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# REDUCTION THEOREMS FOR CHARACTERISTIC FUNCTORS ON FINITE *p*-GROUPS AND APPLICATIONS TO *p*-NILPOTENCE CRITERIA

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**Abstract**

We formalize various properties of characteristic functors on $p$-groups, and discuss relationships between them. Applications to the Thompson subgroup and certain of its analogues are then given.

## 1. Introduction

In a now classical paper ([8]), John Thompson introduced, for $p$ a prime number and $S$ a $p$-group, the subgroup $J_R(S)$ (there denoted by $J(S)$) generated by the abelian subgroups of $S$ of maximal rank:

$$J_R(S) \equiv_{\text{def.}} \left\langle A \in ab(S) \;\middle|\; m(A) = \max_{B \in ab(S)} m(B) \right\rangle, \tag{1.1}$$

where $ab(S)$ denotes the set of all abelian subgroups of $S$, and, for $C$ an abelian group, $m(C)$ denotes the minimal cardinality of a generating system of $C$.

Later on, in [2], Glauberman modified that definition to:

$$J(S) \equiv_{\text{def.}} \left\langle A \in ab(S) \;\middle|\; |A| = \max_{B \in ab(S)} |B| \right\rangle. \tag{1.2}$$

Thompson had formulated a $p$-nilpotence criterion using $J_R$; this work was later built upon by Glauberman ([2]) with his $ZJ$-theorem, and by Thompson himself ([9]). For the prime $p = 2$, it is often more convenient to work with the subgroup $J_e(S)$, defined using *elementary* abelian subgroups instead of abelian ones:

$$J_e(S) \equiv_{\text{def.}} \left\langle A \in ab_e(S) \;\middle|\; |A| = \max_{B \in ab_e(S)} |B| \right\rangle \tag{1.3}$$

where $ab_e(S)$ denotes the set of *elementary* abelian subgroups of $S$.

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The functors $J_e$, $J_R$ and $J$ are *excellently abelian generated characteristic p-functors* in the sense of §3 below. In §4, we shall establish various reduction results concerning such objects; most notably, in certain cases, the normality of $W(S)$ in $G$ (for $S \in \mathcal{S}yl_p(G)$ and $W$ a characteristic $p$-functor) can be inferred from the (apparently much weaker) property of *control of p-nilpotence* by $W$ (see Theorem 4.1 (2)). In the fifth paragraph, we shall specialize our results to the prime $p = 2$ and the functors $J_e$ and $\hat{J}$ (for the definition of the last one of which see [3]), and shall henceforth refine, in a very particular case, Thompson's factorization theorem ([9], Theorem 1 (c)), thus recovering the results of [6].

In the course of the proof some reduction lemmas of independent interest, concerning normality of $p$-subgroups and control of $p$-nilpotence, will be established.

Our notations are standard: for $G$ a (finite) group and $p$ a prime number, $O_p(G)$ will denote the largest normal $p$-subgroup of $G$, $O_{p'}(G)$ the largest normal subgroup of $G$ with order prime to $p$, and $Z(G)$ the center of $G$. We set $o(G) = |G|$, $r_e(G) = m(G)$ if $G$ is an elementary abelian $p$-group for some prime $p$, and $r_e(G) = 0$ else; for $(x, y) \in G^2$:

$$y^x := x^{-1}yx,$$

and, for $A \subseteq G$ and $x \in G$:

$$A^x := \{y^x \mid y \in A\}.$$

As usual, by a slight abuse of language, $G$ will be said to have $p$-length one if $G = O_{p',p,p'}(G)$. By a *class* of groups, we shall mean a family of groups containing every subgroup and every homomorphic image of each of its elements. $\mathcal{A}b$ will denote the class of finite abelian groups, $\mathcal{S}olv$ the class of finite solvable groups, and, for $p$ a prime, $\mathcal{A}b_p$ the class of finite abelian $p$-groups. For $H$ a finite group, $\mathcal{C}'(H)$ will denote the class of finite groups, no section of which is isomorphic to $H$. For $p$ a prime and $n \in \mathbf{N}$, $\mathcal{C}_p^n$ will denote the class of finite groups, one (i.e. all) of whose Sylow $p$-subgroups has (resp. have) nilpotency class at most $n$. As usual, $p$ still denoting a prime number, a finite group $G$ will be termed *p-closed* if it has a normal Sylow $p$-subgroup (equivalently, a unique Sylow $p$-subgroup), i.e. if $G/O_p(G)$ is a $p'$-group, and $G$ will be termed $p$-constrained if, setting $\overline{G} = G/O_{p'}(G)$, one has $C_{\overline{G}}(O_p(\overline{G})) \subseteq O_p(\overline{G})$. A solvable group $G$ is $p$-constrained for all primes $p$.

By $ab(G)$ we shall denote the set of abelian subgroups of a group $G$. Finally, $\Sigma_n$ will denote the symmetric group of degree $n$.

