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ON THE COMMUTATIVITY OF THE RADICAL OF THE GROUP ALGEBRA OF A FINITE GROUP

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Let K be an algebraically closed field of characteristic $p>0$, and G a finite group of order $p^a m$ where $(p, m)=1$ and $a>0$. We denote by $J(KG)$ the radical of the group algebra KG. In case p is odd, D.A.R. Wallace [6] proved that $J(KG)$ is commutative if and only if G is abelian or $G'P$ is a Frobenius group with complement P and kernel G', where P is a Sylow p-subgroup of G and G' the commutator subgroup of G. On the other hand, in case $p=2$, S. Koshitani [1] has recently given a necessary and sufficient condition for $J(KG)$ to be commutative. In this paper, we shall give alternative conditions for $J(KG)$ to be commutative.

If $J(KG)$ is commutative, then G is a p-nilpotent group and a Sylow p-subgroup of G is abelian ([6], Theorem 2). We may therefore restrict our attention to a p-nilpotent group. Now, we put $N=O_{p'}(G)$. For a central primitive idempotent ε of KN, we put $G_\varepsilon=\{g\in G \mid g\varepsilon g^{-1}=\varepsilon\}$. Let a_i $(i=1, 2, \cdots, s)$ be a complete residue system of $G(\text{mod } G_\varepsilon)$

$$G = G_\varepsilon a_1 \cup G_\varepsilon a_2 \cup \cdots \cup G_\varepsilon a_s\,.$$

Then K. Morita [2] proved the following:

Theorem 1. *If G is a p-nilpotent group, then $e=\sum_{i=1}^{s}\varepsilon^{a_i}$ is a central primitive idempotent of KG and KGe is isomorphic to the matrix ring $(KP_\varepsilon)_f$ of degree f over KP_ε for some f, where P_ε is a Sylow p-subgroup of G_ε.*

In what follows, for a subset S of G, we denote by $\hat{S}$ the element $\sum_{x\in S}x$ of KG. By [5], Theorem, it holds that $J(KG)^2=0$ if and only if $p^a=2$. When this is the case, $J(KG)$ is trivially commutative. Therefore we may restrict our attention to the case $p^a\geqq 3$. The following proposition contains [1], Theorem 2.

Proposition. *If G is a non-abelian group and $p^a\geqq 3$, then the following conditions are equivalent:*

(1) *$J(KG)$ is commutative.*
(2) *$(G'P)'=G'$ and $J(KG'P)$ is commutative.*
(3) (i) *G' is a p'-group, and*

(ii) *each block of* $KG'P$, *which is not the principal block, is of defect* 0 *if* $p\neq 2$ *and of defect* 1 *or* 0 *if* $p=2$.

(4) (i) G' *is a* p'*-group, and*

(ii) *for each* $x\in G'-1$, $C_{G'P}(x)$ *is a* p'*-group if* $p\neq 2$ *and its order is not divisible by* 4 *if* $p=2$.

Proof. (1)$\Rightarrow$(2): We put $H=G'P$. Since H is a normal subgroup of G, we have $J(KH)\subset J(KG)$. Hence $J(KH)$ is commutative, and so, by [6], Theorem 2, $|H'|$ is not divisible by p. Since $J(KG)$ is commutative and $J(KG)\supset J(KH)\supset J(KH'P)\supset \hat{H}'J(KP)$, by [6], Lemma 3, we have $\hat{G}'KG\supset J(KG)^2\supset \hat{H}'^2 J(KP)^2=\hat{H}'J(KP)^2\ni \hat{H}'\hat{P}$. Thus, we have $G'\subset H'P$. Since G' is a p'-group by [6], Theorem 2, we have $G'=H'$.

(2)$\Rightarrow$(3): Since $J(KG'P)$ is commutative and $(G'P)'=G'$, G' is a p'-group by [6], Theorem 2. Now, we put $e_1=|G'|^{-1}\hat{G}'$, and $e_2=1-e_1$. Then e_1 and e_2 are central idempotents of $KG'P$. Thus we have $J(KG'P)=e_1J(KG'P)\oplus e_2J(KG'P)$. Since $J(KG'P)$ is commutative, by [6], Lemma 3, we have $J(KG'P)^2=e_1J(KG'P)^2\oplus e_2J(KG'P)^2\subset \widehat{(G'P)'}KG'P=\hat{G}'KG'P=e_1KG'P$. Therefore $e_2J(KG'P)^2=0$, and so by Theorem 1, every non-simple block of $e_2KG'P$ is isomorphic to the matrix ring over KD, where K is of characteristic 2 and D is a group of order 2. Hence $e_2KG'P$ is a direct sum of blocks of defect 0 or of defect 1 or 0 according as p is odd or 2. Since $e_1KG'P(=e_1KP)$ is the principal block, we obtain (3).

(3)$\Rightarrow$(4): This is easy by [3], Theorem 4.

