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## ON DOUBLY TRANSITIVE PERMUTATION GROUPS

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### 1. Introduction

Let  $G$  be a doubly transitive permutation group on a finite set  $\Omega$  and  $\alpha \in \Omega$ . Using the notation of [9], we denote a normal subgroup of  $G_\alpha$  by  $N^\alpha$ . Then, for  $\beta \in \Omega$  other, we define  $N^\beta$  so that  $g^{-1}N^\beta g = N^\gamma$  where  $\gamma = \beta^g$ .

In this paper we shall prove the following:

**Theorem 1.** *Let  $G$  be a doubly transitive permutation group on a finite set  $\Omega$ . Suppose that  $\alpha$  is an element of  $\Omega$ . If  $G_\alpha$  has a normal simple subgroup  $N^\alpha$  which is isomorphic to  $PSL(2, q)$ ,  $Sz(q)$  or  $PSU(3, q)$  with  $q=2^n$ ,  $n \geq 2$ , then one of the following holds:*

- (i)  $|\Omega|=6$ ,  $G \simeq A_6$  or  $S_6$  and  $N^\alpha \simeq PSL(2, 4)$ .
- (ii)  $|\Omega|=11$ ,  $G \simeq PSL(2, 11)$  and  $N^\alpha \simeq PSL(2, 4)$ .
- (iii)  $G$  has a regular normal subgroup.

We introduce some notations: Let  $G$  be a permutation group on  $\Omega$ . For  $X \leq G$  and  $\Delta \subseteq \Omega$ , we define  $F(X) = \{\alpha \in \Omega \mid \alpha^x = \alpha \text{ for all } x \in X\}$ ,  $X(\Delta) = \{x \in X \mid \Delta^x = \Delta\}$ ,  $X_\Delta = \{x \in X \mid \alpha^x = \alpha \text{ for all } \alpha \in \Delta\}$  and  $X^\Delta = X(\Delta)/X_\Delta$ , the restriction of  $X$  on  $\Delta$ . If  $p$  is a prime, we denote by  $O^p(X)$ , the subgroup of  $X$  generated by all  $p'$ -elements in  $X$ . Other notations are standard ([6], [16]).

### 2. Preliminary results

**Lemma 2.1.** *Let  $G$  be a doubly transitive permutation group on  $\Omega$  of even degree and  $N^\alpha$  a nonabelian simple normal subgroup of  $G_\alpha$  with  $\alpha \in \Omega$ . If  $C_G(N^\alpha) \neq 1$ , then  $N_\beta^\alpha = N^\alpha \cap N^\beta$  for  $\alpha \neq \beta \in \Omega$  and  $C_G(N^\alpha)$  is semi-regular on  $\Omega - \{\alpha\}$ .*

*Proof.* Set  $C^\alpha = C_G(N^\alpha)$ . By Corollary B3 and Lemma 2.8 of [17],  $C^\alpha$  is semi-regular on  $\Omega - \{\alpha\}$  or  $N^\alpha$  is a T.I. set in  $G$ . Since  $|\Omega|$  is even and  $N^\alpha$  is  $\frac{1}{2}$ -transitive on  $\Omega - \{\alpha\}$ ,  $|N^\alpha : N_\beta^\alpha|$  is odd for  $\alpha \neq \beta \in \Omega$ . Hence  $N^\alpha$  is not semiregular on  $\Omega - \{\alpha\}$ . By Theorem A of [9],  $N^\alpha$  is not a T.I. set in  $G$ . Hence  $C^\alpha$  is semi-regular on  $\Omega - \{\alpha\}$ .

Set  $\Delta = F(N_\beta^\alpha)$ . Since  $C^\alpha \leq G(\Delta)$ ,  $[C^\alpha, G_\Delta] \leq C^\alpha \cap G_\Delta = 1$ . By Corollary

B1 of [17],  $N_\alpha^\beta \leq G_\Delta$  and so  $[C^\alpha, N_\alpha^\beta] = 1$ . Let  $1 \neq x \in C^\alpha$  and set  $\beta^x = \gamma$ . Then  $N_\alpha^\beta = x^{-1}N_\alpha^\beta x = N_\alpha^\gamma$ . Hence  $N_\alpha^\beta \leq N_\gamma^\beta$ . Since  $\beta \neq \gamma$  and  $G$  is doubly transitive on  $\Omega$ ,  $|N_\alpha^\beta| = |N_\gamma^\beta|$ . Hence  $N_\alpha^\beta = N_\gamma^\beta$ . Similarly we have  $N_\alpha^\gamma = N_\beta^\gamma$ . Hence  $N_\gamma^\beta = N_\beta^\gamma$  and so  $N_\gamma^\beta = N^\beta \cap N^\gamma$ . Since  $G$  is doubly transitive on  $\Omega$ ,  $N_\beta^\alpha = N^\alpha \cap N^\beta$ .

**Lemma 2.2.** *Let  $G$  be a transitive permutation group on a set  $\Omega$ ,  $H$  a stabilizer of a point of  $\Omega$  and  $M$  a nonempty subset of  $G$ . Then*

$$|F(M)| = |N_G(M)| \times |ccl_G(M) \cap H| / |H|.$$

Here  $ccl_G(M) \cap H = \{g^{-1}Mg \mid g^{-1}Mg \subseteq H, g \in G\}$ .

Proof. Set  $W = \{(L, \alpha) \mid L \in ccl_G(M), \alpha \in F(L)\}$  and  $W_\alpha = \{L \mid L \in ccl_G(M), F(L) \ni \alpha\}$ . By the transitivity of  $G$ ,  $|W_\alpha| = |W_\beta|$  holds for every  $\alpha, \beta \in \Omega$ . Counting the number of elements of  $W$  in two ways, we obtain  $|G : N_G(M)| \times |F(M)| = |G : H| \times |ccl_G(M) \cap H|$ . Thus we have Lemma 2.2.

**Lemma 2.3.** *Let  $G \simeq PSL(2, q)$ ,  $Sz(q)$  or  $PSU(3, q)$  with  $q = 2^n > 2$  and suppose that  $G$  is a transitive permutation group on a set  $\Omega$  of odd degree. Let  $H$  be a stabilizer of a point of  $\Omega$ . Then we have the following:*

- (i)  *$H$  has a unique Sylow 2-subgroup  $S$  of  $G$  and  $H = DS$  for a Hall 2'-subgroup  $D$  of  $H$  where  $D \leq Z_{q^2-1}$ .*
- (ii) *Let  $L$  be a subgroup of  $G$  such that  $|L| = |H|$ . Then  $L \in ccl_G(H)$ .*
- (iii)  *$S$  is semi-regular on  $\Omega - F(S)$  and  $|F(S)| = |F(H)| = |N_G(S) : H|$ .*
- (iv) *Set  $D = V \times K$  where  $V \leq Z_{q+1}$ ,  $K \leq Z_{q-1}$ . Then  $K$  acts semiregularly on  $\Omega - F(K)$  and if  $K \neq 1$ ,  $|F(K)| = 2|F(S)|$ .*

Proof. Since  $G$  is generated by its two distinct Sylow 2-subgroups and  $1 \neq |G : H|$  is odd,  $H$  contains a unique Sylow 2-subgroup  $S$  of  $G$  where  $S = O_2(H)$ . By the structure of  $N_G(S)$  we have (i) (cf. § 3 of [2]).

To prove (ii) we may assume that  $S \leq L$ . As above  $S = O_2(L)$  and  $L = D_1 S$  where  $D_1 \leq Z_{q^2-1}$ . Since  $N_G(S)/S$  is cyclic and  $|H| = |L|$ , we get  $H = L$ . Thus (ii) holds.

Let  $t \in I(S)$ . Applying Lemma 2.2,  $|F(t)| = |N_G(t)| \times |ccl_G(t) \cap H| / |H| = (|N_G(t)| \times |ccl_G(t) \cap N_G(S)| / |N_G(S)|) \times (|N_G(S)| / |H|)$ . Since  $N_G(S)$  is a stabilizer of the usual doubly transitive permutation representation of  $G$ , we have  $|N_G(t)| \times |ccl_G(t) \cap N_G(S)| / |N_G(S)| = 1$ , hence  $|F(t)| = |N_G(S) : H|$ . On the other hand,  $|F(S)| = |N_G(S)| \times |ccl_G(S) \cap H| / |H| = |N_G(S) : H|$ . Therefore  $S$  acts semi-regularly on  $\Omega - F(S)$ . As  $N_G(H) = N_G(S)$ , similarly we have  $|F(S)| = |F(H)|$ . Thus (iii) holds.