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## 2. A preliminary lemma

The following result was first stated by Hayashi ([5], Lemma 3.9, p.101), though with an incomplete proof; our own attempt at a proof ([6], Lemme) was not conclusive either (the sentence "$Q$, agissant sans point fixe sur le 2-groupe abélien élémentaire $X$, est donc cyclique" is ambiguous, as in order to thus establish the cyclicity of $Q$, we need to know that *each nonidentity element of* $Q$ acts on $X$ without fixed point, which is not obvious). Here, we shall take the opportunity to clarify the matter once and for all; during the course of the proof, we shall feel free to use some ideas from [5] and [6].

**Lemma 2.1.** *Let $G$ be a* (*solvable*) $\{2,3\}$*-group*; *then the following statements are equivalent*:
(1) *$G$ is $\Sigma_4$-free, and*:
(2) $G = O_{3,2,3}(G)$.

REMARK 2.2. According to Burnside's $p^a q^b$-theorem, the solvability hypothesis is redundant.

Proof. The implication (2) $\Rightarrow$ (1) is obvious, as the condition $G = O_{3,2,3}(G)$ is inherited by all sections of $G$, and $\Sigma_4 \neq O_{3,2,3}(\Sigma_4)$.

Let $G$ denote a minimal counterexample to the statement that (1) $\Rightarrow$ (2); it is clear that $O_3(G) = 1$, that $G$ possesses a unique minimal non-trivial normal subgroup $X$, that $X$ is a 2-group, and that $N_0 = O_{2,3}(G) \subset G$ is the unique maximal normal subgroup of $G$. It follows (as $O_3(G/N_0) = 1$) that $G/N_0$ has order 2; therefore one has $O^3(G) \not\subseteq N_0$, whence $G = O^3(G)$, thus

$$O^3\left(\frac{G}{X}\right) = \frac{O^3(G)X}{X} = \frac{G}{X}.$$

But, by the minimality of $G$, one may write

$$O_{3,2,3}\left(\frac{G}{X}\right) = \frac{G}{X},$$

whence

$$\frac{G}{X} = O_{3,2}\left(\frac{G}{X}\right).$$

Take now $Q \in \mathcal{S}yl_3(G)$; we have just established that $QX \lhd G$, and the Frattini argument yields:

$$G = XN_G(Q).$$

Let $L =_{\text{def.}} N_G(Q)$; then $L \neq G$ and $G = LX$. Let us assume $L \subseteq H \subset G$; then

$$\begin{aligned} H &= H \cap G \\ &= H \cap LX \\ &= L(H \cap X); \end{aligned}$$

but $H \cap X \lhd \langle H, X \rangle = G$, whence $H \cap X = 1$ or $H \cap X = X$. In the second case, $H = LX = G$, a contradiction; therefore $H \cap X = 1$, and $H = L(H \cap X) = L$: $L$ is a maximal subgroup of $G$. Taking now $H = L$ in the above argument yields:

$$L \cap X = 1.$$

Let $C = C_L(X)$; then $C \lhd LX = G$, and $X \not\subseteq C$ (else one would have $G = LX = L$, a contradiction), therefore $C = 1$. As $X \lhd G$, $X \subseteq O_2(G)$, whence $X \lhd O_2(G)$ and $Y = X \cap Z(O_2(G)) \neq 1$; but $Y \lhd G$, therefore $Y = X$, i.e. $X \subseteq Z(O_2(G))$. It follows that

$$\begin{aligned} O_2(G) &\subseteq C_G(X) \\ &= C_G(X) \cap XL \\ &= XC_L(X) \\ &= X. \end{aligned}$$

Therefore $X = O_2(G)$. Let us set $\overline{G} = G/X$; then $O_2(\overline{G}) = O_2(G)/X = 1$, and (as $\overline{G}$ is solvable)

$$(*) \qquad C_{\overline{G}}(O_3(\overline{G})) \subseteq O_3(\overline{G}).$$

Let now $\overline{t} = tX$ denote an element of order 2 in $\overline{G} = G/X$; according to $(*)$, $\overline{t}$ does not centralize $O_3(\overline{G})$, therefore some $\overline{y} \in O_3(\overline{G})$ is not centralized by $\overline{t}$, thus $\overline{z} =_{\text{def.}} [\overline{t}, \overline{y}] \neq 1$, $\overline{z} \in O_3(\overline{G})$, and

$$\begin{aligned} \overline{z}^{\overline{t}} &= \overline{t}^{-1}\overline{z}\overline{t} \\ &= \overline{t}^{-1}\overline{t}^{-1}\overline{y}^{-1}\overline{t}\overline{y}\overline{t} \\ &= \overline{y}^{-1}\overline{t}\overline{y}\overline{t} \\ &= (\overline{t}^{-1}\overline{y}^{-1}\overline{t}\overline{y})^{-1} \\ &= \overline{z}^{-1}. \end{aligned}$$