(4)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(1): Since $G'P$ is a normal subgroup of G and $[G\colon G'P]$ is not divisible by p, we have $J(KG)=J(KG'P)KG$. We put $e_1=|G'|^{-1}\hat{G}'$, and $e_2=1-e_1$. Then e_1 and e_2 are central idempotents of KG and $J(KG)=e_1J(KG'P)\cdot KG\oplus e_2J(KG'P)KG$. Since $e_1J(KG'P)KG\subset \hat{G}'KG$, $e_1J(KG'P)KG$ is a central ideal of KG by [4], Lemma 5. By Theorem 1, every block of $e_2KG'P$ is isomorphic to the matrix ring over KD, where D is a p-group. From our assumption, every non-simple block of $e_2KG'P$ has the radical of square zero. Hence $e_2[J(KG'P)KG]^2=e_2J(KG'P)^2KG=0$, and so $e_2J(KG'P)KG$ is commutative. Thus, $J(KG)$ is commutative.

REMARK. The condition (4) of Proposition for p odd is equivalent to the condition of Wallace's result ([6]) that $G'P$ is a Frobenius group with complement P and kernel G'.

Now, in case $p=2$, we shall give the conditions for $J(KG)$ to be commutative.

Theorem 2. *Assume that* $p=2$, $2^a\geqq 4$ *and* $G'\neq 1$. *Then the following conditions are equivalent*:

(1) *$J(KG)$ is commutative.*

(2) *G' is of odd order and $|P\cap P^h|\leqq 2$ for every $h\in G'P-P$.*

(3) *G' is of odd order and $C_{G'P}(s)/\langle s\rangle$ is either a 2-group or a Frobenius group with complement $P/\langle s\rangle$ for every involution s of P.*

Proof. (1)$\Rightarrow$(2): Suppose that $J(KG)$ is commutative. Then, by Proposition, G' is of odd order. Let h be an arbitrary element of $G'P-P$, and x an arbitrary element of $P\cap P^h$. Then $hxh^{-1}x^{-1}\in P\cap G'=1$, and so $x\in C_{G'P}(h)$. Thus, $P\cap P^h\subset C_{G'P}(h)$. Since we may assume that $h\in G'-1$, we obtain $|P\cap P^h|\leqq 2$ by Proposition.

(2)$\Rightarrow$(3): Let s be an arbitrary involution of P such that $C_{G'P}(s)\neq P$. Then $P\cap P^x=\langle s\rangle$ for $x\in C_{G'P}(s)-P$, and so $C_{G'P}(s)/\langle s\rangle$ is a Frobenius group with complement $P/\langle s\rangle$.

(3)$\Rightarrow$(1): Let x be an element of $G'-1$, and S a Sylow 2-subgroup of $C_{G'P}(x)$. Suppose that $S\neq 1$. Then $S\subset P^u$ for some $u\in G'P$, and $x\in C_{G'P}(S)\subset C_{G'P}(s)$ for every involution s of S. Hence, $C_{G'P}(s)$ is not a 2-group, and so $C_{G'P}(s)/\langle s\rangle$ is a Frobenius group with complement $P^u/\langle s\rangle$. Thus, we have $S\subset P^u\cap P^{ux}=\langle s\rangle$, and hence $|C_{G'P}(x)|$ is not divisible by 4, which implies (1) by Proposition.

Corollary. *Assume that $p=2$, $2^a\geqq 4$ and $G'\neq 1$. If $J(KG)$ is commutative, then a Sylow 2-subgroup of G is a cyclic group or an abelian group of type* $(2, 2^{a-1})$.

Proof. Suppose that $J(KG)$ is commutative. Then, by Theorem 2, $|P\cap P^h|\leqq 2$ for every $h\in G'P-P$. If $P\cap P^h=1$ for all $h\in G'P-P$, then $G'P$ is a Frobenius group with complement P and kernel G'. Hence P is cyclic. On the other hand, if $P\cap P^h=\langle s\rangle$ for some $h\in G'P-P$ and some involution s of P, then $hsh^{-1}s^{-1}\in P\cap G'=1$, and so $h\in C_{G'P}(s)$ and $h\notin P$. Therefore $C_{G'P}(s)$ properly contains P. Hence, $C_{G'P}(s)/\langle s\rangle$ is a Frobenius group with complement $P/\langle s\rangle$ by the condition (3) of Theorem 2. Hence $P/\langle s\rangle$ is cyclic, and so P is a cyclic group or an abelian group of type $(2, 2^{a-1})$.

REMARK. In case G is a non-abelian group and $p^a\geqq 3$, S. Koshitani [1] proved that if $J(KG)$ is commutative, then $N_G(P)$ is abelian. This is included in the following proposition: *Let G be a non-abelian group, and $p^a\geqq 3$. If $J(KG)$ is commutative then G is a semi-direct product of G' by (abelian) $N_G(P)$.*

Proof. It is easy to see $G=G'N_G(P)$. Suppose that $J(KG)$ is commutative. Let x be a p'-element of $N_{G'P}(P)$. Since $G'P$ is a p-nilpotent group, $N_{G'P}(P)$ is the direct product of P and a normal p'-subgroup, and so $C_{G'P}(x)$ contains P. Hence, by Proposition (4), we have $x=1$, which implies that $G'\cap N_G(P)=1$.

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