)$. This indicates a proof that $x$ is $v_m$-torsion by induction on the primitive degrees of $x$. 


TORSION IN BROWN-PETTERSON HOMOLOGY

The "only if" part of the following proposition is due to Miller and Ravenel [11, Lemma 3.2].

**Proposition 2.9.** Let $M$ be an associative $BP\star BP$-comodule. Then $M$ is $v_{n-1}$-torsion if and only if the localization $v_n^{-1}M$ is an associative $BP\star BP$-comodule.

Proof. By Lemma 1.13, we may assume $M$ is finitely-generated. Assuming $M$ is $v_{n-1}$-torsion, there is an $s \geq 0$ such that $I_s^{s+1}M = 0$ (Proposition 2.5). By Lemma 2.1, multiplication by $v_s$ on $M$ is a comodule map. Hence the localization $v_n^{-1}M$, considered as the direct limit of the system

$$M \xrightarrow{v_n^{-1}} M \xrightarrow{v_n^{-1}} M \xrightarrow{v_n^{-1}} \cdots,$$

is an associative $BP\star BP$-comodule. Furthermore, $M \xrightarrow{v_n^{-1}} M$ is a comodule map.

Now assume that $v_n^{-1}M$ is an associative comodule. As a $v_n$-divisible associative comodule, $v_n^{-1}M$ is $v_{n-1}$-torsion by Proposition 2.8. Thus $v_k^{-1}v_n^{-1}M = 0$ for each $k$ satisfying $0 \leq k < n$ by Proposition 2.5. Assume inductively that $M$ is $v_{k-1}$-torsion. By the "only if" part of this proposition, $v_k^{-1}M$ is an associative comodule. Since $v_n^{-1}v_k^{-1}M = v_k^{-1}v_n^{-1}M = 0$, the associative comodule $v_k^{-1}M$ is $v_n$-torsion. By Proposition 2.5, $v_k^{-1}M$ is $v_k$-torsion and thus is zero. So $M$ is $v_n$-torsion.

3. More $BP\star$-module properties of $BP\star BP$-comodules

This section develops some algebraic preliminaries to Section 4. All of the results here are well-known or trivial when $BP\star = BP$. Our point of departure is the $BP\star BP$ version of Landweber’s Filtration Theorem (1.16). A unifying technique is the following.

**Lemma 3.1.** Let $j: \Lambda \to \Gamma$ be a homomorphism of commutative rings with unit. Let $A$ be a right $\Lambda$-module and let $B$ and $C$ be two-sided $\Gamma$-modules such that there is an isomorphism $B \otimes \Gamma C \cong C \otimes \Gamma B$ of left $\Gamma$-modules. Further assume that $B$ is $\Gamma$-flat. If $\text{Tor}^\Lambda_1(A, C) = 0$, then $\text{Tor}^\Gamma_1(A \otimes \Lambda B, C) = 0$.

Proof. If either $B$ or $C$ is $\Gamma$-flat, we have a Kunneth exact sequence

$$\text{Tor}^\Gamma_2(A \otimes \Lambda B, C) \to \text{Tor}^\Lambda_1(A, B) \otimes \Gamma C \to \text{Tor}^\Lambda_1(A, B \otimes \Gamma C) \to \text{Tor}^\Gamma_1(A \otimes \Lambda B, C) \to 0.$$

When $B$ is $\Gamma$-flat (and the roles of $B$ and $C$ are interchanged), this gives an isomorphism $\text{Tor}^\Lambda_1(A, C) \otimes \Gamma B \cong \text{Tor}^\Lambda_1(A, C \otimes \Gamma B)$. The lemma now follows immediately from the isomorphism $B \otimes \Gamma C \cong C \otimes \Gamma B$. 
Throughout this section, let \( J = \{ q_0, \ldots, q_{n-1} \} \) be an invariant regular sequence of length \( n \). Let \( \Lambda = BP_* \) and \( \Gamma = BPJ_* \).

For any commutative ring \( R \) and any \( R \)-module \( M \), we have two dimensions of concern. The projective dimension, \( \text{h. dim}_R M \), is the greatest integer \( k \) such that \( \text{Ext}^k_R(M, N) \neq 0 \) for some \( R \)-module \( N \). The weak dimension, \( \text{w. dim}_R M \), is the greatest integer \( k \) such that \( \text{Tor}_k^R(M, N) \neq 0 \) for some \( R \)-module \( N \). Of course, \( \text{w. dim}_R M \leq \text{h. dim}_R M \).