Let $\omega(\overline{z}) = 3^m$ $(m \geq 1)$, and $\overline{v} =_{\text{def.}} \overline{z}^{3^{m-1}}$; then $\omega(\overline{v}) = 3$ and $\overline{v}^{\overline{t}} = \overline{v}^{-1}$, whence $\langle \overline{t}, \overline{v} \rangle \simeq \Sigma_3$. Set now $V = X\langle t, t^v \rangle$; then $V/X = \langle \overline{t}, \overline{v} \rangle \simeq \Sigma_3$, and $O_3(V) \subseteq C_G(O_2(V)) \subseteq C_G(X) \subseteq X$, whence $O_3(V) = 1$. If $V \neq G$, then (by induction) $V = O_{3,2,3}(V)$, whence $V = O_{2,3}(V)$, $t \in O_2(V)$, $\langle t, t^v \rangle \subseteq O_2(V)$, $V$ is a 2-group, and hence also is $\overline{V}$, a contradiction. Therefore $V = G$ and $L \simeq G/X = \overline{V} \simeq \Sigma_3$. It follows that $G = LX = L \ltimes X$, $X$ (as a minimal normal subgroup of $G$) being a nontrivial irreducible $\mathbf{F}_2 L \simeq \mathbf{F}_2 \Sigma_3$-module. But then $X$ has to be isomorphic to the canonical module $\mathbf{F}_2^2$ for $\Sigma_3 \simeq SL_2(\mathbf{F_2})$, and one obtains $G \simeq \Sigma_3 \ltimes \mathbf{F}_2^2 \simeq \Sigma_4$, a contradiction. □

## 3. Characteristic $p$-functors: generalities

For $p$ a prime number, $\mathcal{G}_p$ will denote the category of finite $p$-groups (morphisms in $\mathcal{G}_p$ being the group isomorphisms in the usual sense).

DEFINITION 3.1 ([2], p.1116). By a characteristic $p$-functor we shall mean a functor $K : \mathcal{G}_p \to \mathcal{G}_p$ such that, for each $P \in \mathcal{G}_p$, $K(P) \subseteq P$ and $K(P) \neq 1$ if $P \neq 1$.

Clearly, whenever $K_1$ and $K_2$ are characteristic $p$-functors, $K_1 \circ K_2$ (simply denoted by $K_1 K_2$), defined by:

$$(K_1 \circ K_2)(P) \equiv_{\text{def.}} K_1(K_2(P))$$

is one. Examples of characteristic $p$-functors include $J_R$, $J$, $\hat{J}$, $J_e$, $Z$, and $\Omega_n$ ($n \in \mathbf{N}$), the last one defined by:

$$\Omega_n(P) \equiv_{\text{def.}} \langle x \in P \mid x^{p^n} = 1 \rangle.$$

A general class of characteristic $p$-functors is obtained *via*:

DEFINITION 3.2. Let $\varphi$ denote a mapping from $\mathcal{A}b_p$ to $\mathbf{N} = \{0, 1, \dots\}$, invariant under isomorphisms, and such that

$$A \neq 1 \implies \varphi(A) \geq 1;$$

then, for $P$ a $p$-group, let

$$K_\varphi(P) \equiv_{\text{def.}} \left\langle A \text{ abelian subgroup of } P \;\middle|\; \varphi(A) = \max_{B \subseteq A;\, B \text{ abelian}} \varphi(B) \right\rangle.$$

It is easily seen that $K_\varphi$ is a characteristic $p$-functor; such characteristic $p$-functors will be termed *excellently abelian generated*. Clearly, $J$, $J_R$ and $J_e$ are such; in fact, $J = K_o$, $J_R = K_m$ and $J_e = K_{r_e}$.

DEFINITION 3.3. The characteristic $p$-functor $W$ is termed excellent if, whenever $G$ is a finite group, $P \in \mathcal{S}yl_p(G)$, $x \in G$, and $W(P) \subseteq Q \subseteq P^x$, then $W(P) = W(Q) = W(P^x)$ ($= W(P)^x$). In particular, $W(P)$ is weakly closed in $P$, and characteristic in any $p$-subgroup of $G$ that contains it.

**Lemma 3.4.** *Any excellently abelian generated characteristic $p$-functor is excellent.*

Proof. For $S$ a $p$-group, let

$$r_\varphi(S) =_{\text{def.}} \max_{A\in ab(S)} \varphi(A).$$

Let us assume that $K_\varphi(P) \subseteq Q \subseteq P^x$, and let $A_0 \in ab(P)$ such that

$$\varphi(A_0) = \max_{A\in ab(P)} \varphi(A) = r_\varphi(P).$$

Obviously,

$$\begin{aligned}
r_\varphi(Q) &\le r_\varphi(P^x)\\
&= \max_{A\in ab(P^x)} \varphi(A)\\
&= \max_{C\in ab(P)} \varphi(C^x)\\
&= \max_{C\in ab(P)} \varphi(C) \quad \text{(as } \varphi \text{ is invariant under isomorphisms)}\\
&= r_\varphi(P)\\
&= \varphi(A_0)\\
&\le r_\varphi(Q) \quad \text{(as } A_0 \subseteq K_\varphi(P) \subseteq Q).
\end{aligned}$$