Let  $x$  be a nontrivial element of  $K$ . Then we have  $|F(\langle x \rangle)| = |N_G(\langle x \rangle)| \times |ccl_G(\langle x \rangle) \cap H| / |H| = (|N_G(\langle x \rangle)| \times |ccl_G(\langle x \rangle) \cap N_G(S)| / |N_G(S)|) (|N_G(S)| / |H|)$ . As before we have  $|N_G(\langle x \rangle)| \times |ccl_G(\langle x \rangle) \cap N_G(S)| / |N_G(S)| = 2$ . Hence  $|F(x)| = 2 \cdot |N_G(S) : H|$  and this is independent of the choice of  $x \in K^\#$ . Thus (iv)

holds.

**Lemma 2.4.** *Let  $G \simeq PSL(2, q)$ ,  $Sz(q)$  or  $PSU(3, q)$  with  $q=2^n > 2$  and  $S$  be a Sylow 2-subgroup of  $G$ ,  $H=N_c(S)$ ,  $t$  an involution outside  $H$ ,  $D=H \cap H^t$ ,  $V=C_D(t)$  and  $K=\{d \in D \mid d^t=d^{-1}\}$ . Then the following hold:*

- (i)  $N_c(\langle k \rangle) = \langle t \rangle D$  whenever  $1 \neq k \in K$ .
- (ii) If  $G \simeq PSU(3, q)$  and  $1 \neq U$  is a subgroup of  $V$ , then  $N_c(U) = C_c(V) = N \times V$  where  $N$  is a subgroup of  $G$  isomorphic to  $PSL(2, q)$ .

Proof. (i) follows from the structure of  $PSL(2, q)$ ,  $Sz(q)$  or  $PSU(3, q)$  (§ 3 of [2]).

We now regard  $PSU(3, q)$  as a usual doubly transitive permutation group on a set  $\Omega$  with  $q^3+1$  points. Then  $V$  is semi-regular on  $\Omega - F(V)$  and  $G(F(U))/G_{F(U)}$  is doubly transitive on  $F(U)=F(V)$ . Clearly  $N_c(U) \leq G(F(U))$  and  $G_{F(U)}=V$ . Hence  $N_c(U) \leq N_c(V)$ . Since  $V$  is cyclic,  $N_c(V) \leq N_c(U)$  and so  $N_c(U)=N_c(V)$ . We now set  $M=O_2'(N_c(V))$ . Then as  $[Z(S), V]=1$  and  $Z(S)$  is a Sylow 2-subgroup of  $N_c(V)$ ,  $M$  centralizes  $V$ . By the Frattini argument  $N_c(V)=(N_c(V) \cap N(Z(S)))M=N_H(V)M=DZ(S) \cdot M \leq C_c(V)$ . Hence  $N_c(V)=C_c(V)$ . By the direct computation, we obtain (ii).

**Lemma 2.5.** *Let  $G \simeq PSL(2, q)$ ,  $Sz(q)$  or  $PSU(3, q)$  with  $q=2^n > 2$  and let  $S$  be a Sylow 2-subgroup of  $G$ .*

- (i) If  $T$  is a maximal subgroup of  $S$ , then  $N_c(T)=S$ .
- (ii) Unless  $G \simeq PSU(3, q)$  where  $q=2^n$  and  $n$  is odd, then by conjugation  $N_c(S)$  acts regularly on the set of all maximal subgroups of  $S$ .

Proof. Since  $N_c(S)$  is strongly embedded in  $G$ ,  $S \leq N_c(T) \leq N_c(S)$  and so  $N_c(T)=RS$  where  $R$  is a Hall  $2'$ -subgroup of  $N_c(T)$ . As  $|S:T|=2$ ,  $R$  centralizes  $S/T \simeq Z_2$  and hence there exists an element  $t \in C_S(R) - T$ . If  $G \simeq PSL(2, q)$  or  $Sz(q)$ , then  $R=1$  (§ 3 of [2]). If  $G \simeq PSU(3, q)$  and  $R \neq 1$ , then by (ii) of Lemma 2.4,  $t \in I(S)=\Omega_1(S) \leq T$ , a contradiction. Thus (i) holds.

Let  $\Gamma$  be the set of all maximal subgroups of  $S$ . Then by conjugation,  $N_c(S)$  acts on  $\Gamma$  and  $(N_c(S))_T=S$  for  $T \in \Gamma$  by (i). Under the assumption of (ii), we can easily verify  $|\Gamma|=|N_c(S):S|$ . From this (ii) follows at once.

**Lemma 2.6.** *Let  $G \simeq PSL(2, q)$ ,  $Sz(q)$  or  $PSU(3, q)$  with  $q=2^n > 2$  and  $A$  be the full automorphism group of  $G$ . Let  $S$  be a Sylow 2-subgroup of  $G$ . Then  $C_A(S)=Z(S)$ . Here we identify  $G$  with the inner automorphism group of  $G$ .*

Proof. Let  $\Omega$  be the set of all Sylow 2-subgroups of  $G$ . Then  $A$  acts faithfully on  $\Omega$  and the action of  $G$  on  $\Omega$  is the same as the usual doubly transitive permutation representation. Hence  $S$  is regular on  $\Omega - \{S\}$  and so  $C_A(S)$  is a 2-subgroup of  $A$ . If  $G \simeq Sz(q)$ ,  $A/G$  is cyclic of odd order and so  $C_A(S) \leq G$ . Hence  $C_A(S)=C_G(S)=Z(S)$ . If  $G \simeq PSL(2, q)$ ,  $S$  is abelian, so that  $C_A(S)=S$

$=Z(S)$ . If  $G \simeq PSU(3, q)$ , there exists a field automorphism such that  $\langle f \rangle S$  is a Sylow 2-subgroup of  $N_A(S)$ . From this  $C_A(S) \leq O_2(N_A(S)) \leq \langle f \rangle S$ . If  $gs \in C_A(S) - S$  where  $g \in \langle f \rangle$  and  $s \in S$ , then  $g$  centralizes  $Z(S)$  and so  $g$  is a field automorphism of order 2 by the structural property of  $A$ . Since  $g$  centralizes  $s$ ,  $s$  must be contained in  $Z(S)$ . Therefore  $g$  centralizes  $S$ , while  $g$  is a field automorphism of order 2. This is a contradiction. Thus  $C_A(S) = S \cap C_A(S) = Z(S)$ .

**Lemma 2.7.** *Let  $G \simeq PSU(3, q)$ ,  $q=2^n$  such that  $n$  is even. Then  $\text{Aut}(G) = \langle f \rangle G$  for a field automorphism  $f$  of  $G$  (see [14]). Let  $B$  be a Borel subgroup and let  $D$  be a diagonal subgroup of  $G$ . Then  $B = DS$  and  $S = O_2(B)$  for some Sylow 2-subgroup  $S$  of  $G$ . Set  $D = V \times K$  with  $V \simeq Z_{q+1}$ ,  $K \simeq Z_{q-1}$ . Then  $C_A(Z(S)) = \langle \tau \rangle VS$  where  $A = \langle f \rangle G$  and  $\{\tau\} = I(\langle f \rangle)$ .*

*Proof.* By the structural properties of  $A$ ,  $[V, Z(S)] = 1$  and  $C_{\langle f \rangle}(Z(S)) = \langle \tau \rangle$ . Since  $N_A(Z(S)) \supset O_2(N_G(Z(S))) = S$ ,  $N_A(Z(S)) = \langle f \rangle N_G(S)$ . Hence  $C_A(Z(S)) = C(Z(S)) \cap \langle f \rangle DS = C_{\langle f \rangle K}(Z(S)) VS$ . Let  $gk \in C_{\langle f \rangle K}(Z(S))$  with  $g \in \langle f \rangle$ ,  $k \in K$ . Since  $g$  is a field automorphism of  $G$ , it centralizes a nontrivial element  $s$  in  $Z(S)$ . Then  $k$  centralizes  $s$  and so  $k=1$ , for otherwise  $s \in C_G(k) = VK$ , a contradiction. So  $C_{\langle f \rangle K}(Z(S)) = C_{\langle f \rangle}(Z(S)) = \langle \tau \rangle$ . Thus  $C_A(Z(S)) = \langle \tau \rangle VS$ .