**Lemma 3.2.** The projective dimension of \( \Gamma \) as a \( \Lambda \)-module is \( n \).

**Proof.** Let \( J_m = \{ q_0, \ldots, q_m \} \subseteq J, m < n \). For \( m < n \), there are short exact sequences of \( \Lambda \)-modules
\[
0 \rightarrow BPJ_* \rightarrow BPJ_{m+1} \rightarrow 0
\]
showing inductively that \( \text{h. dim}_\Lambda BPJ_m^{\ast} \leq m \). The ideal \( \langle q_0, \ldots, q_{n-1} \rangle \) has radical \( \langle q_0, \ldots, q_{n-1} \rangle \) [10, Proposition 2.5]; so \( \Gamma = BPJ_* \) is a \( v_{n-1} \)-torsion module. By the "ideal annihilator estimate" [6, Proposition 4.6], \( \text{h. dim}_\Gamma \Gamma \geq n \).

**Corollary 3.3.** \( \text{Tor}_1^\Lambda(Z(p)[v_1, \ldots, v_m], \Gamma) = 0 \) for all \( m > n \).

**Proof.** Apply Landweber's Theorem 4.2 of [9] to the connective, associative \( BP_* BP \)-comodule \( BPJ_* \Gamma \).

**Lemma 3.4.** For any \( m \) satisfying \( n \leq m \leq n + k + 1 \),
\[
\text{Tor}_1^\Gamma(Z(p)[v_1, \ldots, v_{n+k}], BPJ_* BPJ, BP_*/I_m) = 0.
\]

**Proof.** Recall that \( BP_* BP \) is \( \Gamma \)-free. \( \text{Tor}_1^\Lambda(Z(p)[v_1, \ldots, v_{n+k}], BP_*/I_m) = 0 \) for \( m \leq n + k + 1 \). For \( n \leq m \), \( BP_*/I_m \) is a \( \Gamma \)-module. Apply Lemma 3.1.

Recall that \( E(m)_* = Z(p)[v_1, \ldots, v_{m-1}, v_m, v_m^{-1}] \).

**Lemma 3.5.** Let \( M \) be an associative \( BP_* BP \)-comodule. Let \( B \) be
(i) \( \Gamma \), (ii) \( BP_* BPJ \), or (iii) \( BP_*(v_m^{-1}BPJ) \). Then \( \text{Tor}_1^\Gamma(E(m)_* \otimes_\Lambda B, M) = 0 \).

**Proof.** Both \( \Gamma \) and \( BP_* BP \) are \( \Gamma \)-free. As a direct limit of copies of \( BP_* BPJ, BP_*(v_m^{-1}BPJ) \) is \( \Gamma \)-flat. By Landweber's Exact Functor Theorem [9], \( \text{Tor}_1^\Lambda(E(m)_* \otimes_\Lambda B, BP_*/I_k) = 0, k \geq -1 \). If \( k \geq n, BP_*/I_k \) is a \( \Gamma \)-module and so Lemma 3.1 implies that \( \text{Tor}_1^\Gamma(E(m)_* \otimes_\Lambda B, BP_*/I_k) = 0, k \geq n \). If \( M \) is finitely presented, \( M \) has a finite filtration whose subquotients are isomorphic to suspended copies of \( BP_*/I_k, k \geq n \). If \( M \) is finitely presented, \( M \) has a finite filtration whose subquotients are isomorphic to suspended copies of \( BP_*/I_k, k \geq n \). If \( M \) is finitely presented, \( M \) has a finite filtration whose subquotients are isomorphic to suspended copies of \( BP_*/I_k, k \geq n \). By an induction over \( M \)'s filtration, \( \text{Tor}_1^\Gamma(E(m)_* \otimes_\Lambda B, M) = 0 \) when \( M \) is finitely presented. By (1.13), this suffices to prove the lemma.

**Lemma 3.6.** Let \( M \) be an associative \( BP_* BP \)-comodule. If \( \text{w. dim}_\Gamma \leq n \).
The text contains a proof involving Brown-Peterson homology and torsion modules. The proof involves specific algebraic and homological constructions, and uses notations and terminology typical of algebraic topology and homological algebra. The text is dense and technical, requiring a good understanding of the relevant mathematical concepts to follow. The proof includes references to previous lemmas and corollaries, and uses exact sequences and homomorphisms as key tools in the argument. The goal is to establish certain properties of torsion modules in the context of Brown-Peterson homology.
with \( \pi_*(v_m^{-1}BPJ) \cong v_m^{-1}BPJ \). For a spectrum \( Y \), we have the Boardman map \( Y \to E(m, J) \wedge Y \) which induces a Hurewicz homomorphism \( \pi_*(Y \wedge X) \to E(m, J)_*(Y \wedge X) \). The key topological results of this section compute the kernels of these homomorphisms when \( Y = BPJ \) or \( v_m^{-1}BPJ \). These computations depend on a theorem of Ravenel concerning the right unit of the \( BP \) spectrum.