Therefore $r_\varphi(P) = r_\varphi(Q)$, whence

$$\begin{aligned}
K_\varphi(Q) &= \langle A \in ab(Q) \mid \varphi(A) = r_\varphi(Q)\rangle\\
&= \langle A \in ab(Q) \mid \varphi(A) = r_\varphi(P)\rangle\\
&= \langle A \in ab(P) \mid \varphi(A) = r_\varphi(P)\rangle\\
&= K_\varphi(P)
\end{aligned}$$

(because $A \in ab(P)$ and $\varphi(A) = r_\varphi(P)$ yield $A \subseteq K_\varphi(P) \subseteq Q$).

Incidentally we have shown that $r_\varphi(Q) = r_\varphi(P^x)$, whence $K_\varphi(Q) \subseteq K_\varphi(P^x)$ and $K_\varphi(P) = K_\varphi(Q) \subseteq K_\varphi(P^x) = (K_\varphi(P))^x$, and equality all along follows. □

## 4. A reduction theorem

Let $p$, $W$ and $\mathcal{C}$ denote respectively a prime number, a characteristic $p$-functor, and a class of groups; the following properties of the triple $(W, \mathcal{C}, p)$ will be considered ($S$ denoting a Sylow $p$-subgroup of the group $G$):

(P1) For each $G \in \mathcal{C}$, one has

$$G = N_G(W(S))O_{p'}(G).$$

(P2) For each $p$-solvable $G \in \mathcal{C}$, one has

$$G = N_G(W(S))O_{p'}(G).$$

(P3) For each solvable $G \in \mathcal{C}$, one has

$$G = N_G(W(S))O_{p'}(G).$$

(P4) For each solvable $G \in \mathcal{C}$, all of whose Sylow $q$-subgroups for all primes $q \neq p$ are abelian, one has

$$G = N_G(W(S))O_{p'}(G).$$

(P5) *$W$ controls $p$-length* 1 *in $\mathcal{C}$*, i.e. for each *$p$-solvable* $G \in \mathcal{C}$, if $N_G(W(S))$ has $p$-length one then $G$ has $p$-length one.
(P6) *$W$ controls $p$-nilpotence in $\mathcal{C}$*, i.e. for each $G \in \mathcal{C}$, if $N_G(W(S))$ is $p$-nilpotent then $G$ is $p$-nilpotent.

Stellmacher's result ([7]) asserts the existence of a (non-explicit) characteristic 2-functor $W$ such that (P1) (and hence (P2)–(P6)) hold for $(W, \mathcal{C}'(\Sigma_4), 2)$, where, according to the notations described above, $\mathcal{C}'(\Sigma_4)$ denotes the class of $\Sigma_4$-free groups. In fact, Stellmacher establishes (P1) for $(W, \mathcal{D}, 2)$, where $\mathcal{D}$ denotes the class of $\Sigma_4$-free groups all of whose non-abelian simple sections are isomorphic either to a Suzuki group or to $PSL_2(3^m)$ for some odd integer $m$; but a theorem of Glauberman ([3]), the proof of which can be much simplified using Stellmacher's result, yields that in fact $\mathcal{D} = \mathcal{C}'(\Sigma_4)$.

**Theorem 4.1.** (1) *One has* (P1) $\Rightarrow$ (P2) $\Rightarrow$ (P3) $\Rightarrow$ (P4) $\Rightarrow$ (P6), *and* (P3) $\Rightarrow$ (P5) $\Rightarrow$ (P6).
(2) *If* $p = 2$, $W(S) \subseteq \Omega_1(S)$ *for all* $S$, *and either*

(i) $\mathcal{C} \subseteq \mathcal{C}_{2,2}$ (*the class of* 2-*groups with nilpotency class at most* 2) *and* $W$ *is excellent*,

*or*

(ii) $W$ *is excellently abelian generated*,

*then* (P6) $\Rightarrow$ (P2), *and hence properties* (P2)–(P6) *are equivalent*.

Proof. (1) The implications (P1) $\Rightarrow$ (P2) $\Rightarrow$ (P3) $\Rightarrow$ (P4) are trivial.