### 3. The case $|\Omega|$ is even

Let  $G$  be a doubly transitive permutation group on a finite set  $\Omega$  of even degree satisfying the assumption of our theorem. Let  $\alpha \in \Omega$  and  $\{\alpha\}, \Delta_1, \dots, \Delta_r$  be the set of all  $N^\alpha$ -orbits on  $\Omega$ . Since  $N^\alpha$  is normal in  $G_\alpha$ ,  $|\Delta_i| = |\Delta_j|$  for  $1 \leq i, j \leq r$ . Hence  $|\Omega| = 1 + |\Delta_i| r$  and so both  $|\Delta_i|$  and  $r$  are odd. From this,  $N_\beta^\alpha$  contains a unique Sylow 2-subgroup of  $N^\alpha$  for  $\beta \neq \alpha$  by (i) of Lemma 2.3. Set  $S = O_2(N_\beta^\alpha)$ .

(3.1) The following hold.

- (i) For each  $\Delta_i$  with  $1 \leq i \leq r$ , there exists  $\beta_i \in \Delta_i$  such that  $N_{\beta_i}^\alpha = N_{\beta_i}^\alpha$ .
- (ii)  $F(S) = F(N_\beta^\alpha)$ ,  $|F(S)| = |N_{N^\alpha}(S): N_\beta^\alpha| \times r + 1$  and  $S$  is semi-regular on  $\Omega - F(S)$ .
- (iii) Set  $C^\alpha = C_G(N^\alpha)$ . Then  $C^\alpha = O(G_\alpha)$  and is semi-regular on  $\Omega - \{\alpha\}$ .

*Proof.* Let  $\gamma \in \Delta_i$ . Since  $|N_\beta^\alpha| = |N_\gamma^\alpha|$ , by (ii) of Lemma 2.3,  $N_\beta^\alpha = (N_\gamma^\alpha)^x$  for some  $x \in N^\alpha$ . Put  $\gamma^x = \beta_i$ . Then  $\beta_i \in \Delta_i$  and  $N_\beta^\alpha = N_{\beta_i}^\alpha$ . Thus (i) holds.

Hence by (iii) of Lemma 2.3, for each  $\Delta_i$  with  $1 \leq i \leq r$ ,  $F(S) \cap \Delta_i = F(N_\beta^\alpha) \cap \Delta_i$ ,  $|F(S) \cap \Delta_i| = |N_{N^\alpha}(S): N_\beta^\alpha|$  and  $S$  is semi-regular on  $\Delta_i - (\Delta_i \cap F(S))$ . Thus (ii) holds.

Since  $[O(G_\alpha), N^\alpha] \leq O(G_\alpha) \cap N^\alpha$  and  $N^\alpha$  is a non abelian simple group,  $[O(G_\alpha), N^\alpha] = 1$  and so  $O(G_\alpha) \leq C^\alpha$ . By Lemma 2.1,  $C^\alpha$  is semi-regular on

$\Omega - \{\alpha\}$ . Since  $G_\alpha \triangleright C^\alpha$ ,  $C^\alpha$  is  $\frac{1}{2}$ -transitive on  $\Omega - \{\alpha\}$ . Hence  $|C^\alpha| \mid |\Omega| - 1$ . From this  $C^\alpha$  is of odd order and hence  $C^\alpha \leq O(G_\alpha)$ . Thus  $C^\alpha = O(G_\alpha)$ .

As a Chevalley group,  $N^\alpha$  has a Borel subgroup  $N_{N^\alpha}(S)$ . Let  $D$  be a diagonal subgroup of  $N_{N^\alpha}(S)$ . Then  $N_{N^\alpha}(S) = DS$ . We now denote  $G_\alpha/C^\alpha$  by  $\bar{G}_\alpha$ . By the properties of  $PSL(2, q)$ ,  $Sz(q)$  or  $PSU(3, q)$  ([14]), there exists a field automorphism  $\bar{f}$  such that  $\langle \bar{f} \rangle \bar{N}^\alpha / \bar{N}^\alpha$  is a Sylow 2-subgroup of  $\bar{G}_\alpha / \bar{N}^\alpha$ . Since  $C^\alpha = O(G_\alpha)$ , we may assume  $f$  is a 2-element in  $G_\alpha$ . Since  $DC^\alpha \cap N^\alpha = D$  and  $SC^\alpha \cap N^\alpha = S$ ,  $D$  and  $S$  are  $f$ -invariant. Clearly  $\langle f \rangle S$  is a Sylow 2-subgroup of  $G_\alpha$ . Since  $\langle f \rangle \cap \bar{N}^\alpha = 1$ ,  $\langle f \rangle \cap S \leq C^\alpha$  and so  $\langle f \rangle \cap S = 1$ . Thus we have the following.

(3.1)' There exists a 2-element  $f$  in  $G_\alpha$  satisfying the following.

- (i)  $f$  acts on  $N^\alpha$  as a field automorphism of  $N^\alpha$ .
- (ii)  $D$  and  $S$  are  $f$ -invariant and  $\langle f \rangle \cap S = 1$ .
- (iii)  $\langle f \rangle S$  is a Sylow 2-subgroup of  $G_\alpha$ .

(3.2)  $N_\beta^\alpha / N^\alpha \cap N^\beta$  is cyclic of odd order.

Proof. By Lemma 2.1 and (iii) of (3.1), we may assume that  $C^\alpha = 1$ . First we claim that  $|S : S \cap N^\beta| = 1$  or 2. Since  $S/S \cap N^\beta \cong SN^\beta/N^\beta$  is isomorphic to a 2-subgroup of the outer automorphism group of  $N^\beta$ ,  $S/S \cap N^\beta$  is cyclic. But  $S/S'$  is an elementary abelian 2-group and so  $S/S \cap N^\beta \cong 1$  or  $Z_2$  and hence  $|S : S \cap N^\beta| = 1$  or 2.

To prove (3.2), it suffices to show that  $|S : S \cap N^\beta| \neq 2$ . Assume that  $|S : S \cap N^\beta| = 2$ . Then as  $S$  and  $S \cap N^\beta$  are normal subgroups of  $N_\beta^\alpha$ . Then it follows from (i) of Lemma 2.5 that  $N_\beta^\alpha = S$  and  $|N_\beta^\alpha : N^\alpha \cap N^\beta| = 2$ . Since a Sylow 2-subgroup of  $G_\alpha/N^\alpha$  is cyclic and  $G_{\alpha\beta}/N_\beta^\alpha \cong G_{\alpha\beta}N^\alpha/N^\alpha$ , a Sylow 2-subgroup of  $G_{\alpha\beta}/N_\beta^\alpha$  is cyclic. As  $N_\beta^\alpha N_\alpha^\beta / N_\beta^\alpha$  is a normal subgroup of  $G_{\alpha\beta}/N_\beta^\alpha$  of order 2,  $I(G_{\alpha\beta}) \subseteq N_\beta^\alpha N_\alpha^\beta$ . Let  $f$  be as defined in (3.1)'. Then  $f \neq 1$  as  $N_\beta^\alpha N_\alpha^\beta \not\leq N^\alpha$ . Let  $\tau \in I(\langle f \rangle)$ . Since  $\tau \in N_{G_\alpha}(S)$ ,  $S = N_\beta^\alpha$  and  $|F(S) - \{\alpha\}|$  is odd, there exists  $\gamma$  such that  $\gamma \in F(\tau) \cap F(N_\beta^\alpha)$  and  $\gamma \neq \alpha$ . Clearly  $N_\beta^\alpha \leq N_\gamma^\alpha$ , so that  $N_\beta^\alpha = N_\gamma^\alpha$ . Therefore we may assume  $F(\tau) \ni \beta$  and  $\tau \in G_{\alpha\beta}$ . By Corollary B1 of [17]  $F(N_\beta^\alpha) = F(N_\beta^\alpha)$ . From this  $F(\tau) \supseteq F(N_\beta^\alpha N_\alpha^\beta) = F(N_\beta^\alpha)$  because  $\tau \in I(G_{\alpha\beta}) \subseteq N_\beta^\alpha N_\alpha^\beta$ . So  $\langle \tau \rangle N_\beta^\alpha \leq (\langle \tau \rangle N^\alpha \cap N(N_\beta^\alpha))_{F(N_\beta^\alpha)}$ . Let  $D$  be as defined in (3.1)'. Then  $D \leq N_{N^\alpha}(N_\beta^\alpha)$  and  $D$  is  $\tau$ -invariant. Hence  $[D, \tau] \leq (\langle \tau \rangle N^\alpha \cap N(N_\beta^\alpha))_{F(N_\beta^\alpha)} \cap D = 1$ . Therefore  $\tau$  centralizes  $D$ . Since  $\tau$  is a field automorphism of  $N^\alpha$  of order 2 and  $D$  is a diagonal subgroup of  $N^\alpha$ , this is a contradiction.