**Theorem 4.1** (Ravenel [13]). Let \( \eta_m \) be the composition

\[
\eta_m': BP_* \to BP_*BP \to \mathbb{Z}/p[v_m] \otimes BP_*BP.
\]

Then \( \eta_m'(v_m) = v_m \) and for \( k \geq 1 \),

\[
\eta_m(v_{m+k}) \equiv v_m t^k - v_m^{p^k} t_k \quad \text{modulo } (\eta_m(v_{m+1}), \ldots, \eta_m(v_{m+k-1})).
\]

Let \( G(m, J)_* = \mathbb{Z}[v_1, \ldots, v_m] \otimes BP_*BP \) and let \( \lambda: G(m, J)_* \to v_m^{-1}G(m, J)_* \cong E(m, J)_*BP \) be the localization homomorphism. We have two Hurewicz homomorphisms induced by \( BP \)'s right unit:

\[
h_m^R(\Gamma): \Gamma = BP_* \to BP_*BP \to G(m, J)_*
\]

\[
h_m^L(\Gamma) = \lambda \circ h_m^R(\Gamma): \Gamma \to BP_*BP \to E(m, J)_*BPJ.
\]

For a left \( \Gamma \) module \( M \), we define:

\[
h_m'(M) = h_m'(\Gamma) \otimes 1: M \cong \Gamma \otimes M \to G(m, J)_* \otimes M
\]

\[
h_m(M) = h_m(\Gamma) \otimes 1: M \cong \Gamma \otimes M \to E(m, J)_*BPJ \otimes M.
\]

**Lemma 4.2.** Let \( m \geq n \). Let \( h^i = h_m'(BP_*/I_m) \). For any non-zero element \( y \) in \( BP_*/I_m \), left multiplication by \( h(y) \) in \( G(m, J)_* \otimes BP_*/I_m \) is injective.

**Proof.** The \( m = n = 0 \) case is well-known; so we assume that \( m > 0 \). Since \( \eta_*(v_i) = v_i \) modulo \( I_*BP_*BP \) [2.11.16.1], we have the isomorphisms \( \zeta \) and \( \rho \) in commutative diagram 4.3.

\[
\begin{array}{ccc}
BP_* & \to & \mathbb{Z}/p[v_m] \otimes BP_*BP \otimes 1(n) \otimes j) \to \mathbb{Z}/p[v_m] \otimes BP_*BPJ \\
\downarrow & & \downarrow \zeta \\
BP_*/I_m \to G(m, J)_* \otimes BP_*/I_m = \mathbb{Z}(p)[v_1, \ldots, v_m] \otimes BP_*BPJ \otimes BP_*/I_m \\
\end{array}
\]

A \( \mathbb{Z}/p \)-basis element of \( BP_*/I_m \) is (represented by) a monomial of form \( v_m^{i_0}v_{m+1}^{i_1}v_{m+2}^{i_2} \cdots = v^{i_0} \). Let \( i = i_0 + i_1 + i_2, \ldots \) and let

\[
t^{i_0} = t_0^{i_0}t_1^{i_0}t_2^{i_0}t_2^{i_1} \cdots = t_0^{i_0}t_1^{i_0}t_2^{i_0}t_2^{i_1} \cdots \quad (t_0 = 1).
\]
If \( E=(e_1, e_2, \cdots) \) and \( t^E=t_1^E t_2^E \cdots \), observe that \((j \wedge j) \ast t^E = z^E : 0\) which is a left \( BPJ \)\_\ast\_basis element of \( BPJ \ast BPJ \) (Lemma 5.10). Filter each gradation of the image of \( 1 \otimes (j \wedge j) \ast \) by defining \( v_m \otimes z^E : 0 \) to be of lower filtration than \( v_m \otimes z^E : 0 \) provided that \( e_{s+2}=f_{s+2}, e_{s+1}=f_{s+1} \), but \( e_s<f_s \). We now interpret Theorem 4.1 as saying that \( \xi^{-1} \rho' (v^l) = (1 \otimes (j \wedge j) \ast ) v_m (v^l) \equiv v_m \otimes z^m : 0 \) modulo terms of lower filtration. The result is now evident.

**Corollary 4.4.** Let \( m \gg n \). Then \( h_m (BPJ / I_m ) : BPJ / I_m \to E(m, J) \ast BPJ \otimes BPJ / I_m \) is a monomorphism.

**Proof.** By Lemma 4.2, left multiplication by \( v_m \otimes 1 = h' (v_m ) \) on \( G(m, J) \ast \otimes BPJ / I_m \) is monic; thus the localization map \( \lambda \otimes 1 : G(m, J) \ast \otimes BPJ / I_m \to E(m, J) \ast BPJ \otimes BPJ / I_m \) is monic. A second application of Lemma 4.2 shows that \( h_m (BPJ / I_m ) \) is injective. But \( h_m (BPJ / I_m ) = (\lambda \otimes 1) h_m (BPJ / I_m ) \).