In order to establish that (P3) $\Rightarrow$ (P5), let us assume (P3), let $G$ denote a counterexample to (P5) with minimal order. We shall use arguments similar to Bauman's in

[1], pp.388–389. If $O_{p'}(G) \neq 1$, let $\overline{G} =_{\text{def.}} G/O_{p'}(G)$; then one has:

$$\begin{aligned} N_{\overline{G}}(W(\overline{S})) &= N_{\overline{G}}\left(\frac{W(S)O_{p'}(G)}{O_{p'}(G)}\right) \\ &= \frac{N_G(W(S))O_{p'}(G)}{O_{p'}(G)} \quad \text{(by the Frattini argument)} \\ &\simeq \frac{N_G(W(S))}{N_G(W(S)) \cap O_{p'}(G)}. \end{aligned}$$

Therefore $N_{\overline{G}}(W(\overline{S}))$ has $p$-length one, whence, by induction (as $\overline{G} \in \mathcal{C}$ and $\overline{G}$ is $p$-solvable), $\overline{G}$ has $p$-length one, hence so has $G$, a contradiction. Thus $O_{p'}(G) = 1$, whence (as $G$ is $p$-solvable) $C_G(O_p(G)) \subseteq O_p(G)$; in particular, $O_p(G) \neq \{1\}$. Let now $\overline{G} = G/O_p(G)$, and let $\overline{H} = N_{\overline{G}}(W(\overline{S}))$; if $H = G$, then $W(\overline{S}) \lhd \overline{G}$, thus $W(\overline{S}) \subseteq O_p(\overline{G}) = 1$, $W(\overline{S}) = 1$, $\overline{S} = 1$, $S = O_p(G)$, $W(S) = W(O_p(G)) \lhd G$, and $G = N_G(W(S))$ has $p$-length one, a contradiction. Therefore $H \subset G$; as $N_H(W(S)) \subseteq N_G(W(S))$ has $p$-length one, so has $H$ by induction, hence so has $\overline{H}$, hence so has $\overline{G}$, again by induction ($\overline{G}$ and $H$ both belonging to $\mathcal{C}$). Let $\overline{K} = O_{p'}(\overline{G})$; it appears that $\overline{S}\,\overline{K} \lhd \overline{G}$, hence $SK \lhd G$; if $SK \neq G$, one finds by induction that $SK$ has $p$-length 1; but $SK \lhd G$, whence $O_{p'}(SK) \lhd G$ and $O_{p'}(SK) \subseteq O_{p'}(G) = 1$. Therefore $S \lhd SK$, whence $S = O_p(SK) \lhd G$, and again $W(S) \lhd G$ and $G = N_G(W(S))$, a contradiction. Therefore $G = SK$, and $\overline{G} = \overline{S}\,\overline{K}$.

For $q \in \pi(\overline{K})$, let $\overline{Q}$ denote a Sylow $q$-subgroup of $\overline{K}$; the total number of Sylow $q$-subgroups of $\overline{K}$ is $|\overline{K} : N_{\overline{K}}(\overline{Q})| \not\equiv 0[p]$, therefore one of them, $\overline{K}_q$, is $\overline{S}$-invariant. If, for each $q \in \pi(\overline{K})$, one has $SK_q \neq G$, then, by induction, $SK_q$ has $p$-length one; but $O_{p'}(SK_q) \subseteq C_G(O_p(SK_q)) \subseteq C_G(O_p(G)) \subseteq O_p(G)$, thus $O_{p'}(SK_q) = 1$ and $S \lhd SK_q$, thus $K_q \subseteq N_G(S)$, hence

$$\overline{K} = \langle \overline{K}_q \mid q \in \pi(\overline{K}) \rangle \subseteq \overline{N_G(S)}$$

and $S \lhd SK = G$, a contradiction. Thus for some prime $q$ one has $G = SK_q$, and it appears that $G$ is solvable (in fact, a solvable $\{p, q\}$-group for some prime $q$). But now (P3) yields that $G = N_G(W(S))$, whence $G$ has $p$-length one, a contradiction (in this proof, due to the hypotheses on $\mathcal{C}$, all the groups that appear *belong to* $\mathcal{C}$; such will be the case in all subsequent similar reasonings).

Assuming (P4), let $G$ denote a counterexample to (P6), with minimal order; then Thompson's arguments ([8], pp. 43–44) yield that $O_{p'}(G) = 1$, $O_p(G) \neq 1$ and $G$ is a $\{p, q\}$-group with (elementary) abelian Sylow subgroups for some prime $q \neq p$. But then (P4) yields that $G = N_G(W(S))$, whence $G$ is $p$-nilpotent, a contradiction. Therefore (P4) $\Rightarrow$ (P6) is established.

In order to establish that (P5) $\Rightarrow$ (P6), the same argument works; here, we only need Thompson's reduction up to an earlier point, *viz.* $O_{p'}(G) = 1$ and $G$ $p$-solvable.

(2) Let us assume all the conditions in (2), and let $G$ denote a minimum counterexample to (P6) $\Rightarrow$ (P2); it is clear, as usual, that $O_{2'}(G) = 1$, and then (by the same

reasoning as in (1)) that $O_{2'}(H) = 1$ for any subgroup $H$ of $G$ containing $S$, and therefore that $M :\equiv N_G(W(S))$ is the unique maximal subgroup of $G$ containing $S$. Let $\overline{G} = G/O_2(G)$; then $\overline{G}$ is 2-solvable, and $\overline{M}$ is the unique maximal subgroup of $\overline{G}$ containing $\overline{S}$. By induction, one has

$$\begin{aligned}\overline{G} &= N_{\overline{G}}(W(\overline{S}))O_{2'}(\overline{G})\\ &= N_{\overline{G}}(W(\overline{S}))(\overline{S}O_{2'}(\overline{G}));\end{aligned}$$

the two factors on the right-hand side of this equality contain $\overline{S}$, whence at least one is not contained in $\overline{M}$, i.e. either $N_{\overline{G}}(W(\overline{S})) = \overline{G}$ or $\overline{G} = \overline{S}O_{2'}(\overline{G})$. The first possibility leads to a contradiction as in the proof that (P3) $\Rightarrow$ (P5); therefore $\overline{G} = \overline{S}O_{2'}(\overline{G})$, i.e. $G$ has 2-length one.