(3.3) The following hold.

- (i)  $N^\alpha \cap N^\beta = N^\gamma \cap N^\delta$  for,  $\gamma, \delta \in F(N^\alpha \cap N^\beta)$  with  $\gamma \neq \delta$ .
- (ii)  $G(F(S)) = N_G(N^\alpha \cap N^\beta)$ .
- (iii) Let  $M$  be a subgroup of  $N^\alpha \cap N^\beta$  which contains  $S$ . Then  $F(M) =$

$F(S)$  and  $N_G(M)$  is doubly transitive on  $F(S)$ .

(iv)  $C_{G_\alpha}(S) = Z(S) \times C^\alpha$ .

(v) Let  $M$  be as defined in (iii) and suppose  $C^\alpha \neq 1$ . Then  $O_2(C_G(M))^{F(S)}$  is a regular normal elementary abelian 2-subgroup of  $N_G(M)^{F(S)}$ .

Proof. Let  $\gamma, \delta \in F(N^\alpha \cap N^\beta)$  with  $\gamma \neq \delta$ . We may assume  $\alpha \neq \gamma$ . Since  $G$  is doubly transitive on  $\Omega$ ,  $|N^\alpha \cap N^\beta| = |N^\alpha \cap N^\gamma|$ . By the choice of  $\gamma$ ,  $N^\alpha \cap N^\beta \leq N_\gamma^\alpha$  and  $N_{N^\alpha}(S)/S$  is cyclic. Hence  $N^\alpha \cap N^\beta = N^\alpha \cap N^\gamma$ . Similarly  $N^\gamma \cap N^\alpha = N^\gamma \cap N^\delta$ . Thus (i) holds.

Since  $N_G(N^\alpha \cap N^\beta) \leq N_G(S)$ ,  $N_G(N^\alpha \cap N^\beta) \leq G(F(S))$ . Let  $x \in G(F(S))$ . Then  $\alpha^x, \beta^x \in F(S)$  and  $F(S) = F(N_\beta^\alpha)$  by (ii) of (3.1). Hence  $\alpha^x, \beta^x \in F(N^\alpha \cap N^\beta)$ . Therefore by (i)  $N^{\alpha^x} \cap N^{\beta^x} = N^\alpha \cap N^\beta$  and so  $x \in N_G(N^\alpha \cap N^\beta)$ . Thus (ii) holds.

Suppose  $S \leq M \leq N^\alpha \cap N^\beta$ . If  $M^g \leq G_{\alpha\beta}$  for some  $g \in G_\alpha$ . Then  $M^g \leq N^\alpha \cap G_{\alpha\beta} = N_\beta^\alpha$ . Hence  $M^g = M$  because  $S \leq M$  and  $N_\beta^\alpha/S$  is cyclic of odd order. By the Witt's Theorem  $N_{G_\alpha}(M)$  is transitive on  $F(M) - \{\alpha\}$ . Similarly  $N_{G_\beta}(M)$  is transitive on  $F(M) - \{\beta\}$ . We may assume  $|F(M)| > 2$ . Hence  $N_G(M)$  is doubly transitive on  $F(M)$ . By (ii) of (3.1),  $F(M) = F(S)$ . Thus (iii) holds.

We denote  $G_\alpha/C^\alpha$  by  $\bar{G}_\alpha$ . Clearly  $C_{\bar{G}_\alpha}(\bar{N}^\alpha) = \bar{1}$ . Applying Lemma 2.6,  $C_{\bar{G}_\alpha}(\bar{S}) = Z(\bar{S})$ , hence  $C_{G_\alpha}(S) \leq Z(S) \times C^\alpha$ . The converse implication is obvious. Thus (iv) holds.

Suppose  $C^\alpha \neq 1$ . Then since  $C^\alpha$  is semi-regular on  $\Omega - \{\alpha\}$ ,  $C_G(M)^{F(S)} \geq (C^\alpha)^{F(S)} \neq 1$ . As  $N_G(M)^{F(S)}$  is doubly transitive by (iii),  $C_G(M)^{F(S)}$  is transitive. By (iv),  $(C^\alpha)^{F(S)} \leq C_{G_\alpha}(M)^{F(S)} \leq (Z(S) \times C^\alpha)^{F(S)}$  and so  $C_{G_\alpha}(M)^{F(S)} = (C^\alpha)^{F(S)}$ . Hence  $C_G(M)^{F(S)}$  is a Frobenius group and so  $O_2(C_G(M)^{F(S)}) \neq 1$  because  $|F(S)|$  is even. Since  $C_G(M)^{F(S)} \leq (Z(S) \times C^\alpha)^{F(S)} = Z(S)$ ,  $O_2(C_G(M)^{F(S)}) = O_2(C_G(M))^{F(S)}$  and this is regular on  $F(S)$ . As  $N_G(M)^{F(S)} \supset O_2(C_G(M))^{F(S)}$ ,  $O_2(C_G(M))^{F(S)}$  must be a regular normal elementary abelian 2-subgroup of  $N_G(M)^{F(S)}$ . Thus (v) holds.

(3.4) There exists an involution  $t$  such that  $ccl_G(t) \cap S \neq \phi$ ,  $\alpha^t = \beta$  and  $F(t) \cap F(S) = \phi$ . Set  $\mu = |N_{N^\alpha}(S) : N_\beta^\alpha|$  and  $|S| = q^i$ . Then we have

(i)  $|\Omega| = (q^i + 1)\mu r + 1$ .

(ii)  $|C_S(t)| \geq \sqrt{q}$ ,  $\sqrt{2q}$  or  $q$  according as  $N^\alpha \simeq PSL(2, q)$ ,  $Sz(q)$  or  $PSU(3, q)$ , respectively. Furthermore  $|C_S(t)| \mid |F(S)| = \mu r + 1$ .

(iii) If  $\mu = 1$ , then  $|\Omega| = 6$  and  $G \simeq A_6$  or  $S_6$ .

(iv)  $|\Omega|_2 = |F(S)|_2 \cdot |G : N_G(S)|_2$ .

Proof. Since  $|\Delta_i| = |N^\alpha : N_\beta^\alpha| = |N^\alpha : N_{N^\alpha}(S)| \times |N_{N^\alpha}(S) : N_\beta^\alpha| = (q^i + 1)\mu$  and  $|\Omega| = |\Delta_i|r + 1$ . Hence (i) holds.

Since  $G$  is doubly transitive on  $\Omega$ , there exists an involution  $t$  such that  $ccl_G(t) \cap S \neq \phi$  and  $\alpha^t = \beta$ . Then  $t$  normalizes  $O_2(N^\alpha \cap N^\beta) = S$ . Claim  $F(t) \cap F(S) = \phi$ . Suppose not and let  $\gamma \in F(t) \cap F(S)$ . As  $S \leq N_\gamma^\alpha$ ,  $S \leq N^\alpha \cap N^\gamma$  by (i) of (3.3). Let  $g$  be such that  $t^g \in S$ . Then  $t \in N^\delta \cap G_\gamma = N_\gamma^\delta$  where  $\delta = \alpha^{g^{-1}}$  and

hence  $t \in N^\gamma$ . Since  $t$  normalizes  $S$  and  $\langle t \rangle S \leq N^\gamma$ ,  $t$  must be contained in  $S$ , a contradiction. Hence  $F(t) \cap F(S) = \phi$ . From this  $C_s(t)$  acts semi-regularly on  $F(t)$  and so  $|F(t)|$  is divisibly by  $|C_s(t)|$ . Since  $t^g \in S$ ,  $|F(t)| = |F(t^g)| = |F(S)|$ , hence  $|C_s(t)| \mid |F(S)|$ .