Let us adopt the notation

\[
\tilde{h}_m (\Gamma ) : v_m^{-1} \Gamma = v_m^{-1} BPJ \ast = \pi \ast (v_m^{-1} BPJ) \to E(m, J) \ast (v_m^{-1} BPJ)
\]

for the Hurewicz homomorphism induced by the Boardman map \( v_m^{-1} BPJ \to E(m, J) \wedge v_m^{-1} BPJ \).

For any \( \Gamma \)-module \( M \), we define \( \tilde{h}_m (M) \) by

\[
\tilde{h}_m (M) = \tilde{h}_m (\Gamma ) \otimes 1 : v_m^{-1} M = v_m^{-1} \Gamma \otimes M \to E(m, J) \ast (v_m^{-1} BPJ) \otimes M.
\]

Using the notation \( \lambda (M) : M \to v_m^{-1} M \) for the algebraic localization of \( M \) and \( \lambda : BPJ \to v_m^{-1} BPJ \) for the topological localization of the spectrum \( BPJ \), we have the commutative diagram 4.5.

\[
\begin{array}{ccc}
M & \xrightarrow{\tilde{h}_m (M)} & E(m, J) \ast BPJ \otimes M \\
\lambda (M) & & \quad \downarrow \quad E(m, J) \ast (\lambda) \otimes 1 \\
v_m^{-1} M & \xrightarrow{\tilde{h}_m (M)} & E(m, J) \ast (v_m^{-1} BPJ) \otimes M
\end{array}
\]

**Lemma 4.6.** Let \( m \gg n \). For any associative \( BPJ \ast BPJ \)-comodule \( M \), \( \tilde{h}_m (M) \) is monic.

**Proof.** For a future analogy and some present simplicity, let \( A=BPJ \ast = \Gamma \), \( B=E(m, J) \ast BPJ \), and \( f=h_m (\Gamma ) : A \to B \). We record four essential facts.

(i) By Corollary 4.4, \( f \otimes BPJ / I_m : A \otimes BPJ / I_m \to B \otimes BPJ / I_m \) is monic.

(ii) By Lemma 3.5(ii), \( \text{Tor}_1^{\Gamma} (B, BPJ / I_{j+1}) = 0 \), \( n \leq j+1 \) \( (\leq m+1) \).

(iii) By Lemma 3.5(iii), \( \text{Tor}_1^{\Gamma} (B \otimes v_m^{-1} \Gamma , BPJ / I_j) = 0 \), \( n \leq j \).

(iv) \( A \) is connective.

The lemma is well known when \( m=n=0 \); so we assume \( m \geq 0 \). Multiplication by \( v_j \) on \( BPJ / I_j \) induces commutative diagram 4.7 which has exact rows. We assume \( n \leq j \leq m \) so that the bottom torsion term is zero as indicated.
The vertical $f_i$'s are $f \otimes BP_*|I_j$'s as appropriate. By a downward induction beginning with $j=m$ (i), we assume $f_3$ is monic. By an upward induction on the dimension of elements in connective $A \otimes BP_*|I_j$ (iv), we assume $f_i$ is monic in the dimension of interest. Note that $1 \otimes v_j$ raises dimensions. Thus $f_2$ is monic by the five lemma. By this double induction $f \otimes BP_*|I_j$ is monic for $j$ satisfying $n \leq j \leq m$. Upon $v_m$-localization $f$ induces $f: v_m^{-1}A \rightarrow E(m, J)_*$ $(v_m^{-1}BP)$. Since $(v_m^{-1}A) \otimes BP_*|I_j=0$ for $j>m$, we have that $f \otimes BP_*|I_j; v_m^{-1}BP_*|I_j \rightarrow E(m, J)_*(v_m^{-1}BP) \otimes BP_*|I_j$ is monic for all $j$ satisfying $n \leq j$. To prove $f \otimes M$ is monic for all associative $BP_* BPJ$-comodules, it suffices to prove $\tilde{f} \otimes M$ is monic where $M$ is finitely presented (Lemma 1.13). Such a finitely presented comodule $M$ has a finite filtration by sub-comodules whose subquotients are suspended copies of $BP_*|I_j, j \geq n$. The remainder of the proof is a five-lemma-aided induction over the filtration of $M$ using fact (iii) to have the “bottom-left torsion term” zero.

**Lemma 4.8.** Let $m \geq n$ and let $M$ be an associative $BP_* BPJ$-comodule. Left multiplication by $h_m(M)(v_m)$ acts injectively on $E(m, J)_* BPJ \otimes M$.

Proof. Let $A=B=G(m, J)_*=Z_{(q)}[v_1, \ldots, v_m] \otimes BPJ_* BPJ$. Let $f: A \rightarrow B$ be left multiplication by $h_m(\Gamma)(v_m)$. We record four essential facts.

