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ON DOUBLY TRANSITIVE PERMUTATION GROUPS

YUTAKA HIRAMINE

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1. Introduction

Let $G$ be a doubly transitive permutation group on a finite set $\Omega$ and $\alpha \in \Omega$. Using the notation of [9], we denote a normal subgroup of $G_\alpha$ by $N^\alpha$. Then, for $\beta \in \Omega$ other, we define $N^\beta$ so that $g^{-1}N^\beta g = N^\gamma$ where $\gamma = \beta^g$.

In this paper we shall prove the following:

Theorem 1. Let $G$ be a doubly transitive permutation group on a finite set $\Omega$. Suppose that $\alpha$ is an element of $\Omega$. If $G_\alpha$ has a normal simple subgroup $N^\alpha$ which is isomorphic to $PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ with $q = 2^n$, $n \geq 2$, then one of the following holds:

(i) $|\Omega| = 6$, $G \cong A_6$ or $S_6$ and $N^\alpha \cong PSL(2, 4)$.

(ii) $|\Omega| = 11$, $G \cong PSL(2, 11)$ and $N^\alpha \cong PSL(2, 4)$.

(iii) $G$ has a regular normal subgroup.

We introduce some notations: Let $G$ be a permutation group on $\Omega$. For $X \leq G$ and $\Delta \subseteq \Omega$, we define $F(X) = \{ \alpha \in \Omega \mid \alpha^x = \alpha \text{ for all } x \in X \}$, $X(\Delta) = \{ x \in X \mid \Delta^x = \Delta \}$, $X_\Delta = \{ x \in X \mid \alpha^x = \alpha \text{ for all } \alpha \in \Delta \}$ and $X^\Delta = X(\Delta) / X_\Delta$, the restriction of $X$ on $\Delta$. If $p$ is a prime, we denote by $O_p(X)$, the subgroup of $X$ generated by all $p'$-elements in $X$. Other notations are standard ([16], [16]).

2. Preliminary results

Lemma 2.1. Let $G$ be a doubly transitive permutation group on $\Omega$ of even degree and $N^\alpha$ a nonabelian simple normal subgroup of $G_\alpha$ with $\alpha \in \Omega$. If $C_G(N^\alpha) \neq 1$, then $N^\alpha = N^\alpha \cap N^\beta$ for $\alpha \neq \beta \in \Omega$ and $C_G(N^\alpha)$ is semi-regular on $\Omega - \{ \alpha \}$.

Proof. Set $C^\alpha = C_G(N^\alpha)$. By Corollary B3 and Lemma 2.8 of [17], $C^\alpha$ is semi-regular on $\Omega - \{ \alpha \}$ or $N^\alpha$ is a T.I. set in $G$. Since $|\Omega|$ is even and $N^\alpha$ is $\frac{1}{2}$-transitive on $\Omega - \{ \alpha \}$, $|N^\alpha: N_\beta^\alpha|$ is odd for $\alpha \neq \beta \in \Omega$. Hence $N^\alpha$ is not semiregular on $\Omega - \{ \alpha \}$. By Theorem A of [9], $N^\alpha$ is not a T.I. set in $G$. Hence $C^\alpha$ is semi-regular on $\Omega - \{ \alpha \}$.

Set $\Delta = F(N_\beta^\alpha)$. Since $C^\alpha \leq G(\Delta)$, $[C^\alpha, G_\alpha] \leq C^\alpha \cap G_\alpha = 1$. By Corollary...
B1 of [17], \( N_\theta^\beta \leq G_\lambda \) and so \([C^\alpha, N_\theta^\beta] = 1\). Let \( 1 \neq x \in C^\alpha \) and set \( \beta^x = \gamma \). Then \( N_\theta^\beta = x^{-1}N_\theta^\beta x = N_\theta^\gamma \). Hence \( N_\theta^\beta \leq N_\theta^\gamma \). Since \( \beta \neq \gamma \) and \( G \) is doubly transitive on \( \Omega \), \( |N_\theta^\beta| = |N_\theta^\gamma| \). Hence \( N_\theta^\beta = N_\theta^\gamma \). Similarly we have \( N_\theta^\gamma = N_\theta^\beta \). Hence \( N_\theta^\gamma = N_\theta^\beta \) and so \( N_\theta^\gamma = N_\theta^\beta \). Since \( G \) is doubly transitive on \( \Omega \), \( N_\theta^\beta = N_\theta^\gamma \cap N_\theta^\beta \).

**Lemma 2.2.** Let \( G \) be a transitive permutation group on a set \( \Omega \), \( H \) a stabilizer of a point of \( \Omega \) and \( M \) a nonempty subset of \( G \). Then

\[
|F(M)| = |N_\theta(M)| \times |\text{cl}_\theta(M) \cap H| / |H|.
\]

Here \( \text{cl}_\theta(M) \cap H = \{ g^{-1}Mg | g^{-1}Mg \leq H, g \in G \} \).

Proof. Set \( W = \{(L, \alpha) | L \in \text{cl}_\theta(M), \alpha \in F(L)\} \) and \( W_\alpha = \{L | L \in \text{cl}_\theta(M), F(L) \equiv \alpha\} \). By the transitivity of \( G \), \( |W_\alpha| = |W_\beta| \) holds for every \( \alpha, \beta \in \Omega \). Counting the number of elements of \( W \) in two ways, we obtain \( |G: N_\theta(M)| \times |F(M)| = |G: H| \times |\text{cl}_\theta(M) \cap H| \). Thus we have Lemma 2.2.

**Lemma 2.3.** Let \( G \cong \text{PSL}(2, q), \text{Sz}(q) \) or \( \text{PSU}(3, q) \) with \( q = 2^n > 2 \) and suppose that \( G \) is a transitive permutation group on a set \( \Omega \) of odd degree. Let \( H \) be a stabilizer of a point of \( \Omega \). Then we have the following:

(i) \( H \) has a unique Sylow 2-subgroup \( S \) of \( G \) and \( H = DS \) for a Hall \( 2 \)-subgroup \( D \) of \( H \) where \( D \leq Z_{q^2-1} \).

(ii) Let \( L \) be a subgroup of \( G \) such that \( |L| = |H| \). Then \( L \in \text{cl}_\theta(H) \).

(iii) \( S \) is semi-regular on \( \Omega - F(S) \) and \( |F(S)| = |F(H)| = |N_\theta(S): H| \).

(iv) Set \( D = V \times K \) where \( V \leq Z_{q^2+1}, K \leq Z_{q^1-1} \). Then \( K \) acts semi-regularly on \( \Omega - F(K) \) and if \( K \neq 1 \), \( |F(K)| = 2 |F(S)| \).

Proof. Since \( G \) is generated by its two distinct Sylow 2-subgroups and \( 1 \neq |G: H| \) is odd, \( H \) contains a unique Sylow 2-subgroup \( S \) of \( G \) where \( S = O_2^\beta(H) \). By the structure of \( N_\theta(S) \) we have (i) (cf. § 3 of [2]).

To prove (ii) we may assume that \( S \leq L \). As above \( S = O_2^\beta(L) \) and \( L = D_\lambda S \) where \( D_\lambda \leq Z_{q^2-1} \). Since \( N_\theta(S)/S \) is cyclic and \( |H| = |L| \), we get \( H = L \). Thus (ii) holds.

Let \( t \in I(S) \). Applying Lemma 2.2, \( |F(t)| = |N_\theta(t)| \times |\text{cl}_\theta(t) \cap H| / |H| = (|N_\theta(t)| \times |\text{cl}_\theta(t) \cap N_\theta(S)| / |N_\theta(S)|) \times (|N_\theta(S)| / |H|) \). Since \( N_\theta(S) \) is a stabilizer of the usual doubly transitive permutation representation of \( G \), we have \( |N_\theta(t)| \times |\text{cl}_\theta(t) \cap N_\theta(S)| / |N_\theta(S)| = 1 \), hence \( |F(t)| = |N_\theta(S): H| \). On the other hand, \( |F(S)| = |N_\theta(S)| \times |\text{cl}_\theta(S) \cap H| / |H| = |N_\theta(S): H| \). Therefore \( S \) acts semi-regularly on \( \Omega - F(S) \). As \( N_\theta(H) = N_\theta(S) \), similarly we have \( |F(S)| = |F(H)| \). Thus (iii) holds.

Let \( x \) be a nontrivial element of \( K \). Then we have \( |F(\langle x \rangle)| = |N_\theta(\langle x \rangle)| \times |\text{cl}_\theta(\langle x \rangle) \cap H| / |H| = (|N_\theta(S)| \times |\text{cl}_\theta(S) \cap N_\theta(S)| / |N_\theta(S)|) / |H| \).

As before we have \( |N_\theta(\langle x \rangle)| \times |\text{cl}_\theta(\langle x \rangle) \cap N_\theta(S)| / |N_\theta(S)| = 2 \). Hence \( |F(x)| = 2 \cdot |N_\theta(S): H| \) and this is independent of the choice of \( x \in K^\lambda \). Thus (iv)
holds.

**Lemma 2.4.** Let $G=\mathrm{PSL}(2, q), \mathrm{Sz}(q)$ or $\mathrm{PSU}(3, q)$ with $q=2^n>2$ and $S$ be a Sylow 2-subgroup of $G$. Let $H=N_G(S)$, $t$ an involution outside $H$, $D=H \cap H^t$, $V=C_D(t)$ and $K=\{d \in D \mid d^t=d^{-1}\}$. Then the following hold:

(i) $N_G(\langle k \rangle)=\langle t \rangle \triangleleft D$ whenever $1 \neq k \in K$.

(ii) If $G=\mathrm{PSU}(3, q)$ and $1 \neq U$ is a subgroup of $V$, then $N_G(U)=C_G(V)=N \times V$ where $N$ is a subgroup of $G$ isomorphic to $\mathrm{PSL}(2, q)$.

Proof. (i) follows from the structure of $\mathrm{PSL}(2, q), \mathrm{Sz}(q)$ or $\mathrm{PSU}(3, q)$ (§3 of [2]).

We now regard $\mathrm{PSU}(3, q)$ as a usual doubly transitive permutation group on a set $\Omega$ with $|\Omega|=3^n+1$ points. Then $V$ is semi-regular on $\Omega=\mathrm{F}(V)$ and $G(\mathrm{F}(U))/G(\mathrm{F}(U))$ is doubly transitive on $\mathrm{F}(U)=\mathrm{F}(V)$. Clearly $N_G(U) \leq G(\mathrm{F}(U))$ and $G_\mathrm{F}(U)=V$. Hence $N_G(U) \leq N_G(V)$. Since $V$ is cyclic, $N_G(V) \leq N_G(\langle U \rangle)$ and so $N_G(U)=N_G(V)$. Now we set $M=\mathrm{O}_2^+(N_G(V))$. Then as $[Z(S), V]=1$ and $Z(S)$ is a Sylow 2-subgroup of $N_G(V)$, $M$ centralizes $V$. By the Frattini argument $N_G(V)=(N_G(V) \cap N(S))M=N_H(V)M=\mathrm{DZ}(S) \cdot M \leq C_G(V)$. Hence $N_G(V)=C_G(V)$. By the direct computation, we obtain (ii).

**Lemma 2.5.** Let $G=\mathrm{PSL}(2, q), \mathrm{Sz}(q)$ or $\mathrm{PSU}(3, q)$ with $q=2^n>2$ and let $S$ be a Sylow 2-subgroup of $G$.