If  $N^\alpha \simeq PSL(2, q)$ , then  $|\Omega_1(S/S')| = |S| = q$  and by Lemma 1 of [7],  $|C_s(t)| \geq \sqrt{q}$ . If  $N^\alpha \simeq Sz(q)$ , then  $|\Omega_1(S/S')| = q$ . Since  $q$  is an odd power of 2 in this case, similarly  $|C_s(t)| \geq \sqrt{2q}$ . If  $N^\alpha \simeq PSU(3, q)$ , then  $|\Omega_1(S/S')| = q^2$  and so similarly  $|C_s(t)| \geq q$ . Thus we have (ii).

Suppose  $\mu = 1$ . Then  $N^\alpha$  is doubly transitive on each  $N^\alpha$ -orbit  $\neq \{\alpha\}$ . Applying Theorem D of [10],  $r = 1$ . Therefore,  $|F(S)| = \mu r + 1 = 2$  and so by (i) and (ii),  $q = 4$ ,  $N^\alpha \simeq PSL(2, 4)$  and  $|\Omega| = 6$ . Thus (iii) holds.

Since  $|\Omega| = |G : N_G(S)| \times |N_G(S) : N_{G_\alpha}(S)| / |G_\alpha : N_{G_\alpha}(S)|$  and  $|G_\alpha : N_{G_\alpha}(S)|$  is odd, (iv) holds.

(3.5) Let  $\pi$  be the set of primes which divides  $q-1$  and  $K$  a Hall  $\pi$ -subgroup of  $N^\alpha \cap N^\beta$ . If  $K \neq 1$ , then  $C^\alpha = 1$ .

Proof. Suppose  $K \neq 1$  and  $C^\alpha \neq 1$ . Set  $\Gamma_i = \Delta_i \cap F(S)$  and  $\Lambda_i = \Delta_i \cap F(K)$ . Then by (i) of (3.1) and Lemma 2.3, for each  $i$  with  $1 \leq i \leq r$   $|\Lambda_i| = 2|\Gamma_i| = 2|N_{N^\alpha}(S) : N_{\beta_i}^\alpha| = 2|N_{N^\alpha}(S) : N_\beta^\alpha|$  and  $K$  is semi-regular on  $\Delta_i - \Lambda_i$ . By (v) of (3.3),  $O_2(C_G(KS))^{F(S)}$  is a regular normal elementary abelian 2-subgroup of  $N_G(KS)^{F(S)}$ . Set  $E = O_2(C_G(KS))$ . It follows from (iv) of (3.3) that  $E_{F(S)} \leq (Z(S) \times C^\alpha)_{F(S)}$ . Since  $F(Z(S)) = F(S)$  by (ii) of (3.1) and  $(C^\alpha)_{F(S)} = 1$  by (iii) of (3.1),  $(Z(S) \times C^\alpha)_{F(S)} = Z(S)$ . On the other hand  $Z(S) \cap C(K) = 1$  (cf. § 3 of [2]) and so  $E_{F(S)} = 1$ . Hence  $E \simeq E^{F(S)}$ . Since  $E$  is regular on  $F(S)$ ,  $|F(S)| = |E^{F(S)}|$  and so we have  $|F(S)| = |E|$ . Since  $KS$  is a subgroup of  $N_\beta^\alpha$  which contains  $S$ , by (ii) of (3.1) we have  $F(S) = F(KS)$ . From this  $F(S)$  is a subset of  $F(K)$ . Hence  $|F(K) - F(S)| = |F(K) - \{\alpha\}| - |F(S) - \{\alpha\}| = \sum_{i=1}^r |\Lambda_i| - \sum_{i=1}^r |\Gamma_i| = r \times |N_{N^\alpha}(S) : N_\beta^\alpha|$ . Since  $r$  is odd,  $|F(K) - F(S)|$  is odd. On the other hand  $E$  fixes  $F(K) - F(S)$  setwise because  $E$  centralizes  $S$  and  $K$ . Therefore  $E$  fixes an element  $\gamma \in F(K) - F(S)$  as  $E$  is a 2-subgroup of  $G$ . Since  $N_\gamma^\alpha / O_2(N_\gamma^\alpha)$  is cyclic of odd order,  $K \leq N_\gamma^\alpha$  and  $|K \cdot O_2(N_\gamma^\alpha)| \mid |N^\alpha \cap N^\gamma|$ , we have  $K \cdot O_2(N_\gamma^\alpha) \leq N^\alpha \cap N^\gamma$ . Hence  $K \leq N^\gamma$  and so  $|C_{N^\gamma}(K)|$  is odd by (i) of Lemma 2.4. Since  $C_{G_\gamma}(K) / C_{N^\gamma}(K) C^\gamma \simeq C_{G_\gamma}(K) N^\gamma C^\gamma / N^\gamma C^\gamma$ , a Sylow 2-subgroup of  $C_{G_\gamma}(K)$  is cyclic. But  $E \leq C_{G_\gamma}(K)$  and hence  $|E| = |F(S)| = 2 = \mu r + 1$ . From this  $\mu = r = 1$ . By (iii) of (3.4)  $C^\alpha = 1$ , which is contrary to the assumption  $C^\alpha \neq 1$ . So (3.5) holds.

(3.6) Suppose  $K \neq 1$  and let  $S_1$  be a subgroup of  $S$ . If  $S_1^g \leq N_G(S)$  and  $S_1^g \not\leq S$  for some  $g \in G$ , then  $S_1 \leq Z_2 \times Z_4$  and  $|S_1| \mid 2|G_\alpha / N^\alpha|$ .

Proof. Set  $S_1^g = T$ . By (ii) of (3.1),  $T$  is semi-regular on  $\Omega - F(T)$ . Claim



$F(T) \cap F(S) = \phi$ . Suppose not and let  $\gamma \in F(T) \cap F(S)$ . Then  $T \leq N_\gamma^{\alpha^\varepsilon}$  and  $S \leq N_\gamma^\alpha$ . By (3.2)  $T \leq N^{\alpha^\varepsilon} \cap N^\gamma$  and  $S \leq N^\alpha \cap N^\gamma$  and so  $TS \leq N^\gamma$ . Since  $S$  is a Sylow 2-subgroup of  $N^\gamma$ ,  $TS = S$ . Hence  $T \leq S$ , a contradiction. Thus  $F(T) \cap F(S) = \phi$ . From this  $T$  acts semi-regularly on  $F(S)$ . By (ii) of (3.3),  $T$  normalizes  $N^\alpha \cap N^\beta$  and so  $T \leq N_G(S) \cap N_G(KS)$ . By the Frattini argument  $KST = N_{KST}(K) \cdot KS = N_{ST}(K) \cdot KS$ , so that  $N_{ST}(K)^{F(S)} = T^{F(S)}$  as  $F(S) = F(KS)$ . For an arbitrary  $\gamma \in F(S)$ ,  $N_{ST}(K)_\gamma = N_S(K) = C_S(K) = 1$ , whence  $N_{ST}(K) \simeq N_{ST}(K)^{F(S)}$ . Hence  $T \simeq N_{ST}(K)$ . Now  $N_{ST}(K)$  acts on  $F(K) - F(S)$  and  $|F(K) - F(S)|$  is odd. Hence  $N_{ST}(K)$  fixes some  $\delta \in F(K) - F(S)$ . Since  $K \leq N_\delta^\alpha$  and  $|K \cdot O_2(N_\delta^\alpha)| \mid |N^\alpha \cap N^\delta|$ , we have  $K \leq N^\alpha \cap N^\delta$  as in the proof of (3.5). By (i) of Lemma 2.4,  $N_{N^\delta}(K) = D \langle u \rangle \triangleright D$  where  $u$  is an involution and  $D$  is a cyclic subgroup of  $N^\delta$  of odd order. Since  $N_{G_\delta}(K)/N_{N^\delta}(K) \simeq N_{G_\delta}(K)N^\delta/N^\delta$  and a Sylow 2-subgroup of  $G_\delta/N^\delta$  is cyclic, a Sylow 2-subgroup of  $N_{G_\delta}(K)$  is isomorphic to a subgroup of  $Z_2 \times Z_m$  for some integer  $m$ . Since  $T \leq S^\varepsilon$  and  $S$  is of exponent at most 4, (3.6) follows immediately.