As $\overline{S}$ is contained into a unique maximal subgroup of $\overline{G}$ ($\overline{M}$), $O_{2'}(\overline{G})$ possesses a unique maximal $\overline{S}$-invariant proper subgroup: $O_{2'}(\overline{G}) \cap \overline{M}$. It follows, firstly, that $O_{2'}(\overline{G})$ is a $q$-group for some prime $q \neq 2$: $O_{2'}(\overline{G}) = \overline{Q}$ ($Q \in \mathcal{S}yl_q(G)$), and therefore that $G = SQ$ is a solvable $\{2, q\}$-group, and secondly that $\overline{S}$ acts irreducibly on $\overline{Q}/\Phi(\overline{Q})$; in particular, $Z(\overline{S})$ is cyclic.

Let $N \equiv_{\text{def.}} \langle W(S)^G\rangle \lhd G$; then $O_{2'}(N) = 1$, and $S \cap N \in \mathcal{S}yl_2(N)$. If $N < G$, the minimality of $G$ yields:

$$\begin{aligned}N &= N_N(W(S \cap N))O_{2'}(N)\\ &= N_N(W(S \cap N)).\end{aligned}$$

But $W(S) \subseteq S \cap N \subseteq S$, whence $W(S) = W(S \cap N)$, as $W$ is excellent (in case (i) by assumption, and in case (ii) by Lemma 3.4). The Frattini argument now yields that:

$$\begin{aligned}G &= N N_G(S \cap N)\\ &\subseteq N N_G(W(S \cap N))\\ &\subseteq N_G(W(S \cap N))\\ &\subseteq G,\end{aligned}$$

whence $G = N_G(W(S \cap N)) = N_G(W(S))$ is 2-nilpotent, a contradiction. Therefore $N = G$, i.e. $G = \langle W(S)^G\rangle$; thence

$$\begin{aligned}\overline{G} &= \left\langle \overline{W(S)}^{\overline{G}} \right\rangle\\ &= \left\langle \overline{W(S)}^{\overline{S}\,\overline{Q}} \right\rangle\\ &= \left\langle \overline{W(S)}^{\overline{Q}} \right\rangle\\ &\subseteq \overline{W(S)}\,\overline{Q} \quad (\text{as } \overline{Q} \lhd \overline{G}),\end{aligned}$$

and $\overline{S} = \overline{S} \cap \overline{W(S)}\,\overline{Q} = \overline{W(S)}(\overline{S} \cap \overline{Q}) = \overline{W(S)}$, i.e. $S = W(S)O_2(G)$.

In case (ii), let $W = K_\varphi$; then $W(S) \nsubseteq O_2(G)$ (else one would have $S = W(S)O_2(G) = O_2(G) \lhd G$), whence there is an abelian subgroup $A$ of $S$ with $\varphi(A) = r_\varphi(P)$ and $A \nsubseteq O_2(G)$. Let $N = \langle A^G \rangle \lhd G$; if $N \neq G$, then, by induction, it follows as above that $W(S \cap N) \lhd N$ whence $W(S \cap N) \subseteq O_2(N) \subseteq O_2(G)$. But

$$\varphi(A) \leq r_\varphi(S \cap N) \leq r_\varphi(S) = \varphi(A)$$

whence $\varphi(A) = r_\varphi(S \cap N)$ and $A \subseteq K_\varphi(S \cap N) = W(S \cap N) \subseteq O_2(N) \subseteq O_2(G)$, a contradiction. Therefore $G = \langle A^G \rangle$, whence

$$\begin{aligned} \overline{G} &= \langle \overline{A}^{\overline{G}} \rangle \\ &= \langle \overline{A}^{\overline{S}\overline{Q}} \rangle \\ &= \langle \overline{A}^{\overline{S}} \rangle \overline{Q} \quad (\text{as } \overline{Q} \lhd \overline{G}); \end{aligned}$$

therefore

$$\begin{aligned} \overline{S} &= \overline{S} \cap \overline{G} \\ &= \overline{S} \cap \langle \overline{A}^{\overline{S}} \rangle \overline{Q} \\ &= \langle \overline{A}^{\overline{S}} \rangle (\overline{S} \cap \overline{Q}) \\ &= \langle \overline{A}^{\overline{S}} \rangle. \end{aligned}$$

By a well-known property of $p$-groups, it follows that $\overline{S} = \overline{A}$; in particular, $\overline{S}$ is abelian.