(i) If $T$ is a maximal subgroup of $S$, then $N_G(T)=S$.

(ii) Unless $G=\mathrm{PSU}(3, q)$ where $q=2^n$ and $n$ is odd, then by conjugation $N_G(S)$ acts regularly on the set of all maximal subgroups of $S$.

Proof. Since $N_G(S)$ is strongly embedded in $G$, $S \leq N_G(T) \leq N_G(S)$ and so $N_G(T)=RS$ where $R$ is a Hall 2-subgroup of $N_G(T)$. As $|S:T|=2$, $R$ centralizes $S/T=Z_2$ and hence there exists an element $t \in C_R(R)-T$. If $G=\mathrm{PSL}(2, q)$ or $\mathrm{Sz}(q)$, then $R=1$ (§3 of [2]). If $G=\mathrm{PSU}(3, q)$ and $n \neq 1$, then by (ii) of Lemma 2.4, $t \in \Omega(S)=\Omega(S) \leq T$, a contradiction. Thus (i) holds.

Let $\Gamma$ be the set of all maximal subgroups of $S$. Then by conjugation, $N_G(S)$ acts on $\Gamma$ and $(N_G(S))_T=S$ for $T \in \Gamma$ by (i). Under the assumption of (ii), we can easily verify $|\Gamma|=|N_G(S):S|$. From this (ii) follows at once.

**Lemma 2.6.** Let $G=\mathrm{PSL}(2, q), \mathrm{Sz}(q)$ or $\mathrm{PSU}(3, q)$ with $q=2^n>2$ and $A$ be the full automorphism group of $G$. Let $S$ be a Sylow 2-subgroup of $G$. Then $C_A(S)=Z(S)$.

Here we identify $G$ with the inner automorphism group of $G$.

Proof. Let $\Omega$ be the set of all Sylow 2-subgroups of $G$. Then $A$ acts faithfully on $\Omega$ and the action of $G$ on $\Omega$ is the same as the usual doubly transitive permutation representation. Hence $S$ is regular on $\Omega=\{S\}$ and so $C_A(S)$ is a 2-subgroup of $A$. If $G=\mathrm{Sz}(q)$, $A/G$ is cyclic of odd order and so $C_A(S) \leq G$. Hence $C_A(S)=C_G(S)=Z(S)$. If $G=\mathrm{PSL}(2, q)$, $S$ is abelian, so that $C_A(S)=S$. 

If $G \cong PSU(3, q)$, there exists a field automorphism such that $\langle f \rangle S$ is a Sylow 2-subgroup of $N_A(S)$. From this $C_A(S) \leq O_2(N_A(S)) \leq \langle f \rangle S$. If $g \in C_A(S) - S$ where $g \in \langle f \rangle$ and $s \in S$, then $g$ centralizes $Z(S)$ and so $g$ is a field automorphism of order 2 by the structural property of $A$. Since $g$ centralizes $s$, $s$ must be contained in $Z(S)$. Therefore $g$ centralizes $S$, while $g$ is a field automorphism of order 2. This is a contradiction. Thus $C_A(S) = S \cap C_A(S) = Z(S)$.

**Lemma 2.7.** Let $G \cong PSU(3, q)$, $q = 2^e$ such that $n$ is even. Then $\text{Aut}(G) = \langle f \rangle G$ for a field automorphism $f$ of $G$ (see [14]). Let $B$ be a Borel subgroup and let $D$ be a diagonal subgroup of $G$. Then $B = DS$ and $\text{Sylow } 2$-subgroup $S$ of $G$. Set $D = V \times K$ with $V = Z_{q+1}$, $K = Z_{q-1}$. Then $C_A(Z(S)) = \langle \tau \rangle VS$ where $A = \langle f \rangle G$ and $\{\tau\} = I(\langle f \rangle)$.

Proof. By the structural properties of $A$, $[V, Z(S)] = 1$ and $C_{\langle f \rangle}(Z(S)) = \langle \tau \rangle$. Since $N_A(Z(S)) \supset O_2(N_A(Z(S))) = S$, $N_A(Z(S)) = \langle f \rangle N_G(S)$. Hence $C_A(Z(S)) = C(Z(S)) \cap \langle f \rangle DS = C_{\langle f \rangle}(Z(S)) VS$. Let $gh \in C_{\langle f \rangle}(Z(S))$ with $g \in \langle f \rangle$, $k \in K$. Since $g$ is a field automorphism of $G$, it centralizes a nontrivial element $s$ in $Z(S)$. Then $k$ centralizes $s$ and so $k = 1$, for otherwise $s \in C_A(k) = VK$, a contradiction. So $C_{\langle f \rangle}(Z(S)) = C_{\langle f \rangle}(Z(S)) = \langle \tau \rangle$. Thus $C_A(Z(S)) = \langle \tau \rangle VS$.

3. The case $|\Omega|$ is even

Let $G$ be a doubly transitive permutation group on a finite set $\Omega$ of even degree satisfying the assumption of our theorem. Let $\alpha \in \Omega$ and $\{\alpha\}, \Delta_1, \ldots, \Delta_r$ be the set of all $\mathbb{N}^\ast$-orbits on $\Omega$. Since $\mathbb{N}^\ast$ is normal in $G_\alpha$, $|\Delta_i| = |\Delta_j|$ for $1 \leq i, j \leq r$. Hence $|\Omega| = 1 + |\Delta_i|$ and so both $|\Delta_i|$ and $r$ are odd. From this, $\mathbb{N}^\ast_{\beta}$ contains a unique Sylow 2-subgroup of $\mathbb{N}^\ast$ for $\beta = \alpha$ by (i) of Lemma 2.3. Set $S = O_2(\mathbb{N}^\ast_{\beta})$.

(3.1) The following hold.

(i) For each $\Delta_i$ with $1 \leq i \leq r$, there exists $\beta_i \in \Delta_i$ such that $\mathbb{N}^\ast_{\beta_i} = \mathbb{N}^\ast_{\beta_i}$.  
(ii) $F(S) = F(\mathbb{N}^\ast_{\beta_i})$, $|F(S)| = |\mathbb{N}^\ast(\mathbb{N}^\ast_{\beta_i}) : \mathbb{N}^\ast_{\beta_i}| \times r + 1$ and $S$ is semi-regular on $\Omega - F(S)$.  
(iii) Set $C^\ast = C_G(\mathbb{N}^\ast)$. Then $C^\ast = O(G_\alpha)$ and is semi-regular on $\Omega - \{\alpha\}$.

Proof. Let $\gamma \in \Delta_i$. Since $|\mathbb{N}^\ast_{\beta_i}| = |\mathbb{N}^\ast_{\beta_i}|$, by (ii) of Lemma 2.3, $\mathbb{N}^\ast_{\beta_i} = (\mathbb{N}^\ast_{\beta_i})^\ast$ for some $x \in \mathbb{N}^\ast$. Put $\gamma^\ast = \beta_i$. Then $\beta_i \in \Delta_i$ and $\mathbb{N}^\ast_{\beta_i} = \mathbb{N}^\ast_{\beta_i}$. Thus (i) holds.

Hence by (iii) of Lemma 2.3, for each $\Delta_i$ with $1 \leq i \leq r$, $F(S) \cap \Delta_i = F(\mathbb{N}^\ast_{\beta_i}) \cap \Delta_i$, $|F(S) \cap \Delta_i| = |\mathbb{N}^\ast_{\beta_i}(\mathbb{N}^\ast_{\beta_i}) : \mathbb{N}^\ast_{\beta_i}|$ and $S$ is semi-regular on $\Delta_i - (\Delta_i \cap F(S))$. Thus (ii) holds.

Since $[O(G_\alpha), \mathbb{N}^\ast] \leq O(G_\alpha) \cap \mathbb{N}^\ast$ and $\mathbb{N}^\ast$ is a non abelian simple group, $[O(G_\alpha), \mathbb{N}^\ast] = 1$ and so $O(G_\alpha) \leq C^\ast$. By Lemma 2.1, $C^\ast$ is semi-regular on
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\[ \Omega - \{ \alpha \} \] Since \( G_\alpha \triangleright C^\alpha, C^\alpha \) is \( \frac{1}{2} \)-transitive on \( \Omega - \{ \alpha \} \). Hence \( |C^\alpha| \mid |\Omega| - 1 \). From this \( C^\alpha \) is of odd order and hence \( C^\alpha \cong O(G_\alpha) \). Thus \( C^\alpha = O(G_\alpha) \).

As a Chevalley group, \( N^* \) has a Borel subgroup \( N_\alpha^*(S) \). Let \( D \) be a diagonal subgroup of \( N_\alpha^*(S) \). Then \( N_\alpha^*(S) = DS \). We now denote \( G_\alpha \) by \( G^\alpha \). By the properties of \( PSL(2, q) \), \( Sz(q) \) or \( PSU(3, q) \) ([14]), there exists a field automorphism \( f \) such that \( \langle f \rangle \triangleright N^*/N^\alpha \) is a Sylow 2-subgroup of \( N^*/N^\alpha \). Since \( C^\alpha = O(G_\alpha) \), we may assume \( f \) is a 2-element in \( G_\alpha \). Since \( DC^\alpha \cap N^\alpha = D \) and \( SC^\alpha \cap N^\alpha = S \), \( D \) and \( S \) are \( f \)-invariant. Clearly \( \langle f \rangle \) is a Sylow 2-subgroup of \( G_\alpha \). Since \( \langle f \rangle \cap N^\alpha = 1 \), \( \langle f \rangle \cap S \leq C^\alpha \) and so \( \langle f \rangle \cap S = 1 \). Thus we have the following.

(3.1) There exists a 2-element \( f \) in \( G_\alpha \) satisfying the following.

(i) \( f \) acts on \( N^* \) as a field automorphism of \( N^* \).
(ii) \( D \) and \( S \) are \( f \)-invariant and \( \langle f \rangle \cap S = 1 \).
(iii) \( \langle f \rangle \) is a Sylow 2-subgroup of \( G_\alpha \).

(3.2) \( N^\alpha \cap N^\beta \) is cyclic of odd order.