(3.7) One of the following holds.

- (i)  $|\Omega| = 6$  and  $G \simeq A_6$  or  $S_6$ .
- (ii)  $N^\alpha \cap N^\beta$  is a  $\pi'$ -group.

**Proof.** Let  $K$  be a Hall  $\pi$ -subgroup of  $N^\alpha \cap N^\beta$  and suppose  $G \neq A_6, S_6$  and  $K \neq 1$ . Let  $t$  be an involution as in (3.4) and  $Q$  a Sylow 2-subgroup of  $G$  containing  $\langle t \rangle S$ . Then  $Q \triangleright S$ . For otherwise, let  $x \in N_Q(N_Q(S)) - N_Q(S)$ , then  $S^x \neq S$  and  $S^x$  normalizes  $S$ . Applying (3.6) to  $S^x$ ,  $S \simeq Z_2 \times Z_2$  and  $N^\alpha \simeq PSL(2, 4)$ . But since  $K \neq 1$ ,  $|N^\alpha \cap N^\beta| = 12$  and hence  $\mu = 1$ . It follows from (iii) of (3.4) that  $G \simeq A_6$  or  $S_6$ , which is contrary to the assumption.

Since  $Q \triangleright S$  and all involutions in  $S$  are conjugate in  $G$ ,  $t$  is conjugate to  $s$  for an involution  $s \in Z(Q) \cap S$ . As  $s$  is an extremal element in  $Q$ , there is an element  $g \in G$  such that  $t^g = s$  and  $(C_Q(t))^g \leq Q$ . Set  $T = (C_S(t))^g$ . If  $T \leq S$ , as  $S$  is semi-regular on  $\Omega - F(S)$ ,  $F(S)^g = F(S)$ . Hence  $F(t) = F(s)^{g^{-1}} = F(S)$ , contrary to the choice of  $t$ . Therefore  $T \not\leq S$ . Applying (3.6) again,  $C_S(t) \leq Z_2 \times Z_4$ ,  $|C_S(t)| \mid 2 \cdot |G_\alpha/N^\alpha|$ .

If  $N^\alpha \simeq PSL(2, q)$ , by (ii) of (3.4),  $\sqrt{q} \leq |C_S(t)| \mid 2 \cdot |G_\alpha/N^\alpha|$  and so  $q = 2^2$  or  $2^4$ . As before,  $q \neq 2^2$ , hence  $q = 2^4$ ,  $N^\alpha \simeq PSL(2, 2^4)$ . Then  $r = 1$  because the outer automorphism group of  $PSL(2, 2^4)$  is cyclic of order 4. Since  $\mu \neq 1$  and  $K \neq 1$ ,  $(\mu, |K|, |F(K)|, |\Omega|)$  is  $(3, 5, 7, 52)$  or  $(5, 3, 11, 86)$  by (iv) of Lemma 2.3 and (i) of (3.4). By the Witt's Theorem,  $N_G(K)$  is doubly transitive on  $F(K)$ . Hence  $|G|$  is divisible by  $|F(K)|$ . Since  $C^\alpha = 1$  by (3.5), we have  $|G| \mid |\Omega| \cdot |\text{Aut}(PSL(2, 2^4))|$ . Hence we can verify  $|F(K)| \nmid |G|$  in both cases. This is a contradiction.

If  $N^\alpha \simeq Sz(q)$ , similarly we obtain  $\sqrt{2q} < |C_S(t)| \mid 2 |G_\alpha/N^\alpha|$ . But in this case since the outer automorphism group of  $N^\alpha$  is cyclic of odd order,  $|G_\alpha/N^\alpha|$

is odd and so  $\sqrt{2q} \leq 2$ . Hence  $q \leq 2$ , a contradiction.

If  $N^\alpha \simeq PSU(3, q)$ , similarly  $q \leq |C_s(t)| |2| G_\alpha / N^\alpha |$ . Hence  $q = 2^2$ ,  $N^\alpha \simeq PSU(3, 2^2)$ . As in the first case,  $r = 1$  and  $(\mu, |K|, |F(K)|, |\Omega|) = (5, 3, 11, 326)$  and so  $11 = |F(K)| | |\Omega| \cdot |\text{Aut}(PSU(3, 2^2))|$ , a contradiction.

In (3.8)–(3.11), we shall prove that  $N_\beta^\alpha = N^\alpha \cap N^\beta$ . First we note the following.

(3.8) If  $C^\alpha \neq 1$ ,  $N_\beta^\alpha = N^\alpha \cap N^\beta$ .

Proof. Since  $N^\alpha$  is a nonabelian simple group, (3.8) follows immediately from Lemma 2.1.

(3.9) Let  $p$  be a prime with  $p \mid |N_\beta^\alpha : N^\alpha \cap N^\beta|$  and assume the following:

(\*)  $p \neq 3$  if  $N^\alpha \simeq PSU(3, 2^n)$  and  $n$  is odd.

Then  $\mu = p$ .

Proof. It follows from (3.8) that  $C^\alpha = 1$ . Hence  $G_\alpha / N^\alpha$  is isomorphic to a subgroup of the outer automorphism group of  $N^\alpha$  and so under the hypothesis (\*), a Sylow  $p$ -subgroup of  $G_\alpha / N^\alpha$  is normal and cyclic ([14]). Set  $W = N_G(S)_{F(S)}$ . Since  $W / N_\beta^\alpha \leq G_{\alpha\beta} / N_\beta^\alpha \simeq G_{\alpha\beta} N^\alpha / N^\alpha$ , a Sylow  $p$ -subgroup of  $W / N_\beta^\alpha$  is normal and cyclic. Hence all elements in  $W$  of order  $p$  is contained in  $N_\beta^\alpha N_\alpha^\beta$  because  $|N_\beta^\alpha N_\alpha^\beta / N_\beta^\alpha| = |N_\beta^\alpha : N^\beta \cap N^\alpha| = |N_\beta^\alpha : N^\alpha \cap N^\beta|$  and  $p \mid |N_\beta^\alpha : N^\alpha \cap N^\beta|$ . Let  $P$  be a Sylow  $p$ -subgroup of  $W$ . Then  $\Omega_1(P) \leq N_\beta^\alpha N_\alpha^\beta$ . Set  $Q = \Omega_1(P)$ . Since  $N_\beta^\alpha N_\alpha^\beta / N_\beta^\alpha \simeq N_\beta^\alpha / N^\alpha \cap N^\beta$ , by (3.2)  $N_\beta^\alpha N_\alpha^\beta / N_\beta^\alpha$  is cyclic and so  $Q'$  is a cyclic subgroup of  $N_\beta^\alpha$ , similarly  $Q' \leq N_\alpha^\beta$ . Hence  $Q' \leq N^\alpha \cap N^\beta$  and the  $p$ -rank of  $Q/Q'$  is at most 2.

By the Frattini argument,  $N_G(S) = (N_G(S) \cap N(P))W$ . Let  $M$  be a normal subgroup of  $N_G(S) \cap N(P)$  such that  $M^{F(S)}$  is a minimal normal subgroup of  $N_G(S)^{F(S)}$ . We choose  $M$  so that its order is minimal. Since  $N_G(S)^{F(S)}$  is doubly transitive,  $M^{F(S)}$  is an elementary abelian 2-subgroup or a direct product of isomorphic non abelian simple groups. As  $Q'$  is cyclic,  $M/C_M(Q')$  is abelian and its Sylow 2-subgroup is cyclic. Hence by the minimality of  $M$ ,  $M = C_M(Q')$ .

Set  $\bar{Q} = Q/Q'$ . We argue that  $C_M(\bar{Q}) \leq W$ . To prove this, it suffices to show that  $M \neq C_M(\bar{Q})$ . If  $M = C_M(\bar{Q})$ ,  $M$  stabilizes the normal series  $Q \triangleright Q' \triangleright 1$  and hence  $O^p(M)$  centralizes  $P$  by Theorem 5.3.2 and Theorem 5.3.1 of [6]. Obviously  $O^p(M) \not\leq W$  and so  $O^p(M) = M$  by the minimality of  $M$ . Therefore  $M$  centralizes  $P$ . Let  $x$  be an element of  $M$  such that  $\alpha^x = \beta$ , then  $P \cap N_\beta^\alpha \leq N^\alpha \cap N^{\alpha^x} = N^\alpha \cap N^\beta$ . But since  $P \cap N_\beta^\alpha$  is a Sylow  $p$ -subgroup of  $N_\beta^\alpha$ ,  $p \nmid |N_\beta^\alpha : N^\alpha \cap N^\beta|$ , a contradiction.