In case (i), $\mathcal{C} \subseteq \mathcal{C}_{2,2}$, i.e. $cl(S) \leq 2$, whence

$$\begin{aligned} [S, S] &\subseteq Z(S) \\ &\subseteq C_G(O_2(G)) \\ &\subseteq O_2(G) \end{aligned}$$

(by the solvability of $G$ and the Hall-Higman lemma), whence, again, $\overline{S}$ is abelian. Therefore, $\overline{S}$ is abelian in both cases, (i) and (ii). Now, from the fact that $Z(\overline{S})$ is cyclic, follows that $\overline{S}$ itself is. But $\overline{S} = \overline{W(S)} \subseteq \overline{\Omega_1(S)} \subseteq \Omega_1(\overline{S})$ (by the hypothesis); therefore $\overline{S}$ has order 2.

Now, as $\overline{S}$ acts irreducibly on the $\mathbf{F}_q$-module $M = \overline{Q}/\Phi(\overline{Q})$, the nontrivial element $\overline{t}$ of $\overline{S}$ either centralizes each element of $M$, or inverts each element of $M$; now, irreducibility forces $|M| = q$, i.e. $\overline{Q}/\Phi(\overline{Q}) = M$ is cyclic; but then so are $\overline{Q}$, and $Q \simeq \overline{Q}$.

Let now $\overline{H} = \overline{S}\Phi(\overline{Q})$; then $H < G$ (in fact, $|G : H| = q$), and $S \subseteq H$. Therefore $H$ is contained in $M = N_G(W(S))$, whence

$$\begin{aligned}[\overline{S}, \Phi(\overline{Q})] &= [\overline{W(S)}, \Phi(\overline{Q})] \\ &\subseteq [\overline{W(S)}, \overline{H}] \cap \Phi(\overline{Q}) \\ &\subseteq [\overline{W(S)}, \overline{M}] \cap \Phi(\overline{Q}) \\ &\subseteq \overline{W(S)} \cap \Phi(\overline{Q}) \\ &= 1,\end{aligned}$$

i.e. $\overline{S}$ centralizes $\Phi(\overline{Q})$. If $|\overline{Q}| \geq q^2$, then $\Omega_1(\overline{Q}) \subseteq \Phi(\overline{Q})$, whence $\overline{S}$ centralizes $\Omega_1(\overline{Q})$, and therefore $\overline{S}$ centralizes $\overline{Q}$, a contradiction. Thus $|\overline{Q}| = q$, and $\overline{G} = \overline{S}\overline{Q}$ is dihedral of order $2q$; it follows that $\overline{S}$ is a maximal subgroup of $\overline{G}$, i.e. $S$ is a maximal subgroup of $G$. Therefore $S = M = N_G(W(S))$, and $N_G(W(S))$ is 2-nilpotent; but now (P6) yields that $G$ itself is 2-nilpotent, a contradiction. □

## 5. Of $J_e$ and $\hat{J}$

By a well-known variation ([4], Proposition 4.162, p.253) on Thompson's factorization ([9], Theorem 1 (c)), any solvable $\Sigma_3$-free finite group $G$ with Sylow 2-subgroup $S$ satisfies:

$$G = N_G(J_e(S))C_G(Z(S))O_{2'}(G). \tag{5.1}$$

In [3] Glauberman introduced a new characteristic functor $\hat{J}$ having the property that, for each 2-group $S$, one has:

$$J_e(S) \subseteq \hat{J}(S) \subseteq S. \tag{5.2}$$

For this functor he was able to prove ([3], Theorem 7.4, p.48) that, for each 2-constrained $\Sigma_4$-free finite group $G$ and each $S \in \mathcal{S}yl_2(G)$, one had:

$$G = N_G(\hat{J}(S))C_G(Z(S))O_{2'}(G). \tag{5.3}$$

By (5.2) one finds $J_e(S) = J_e(\hat{J}(S))$ char $\hat{J}(S)$ whence

$$N_G(\hat{J}(S)) \subseteq N_G(J_e(S));$$

(5.3) is therefore stronger than (5.1).

In the particular case that $S$ has nilpotence class at most two, we can state

**Theorem 5.1.** *Let $G$ be a 2-constrained, $\Sigma_4$-free finite group with Sylow 2-subgroup $S$ of nilpotence class at most two; then one has:*

$$G = N_G(\hat{J}(S))O_{2'}(G).$$

By the above remark follows

**Corollary 5.2.** *In the situation of the theorem,*

$$G = N_G(J_e(S))O_{2'}(G).$$

Thus one can assert

**Corollary 5.3.** *Let $G$ be a finite solvable $\Sigma_4$-free group with Sylow 2-subgroup $S$ of class at most two; then:*

$$G = N_G(J_e(S))O_{2'}(G).$$

*In other words,* $(J_e, \mathcal{C}'(\Sigma_4) \cap \mathcal{S}olv, 2)$ *satisfies* (P1), *and hence* (P2)–(P6).