Proof. By Lemma 2.1 and (iii) of (3.1), we may assume that \( C^\alpha = 1 \). First we claim that \( |S|: S \cap N^\beta| = 1 \) or 2. Since \( S/S \cap N^\beta \) is isomorphic to a 2-subgroup of the outer automorphism group of \( N^\beta \), \( S/S \cap N^\beta \) is cyclic. But \( S/S' \) is an elementary abelian 2-group and so \( S/S \cap N^\beta \) is cyclic. As \( S/S \cap N^\beta \) is cyclic and \( G^\alpha \cap N^\beta \) is cyclic, \( S/S \cap N^\beta \) is cyclic. As \( S/S \cap N^\beta \) is a normal subgroup of \( G^\alpha \cap N^\beta \) of order 2, \( I(G^\alpha) \subseteq N^\beta \cap N^\alpha \). Let \( f \) be as defined in (3.1). Then \( f \neq 1 \) as \( N^\beta \cap N^\alpha \leq N^* \). Let \( \tau \in I(\langle f \rangle) \). Since \( \tau \in N_\alpha^*(S), S = N^\alpha \) and \( I(S) = \{ \alpha \} \) is odd, there exists \( \gamma \) such that \( \gamma \in F(\tau) \cap F(N^\alpha) \) and \( \gamma \neq \alpha \). Clearly \( N^\alpha \leq N^\gamma \), so that \( N^\beta \leq N^\gamma \). Therefore we may assume \( F(\tau) \equiv \beta \) and \( \tau \in G_{\alpha \beta} \). By Corollary B1 of [17] \( F(N^\beta) = F(N^\alpha) \). From this \( F(\tau) \equiv F(N^\beta) \equiv F(N^\alpha) \) because \( \tau \in I(G_{\alpha \beta}) \subseteq N^\beta \cap N^\alpha \). So \( \langle \tau \rangle N^\beta \leq \langle \tau \rangle N^\alpha \cap (N(N^\beta)_{F(N^\beta)}) \). Let \( D \) be as defined in (3.1). Then \( D \leq N_\alpha^*(N^\beta) \) and \( D \) is \( \tau \)-invariant. Hence \( [D, \tau] \leq \langle \tau \rangle N^\alpha \cap (N(N^\beta)_{F(N^\beta)}) \) \( \cap D = 1 \). Therefore \( \tau \) centralizes \( D \). Since \( \tau \) is a field automorphism of \( N^\alpha \) of order 2 and \( D \) is a diagonal subgroup of \( N^\alpha \), this is a contradiction.

(3.3) The following hold.

(i) \( N^\alpha \cap N^\beta = N^\gamma \cap N^\delta \) for, \( \gamma, \delta \in F(N^\alpha \cap N^\beta) \) with \( \gamma \neq \delta \).
(ii) \( G(F(S)) = N_\alpha(N^\alpha \cap N^\beta) \).
(iii) Let \( M \) be a subgroup of \( N^\alpha \cap N^\beta \) which contains \( S \). Then \( F(M) = \langle f \rangle \cap S \).
$F(S)$ and $N_G(M)$ is doubly transitive on $F(S)$.

(iv) $C_{G_{a}}(S)=Z(S) \times C^a$.

(v) Let $M$ be as defined in (iii) and suppose $C^a \neq 1$. Then $O_2(C_{G}(M))_{F(S)}$ is a regular normal elementary abelian 2-subgroup of $N_{G}(M)_{F(S)}$.

Proof. Let $\gamma, \delta \in F(N^a \cap N^\beta)$ with $\gamma \neq \delta$. We may assume $\alpha \neq \gamma$. Since $G$ is doubly transitive on $\Omega$, $|N^\alpha \cap N^\beta| = |N^\gamma \cap N^\beta|$. By the choice of $\gamma, N^\gamma \cap N^\beta \leq N^\gamma$ and $N_{N^a}(S)/S$ is cyclic. Hence $N^\gamma \cap N^\beta = N^\gamma \cap N^\beta$. Similarly $N^\gamma \cap N^\alpha = N^\gamma \cap N^\alpha$. Thus (i) holds.

Since $N_{G}(N^\alpha \cap N^\beta) \leq N_{G}(S), N_{G}(N^a \cap N^\beta) \leq G(F(S))$. Let $x \in G(F(S))$. Then $\alpha^x, \beta^x \in F(S)$ and $F(S)=F(N^a)$ by (ii) of (3.1). Hence $\alpha^x, \beta^x \in F(N^\alpha \cap N^\beta)$. Therefore by (i) $N^\alpha \cap N^\beta = N^\alpha \cap N^\beta$ and so $x \in N_{G}(N^\alpha \cap N^\beta)$. Thus (ii) holds.

Suppose $S \leq M \leq N^\alpha \cap N^\beta$. If $M^f \leq G_{a\beta}$ for some $g \in G_a$. Then $M^f = M$ because $S < M$ and $N^\alpha \cap S$ is cyclic of odd order. By the Witt’s Theorem $N_{G_{a}}(M)$ is transitive on $F(M) - \{a\}$. Similarly $N_{G_{a}}(M)$ is transitive on $F(M) - \{\beta\}$. We may assume $|F(M)| > 2$. Hence $N_{G_{a}}(M)$ is doubly transitive on $F(M)$. By (ii) of (3.1), $F(M) = F(S)$. Thus (iii) holds.

We denote $G_{a}/C^a$ by $G_a$. Clearly $C_{G}(S) = Z(S) = Z(S) \times C^a$. The converse implication is obvious. Thus (iv) holds.

Suppose $C^a \neq 1$. Then since $C^a$ is semi-regular on $\Omega - \{a\}$, $C_{G}(M)_{F(S)} \cong (C^a)^{F(S)} \cong 1$. As $N_{G}(M)_{F(S)}$ is doubly transitive by (iii), $C_{G}(M)_{F(S)}$ is transitive. By (iv), $(C^a)^{F(S)} \leq C_{G_a}(M)_{F(S)} \leq (Z(S) \times C^a)^{F(S)}$ and so $C_{G_a}(M)_{F(S)} = (C^a)^{F(S)}$. Hence $C_{G}(M)_{F(S)}$ is a Frobenius group and so $O_2(C_{G}(M)_{F(S)}) \neq 1$ because $|F(S)|$ is even. Since $C_{G}(M)_{F(S)} \cong (Z(S) \times C^a)^{F(S)} = Z(S), O_2(C_{F}(M)_{F(S)}) = O_2(C_{G}(M)_{F(S)})$ and this is regular on $F(S)$. As $N_{G}(M)_{F(S)} \geq C_{G}(M)_{F(S)}, O_2(C_{G}(M)_{F(S)})$ must be a regular normal elementary abelian 2-subgroup of $N_{G}(M)_{F(S)}$. Thus (v) holds.

(3.4) There exists an involution $t$ such that $ccl_{G}(t) \cap S = \phi, \alpha^t = \beta$ and $F(t) \cap F(S) = \phi$. Set $\mu = |N_{N^a}(S) : N^a_{\phi}|$ and $|S| = q^f$. Then we have

(i) $|\Omega| = (q^f + 1)\mu r + 1$.

(ii) $|C_{S}(t)| \geq \sqrt{q}, \sqrt{2q}$ or $q$ according as $N^a = PSL(2, q), Sz(q)$ or $PSU(3, q)$, respectively. Furthermore $|C_{S}(t)|$ and $|F(S)| = \mu r + 1$.

(iii) If $\mu = 1$, then $|\Omega| = 6$ and $G = A_6$ or $S_6$.

(iv) $|\Omega| = |F(S)|_{2^*} |G : N_G(S)|_{2^*}$.

Proof. Since $|\Delta_t| = |N^a : N^a_{\phi}| = |N^a : N_{N^a}(S) : N^a_{\phi}| = (q^f + 1)\mu$ and $|\Delta_t| = |\Delta_t| \mu r + 1$. Hence (i) holds.

Since $G$ is doubly transitive on $\Omega$, there exists an involution $t$ such that $ccl_{G}(t) \cap S \neq \phi$ and $\alpha^t = \beta$. Then $t$ normalizes $O_2(N_{N^a} \cap N^\beta) = S$. Claim $F(t) \cap F(S) = \phi$. Suppose not and let $\gamma \in F(t) \cap F(S)$. As $S \leq N^a_{\gamma}$, $S \leq N^\alpha \cap N^\beta$ by (i) of (3.3). Let $g$ be such that $t^g \in S$. Then $t \in N^a \cap G_{\gamma} = N^a_{\gamma}$ where $\delta = \alpha^{t-1}$ and
hence \( t \in N' \). Since \( t \) normalizes \( S \) and \( \langle t \rangle S \leq N' \), \( t \) must be contained in \( S \), a contradiction. Hence \( F(t) \cap F(S) = \phi \). From this \( C_S(t) \) acts semi-regularly on \( F(t) \) and so \( |F(t)| \) is divisibly by \( |C_S(t)| \). Since \( t^S \in S \), \( |F(t)| = |F(t^S)| = |F(S)| \), hence \( |C_S(t)| | F(S)|. 

If \( N^\ast \cong PSL(2, q) \), then \( |\Omega_1(S'/S^\ast)| = |S^\ast| = q \) and by Lemma 1 of [7], \( |C_S(t)| \geq \sqrt{q} \). If \( N^\ast = Sz(q) \), then \( |\Omega_1(S'/S^\ast)| = q \). Since \( q \) is an odd power of 2 in this case, similarly \( |C_S(t)| \geq 2\sqrt{q} \). If \( N^\ast \cong PSU(3, q) \), then \( |\Omega_1(S'/S^\ast)| = q^2 \) and so similarly \( |C_S(t)| \geq q^2 \). Thus we have (ii).

Suppose \( \mu = 1 \). Then \( N^\ast \) is doubly transitive on each \( N^\ast \)-orbit \( \neq \{\alpha\} \). Applying Theorem D of [10], \( r = 1 \). Therefore, \( |F(S)| = \mu r + 1 = 2 \) and so by (i) and (ii), \( N^\ast \cong PSL(2, 4) \) and \(|\Omega| = 6 \). Thus (iii) holds.

(3.5) Let \( \pi \) be the set of primes which divides \( q - 1 \) and \( K \) a Hall \( \pi \)-subgroup of \( N^\ast \cap N^\# \). If \( K \neq 1 \), then \( C^\ast = 1 \).

Proof. Suppose \( K \neq 1 \) and \( C^\ast \neq 1 \). Set \( \Gamma_i = \Delta_i \cap F(S) \) and \( \Lambda_i = \Delta_i \cap F(K) \). Then by (i) of (3.1) and Lemma 2.3, for each \( i \) with \( 1 \leq i \leq r \) \( |\Lambda_i| = 2 |\Gamma_i| = 2 |N^\ast(S) : N^\#(S) : N^\ast(S) \cap N^\#(S)| \) and \( K \) is semi-regular on \( \Delta_i - \Lambda_i \).

By (v) of (3.3), \( O_2(C_G(KS))^{F(S)} \) is a regular normal elementary abelian 2-subgroup of \( N^\ast(KS)^{F(S)} \). Set \( E = O_2(C_G(KS)) \). It follows from (iv) of (3.3) that \( E^{F(S)} \leq (Z(S) \times C^\ast)^{F(S)} \). Since \( F(Z(S)) = F(S) \) by (ii) of (3.1) and \( (C^\ast)^{F(S)} = 1 \) by (iii) of (3.1), \( (Z(S) \times C^\ast)^{F(S)} = Z(S) \). On the other hand \( Z(S) \cap C(K) = 1 \) (cf. § 3 of [21]) and so \( E = E^{F(S)} \). Hence \( E = E^{F(S)} \) and so we have \( |F(S)| = |E| \). Since \( KS \) is a subgroup of \( N^\#_S \) which contains \( S \), by (ii) of (3.1) we have \( F(S) = F(KS) \). From this \( F(S) \) is a subset of \( F(K) \). Hence \( |F(K) - F(S)| = |F(K) - \{\alpha\}| - |F(S) - \{\alpha\}| = \sum \limits_{i=1}^{r} |\Lambda_i| - \sum \limits_{i=1}^{r} |\Gamma_i| = r \times |N^\ast(S) : N^\#(S)|. \) Since \( r \) is odd, \( |F(K) - F(S)| \) is odd. On the other hand \( E \) fixes \( F(K) - F(S) \) setwise because \( E \) centralizes \( S \) and \( K \). Therefore \( E \) fixes an element \( \gamma \in F(K) - F(S) \) as \( E \) is a 2-subgroup of \( G \). Since \( N^\#/O_2(N^\#) \) is cyclic of odd order, \( K \leq N^\# \) and \( |K : O_2(N^\#)| = |N^\# |N^\# \cap N^\ast| \), we have \( K \cdot O_2(N^\#) \leq N^\# \cap N^\ast \). Hence \( K \leq N^\ast \) and so \( |C_G(K)| \) is odd by (i) of Lemma 2.4. Since \( C_G(K)/C_N(K) \cong C_{g}(K)N^\#/N^\#C^\ast \), a Sylow 2-subgroup of \( C_g(K) \) is cyclic. But \( E \leq C_g(K) \) and hence \( |E| = |F(S)| = 2 = \mu r + 1 \). From this \( \mu = r = 1 \). By (iii) of (3.4) \( C^\ast = 1 \), which is contrary to the assumption \( C^\ast \neq 1 \). So (3.5) holds.