(i) By Lemma 4.2, $f \otimes BP_*|I_m$ is monic.

(ii) By Lemma 3.4, $\text{Tor}_1^F(B, BP_*|I_{j+1})=0, n \leq j+1 \leq m+1$.

(iii) By Lemma 3.5 (ii), $\text{Tor}_1^F(v_m^{-1}B, BP_*|I_j)=0, n \leq j$.

(iv) $A$ is connective.

Follow the pattern of the proof of Lemma 4.6.

**Theorem 4.9.** Let $J=\{q_0, \ldots, q_n-1\}$ be an invariant regular sequence of length $n$ and let $m \geq n$. Let $M$ be an associative $BP_* BPJ$-comodule. The kernel of $h_m(M): M \rightarrow E(m, J)_* BPJ \otimes_{BPJ} M$ is the $v_m$-torsion subgroup of $M$.

Proof. In diagram 4.5, $\tilde{h}_m(M)$ is monic by Lemma 4.6. By Lemma 4.8, left multiplication by $h_m(M)(v_m)$—i.e. right multiplication by $v_m \otimes 1$—is monic on $E(m, J)_* BPJ \otimes M$. Thus the localization map $E(m, J)_* \lambda(\lambda) \otimes 1$ is monic in 4.5. Thus the kernel of $h_m(M)$ coincides with that of $\lambda(M)$ which is the $v_m$-torsion subgroup of $M$. 

(4.7) $\text{Tor}_1^F(A, BP_*|I_{j+1}) \rightarrow A \otimes BP_*|I_j \rightarrow A \otimes BP_*|I_j \rightarrow 0$
**Theorem 4.10.** Let $J = \{q_0, \cdots, q_{n-1}\}$ be an invariant regular sequence of length $n$ and let $m \geq n$. Let $X$ be any CW spectrum.

(i) The Boardman map $v_m^{-1}BPJ \rightarrow E(m, J) \wedge v_m^{-1}BPJ$ induces a Hurewicz monomorphism

$$h_m(X) : \pi_*(v_m^{-1}BPJ \wedge X) \rightarrow E(m, J)_*(v_m^{-1}BPJ \wedge X).$$

(ii) The Boardman map $BPJ \rightarrow E(m, J) \wedge BPJ$ induces a Hurewicz homomorphism

$$h_m(X) : \pi_*(BPJ \wedge X) \rightarrow E(m, J)_*(BPJ \wedge X)$$

whose kernel is precisely the $v_m$-torsion subgroup of $BPJ_*X = \pi_*(BPJ \wedge X)$.

**Proof.** The latter part follows immediately from Theorem 4.9 and the isomorphism $E(m, J)_*(BPJ \wedge X) \cong E(m, J)_*BPJ \otimes BPJ_*X$ (Lemma 3.5(ii)). Similarly, the first part follows from Lemma 4.6.

**Corollary 4.11.** Let $J$ continue to be an invariant regular sequence of length $n$ and let $m \geq n$. For any CW spectrum $X$, $(v_m^{-1}BPJ)_*X = 0$ if and only if $E(m, J)_*X = 0$.

**Proof.** The "only if" statement follows from a Conner-Floyd type isomorphism:

$$E(m, J)_*X \cong E(m)_* \otimes_{\eta_m} BPJ_*,BPJ_*X \cong E(m)_* \otimes_{v_m^{-1}BPJ} BPJ_*X.$$

Its converse follows from Theorem 4.10(i) and the isomorphisms

$$E(m, J)_*X \otimes_{BPJ_*} BPJ_*(v_m^{-1}BPJ) \cong E(m, J)_*(X \wedge v_m^{-1}BPJ) \cong E(m, J)_*(v_m^{-1}BPJ \wedge X).$$

**Corollary 4.12.** Let $J$ be a finite invariant regular sequence of length $n$. Let $m \geq n$. Let $X$ be a connective CW spectrum. If $w.\text{dim}_{BPJ_*} BPJ_*X \leq m-n$, then the Hurewicz homomorphism

$$h_m(X) : \pi_*(BPJ \wedge X) \rightarrow E(m, J)_*(BPJ \wedge X)$$

is injective. (Cf. [6, Theorem 6.1].)

**Proof.** Proposition 3.7 and Theorem 4.10 (ii).

5. **BPJ*BPJ and BPJ*BPJ**

Let $A$ be an algebra over the ground ring $R$, $N$ be an $R$-module, and $M$ be an associative $A$-module. Then there is an isomorphism

$$\theta : \text{Hom}_A (A \otimes_R N, M) \rightarrow \text{Hom}_R (N, M)$$

defined by $\theta(f) = f(\eta \otimes 1)$ where $\eta : R \rightarrow A$ is the unit map. $\theta^{-1}(g) = \varphi(1 \otimes g)$.
where \( \varphi: A \otimes_R M \to M \) gives \( M \)'s \( A \)-module structure. (Adams [1, p. 320].)