This Corollary was first proved by the author in [6].

Proof of Theorem 5.1. Let $G$ be a counterexample of minimal order.

(1) $\boldsymbol{O_{2'}(G) = 1}$. If not, $\overline{G} = G/O_{2'}(G)$ is of smaller order than $G$ and satisfies the hypothesis, whence

$$\overline{G} = N_{\overline{G}}(\hat{J}(\overline{S}))O_{2'}(\overline{G}) = N_{\overline{G}}(\hat{J}(\overline{S})).$$

But the canonical map $S \to SO_{2'}(G)/O_{2'}(G) = \overline{S}$ is an isomorphism, whence $\hat{J}(\overline{S}) = \hat{J}(S)O_{2'}(G)/O_{2'}(G)$ and

$$N_{\overline{G}}(\hat{J}(\overline{S})) = \frac{N_G(\hat{J}(S)O_{2'}(G))}{O_{2'}(G)} = \frac{N_G(\hat{J}(S))O_{2'}(G)}{O_{2'}(G)},$$

by the Frattini argument. Thus we get $G = N_G(\hat{J}(S))O_{2'}(G)$, a contradiction.

(2) $\boldsymbol{C_G(O_2(G)) \subseteq O_2(G)}$. Obvious, because $G$ is 2-constrained and $O_{2'}(G) = 1$.

(3) $\boldsymbol{M = N_G(\hat{J}(S))}$ **is the unique maximal subgroup of** $\boldsymbol{G}$ **that contains** $\boldsymbol{S}$. By hypothesis $M \subset G$. Let $H$ be a proper subgroup of $G$ containing $S$; one has $O_2(G) \subseteq S \subseteq H$, whence (as in the proof of Theorem 4.1 (1))

$$O_2(G) \subseteq O_2(H)$$

and:

$$\begin{aligned} C_H(O_2(H)) &= H \cap C_G(O_2(H)) \\ &\subseteq H \cap C_G(O_2(G)) \\ &\subseteq H \cap O_2(G) \quad \text{(by (2))} \\ &\subseteq O_2(H). \end{aligned}$$

Therefore $O_{2'}(H) = 1$ and $H$ is 2-constrained with Sylow 2-subgroup $S$; the minimality of $G$ now yields:

$$\begin{aligned} H &= N_H(\hat{J}(S))O_{2'}(H) = N_H(\hat{J}(S)) \\ &\subseteq N_G(\hat{J}(S)) = M. \end{aligned}$$

Thus $M$ is a proper subgroup of $G$ that contains any proper subgroup of $G$ containing $S$; the result follows.

(4) $\boldsymbol{Z(S) \subseteq Z(G)}$. By (5.3) one has

$$G = N_G(\hat{J}(S))C_G(Z(S))O_{2'}(G) = MC_G(Z(S));$$

thus $S \subseteq C_G(Z(S)) \nsubseteq M$, whence $C_G(Z(S)) = G$ by (3).

(5) **$\boldsymbol{G}$ centralizes $\boldsymbol{O_2(G)/Z(G)}$.** Let $C = C_G(O_2(G)/Z(G)) \triangleleft G$; then

$$[S, O_2(G)] \subseteq [S, S] \subseteq Z(S) \subseteq Z(G)$$

(by (4) and the hypothesis on $S$). It follows that $S \subseteq C$, whence

$$G = CN_G(S),$$

again by the Frattini argument. If $C$ were different from $G$, one would have $C \subseteq M$ (because of (3)) and

$$G = CN_G(S) \subseteq MN_G(S) \subseteq M.M = M,$$

a contradiction. Thus $C = G$.

(6) **The End**. By (5) one has $[G, O_2(G)] \subseteq Z(G)$, i.e.

$$[G, O_2(G), G] = [O_2(G), G, G] = 1.$$

Philip Hall's three subgroups lemma now yields

$$[G, G, O_2(G)] = 1,$$

that is:

$$G' \subseteq C_G(O_2(G)),$$

whence $G' \subseteq O_2(G)$ by (2). Therefore $H = G/O_2(G)$ is an abelian group with $O_2(H) = 1$, i.e. an abelian $2'$-group; it appears that $S = O_2(G) \triangleleft G$, whence $\hat{J}(S) \triangleleft G$, thus $G = M$ and again a contradiction ensues. This concludes the proof. □

REMARK 5.4. It seems difficult to generalize directly Corollary 5.2, and even Corollary 5.3, as the counter-examples to the $ZJ$-theorem for $p = 2$ given by Glauberman in the last paragraph of [2] show. Such a counterexample $G$ is solvable, with Sylow 2-subgroup $S$ of nilpotence class 3 (this is not difficult to see), and $S$ possesses a unique abelian subgroup of maximal order $A$, that is elementary abelian. Therefore $J_e(S)$, $J_R(S)$, $J(S)$ and $ZJ(S)$ all coincide with $A$, and neither is normal in $G$.

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