(3.6) Suppose \( K \neq 1 \) and let \( S_i \) be a subgroup of \( S \). If \( S_i \leq N_G(S) \) and \( S_i \leq S \) for some \( g \in G \), then \( S_i \leq Z_3 \times Z_4 \) and \( |S_i| \leq |G|/|N^\ast| \).

Proof. Set \( S_i = T \). By (ii) of (3.1), \( T \) is semi-regular on \( \Omega - F(T) \). Claim
\( F(T) \cap F(S) = \emptyset \). Suppose not and let \( \gamma \in F(T) \cap F(S) \). Then \( T \leq N^S_T \) and \( S \leq N^T_S \). By (3.2) \( T \leq N^S_T \cap N^T_S \) and \( S \leq N^S_T \cap N^T_S \) and so \( TS \leq N^T_S \). Since \( S \) is a Sylow 2-subgroup of \( N^S_T \), \( TS = S \). Hence \( T \leq S \), a contradiction. Thus \( F(T) \cap F(S) = \emptyset \). From this \( T \) acts semi-regularly on \( F(S) \). By (ii) of (3.3), \( T \) normalizes \( N^\gamma_S \cap N^\gamma_T \) and so \( TS \leq N^\gamma_S \cap N^\gamma_T \). Since \( S \) is a Sylow 2-subgroup of \( N^\gamma_S \cap N^\gamma_T \), \( TS = S \). Hence \( T \subseteq S \), a contradiction. Thus \( F(T) \cap F(S) = \emptyset \). From this \( T \) acts semi-regularly on \( F(S) \). By (ii) of (3.3), \( T \) normalizes \( N^S_T \cap N^T_S \) and so \( TS \leq N^S_T \cap N^T_S \). Hence \( T = N^S_T \cap N^T_S \). Now \( N^S_T \cap N^T_S \) acts on \( F(S) \) and \( F(K) \). But in this case since the outer automorphism group of \( N^S_T \cap N^T_S \) is cyclic of odd order, \( |G|/|N^S_T \cap N^T_S| \) is a prime in both cases. This is a contradiction.

If \( N^S_T \cap N^T_S \neq \emptyset \), similarly we obtain \( \sqrt{2}q < |G_S(t)| \cdot |2 \cdot G_a/N^a| \). But in this case since the outer automorphism group of \( N^a \) is cyclic of odd order, \(|G_a/N^a| \)
is odd and so $\sqrt{2q} \leq 2$. Hence $q \leq 2$, a contradiction.

If $N^\ast \simeq PSU(3, g)$, similarly $q \leq |C_4(t)|/|2|G_4/N^\ast|$. Hence $q = 2^2$, $N^\ast \simeq PSU(3, 2^2)$. As in the first case, $r = 1$ and $(\mu, |K|, |F(\Omega)|, |\Omega|) = (5, 3, 11, 326)$ and so $11 = |F(\Omega)|/|\Omega| \cdot |\Aut(PSU(3, 2^2))|$, a contradiction.

In (3.8)-(3.11), we shall prove that $N^\ast = N^\ast \cap N^\beta$. First we note the following.

(3.8) If $C^\ast = 1$, $N^\ast = N^\ast \cap N^\beta$.

Proof. Since $N^\ast$ is a nonabelian simple group, (3.8) follows immediately from Lemma 2.1.

(3.9) Let $p$ be a prime with $p | |N^\ast_\beta: N^\ast \cap N^\beta|$ and assume the following:

\[ (*) \quad p \neq 3 \text{ if } N^\ast \simeq PSU/(3, 2^2) \text{ and } n \text{ is odd.} \]

Then $\mu = p$.

Proof. It follows from (3.8) that $C^\ast = 1$. Hence $G_4/N^\ast$ is isomorphic to a subgroup of the outer automorphism group of $N^\ast$ and so under the hypothesis $(*)$, a Sylow $p$-subgroup of $G_4/N^\ast$ is normal and cyclic ([14]). Set $N^\ast G = N^\ast (S)$. Since $W/N^\ast \leq G_4/N^\ast = G_4/N^\ast$, a Sylow $p$-subgroup of $W/N^\ast$ is normal and cyclic. Hence all elements in $W$ of order $p$ are contained in $N^\ast$ because $|N^\ast_\beta N^\ast_\beta/N^\ast| = |N^\ast_\beta: N^\ast \cap N^\beta| = |N^\ast_\beta: N^\ast \cap N^\beta|$ and $p | |N^\ast_\beta: N^\ast \cap N^\beta|$. Let $P$ be a Sylow $p$-subgroup of $W$. Then $\Omega_2(P) \leq N^\ast N^\ast_\beta$. Set $Q = \Omega_2(P)$. Since $N^\ast N^\ast_\beta / N^\ast_\beta \simeq N^\ast_\beta / N^\ast \cap N^\beta$, by (3.2) $N^\ast_\beta N^\ast_\beta / N^\ast_\beta$ is cyclic and so $Q'$ is a cyclic subgroup of $N^\ast_\beta$, similarly $Q' \leq N^\ast_\beta$. Hence $Q' \leq N^\ast \cap N^\beta$ and the $p$-rank of $Q/Q'$ is at most 2.

By the Frattini argument, $N^\ast G = (N^\ast G \cap N(P))W$. Let $M$ be a normal subgroup of $N^\ast G \cap N(P)$ such that $M F^G$ is a minimal normal subgroup of $N^\ast G F^G$. We choose $M$ so that its order is minimal. Since $N^\ast G F^G$ is doubly transitive, $M F^G$ is an elementary abelian 2-subgroup or a direct product of isomorphic nonabelian simple groups. As $Q'$ is cyclic, $M/C_M(Q')$ is abelian and its Sylow 2-subgroup is cyclic. Hence by the minimality of $M$, $M = C_M(Q')$.

Set $\overline{Q} = Q/Q'$. We argue that $C_M(\overline{Q}) \leq W$. To prove this, it suffices to show that $M \cap C_M(\overline{Q})$ stabilizes the normal series $\overline{Q} \leq Q' \leq 1$ and hence $O_p(M)$ centralizes $\overline{Q}$ by Theorem 5.3.2 and Theorem 5.3.1 of [6]. Obviously $O_p(M) \leq W$ and so $O_p(M) = M$ by the minimality of $M$. Therefore $M$ centralizes $P$. Let $x$ be an element of $M$ such that $x^2 = \beta$, then $P \cap N^\ast_\beta \leq N^\ast \cap N^\ast = N^\ast \cap N^\beta$. But since $P \cap N^\ast_\beta$ is a Sylow $p$-subgroup of $N^\ast_\beta$, $P \cap N^\ast \cap N^\beta$, a contradiction.

Set $C = C_M(\Omega_2(\overline{Q}))$. Then $M/C \leq GL(2, p)$ because the $p$-rank of $\overline{Q}$ is at most 2. By the minimality of $M$, $M/C \leq SL(2, p)$. On the other hand $O_p(C) \leq C_M(\overline{Q}) \leq W$. Therefore $C F^G$ is a normal $p$-subgroup of $N^\ast G F^G$. Since
\(p \neq 2, C^{F(S)} = 1\) and so \(C \leq W\). Hence \(M^{F(S)}\) is isomorphic to a homomorphic image of a subgroup of \(SL(2, p)\).

Hence if \(M^{F(S)}\) is an elementary abelian 2-group, we have \(M^{F(S)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2\) and \(|F(S)| = 4\). From (ii) and (iii) of (3.4), \(\mu = 3\) and \(r = 1\). By (ii) of (3.4), \(N^a \cong PSL(2, 4), PSL(2, 16)\) or \(PSU(3, 4)\) and hence \(|G_a : N^a| = 1, 2 \text{ or } 4\), which is contrary to \(p||N^a : N^a \cap N^a|| = |N^a_a/N^a_a/N^a_a|\).

If \(M^{F(S)}\) is a direct product of isomorphic non abelian simple groups by Dickson's Theorem (Hauptsatz 8.27 [8]) \(M^{F(S)} = PSL(2, p)\) with \(p > 5\) or \(A_5\). Claim \(M^{F(S)} = PSL(2, 3)\) or \(A_5\). Suppose \(M^{F(S)} = PSL(2, p)\) with \(p > 5\). Then \(|F(S)| = 6, \mu = 5\) and \(r = 1\). By (ii) of (3.4), we obtain \(q = 2^2\) and \(N^a = PSL(2, 4)\). Hence \(5, |N^a : N^a_a| = 5\), a contradiction. Thus \(M^{F(S)} = PSL(2, p)\) with \(p > 5\). Hence \(|N^a : F(S)| = M^{F(S)} = 1\) or 2. From this as \(|F(S)|\) is even, \(M^{F(S)}\) is also doubly transitive. Again by Dickson's Theorem, we know all maximal subgroups of \(PSL(2, p)\) with \(p > 5\) and hence \(PSL(2, p)\) with \(p > 5\) has a unique doubly transitive permutation representation of even degree, which is the known one. From this \(|F(S)| = p + 1\). Since \(|F(S)| = \mu r + 1 = \mu + 1\), we obtain \(\mu = p\).

(3.10) If \(N^a = PSU(3, q)\) and \(n\) is odd, then \(3, |N^a : N^a \cap N^b|\).

Proof. By (3.8), we may assume \(C^a = 1\). Set \(W = N_G(S)_{F(S)}\) and let \(P\) be a Sylow 3-subgroup of \(W\). As \(G_a = N^a \cap N^b/N^a \leq G_a/N^a\), a Sylow 3-subgroup of \(W/N^a\) is an abelian 3-group of rank at most 2, so that \(P' \leq N^a\) and similarly \(P' \leq N^b\). Hence \(P' \leq N^a \cap N^b\) and \(P'\) is cyclic.

Similarly as in the proof of (3.9) we can choose a normal subgroup \(M\) of \(N_G(S) \cap N(P)\). Denote \(P/P'\) by \(\hat{P}\). Then \(\Omega^a_{\hat{P}}\) is an elementary abelian 3-subgroup of \(W/N^a\) at rank at most 3. Then as in the proof of (3.9), \(M\) centralizes \(P'\) and \(C_{M}(\hat{P})\) is contained in \(W\). Hence \(M/C \leq SL(3, 3)\) where \(C = C_{M}(\hat{P})\).