**Lemma 5.1.** Let \( h: M \to A \otimes_R N \) be an \( A \)-module homomorphism. If \( h \) is split epic as an \( R \)-module homomorphism, then it is also split epic as an \( A \)-module map.

Proof. Let the \( R \)-module map \( f: A \otimes_R N \to M \) be a right inverse for \( h \). Then \( \theta^{-1}(f(\varphi \otimes 1)) \) is the desired \( A \)-module splitting of \( h \).

**Lemma 5.2.** Let \( C \) be a coalgebra over \( R \), \( N \) be an \( R \)-module, and \( M \) be an associative \( C \)-comodule. Let \( h: C \otimes_R N \to M \) be a \( C \)-comodule map. If \( h \) is split monic as an \( R \)-module homomorphism, then it is also split monic as a \( C \)-comodule map.

Proof. This is the formal dual of Lemma 5.1.

Let \( J = \{ q_0, \ldots, q_{n-1} \} \) be a finite invariant regular sequence in \( BP_* \). By Baas [4], there exists an associative left \( BP \)-module spectrum \( BPJ \) with pairing \( \varphi: BP \wedge BPJ \to BPJ \) such that \( \pi_* BPJ = BPJ_* \cong BPJ_*(q_0, \ldots, q_{n-1}) \). Let \( J_{m+1} = \{ q_0, \ldots, q_m \}, m < n. \) \( BPJ_m \) and \( BPJ_{m+1} \) are related by a cofibration of \( BP \)-module spectra

\[
S^d BPJ_m \xrightarrow{\varphi_m(q_m \wedge 1)} BPJ_m \xrightarrow{j_m} BPJ_{m+1} \xrightarrow{k_m} S^{d+1} BPJ_m.
\]

Here \( d \) is the dimension of \( q_m \) in \( BP_* \). \( \varphi_m: BP \wedge BPJ_m \to BPJ_m \) defines \( BPJ_m \)'s \( BP \)-module structure; \( \varphi_m(1 \wedge j_{m-1}) = j_{m-1} \circ \varphi_{m-1} \) and \( \varphi_n = \varphi \). \( BPJ_0 = BP \) and \( \varphi_0 = m: BP \wedge BP \to BP \). Let

\[
j_{m+s,m} = j_{m+s-1} \circ \cdots \circ j_m: BPJ_m \to BPJ_{m+s}.
\]

Let \( \iota_m = j_{m,0} \circ \iota: S^0 \to BPJ_m \) where \( \iota: S^0 \to BP \) is the unit for the Brown-Peterson spectrum.

The homomorphism

\[
\psi: BPJ^*(BPJ_1 \wedge BPJ_m) \xrightarrow{(\varphi_1 \wedge \varphi_m)^*(1 \wedge T \wedge 1)^*} BPJ^*(BP \wedge BP \wedge BPJ_1 \wedge BPJ_m) \cong BPJ^*(BP \wedge BP) \hat{\otimes}_{BPJ^*} BPJ^*(BPJ_1 \wedge BPJ_m)
\]

makes \( BPJ^*(BPJ_1 \wedge BPJ_m) \) into an associative \( BPJ^*(BP \wedge BP) \)-comodule. (See Würgler [16].) \( \varphi(1 \wedge j_{n,0}) = j_{n,0} \circ \varphi_0 \) gives a distinguished element of \( BPJ^*(BP \wedge BP) \). A map \( f: BPJ_1 \wedge BPJ_m \to BPJ \) is said to be primitive if \( \psi[f] = [j_{n,0} \circ \varphi_0] \otimes [f] \). In other words, \( f(\varphi_1 \wedge \varphi_m)(1 \wedge T \wedge 1) = \varphi(\varphi_0 \wedge 1)(1 \wedge 1 \wedge f) \). We follow Würgler in denoting the set of primitives of \( BPJ^*(BPJ_1 \wedge BPJ_m) \) by \( \text{Pr} \ BPJ^*(BPJ_1 \wedge BPJ_m) \).

**Remark 5.3.** When \( BPJ_1 = BPJ_m = BPJ \), a multiplication \( \mu: BPJ \wedge BPJ \to
BPJ in $BPJ^*(BPJ \wedge BPJ)$ is primitive if and only if the following three conditions hold.

(i) $\mu(\varphi \wedge 1) = \varphi(1 \wedge \mu) : BPJ \wedge BPJ \rightarrow BPJ$

(ii) $\mu(\varphi \wedge 1)(T \wedge 1) = \mu(1 \wedge \varphi) : BPJ \wedge BPJ \rightarrow BPJ$

(iii) $\mu(1 \wedge \varphi)(1 \wedge T) = \varphi \circ T(\mu \wedge 1) : BPJ \wedge BPJ \wedge BP \rightarrow BPJ$

(The first two conditions imply the third.) Conditions (i), (ii), and (iii) give Araki-Toda's characterization of a quasi-associative multiplication [3].