Set  $C = C_M(\Omega_1(\bar{Q}))$ . Then  $M/C \leq GL(2, p)$  because the  $p$ -rank of  $\bar{Q}$  is at most 2. By the minimality of  $M$ ,  $M/C \leq SL(2, p)$ . On the other hand  $O^p(C) \leq C_M(\bar{Q}) \leq W$ . Therefore  $C^{F(S)}$  is a normal  $p$ -subgroup of  $N_G(S)^{F(S)}$ . Since

$p \neq 2$ ,  $C^{F(S)} = 1$  and so  $C \leq W$ . Hence  $M^{F(S)}$  is isomorphic to a homomorphic image of a subgroup of  $SL(2, p)$ .

Hence if  $M^{F(S)}$  is an elementary abelian 2-group, we have  $M^{F(S)} \simeq Z_2 \times Z_2$  and  $|F(S)| = 4$ . From (ii) and (iii) of (3.4),  $\mu = 3$  and  $r = 1$ . By (ii) of (3.4),  $N^\alpha \simeq PSL(2, 4)$ ,  $PSL(2, 16)$  or  $PSU(3, 4)$  and hence  $|G_\alpha : N^\alpha| = 1, 2$  or  $4$ , which is contrary to  $p \mid |N_\alpha^\beta : N^\beta \cap N^\alpha| = |N_\alpha^\beta N^\alpha / N^\alpha|$ .

If  $M^{F(S)}$  is a direct product of isomorphic non abelian simple groups by Dickson's Theorem (Hauptsatz 8.27 [8])  $M^{F(S)} \simeq PSL(2, p)$  with  $p > 5$  or  $A_5$ . Claim  $M^{F(S)} \neq A_5$ . Suppose  $M^{F(S)} \simeq A_5$ , then  $N_G(S)^{F(S)} \simeq A_5$  or  $S_5$  and so  $|F(S)| = 6$ ,  $\mu = 5$  and  $r = 1$ . By (ii) of (3.4), we obtain  $q = 2^2$  and  $N^\alpha \simeq PSL(2, 4)$ . Hence  $5 \nmid |N_{N^\alpha}(S) : N_\beta^\alpha| = \mu = 5$ , a contradiction. Thus  $M^{F(S)} \simeq PSL(2, p)$  with  $p > 5$ . Hence  $|N_G(S)^{F(S)} : M^{F(S)}| = 1$  or  $2$ . From this as  $|F(S)|$  is even,  $M^{F(S)}$  is also doubly transitive. Again by Dickson's Theorem, we know all maximal subgroups of  $PSL(2, p)$  with  $p > 5$  and hence  $PSL(2, p)$  with  $p > 5$  has a unique doubly transitive permutation representation of even degree, which is the known one. From this  $|F(S)| = p + 1$ . Since  $|F(S)| = \mu r + 1 = \mu + 1$ , we obtain  $\mu = p$ .

(3.10) If  $N^\alpha \simeq PSU(3, q)$  and  $n$  is odd, then  $3 \nmid |N_\beta^\alpha : N^\alpha \cap N^\beta|$ .

Proof. By (3.8), we may assume  $C^\alpha = 1$ . Set  $W = N_G(S)_{F(S)}$  and let  $P$  be a Sylow 3-subgroup of  $W$ . As  $G_{\alpha\beta} / N_\beta^\alpha \simeq G_{\alpha\beta} N^\alpha / N^\alpha \leq G_\alpha / N^\alpha$ , a Sylow 3-subgroup of  $W / N_\beta^\alpha$  is an abelian 3-group of rank at most 2, so that  $P' \leq N_\beta^\alpha$  and similarly  $P' \leq N_\alpha^\beta$ . Hence  $P' \leq N^\alpha \cap N^\beta$  and  $P'$  is cyclic.

Similarly as in the proof of (3.9) we can choose a normal subgroup  $M$  of  $N_G(S) \cap N(P)$ . Denote  $P/P'$  by  $\bar{P}$ . Then  $\Omega_1(\bar{P})$  is an elementary abelian 3-subgroup of rank at most 3. Then as in the proof of (3.9),  $M$  centralizes  $P'$  and  $C_M(\Omega_1(\bar{P}))$  is contained in  $W$ . Hence  $M/C \leq SL(3, 3)$  where  $C = C_M(\Omega_1(\bar{P}))$ .

If  $M^{F(S)}$  is an elementary abelian 2-group, by the structure of  $SL(3, 3)$ ,  $M^{F(S)} \simeq Z_2 \times Z_2$  and so  $|F(S)| = 4$ ,  $\mu = 3$  and  $r = 1$ . Let  $p_1 \in \pi$ . Since  $n$  is odd,  $3 \notin \pi$ . Therefore  $p_1 \neq 3$ . By (3.7),  $p_1 \nmid |N^\alpha \cap N^\beta|$ . Hence  $p_1 \mid |N_\beta^\alpha : N^\alpha \cap N^\beta|$  and applying (3.9) to  $p_1$ , we have  $\mu = p_1 = 3$ , a contradiction.

If  $M^{F(S)}$  is a direct product of isomorphic non abelian simple groups, we have  $M^{F(S)} \simeq SL(3, 3)$  because every proper subgroup of  $SL(3, 3)$  is solvable. Hence  $|N_G(S)^{F(S)} : M^{F(S)}| = 1$  or  $2$  and so  $M^{F(S)}$  is also doubly transitive. By (ii) of (3.1),  $N_{N^\alpha}(S)_{F(S)} = N_\beta^\alpha$ . Therefore,  $N_{N^\alpha}(S)^{F(S)}$  is cyclic of order  $\mu$ . Since  $|SL(3, 3)| = 2^4 3^3 13$ ,  $\mu = 3$  or  $13$ . If  $\mu = 3$ , applying (3.7) and (3.9),  $\pi$  is empty, a contradiction. If  $\mu = 13$ , then  $(M_\alpha)^{F(S)} \triangleright N_{N^\alpha}(S)^{F(S)} \simeq Z_{13}$ . Hence  $(M_\alpha)^{F(S)}$  is isomorphic to the normalizer of a Sylow 13-subgroup in  $SL(3, 3)$ , while this permutation representation of  $SL(3, 3)$  is not doubly transitive. Thus (3.10) is proved.

(3.11)  $N_\beta^\alpha = N^\alpha \cap N^\beta$ .

Proof. Suppose not and let  $p$  be a prime with  $p \mid |N_\beta^\alpha : N^\alpha \cap N^\beta|$ . Then it follows from (3.7), (3.9) and (3.10) that  $q-1=p^e$  for some integer  $e \geq 2$ . If  $e$  is even,  $p^e \equiv 1 \pmod{4}$ , while  $q-1 \equiv -1 \pmod{4}$ , a contradiction. If  $e$  is odd,  $2^n = q = c(p+1)$  where  $c = p^{e-1} - p^{e-2} + \dots - p + 1$ . We note that  $e \geq 3$ . Since  $c$  is odd,  $c=1$ , a contradiction. Thus  $N_\beta^\alpha = N^\alpha \cap N^\beta$ .

(3.12) Suppose  $N^\alpha \simeq PSL(2, q)$  or  $Sz(q)$  and  $G \neq A_6, S_6$ . Then

- (i)  $N_\beta^\alpha = N^\alpha \cap N^\beta$  is a Sylow 2-subgroup of  $N^\alpha$ .
- (ii) If  $N^\alpha \simeq PSL(2, q)$ , then  $|F(S)| = q$  and  $|\Omega| = q^2$ .
- (iii) If  $N^\alpha \simeq Sz(q)$ , then  $|F(S)| = q^2$  and  $|\Omega| = q^4$ .
- (iv) There is an element  $x$  in  $G$  such that  $S \neq S^x$ ,  $[S, S^x] = 1$  and  $F(S) \cap F(S^x) = \phi$ .