If \(M^{F(S)}\) is an elementary abelian 2-group, by the structure of \(SL(3, 3)\), \(M^{F(S)} = \mathbb{Z}_2 \times \mathbb{Z}_2\) and so \(|F(S)| = 4, \mu = 3\) and \(r = 1\). Let \(p_1 \in \pi\). Since \(n\) is odd, \(3 \notin \pi\). Therefore \(p_1 = 3\). By (3.7), \(\pi_{F(S)} \cap |N^a \cap N^b|\). Hence \(p_1 ||N^a : N^a \cap N^b||\) and applying (3.9) to \(p_1\), we have \(\mu = p_1 = 3\), a contradiction.

If \(M^{F(S)}\) is a direct product of isomorphic non abelian simple groups, we have \(M^{F(S)} = SL(3, 3)\) because every proper subgroup of \(SL(3, 3)\) is solvable. Hence \(|N_G(S)_{F(S)} : M^{F(S)}| = 1\) or 2 and so \(M^{F(S)}\) is also doubly transitive. By (ii) of (3.1), \(N_{N^a(S)_{F(S)}} = N^a\). Therefore, \(N_{N^a(S)_{F(S)}} = N^a\) is cyclic of order \(\mu\). Since \(|SL(3, 3)| = 2^3 \cdot 3^2\cdot 13, \mu = 3\) or 13. If \(\mu = 3\), applying (3.7) and (3.9), \(\pi\) is empty, a contradiction. If \(\mu = 13\), then \((M_a)^{F(S)} \triangleright N_{N^a(S)_{F(S)}} = Z_{13}\). Hence \((M_a)^{F(S)}\) is isomorphic to the normalizer of a Sylow 13-subgroup in \(SL(3, 3)\), while this permutation representation of \(SL(3, 3)\) is not doubly transitive. Thus (3.10) is proved.

(3.11) \(N^a = N^a \cap N^b\).
Proof. Suppose not and let $p$ be a prime with $p | N^*_\beta | N^*_\alpha \cap N^\beta |$. Then it follows from (3.7), (3.9) and (3.10) that $q-1=p^e$ for some integer $e \geq 2$. If $e$ is even, $p^e \equiv 1 \pmod{4}$, while $q-1 \equiv -1 \pmod{4}$, a contradiction. If $e$ is odd, $2^e=q=c(p+1)$ where $c=p^{e-1}p^{e-2}+\cdots+p+1$. We note that $e \geq 3$. Since $c$ is odd, $c=1$, a contradiction. Thus $N^*_\beta = N^*_\alpha \cap N^\beta$.

(3.12) Suppose $N^* = PSL(2, q)$ or $Sz(q)$ and $G \neq A_6, S_6$. Then
(i) $N^*_\beta = N^*_\alpha \cap N^\beta$ is a Sylow 2-subgroup of $N^*_\alpha$.
(ii) If $N^* = PSL(2, q)$, then $|F(S)| = q$ and $|\Omega| = q^2$.
(iii) If $N^* = Sz(q)$, then $|F(S)| = q^2$ and $|\Omega| = q^4$.
(iv) There is an element $x$ in $G$ such that $S \neq S^x$, $[S, S^x] = 1$ and $F(S) \cap F(S^x) = \phi$.

Proof. By assumption, $N^*_N(S) = (q-1)q^i$ where $|S| = q^i$. Hence (i) follows immediately from (3.7) and (3.11).

We now argue that $|F(S)|$ is a power of 2. By (v) of (3.3), it suffices to consider the case $C^* = 1$. Applying (ii) of (3.4), $q | |F(S)|^2$. By (i), $\mu = |N^*_N(S): N^*_\beta| = q-1$ and so $|F(S)| = \mu r+1=(q-1)r+1$. Hence $q | (r-1)^3$, while $r$ is a divisor of $n$ where $2^r = q$ because $C^* = 1$ and $G_{a\alpha}/N^*$ is isomorphic to a subgroup of the outer automorphism group of $N^*_\alpha$. Therefore $r=1$ and $|F(S)| = q$, a power of 2.

Hence by (iv) of (3.4), $|F(S)| = (q-1)r+1 | |\Omega| = (q+1)(q-1)r+1$ and so $q | (q-1)r+1$ and $(q-1)r+1 | q^i$. From this, $(i, r) = (1, 1), (2, 1)$ or $(2, q+1)$. If $(i, r) = (1, 1)$ or $(2, q+1)$, we obtain (ii) or (iii), respectively. We argue $(i, r) = (2, 1)$. Suppose $(i, r) = (2, 1)$. Then $N^* = Sz(q)$, $|F(S)| = q$ and $|\Omega| = q(q^2-q+1)$. In this case, $|G_{a\alpha}/C^*N^*|$ is odd, we have $I(G_{a\alpha}) = I(N^*_\alpha \cap N^\beta)$. From this, all involutions in a fixed Sylow 2-subgroup of $G_{a\alpha}$ have a common fixed point set. By [12], $G$ has a regular normal subgroup and so $q^2-q+1=1$, a contradiction.

Since by (iv) of (3.4) $|\Omega| = |F(S)| \times |G: N^*_S(S)|$, $|G: N^*_S(S)|$ is divisible by 2. Let $S_1$ be a Sylow 2-subgroup of $N^*_S(S)$ and $S_2$ a Sylow 2-subgroup of $N^*_S(S)$. Since $2 | |G: N^*_S(S)|$, $S_1 \neq S_2$. Let $x \in S_2 \setminus S_1$, then $S \neq S^x$ and $S_1 \nmid S \cap S^x$. Suppose $\gamma \in F(S) \cap F(S^x)$. Then by (i), $SS^x \leq N^\gamma$ and so $S = S^x$, a contradiction. Therefore $F(S) \cap F(S^x) = \phi$ and hence $[S, S^x] = 1$ by (ii) of (3.1). Thus (iii) holds.

(3.13) The following hold.
(i) $N^*_\alpha \neq Sz(q)$.
(ii) Suppose $N^* = PSL(2, q)$ and let $S^x$ be as defined in (3.12). Then $O_2(C_{a\alpha}(S))$ is a Sylow 2-subgroup of $C_{a\alpha}(S)$ and $O_2(C_{a\alpha}(S)) = S \times S^x$.

Proof. Suppose $N^* = PSL(2, q)$ or $Sz(q)$. If $C^* = 1$, $O_2(C_{a\alpha}(S))^{p(S)}$ is a regular normal subgroup of $N^*_S(S)^{p(S)}$ by (v) of (3.3). If $C^* = 1$, by (iv) of (3.3)
\( C_G(S) = Z(S) \) and so \( C_G(S) \cap F(S) = Z(S) \). By (3.12), \( C_G(S) \cap F(S) \geq (S^*)^F(S) \neq 1 \), and \( |F(S)| = q = |S| \) and so \( C_G(S) = Z(S) \times S^* \). Hence in both cases \( O_2(C_G(S)) \) is regular on \( F(S) \).

Since by (iv) of (3.3) \( C_G(S) \cap F(S) = C_G(S) = Z(S) \) and by (ii), (iii) of (3.12) \( q' \mid S^* \mid = F'(S) = |C_G(S) : C_G(S)| \), we have \( |O_2(C_G(S))| = Z(S) \times S^* \). Hence \( O_2(C_G(S)) \) is a \( S \)-subgroup of \( C_G(S) \).

Since \( Z(O_2(C_G(S))) = Z(S) \times S^* \), and \( |F(S)| = |S| \), \( Z(S) \times S^* \) and this is a \( S \)-subgroup of \( C_G(S) \).

Since \( Z(O_2(C_G(S))) \cap F(S) = Z(S) \times S^* \) and \( |F(S)| = |S| \), \( Z(S) \times S^* \) and \( Z(S) \times S^* \) is abelian. So (3.13) follows.

(3.14) Suppose \( N^* \cong PSL(2, q) \) and \( G \neq A_6, S_6 \). Put \( E = O_2(C_G(S)) = S \times S^* \), \( W = \{ T | T \in ccI(S), T \leq E \} \). Then we have the following:

(i) \( |W| = q \) and \( \Omega = \bigcup F(T) \) where \( T \) runs over every element of \( W \).

(ii) \( N_G(E) \cap ccI(S) \subseteq E \) for all \( s \in I(S) \).

(iii) If \( E \cap E^* \cap ccI(s) = \phi \) for some \( g \in G \), then \( g \in N_G(E) \).

Proof. Let \( D \) be a Hall 2'-subgroup of \( N_{2^m}(S) \). Then \( D \cong Z_{d-1} \) and by (i) of (3.12) \( D \) is semi-regular on \( \Omega = \{ \alpha \} \). If \( d \in N_{2^m}(S^*) \), \( \langle d \rangle \) acts semi-regularly on \( F(S^*) \) since \( \alpha \in F(S^*) \). Hence the order of \( d \) divides \( |F(S)| \). But \( |F(S)| = q \) by (ii) of (3.12), hence \( |d| (q, q - 1) = 1 \) and so \( d = 1 \). Therefore \( N_{2^m}(S^*) = 1 \).

Hence \( |\{ S^* | d \in D \}| = q - 1 \) and \( |\{ S^* | d \in D \}| \leq W \) as \( D \) normalizes \( E \). If \( S = S^* \) for some \( d \in D \), \( S = S^* = S_{d, 1} \), a contradiction. Hence \( |W| \geq q \).

If there exist \( S_1, S_2 \in W \) such that \( S_1 \neq S_2 \) and \( F(S_1) \cap F(S_2) \neq \phi \) \( \gamma \in F(S_1) \cap F(S_2) \). Then \( S_1, S_2 \leq N^* \) by (i) of (3.12) and so \( <S_1, S_2> = N^* \), which is contrary to \( <S_1, S_2> \leq E \). Hence \( F(S_1) \cap F(S_2) = \phi \) for \( S_1, S_2 \in W \) such that \( S_1 \neq S_2 \).

Since \( |F(S)| = q \) and \( |\Omega| = q^2 \) by (ii) of (3.12), we have \( |W| \leq q \). Thus (i) holds.

Let \( s \in I(S) \) and suppose \( s \in N_G(E) - E \) for some \( g \in G \). Then \( s \in N^* \) where \( \gamma = \alpha^g \). By (i) we choose \( T \in W \) so that \( \gamma \in F(T) \). Then \( \langle \gamma \rangle, \langle T \rangle = N^* \) as \( s \in T \) and \( T \) is a Sylow 2'-subgroup of \( N^* \). On the other hand \( \langle \gamma \rangle, \langle T \rangle \leq \langle s \rangle \), which is a 2-subgroup of \( N_G(E) \), a contradiction. Thus (ii) holds.

Let \( 1 \neq T \in E \cap E^* \cap ccI(s) \) for \( g \in G \) and \( s \in I(S) \). Then there are \( S_1 \leq E \) and \( S_2 \leq E^* \) such that \( T \in S_1 \cap S_2 \) and \( S_1, gS_2g^{-1} \in W \). Since \( F(S_1) = F(T) = F(S_2) \) by (ii) of (3.1), \( <S_1, S_2> \leq N^* \cap N^* \) for \( g \in F(T) \). Hence \( S_1 = S_2 \) by (i) of (3.12). Applying (ii) of (3.13) to \( S_1 \), we obtain \( E = O_2(C_G(S_1)) = O_2(C_G(S_2)) = E^* \). Thus (iii) holds.

(3.15) Suppose \( N^* \cong PSL(2, q) \) and \( G \neq A_6, S_6 \). Then \( G \) has a regular normal subgroup.