Lemma 5.4. Let a multiplication $\mu : BPJ \wedge BPJ \rightarrow BPJ$ be quasi-associative and let $\iota_n : S^0 \rightarrow BPJ_n = BPJ$ be a unit for $\mu$. Then the following diagram commutes.

\[
\begin{array}{ccc}
BPJ \wedge BPJ & \xrightarrow{\iota_n \wedge 1} & BPJ \wedge BPJ \\
\varphi & \downarrow & \mu \\
BPJ & \xleftarrow{\varphi} & BPJ \wedge BPJ
\end{array}
\]

Proof. Routine.

Proposition 5.5 (Würzler [16, Theorem 5.1]). Let $J = \{q_0, \ldots, q_{n-1}\}$ be an invariant regular sequence of $BPJ_\ast$. For $0 \leq m \leq n$, there is a quasi-associative multiplication $\mu_m : BPJ_m \wedge BPJ_m \rightarrow BPJ_m$ with unit $\iota_m : S^0 \rightarrow BPJ_m$ such that $j_m \circ \mu_{m-1} = \mu_m(j_m \wedge j_m)$ as maps $BPJ_{m-1} \wedge BPJ_{m-1} \rightarrow BPJ_m$.

Proof. For $0 \leq m \leq n$, we construct primitive maps $\mu_m' : BPJ_{m-1} \wedge BPJ_m \rightarrow BPJ_m$ and $\mu_m : BPJ_m \wedge BPJ_m \rightarrow BPJ_m$ such that all of the obvious compositions commute:

(i) $\mu_m'(1 \wedge j_m) = j_m \circ \mu_{m-1}$;

(ii) $\mu_m'(j_m \wedge 1) = \phi_m$;

(iii) $\mu_m(j_m \wedge 1) = \mu_m'$;

(iv) $\mu_m(1 \wedge j_m) = \phi_m \circ T$

where $T : BPJ_m \wedge BP \rightarrow BP \wedge BP_m$ is the switching map. (Compare Lemma 5.4.)

The proof is by induction on $m$. We sketch the inductive step.

Since $\eta_m(q_i) \equiv (q_0, \ldots, q_i) \cdot BP \wedge BP$, the cofibration

\begin{equation}
BPJ_j \xrightarrow{j_l} BPJ_{j+l} \xrightarrow{k_l} S^{l+1}BPJ_{j+l}
\end{equation}

induces a split short exact sequence of $BPJ^*_m$-modules, $0 \leq k, l + 1 \leq m \leq n$.

\begin{equation}
0 \rightarrow BPJ^*_m(BPJ_k \wedge BPJ_l) \rightarrow BPJ^*_m(BPJ_k \wedge BPJ_{l+1}) \xrightarrow{(1 \wedge j_l)^*} BPJ^*_m(BPJ_k \wedge BPJ_l) \rightarrow 0.
\end{equation}

We assume inductively that $BPJ^*_m(BPJ_k \wedge BPJ_l) \cong BPJ^*_m(BP \wedge BP) \otimes_{BPJ^*_m} N$.
for some $BPJ_m$-module $N$. By Lemma 5.1, (5.7) splits as $BPJ_m^*(BP\wedge BP)$-modules and the middle term of (5.7) has the desired inductive structure. By Lemma 5.2, (5.7) splits as $BPJ_m^*(BP\wedge BP)$-comodules. Hence the functor $Pr(\_)$ preserves the exactness of (5.7). We first pick $\mu_0' \in Pr BPJ_m^*(BPJ_{m-1}\wedge BPJ_m)$ satisfying (i) and (ii), and next $\mu_1 \in Pr BPJ_m^*(BPJ_m \wedge BPJ_m)$ satisfying (iii) and (iv).

Quasi-associative multiplications $\mu: BPJ_\Lambda BPJ \rightarrow BPJ$ with unit $i_*: S^0 \rightarrow BPJ$ exist by Proposition 5.5. We assume that a choice of such a $\mu$ is fixed throughout this paper.