Proof. We count the set \( \{ \langle \gamma \rangle, T | \gamma \in F(T), T \in ccI(S) \} \) in two ways and we have \( q^2 \times (q + 1) = |ccI(S)| \times q \) by (3.12). Hence \( |ccI(S)| = q(q + 1) \). On the other hand we have \( |ccI(S)| = |G: N_G(E)| \times q \) by (i), (ii) of (3.14). From this, \( |G: N_G(E)| = q + 1 \).
Set $\Gamma = \text{cl}_G(E)$. We now consider the action of $G$ on $\Gamma$. By definition, $G$ is transitive on $\Gamma$ and $N_G(E)$ is a stabilizer of $E \in \Gamma$. We argue that $G$ is regular on $\Gamma$. Suppose not and let $s \in S$ such that $s \neq E \in G$. Then $gsg^{-1} \in N_G(E)$. By (ii) of (3.14), $gsg^{-1} \in E \cap E = N_G(E)$. Hence $E = gEg^{-1}$. By (iii) of (3.14), $E \neq gEg^{-1}$. Hence $E = gEg^{-1}$, a contradiction.

Since $S \leq N_G(E)$ and $|S| = \frac{|\Gamma|}{2}$, $\Gamma$ is regular on $\Gamma$ and $G\Gamma$ is doubly transitive. Since $S$ is abelian and regular on $\Gamma$, $G\Gamma$ is a Frobenius group on $\Omega$ by (ii) of (3.1).

We now consider the case that $N_G(E) = \text{PSU}(3, q)$. By (3.7) and (3.11), $N_G(E) = U$ where $U$ is a Hall 2'-subgroup of $N_G(E)$ and $U \leq Z_{q+1}$ with $\varepsilon = (q+1, 3)$. As in the proof of (3.1)', we set $N_{G^*}(S) = DS$ and $D = V \times K$. Here $V = Z_{q+1}$ and $K = Z_{q-1}$. Since $N_{G^*}(S) \geq N_G(E)$, we assume $U = V \cap N_G(E)$.

(3.16) Suppose $N_G(E) = \text{PSU}(3, q)$. Then $N_G(E) \leq N^* \cap N^\beta$ is a Sylow 2-subgroup of $N^*$. In particular $\mu = q^2 - 1$. We proceed as follows:

Proof. Suppose not and $U \neq 1$. If $U \neq G_{a\beta}$ for $g \in G$, $U = N^a \cap N^\beta = N^a \cap N^\beta \cap N^\beta \leq N^a \cap N^\beta$. Hence $U$ is conjugate to $U^g$ in $N^a \cap N^\beta \leq G_{a\beta}$.

By the Witt's Theorem $N_G(U)$ is doubly transitive on $F(U)$. By (ii) of Lemma 2.4, $N_{N^*}(U) = N \times V$ where $N \simeq \text{PSL}(2, q)$. Hence $N_G(U)$ satisfies the assumption of Theorem 1. By (i) of (3.1), the number of fixed points of $U$ on $\Delta$, is constant for each $N^*$-orbit $\Delta$, and so $|F(U)| = |F(U) \cap \Delta_1| \times r + 1 = (|N_{N^*}(U)| \times |N^a \cap N_{N^*}(U)|) \times |N^\beta_0 : N_G(U)| \times |N^\beta_0 : N_G(U)| \times r + 1 = (|PSL(2, q)| \times |V|) \times |Z(S)| \times |U| \times r + 1 = (q^2 - 1) \times r \times |V| \times |U| + 1$. Hence $|F(U)|$ is even and $|F(U)| = 6$. Applying (3.12) to $N_G(U)$, we obtain $|F(U)| = q^2$, $|F(U) \cap F(Z(S))| = q$. Hence $r = 1$, $U = V$, $N^a_0 = V \times S$ and $|F(U)| = q^2$ and so $\mu = |N_G(S) : N^*| = q - 1$. Since by (ii) of (3.1) $F(U) \equiv F(S)$, $|F(Z(S))| = |F(S)| = q$. Furthermore by (3.15), $N_G(V)^F$ has a regular normal elementary abelian 2-subgroup, say $E^F(V)$. Clearly $E^F(V) \leq C_G(V)^F$. Hence we may assume that $E$ is a 2-subgroup of $C_G(V)$. Put $P = E^F(V)$. Then $|E| = |P|q^2$. By (i) of (3.4), $|\Omega| = q^2 - q^3 + q$ and so $2q^2 \not{|} \Omega - F(V)|$. Hence there exists $\gamma \in \Omega - F(V)$ such that $|E : E\gamma| \leq q$. Let $T$ be a Sylow 2-subgroup of $\gamma$ containing $E\gamma$. Since $E\gamma | T \cap N^\gamma$ is isomorphic to a subgroup of $T/T \cap N^\gamma$ and $T/T \cap N^\gamma \simeq TN^\gamma$ is cyclic. If $E\gamma \cap T \cap N^\gamma = 1$, $E\gamma$ is cyclic and so $|E\gamma | E\gamma \cap P| \leq 2$. Then $|E\gamma \cap P| \geq |E\gamma|/2 \geq |P|/2 \geq |P|$, a contradiction. Hence $E\gamma \cap T \cap N^\gamma = 1$. Let $z \in E\gamma \cap T \cap N^\gamma$ with $z \neq 1$. Since $|F(z)| = q < |F(P)|$, $z \in E$ and $E^F(V)$ is regular, we have $F(z) \cap F(V) = \phi$. Hence $V$ acts semi-regularly on $F(z)$. From this, $q = |F(z)| = (q+1)/k$ for some integer $k \geq 1$. Since $q$ is a power
of 2, \( q + 1 / \varepsilon - 1 = 0 \), a contradiction.

(3.17) Suppose \( N^* = PSU(3, q) \). Then the following hold.

(i) \( |\Omega| = q^2 - q + q^2, |F(S)| = q^2 \).

(ii) \( N_G(S)^F(S) \) has a regular normal subgroup.

Proof. If \( C^* \neq 1 \), (ii) follows from (v) of (3.3) and so \( |F(S)| \) is a power of 2. By (3.4) and (3.16), \( |F(S)| = (q^2 - 1)\varepsilon + 1 \) and \( (q^2 - 1)\varepsilon + 1 | (q^3 + 1)(q^2 - 1)\varepsilon + 1 \), hence \( (q^2 - 1)\varepsilon + 1 | q^2 \). By calculation, we obtain \( r = \varepsilon \). So (i) follows.

We now assume \( C^* = 1 \). By (ii) of (3.4), \( q | |F(S)| = (q^2 - 1)\varepsilon + 1 \) so that \( r = qk + \varepsilon \) for an integer \( k \geq 0 \). Since \( C^* = 1 \), \( r \) is a divisor of \( |G_2/N^*| \). Hence \( r | 2n\varepsilon \), so that \( r n\varepsilon \). Therefore \( n\varepsilon \geq r = qk + \varepsilon = 2^a k + \varepsilon \). Hence \( k = 0 \) and \( r = \varepsilon \). From this (i) follows.

Let \( f \) be a field automorphism as defined in (3.1) and let \( T \) be a Sylow 2-subgroup of \( N_G(S) \) which contains \(<f> S\). Since \( |N_G(S): N_{G_2}(S)| = |F(S)| = q^2 \) by (i), \( |T| = 2^m q^2 \) where \( |<f>| = 2^m \). Since \( T \triangleright S \) and \( \Omega - F(S) = q^2(q^2 - 1) \) there exists \( \gamma \in \Omega - F(S) \) such that \( |T: T_\gamma| = q^2 \), hence \( |T_\gamma| = 2^a q^2 \) and \( T = ST_\gamma \). Set \( W = T_\gamma \cap N^2 \). Then \( W \) is semi-regular on \( F(S) \) because \( \gamma \in \Omega - F(S) \). In particular \( |W| \leq |F(S)| = q^2 \). We note that \( |T_\gamma N^2/|N^2| \leq 2^m \). Since \( T_\gamma W = T_\gamma N^2/N^2 \), we have \( |W| \geq q^2 \). Hence \( |W| = q^2 \) and \( W \) is regular on \( F(S) \). Moreover \( |T_\gamma : W| = 2^m \).

Since \( N_{G_{2g}}(S)/S \cong N_{G_{2g}}(S)N^2/S^2 \) by (3.16), \( N_{G_{2g}}(S)^F(S) \) is isomorphic to a homomorphic image of a subgroup of the outer automorphism group of \( N^* \). Hence \( N_{G_{2g}}(S)^F(S) \) is abelian when \( n \) is even or \( f = 1 \). In this case by [1], (ii) holds because \( |F(S)| = q^2 \). We now assume \( n \) is odd and \( |<f>| = 2^m \). Since \( T = ST_\gamma \) and \( |T : W| = 2^m q^2 \), \( |F(S)| = q^2 \) by (i). For otherwise \( f \in N_G(S)^F(S) \) and \( [f, D] \leq N_G(S)^F(S) \cap D = 1 \) as \( D \) is \( f \)-invariant and \( D \leq N_G(S) \). But since \( f \neq 1 \), \( f \) does not centralize \( D \). Therefore \( f^{F(S)} \neq 1 \). As \( f \in G_2 \), \( f^{F(S)} \in W^{F(S)} \). Hence \( T^{F(S)} = <f>^{F(S)} W^{F(S)} = W^{F(S)} \). Since \( W^{F(S)} \) is regular, \( f^{F(S)} \) is not conjugate to any element in \( W^{F(S)} \). Hence \( f^{F(S)} \) is not contained in \( O^2(N_G(S)^F(S)) \) by Lemma 2 of [3]. Since \( <f^{F(S)}> \) is a Sylow 2-subgroup of \( (N_G(S)^F(S))_aB \), \( O^2(N_G(S)^F(S))_aB \) is of odd order. As before \( (N_G(S)^F(S))_aB \) is isomorphic to a homomorphic image of a subgroup of the outer automorphism group of \( N^* \), \( O^2(N_G(S)^F(S))_aB \) is abelian. Again by [1], \( O^2(N_G(S)^F(S)) \) has a regular normal subgroup as \( |F(S)| = q^2 \). Thus (ii) also holds in this case.

(3.18) \( N^* \neq PSU(3, q) \).

Proof. Let \( f \) be as in (3.1). By the same argument as in the proof of (ii) of (3.17), we have \( I< <f> > \neq N_G(S)^F(S) \) and so \( S \) is a Sylow 2-subgroup of \( N_G(S)^F(S) \).

By (ii) of (3.17), there is a normal subgroup \( L \) of \( N_G(S) \) such that \( L \geq N_G(S)^F(S) \) and \( L^{F(S)} \) is an elementary abelian 2-group of \( N_G(S)^F(S) \). Let \( T \) be a Sylow 2-subgroup of \( <f>L \) which contains \( f \). Set \( E = T \cap L \). Then \( E \)
is a Sylow 2-subgroup of $L$. Since $S$ is a unique Sylow 2-subgroup of $N_G(S)_{F(S)}$, $E/S \cong L^F(S)$ is an elementary abelian 2-subgroup of order $q^2$. As $\langle f \rangle \cap E = \langle f \rangle \cap \langle e \rangle \cap S = 1$, $T = \langle f \rangle E \triangleright E$.

Since $T \triangleright S$ and $|\Omega - F(S)| = q^2(q^2 - 1)$ by (i) of (3.17), we can choose $\gamma \in \Omega - F(S)$ such that $|T: T_\gamma| = q^2$. Hence $|T_\gamma| = 2^n q^2$ where $2^n$ is the order of $f$. Since $T_\gamma | T_\gamma \cap C\gamma N^\gamma = T_\gamma N^\gamma C\gamma / C\gamma N^\gamma$ is cyclic of order at most $2^n$, $|T_\gamma \cap C\gamma N^\gamma| = |T_\gamma \cap N^\gamma| \geq q^2$. Moreover $T_\gamma \cap N^\gamma / T_\gamma \cap N^\gamma \cap E = (T_\gamma \cap N^\gamma)E/E$ is cyclic of order at most $2^m$, we have $|T_\gamma \cap N^\gamma \cap E| \geq q^2 / 2^m$. Since the order of $f$ is a divisor of $2n$, we have $|T_\gamma \cap N^\gamma \cap E| \geq q(2^n / 2^m) \geq q$.

If $T_\gamma \cap N^\gamma \cap E$ contains no element of order 4, then $T_\gamma \cap N^\gamma \cap E$ is an elementary abelian 2-subgroup of $N^\gamma$ of order $q$ and hence $T_\gamma \cap N^\gamma / T_\gamma \cap N^\gamma \cap E$ is an elementary abelian 2-group. Therefore $|(T_\gamma \cap N^\gamma)E/E| \leq 2$ and so $|T_\gamma \cap N^\gamma \cap E| \geq q^2 / 2 > q$, a contradiction.

If $T_\gamma \cap N^\gamma \cap E$ contains an element $x$ of order 4, then $1 \neq x \in S$ because $E/S$ is an elementary abelian 2-group. Since $\gamma \in F(x^2)$, by (ii) of (3.1) we have $\gamma \in F(S)$, which is contrary to $\gamma \in \Omega - F(S)$. Thus (3.18) holds.

In this section we have proved the following:

**Theorem 2.** Suppose $G^\alpha$ satisfies the hypothesis of Theorem 1 and $|\Omega|$ is even. Then $N^\alpha \neq S_2(q)$, $PSU(3, q)$, $N^\alpha = PSL(2, q)$ and either

(i) $G^\alpha \cong A_5$ or $S_6$ or

(ii) $|\Omega| = q^2$, $|N^\alpha| = |N^\alpha \cap N^\beta| = q$ and $G$ has a regular normal subgroup.

4. The case $|\Omega|$ is odd

Let $G$ be a doubly transitive permutation group on $\Omega$ of odd degree satisfying the assumption of Theorem 1. By Theorem A of [10] and Theorem B of [11], we may assume $C_G(N^\alpha) = 1$. Hence $G_{\alpha}/N^\alpha$ is isomorphic to a subgroup of the outer automorphism group of $N^\alpha$. Let $\{\alpha_1, \Delta_1, \Delta_2, \ldots, \Delta_r\}$ be the set of all $N^\alpha$-orbits on $\Omega$. Clearly $r$ is a divisor of $|G_{N^\alpha}/N^\alpha|$.

From now on we assume that $G$ has no regular normal subgroup and prove that $G \cong PSL(2, 11)$. Let $M$ be a minimal normal subgroup of $G$. Then by assumption, $M_{\alpha} \neq 1$.

(4.1) $M$ is simple and $N^\alpha \leq M$.

Proof. Since $G$ is doubly transitive and $M_{\alpha} \neq 1$, $M$ is a simple group (cf. Exercise 12.4 of [16]). If $N^\alpha \leq M$, then $M_{\alpha} \cap N^\alpha = 1$ as $N^\alpha$ is simple and hence $M_{\alpha} \leq C_G(N^\alpha) = 1$, a contradiction. Thus $N^\alpha \leq M$.

As in (3.1), there is a 2-element $f$ of $M_{\alpha}$ such that $f$ acts on $N^\alpha$ as a field automorphism, $\langle f \rangle S \triangleright S$, $\langle f \rangle \cap S = 1$ and $\langle f \rangle S$ is a Sylow 2-subgroup of $M_{\alpha}$, where $N_{N^\alpha}(S) = DS$ is a Borel subgroup of $N^\alpha$, $S$ is a unipotent subgroup of $N^\alpha$, and $D$ is a diagonal subgroup of $N^\alpha$. 
(4.2) If $f \neq 1$, then $I(N^\alpha) \neq N^\alpha \cap N^\beta$ for $\beta \neq \alpha$.

Proof. Suppose $f \neq 1$ and $\tau \in I(\langle f \rangle)$. Since $M$ is a simple group with a Sylow $2$-subgroup $\langle f \rangle S$, $\tau^f \in S$ for some $g \in M$. By Lemma 2 of [3], set $\gamma = \alpha^{-1}$. Then $\tau \in N^\gamma$ and clearly $\tau \in N^\gamma \cap N^\alpha$, so that $I(N^\gamma) \neq N^\gamma \cap N^\alpha$. By the transitivity of $G$, we obtain $I(N^\gamma) \neq N^\alpha \cap N^\beta$ for any $\beta \neq \alpha$.

(4.3) Suppose $f = 1$. Then $N^\alpha \neq Sz(q), PSU(3, q)$.

Proof. If $N^\alpha = Sz(q)$, $|G_a/N^\alpha| = 2$ and hence $f = 1$, a contradiction. Therefore $N^\alpha \neq Sz(q)$. Suppose $N^\alpha = PSU(3, q)$ and let $s \in Z(\langle f \rangle S) \cap I(S)$. As in the proof of (4.2), $ccl(\tau) \cap S \neq \phi$. Then since $\tau$ is an extremal element there exists $g \in M$ such that $\tau^g = s$. Let $\tau_1 \in N^\alpha$ and $\tau_2 \in N^\alpha$. By Lemma 2.4, $\tau \in N^\gamma$ and $\tau \in N^\gamma$ is a strongly embedded subgroup of $N^\alpha$. Since $|N^\alpha/N^\beta| = 2$, $N^\alpha/N^\beta$ is solvable, while $N^\alpha/N^\beta$ is solvable, a contradiction.

Let $V_1$ be a $\tau$-invariant Hall $2$-subgroup of $N^\alpha$. Then since $V_1$ normalizes $\Omega_2(O_2(N^\alpha)) \leq Z(S)$, $V_1$ centralizes $Z(S)/Z(S) \cap N^\alpha \cap N^\gamma = Z(S)$. Hence by (i) of Lemma 2.4, $V_1 \leq Z(S)$ and so $\tau \in I(S)$. Therefore $I(N^\gamma) \leq Z(S)$. Similarly $I(N^\alpha) \leq Z(S)$. Since $\tau \in I(N^\gamma)$, we have $N^\alpha \cap N^\gamma \leq C(\tau) \cap N^\alpha$. Since $\tau$ is a field automorphism of order $2$, $C(\tau) \cap N^\alpha = K \cap N^\alpha$. Hence $N^\alpha \cap N^\gamma \leq KZ(S) \cap N^\alpha = Z(S) \cap N^\alpha$. By Bender's Theorem ([2]), $N^\alpha/N^\beta \cap N^\gamma$ is not solvable, while $N^\alpha/N^\beta \cap N^\gamma$ is solvable, a contradiction.

We claim that $F(z) = F(Z(S))$ for $z \in I(N^\alpha)$. Set $\Delta_1$ to be an arbitrary $N^\alpha$-orbit on $\Omega - \{\alpha\}$. Since all elementary abelian $2$-subgroups of $N^\alpha$ of order $q$ are conjugate in $N^\alpha$, there exists $\gamma \in \Delta_1$ with $Z(S) \subseteq N^\gamma$. Hence by Lemma 2.2, $|F(z) \cap \Delta_1| = |C_\gamma(z)| \times |Z(S)|/|N^\gamma| = (q + 1)/\epsilon \times q^2(q-1)/|N^\gamma|$ for $z \in I(N^\gamma)$. On the other hand $|F(Z(S)) \cap \Delta_1| = |N^\gamma(Z(S))/|N^\gamma| = (q^2 - 1)/\epsilon \times q^2/|N^\gamma|$. Hence $F(z) \cap S = F(Z(S)) \cap S$, and so $F(z) = F(Z(S))$. In particular $F(\tau) = F(Z(S))$ for $\tau \in I(N^\alpha)$ and $N^\alpha \cap N^\gamma = \{\beta\}$.

We claim that $(V_1)_F(\langle z \rangle) = 1$. Set $S_1 = O_2(N^\alpha)$. Let $d \in V_1$ with $d \neq 1$, $\Delta_1$ be a $N^\alpha$-orbit which contains $\beta$ and let $D_1$ be a $\tau$-invariant Hall $2$-subgroup of $N^\alpha$. Then by Lemma 2.2, $|F(X) \cap \Delta_1| = |N^\alpha(X)| \times |Z(S)|/|N^\gamma| = |D_1(Z(S))/|N^\alpha| = q^2 - 1)/\epsilon \times q^2/|N^\gamma|$. Since $S_1/N^\alpha \cap N^\gamma$ is cyclic and $N^\alpha \cap N^\gamma \leq Z(S)$, $S \subseteq S_1$. Therefore $F(X) = F(Z(S))$ and so $(V_1)_F(\langle z \rangle) = 1$.

Since $D_1 \leq N^\alpha(Z(S))$ and $\tau \in N^\alpha(Z(S))$, $(\tau, D_1) \leq N^\alpha(Z(S))_F(\langle z \rangle) \cap D_1$
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1. Hence \( D_1 \leq C(\tau) \cap N_{\alpha}(S) = KZ(S) \) with \( K \simeq \mathbb{Z}_{q-1} \), which is contrary to \( |D_1| = (q^2-1)/6 \). So (4.3) is proved.

(4.4) Suppose \( N^\sigma \simeq PSL(2, q) \) and \( f \neq 1 \). Then the following hold.

(i) \( N^\sigma_\beta \) is a 2-subgroup of \( N^\sigma \) and \( |N^\sigma_\beta : N^\sigma \cap N^\beta| = 2 \).

(ii) Let \( \tau \in \langle f \rangle \). Then for some \( \beta \neq \alpha \) and \( \tau \in N^\sigma_\alpha - N^\sigma_\beta \), \( |C_{\beta}(\tau)| = \sqrt{q} \) and \( N^\sigma \cap N^\beta \leq C_{\beta}(\tau) \leq N^\sigma_\beta \).

Proof. As in the proof of (4.3), there exist \( s \in I(S) \) and \( g \in M \) such that \( \tau^s = s \) and \( (C_{\langle f \rangle}(\tau))^s \leq \langle f \rangle S \). Put \( \beta = \alpha s^{-1} \). Then \( \tau \in N^\sigma_\alpha - N^\sigma_\beta \) and \( C_{\beta}(\tau) \leq N^\sigma_\beta \). Since \( \tau \) is a field automorphism of \( N^\sigma \) of order 2, \( |C_{\beta}(\tau)| = \sqrt{q} \). Claim \( N^\sigma_\beta \leq N_{N^\sigma}(S) \). If \( q \neq 2^a \), as \( C_{\beta}(\tau) \leq N^\sigma_\beta \), a Sylow 2-subgroup of \( N^\sigma \) is non cyclic. Hence as in the proof of (4.3), \( N^\sigma_\beta \leq N_{N^\sigma}(S) \). If \( q = 2^a \), \( N^\sigma_\beta \simeq \mathbb{A}_5 \) and so \( \langle \tau \rangle N^\sigma \simeq \mathbb{A}_5 \) and \( M = G_\alpha = S_5 \). In particular \( r = 1 \). Hence \( N^\sigma_\beta \leq N_{N^\sigma}(S) \).