The cofibration (5.6) induces two split short exact sequences of $BPJ_\ast \simeq BPJ_\ast$-modules.

\begin{equation}
0 \rightarrow BPJ_\ast BPJ_1 \rightarrow BPJ_\ast BPJ_{i+1} \rightarrow BPJ_\ast BPJ_1 \rightarrow 0
\end{equation}

\begin{equation}
0 \rightarrow BPJ_\ast BPJ_1 \rightarrow BPJ_\ast BPJ_{i+1} \rightarrow BPJ_\ast BPJ_1 \rightarrow 0
\end{equation}

Recall that $BP_\ast BP \cong BP_\ast [t_1, t_2, \ldots]$ where the indeterminate $t_i$ is of dimension $2p^i - 2$. (Let $t_0 = 1$.) Thus $BP_\ast BP \cong BP_\ast [t_1, t_2, \ldots]$. An argument using Lemma 5.1, similar to that of the proof of Proposition 5.5, shows that $BPJ_\ast BPJ_i$ is a free left $BPJ_\ast BP$-module. Let $A = (a_0, \ldots, a_{t-1})$ be an $l$-tuple of 0's and 1's. A free left $BPJ_\ast BP$-basis of $BPJ_\ast BPJ_1$ is given by the symbols

$$\partial^A = \partial_0^{a_0} \cdots \partial_{t-1}^{a_{t-1}}$$

of dimension $\sum_i a_i$ (dimension $(q_i) + 1$). In (5.8), $j_*$ sends $\partial^A$ to a symbol of the same name. We choose elements $\partial^A \partial_i \in BPJ_\ast BPJ_i$ so that $k_*(\partial^A \partial_i) = \partial^A$. Let $z^{E,A} \in BPJ_\ast BPJ$ be the element corresponding to $t_i \cdots t_{m-1} \partial^A$ where $E = (e_1, \ldots, e_m, 0, \ldots)$. Let $c: BPJ_\ast BPJ \rightarrow BPJ_\ast BPJ$ be the conjugation induced by interchange of the $BPJ$ factors of $BPJ\Lambda BPJ$.

**Lemma 5.10.** Let $J = \{q_0, \ldots, q_{t-1}\}$ be an invariant regular sequence of $BP_\ast$. A free $BPJ_\ast$ basis for $BPJ_\ast BPJ$ is given by the elements $z^{E,A}$ where $A = (a_0, \ldots, a_{t-1})$ is a sequence of 0's and 1's. The left action of $BPJ_\ast BP$ on $BPJ_\ast BPJ$ is given by

\begin{equation}
t^F z^{E,A} = z^{E+F,A}.
\end{equation}

$BPJ_\ast BPJ$ is free as a right $BPJ_\ast$-module on the basis $c(z^{E,A})$.

As explained in §1, $1 \Lambda i_*: BPJ \Lambda S^0 \rightarrow BPJ \Lambda BPJ$ induces a coaction

$$\psi_X: BPJ_\ast X \rightarrow BPJ_\ast (BPJ \Lambda X) \cong BPJ_\ast BPJ \otimes BPJ_\ast BPJ \Lambda X$$

and we define elementary $BPJ$ operations $s_{E,A}$ by the formula
(5.12) \[ \psi_X(x) = \sum_{n,A} c(z^{E_A}) \otimes s_{E_A}(x). \]

Since \( BP^*BP \) is Hausdorff, each elementary \( BP \) operation

\[ s_{E_A}: BP^*(\rightarrow BP^*+(\rightarrow BP^* \to S^*BP). \]

is induced by a unique map of spectra \( S_{E_A}: BP^* \to S^*BP \). By an induction (over \( l \leq n \)) using (5.9), one can prove:

**Lemma 5.12.** \( BP^*BP \) is a direct product of copies of \( BP^* \) indexed by the maps \( S_{E_A} \). Each element \( \theta \in BP^*BP \) has a unique representation as a convergent infinite sum

\[ \theta = \sum_{n,A} q_{E,A}S_{E,A}, \quad q_{E,A} \in BP^{l-d} \]

where \( d \) is the dimension of \( S_{E,A} \).

**Remark 5.13.** Another induction using the exactness of (5.9) shows that

\[ S_{0,0} j_{n,l} - j_{n,l} = \sum_{A \geq B} q_{A} S_{0,A} \circ j_{n,l} \]

for \( 0 \leq l \leq n \) and \( q_{A} \in BP^* \). This establishes (1.8).

**Remark 5.14.** The composition \( S_{E,A} \circ S_{F,B} \) has a representation \( \sum q_{G,C} S_{G,C} \) by Lemma 5.12. Here the dimension of \( S_{G,C} \) must be greater than or equal to the sum of the dimensions of \( S_{E,A} \) and \( S_{F,B} \). Since \( S_{E,A} \circ S_{F,B} = S_{E,A} \circ S_{E,B} = (S_{E,A} \circ S_{F,B})^{\ast} \), (1.9) is established. In general, the relations given by \( S_{E,A} \circ S_{F,B} \) will be even more frightful than the ones given by \( r_{E} \circ r_{F} \) in \( BP \) theory. In particular: \( S_{0,0} \circ S_{0,0} \) will not be \( S_{0,0} \) unless the \( q_{A} \)'s in Remark 5.13 happen to be all zero.

**References**


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