If \( r = 1 \), as \( N^\sigma_\beta \leq N_{N^\sigma}(S) \).

(4.5) Suppose \( N^\sigma \simeq PSL(2, q) \) and \( f \neq 1 \). Let \( T = N^\sigma_\alpha N^\sigma_\beta \). Then

(i) \( N^\sigma_\gamma(T) \) is doubly transitive on \( F(T) \).

(ii) \( N^\sigma_\gamma(T) = S \) and \( S_T = N^\sigma_\gamma \) for every \( \gamma \in F(T) \).

Proof. Since \( G_{ab}/N^\sigma_\gamma \) is cyclic and by (i) of (4.4) \( T/N^\sigma_\beta = Z_2 \), \( I(G_{ab}) \subseteq T \). Clearly \( \langle I(G_{ab}) \rangle = T \). Hence by the Witt's Theorem, we have (i).

Let \( K_1 \) be a Hall 2\(^{-}\)-subgroup of \( N_{N^\sigma}(T) \). Then \( K_1 \) normalizes \( T \cap N^\sigma = N^\sigma_\beta \). Since \( T/N^\sigma_\beta = Z_2 \), \( K_1, T/N^\sigma_\beta = 1 \) and so \( T = C_T(K_1)N^\sigma_\beta \). If \( K_1 \neq 1 \), by (i) of Lemma 2.4 \( C_T(K_1) = 1 \). Hence \( K_1 = 1 \) and \( N_{N^\sigma}(T) = S \).

Let \( \gamma \in F(T) - \{ \alpha \} \). Then obviously \( N^\sigma_\gamma \leq S_T = N^\sigma_\gamma \). Since \( G \) is doubly transitive on \( \Omega, |N^\sigma_\gamma| = |N^\sigma_\gamma| \), so that \( N^\sigma_\gamma = S_T = N^\sigma_\gamma \). Thus (ii) holds.

(4.6) Suppose \( N^\sigma \simeq PSL(2, q) \) and \( f \neq 1 \). Put \( q = 2^a \). Then

(i) \( (n, |N^\sigma_\beta|) = (2, 2), (2, 2^2), (4, 2^3) \) or \( (6, 2^4) \).

(ii) If \( (n, |N^\sigma_\beta|) = (6, 2^4), N^\sigma_\gamma(T)^{F(T)} = A_5 \).

Proof. \( |G_{ab}/N^\sigma| = n \) and \( f \neq 1 \), \( n \) is even and so we set \( n = 2m \). By (ii) of (4.4), \( |N^\sigma_\beta| = 2^{m+1} \) where \( \epsilon = 0 \) or 1. Since \( N_{G_{ab}}(T)/T \leq G_{ab}/T \simeq (G_{ab}/N^\sigma_\beta)(T/N^\sigma_\beta) \) and \( N_{G_{ab}}(T)^{F(T)} \) is cyclic and \( |N_{G_{ab}}(T)^{F(T)}| = m \). By (4.5), \( N^\sigma_\beta(T)^{F(T)} \) is doubly transitive and \( S_T = S/N^\sigma_\beta \) is semi-regular on \( F(T) - \{ \alpha \} \). Since \( N_{G_{ab}}(T)^{F(T)} \) is cyclic, by [1] \( N_{G_{ab}}(T)^{F(T)} \simeq PSL(2, q_1) \) where \( q_1 \) is a power of 2 or \( N_{G}(T)^{F(T)} \) has a regular normal subgroup. If \( (n, |N^\sigma_\beta|) \neq (2, 2), (2, 2^2) \) and \( (4, 2^3) \), \( S_T \) contains a four-group, which is semi-regular on \( F(T) - \{ \alpha \} \). Hence \( N_{G}(T)^{F(T)} \) contains no regular normal subgroup and so
\( N_\sigma(T)^{F(T)} \cong PSL(2, q_1) \). Since \( N_{N^*}(T)^{F(T)} = S^{F(T)} = S/N_\sigma^* \) and \( N_{G_\alpha}(T)^{F(T)} \supset \ N_{N^*}(T)^{F(T)}, q_1 = 2^{m-e} > 2 \). Hence \( 2^{m-e} - 1 = |N_{G_\alpha}(T)^{F(T)}| \), so that \( 2^{m-e} - 1 | m \). From this, \( e=1, m=3 \) and \( N_\sigma(T)^{F(T)} \simeq A_5 \). Thus (4.6) holds.

(4.7) \( f=1 \).

Proof. Suppose \( f \neq 1 \). Then by (4.3) and (4.6), it suffices to consider the case (i) of (4.6).

If \( N^* \cong PSL(2, 2^2) \) and \( |N_\sigma^*| = 2 \), \( G_\sigma = N_\sigma^* N^* \cong Aut(N^*) = S_6 \). Hence \( r=1 \). Therefore \( |\Omega| = 1+ |N^*: N_\sigma^*| = 31 \) and \( |G| = |\Omega| \). Therefore, \( |G_\sigma| = 2^3 \cdot 3 \cdot 5 \cdot 31 \).

By the Sylow’s theorem, \( G \) has a regular normal subgroup of order 31. But this is a contradiction as \( G \supset N^* \).

If \( N^* \cong PSL(2, 2^3) \) and \( |N_\sigma^*| = 2^4 \), as above \( G \in \mathfrak{S}_6 \) and \( |N^*: N_\sigma^*| = 16 \), a contradiction.

If \( N^* \cong PSL(2, 2^4) \) and \( |N_\sigma^*| = 2^5 \), \( |Aut(N^*)| : N^*| = 4 \) and so \( |G_\sigma: N_\sigma^* N^*| \leq 2 \). Hence \( r=1 \) or 2 and \( |\Omega| = 511 \) or 1021 respectively. By Lemma 2.2, for \( s \in N_\sigma^* \setminus \{1\} \), \( |F(s) \setminus \{\alpha\}| = 14 \) or 28 respectively. Let \( \tau \) be a field automorphism of \( N^* \) of order 2 as in (4.4). Then \( C_{N^*}(\tau) \cong PSL(2, 2^2) \) and \( |F(\tau) \setminus \{\alpha\}| = 14 \) or 28 since \( \tau \) is conjugate to \( \sigma \). From this an element \( \chi \in C_{N^*}(\tau) \) of order 5 fixes at least four points in \( \Omega \). Since \( 5 \not| \displaystyle \sum \Omega \), \( \langle \chi \rangle \) is a Sylow 5-subgroup of \( G \) and so \( x^\chi \in N^* \) for some \( g \in G \). But \( F(x^\chi) = \{\alpha\} \) because \( |N_\sigma^*| = |N_\sigma^*| = 2^3 \) for all \( \gamma \neq \alpha \).

Therefore \( |F(x^\chi)| = 1 \), which is contrary to \( |F(x^\chi)| \geq 4 \).

If \( N^* \cong PSL(2, 2^3) \) and \( |N_\sigma^*| = 2^4 \), by (ii) of (4.6), \( |N_{G_\alpha}(T)^{F(T)}| = 3 \). Hence \( 3 \not| \displaystyle \sum \Omega \), \( \langle \chi \rangle \) is a Sylow 3-subgroup of \( G \). Since \( G_{G_\alpha: N_\sigma^* N^*} = G_\sigma \cong Aut(N^*) \) and \( N_\sigma^* N^*: N^*| = 2 \) by (i) of (4.4), we have \( G_{G_\alpha: N_\sigma^* N^*} = G_\sigma \cong Aut(N^*) \). In particular \( r=1 \) and \( |\Omega| = 16381 \). Moreover \( |F(s) \setminus \{\alpha\}| = 60 \). As before \( |F(\tau) \setminus \{\alpha\}| = 60 \), \( C_{N^*}(\tau) \cong PSL(2, 2^3) \) and an element of \( C_{N^*}(\tau) \) of order 7 fixes at least five points. But since \( 7 \not| \displaystyle \sum \Omega \) and \( 7 \not| \displaystyle \sum N_\sigma^* \), every element of order 7 fixes exactly one point, a contradiction.

(4.8) \( G^\Omega \cong PSL(2, 11) \), \( |\Omega| = 11 \).

Proof. By (4.7), \( |M_\sigma: N^*| \) is odd and so a Sylow 2-subgroup of \( N^* \) is also that of \( M \). By [4], [5] and [15], it suffices to consider the following cases:

(i) \( N^* \cong PSL(2, 2^2) \), \( M \cong PSL(2, q_1) \), \( q_1 \equiv 3 \) or 5 (mod 8), \( q_1 > 3 \).

(ii) \( N^* \cong PSL(2, 2^3) \), \( C_M(t) \supset Z_2 \times PSL(2, 3^{3m+1}), t \in I(M) \) (\( m \geq 1 \)).

(iii) \( N^* \cong PSL(2, 2^4) \), \( M \cong J_1 \), the smallest Janko group.

First we consider the case (i). If \( |N_\sigma^*| \) is odd, every involution in \( M \) has a unique fixed point and so \( M \cong PSL(2, 5) \) by [2]. But then \( M = N^* \), a contradiction. Hence \( |N_\sigma^*| = 2, 4, 6, 10 \) or 12. On the other hand \( r=1 \) or 2 because \( |Aut(N^*)| : N^*| = 2 \). From this \( |\Omega| = 1+ |N^*: N_\sigma^*| = 7, 11, 13, 21, 31 \) or 61. Since \( M \cong PSL(2, q_1) \) and \( |M| = |\Omega||N^*| \), we get \( |\Omega| = 11, |N_\sigma^*| = 6 \) and \( M \cong PSL(2, 11) \). Thus \( |\Omega| = 11 \) and \( G \cong PSL(2, 11) \).
Next we consider the case (ii). As in the case (i), $|N_\alpha^*|$ is even. Let $t \in I(N_\alpha^*)$. Since $|M_\alpha: N_\alpha^*|=1$ or 3, $I(M_\alpha) = \{ t^g | g \in M_\alpha \}$ and so $C_M(t)$ is transitive on $F(t)$. Hence $|F(t)| = |C_M(t): C_{M_\alpha}(t)|$. Since $|C_{M_\alpha}(t)| = |C_{M_\alpha}(t)N_\alpha^* : N_\alpha^*| |C_{N_\alpha^*}(t)|$, $|F(t)| \geq (3^{2m+1} - 1)3^{2m+1}(3^{2m+1} + 1)/24$. Since $|M_\alpha: N_\alpha^*| = 1$ or 3, $r = 1$ or 3. Therefore $|F(t)| = 1 + (|C_{N_\alpha^*}(t)|/|N_\alpha^*|) \cdot r < 1 + 8 \times 3 = 25$. Hence $25 > (3^{2m+1} - 1)^{3/24}$ and so $3^{2m+1} < 11$, a contradiction.

Finally we consider the case (iii). Since $N_\alpha^* = PSL(2, 2^3)$, $3^3 | N_\alpha^*$. But $3^2 \not| |M| = |J_1| = 2^6 \cdot 3 \cdot 7 \cdot 11 \cdot 19$, a contradiction.

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References