Proof. By assumption,  $N_{N^\alpha}(S) = (q-1)q^i$  where  $|S| = q^i$ . Hence (i) follows immediately from (3.7) and (3.11).

We now argue that  $|F(S)|$  is a power of 2. By (v) of (3.3), it suffices to consider the case  $C^\alpha = 1$ . Applying (ii) of (3.4),  $q \mid |F(S)|^2$ . By (i),  $\mu = |N_{N^\alpha}(S) : N_\beta^\alpha| = q-1$  and so  $|F(S)| = \mu r + 1 = (q-1)r + 1$ . Hence  $q \mid (r-1)^2$ , while  $r$  is a divisor of  $n$  where  $2^n = q$  because  $C^\alpha = 1$  and  $G_\alpha/N^\alpha$  is isomorphic to a subgroup of the outer automorphism group of  $N^\alpha$ . Therefore  $r=1$  and  $|F(S)| = q$ , a power of 2.

Hence by (iv) of (3.4),  $|F(S)| = (q-1)r + 1 \mid |\Omega| = (q^i + 1)(q-1)r + 1$  and so  $q \mid (q-1)r + 1$  and  $(q-1)r + 1 \mid q^i$ . From this,  $(i, r) = (1, 1), (2, 1)$  or  $(2, q+1)$ . If  $(i, r) = (1, 1)$  or  $(2, q+1)$ , we obtain (ii) or (iii), respectively. We argue  $(i, r) \neq (2, 1)$ . Suppose  $(i, r) = (2, 1)$ . Then  $N^\alpha \simeq Sz(q)$ ,  $|F(S)| = q$  and  $|\Omega| = q(q^2 - q + 1)$ . In this case, since  $|G_\alpha/C^\alpha N^\alpha|$  is odd, we have  $I(G_{\alpha\beta}) = I(N^\alpha \cap N^\beta)$ . From this, all involutions in a fixed Sylow 2-subgroup of  $G_{\alpha\beta}$  have a common fixed point set. By [12],  $G$  has a regular normal subgroup and so  $q^2 - q + 1 = 1$ , a contradiction.

Since by (iv) of (3.4)  $|\Omega| = |F(S)| \times |G : N_G(S)|_2$ ,  $|G : N_G(S)|_2$  is divisible by 2. Let  $S_1$  be a Sylow 2-subgroup of  $N_G(S)$  and  $S_2$  a Sylow 2-subgroup of  $N_G(S_1)$ . Since  $2 \mid |G : N_G(S)|$ ,  $S_1 \neq S_2$ . Let  $x \in S_2 - S_1$ , then  $S \neq S^x$  and  $S_1 \triangleright S, S^x$ . Suppose  $\gamma \in F(S) \cap F(S^x)$ . Then by (i),  $SS^x \leq N^\gamma$  and so  $S = S^x$ , a contradiction. Therefore  $F(S) \cap F(S^x) = \phi$  and hence  $[S, S^x] = 1$  by (ii) of (3.1). Thus (iii) holds.

(3.13) The following hold.

- (i)  $N^\alpha \neq Sz(q)$ .
- (ii) Suppose  $N^\alpha \simeq PSL(2, q)$  and let  $S^x$  be as defined in (3.12). Then  $O_2(C_G(S))$  is a Sylow 2-subgroup of  $C_G(S)$  and  $O_2(C_G(S)) = S \times S^x$ .

Proof. Suppose  $N^\alpha \simeq PSL(2, q)$  or  $Sz(q)$ . If  $C^\alpha \neq 1$ ,  $O_2(C_G(S))^{F(S)}$  is a regular normal subgroup of  $N_G(S)^{F(S)}$  by (v) of (3.3). If  $C^\alpha = 1$ , by (iv) of (3.3)

$C_{G\alpha}(S) = Z(S)$  and so  $C_G(S)_{F(S)} = Z(S)$ . By (3.12),  $C_G(S)^{F(S)} \geq (S^x)^{F(S)} \neq 1$ , and  $|F(S)| = q^i = |S|$  and so  $C_G(S) = Z(S) \times S^x$ . Hence in both cases  $O_2(C_G(S))$  is regular on  $F(S)$ .

Since by (iv) of (3.3)  $C_G(S)_{F(S)} = C_{G\alpha\beta}(S) = Z(S)$  and by (ii), (iii) of (3.12)  $q^i = |S^x| = |F(S)| = |C_G(S) : C_{G\alpha}(S)|$ , we have  $O_2(C_G(S)) = Z(S) \times S^x$  and this is a Sylow 2-subgroup of  $C_G(S)$ . Since  $Z(O_2(C_G(S)))^{F(S)} = Z(S^x)^{F(S)}$ ,  $N_G(S) \triangleright Z(O_2(C_G(S)))$  and  $|F(S)| = |S|$ ,  $|Z(S^x)^{F(S)}| = |S|$ . Hence  $|Z(S)| = |S|$  and  $S$  is abelian. So (3.13) follows.

(3.14) Suppose  $N^\alpha \simeq PSL(2, q)$  and  $G \neq A_6, S_6$ . Put  $E = O_2(C_G(S)) = S \times S^x$ ,  $W = \{T \mid T \in ccl_G(S), T \leq E\}$ . Then we have the following:

- (i)  $|W| = q$  and  $\Omega = \bigcup_T F(T)$  where  $T$  runs over every element of  $W$ .
- (ii)  $N_G(E) \cap ccl_G(s) \subseteq E$  for all  $s \in I(S)$ .
- (iii) If  $E \cap E^g \cap ccl_G(s) \neq \phi$  for some  $g \in G$ , then  $g \in N_G(E)$ .

Proof. Let  $D$  be a Hall  $2'$ -subgroup of  $N_{N^\alpha}(S)$ . Then  $D \simeq Z_{q-1}$  and by (i) of (3.12)  $D$  is semi-regular on  $\Omega - \{\alpha\}$ . If  $d \in N_D(S^x)$ ,  $\langle d \rangle$  acts semi-regularly on  $F(S^x)$  since  $\alpha \notin F(S^x)$ . Hence the order of  $d$  divides  $|F(S)|$ . But  $|F(S)| = q$  by (ii) of (3.12), hence  $|\langle d \rangle| \mid (q, q-1) = 1$  and so  $d = 1$ . Therefore  $N_D(S^x) = 1$ . Hence  $|\{S^{xd} \mid d \in D\}| = q-1$  and  $\{S^{xd} \mid d \in D\} \subseteq W$  as  $D$  normalizes  $E$ . If  $S = S^{xd}$  for some  $d \in D$ ,  $S^x = S^{d^{-1}} = S$ , a contradiction. Hence  $|W| \geq q$ . If there exist  $S_1, S_2 \in W$  such that  $S_1 \neq S_2$  and  $F(S_1) \cap F(S_2) \neq \phi$ . Let  $\gamma \in F(S_1) \cap F(S_2)$ . Then  $S_1, S_2 \leq N^\gamma$  by (i) of (3.12) and so  $\langle S_1, S_2 \rangle = N^\gamma$ , which is contrary to  $\langle S_1, S_2 \rangle \leq E$ . Hence  $F(S_1) \cap F(S_2) = \phi$  for  $S_1, S_2 \in W$  such that  $S_1 \neq S_2$ . Since  $|F(S)| = q$  and  $|\Omega| = q^2$  by (ii) of (3.12), we have  $|W| \leq q$ . Thus (i) holds.

Let  $s \in I(S)$  and suppose  $s^g \in N_G(E) - E$  for some  $g \in G$ . Then  $s^g \in N^\gamma$  where  $\gamma = \alpha^g$ . By (i) we choose  $T \in W$  so that  $\gamma \in F(T)$ . Then  $\langle s^g, T \rangle = N^\gamma$  as  $s^g \notin T$  and  $T$  is a Sylow 2-subgroup of  $N^\gamma$ . On the other hand  $\langle s^g, T \rangle \leq \langle s^g \rangle E$ , which is a 2-subgroup of  $N_G(E)$ , a contradiction. Thus (ii) holds.

Let  $1 \neq t \in E \cap E^g \cap ccl_G(s)$  for  $g \in G$  and  $s \in I(S)$ . Then there are  $S_1 \leq E$  and  $S_2 \leq E^g$  such that  $t \in S_1 \cap S_2$  and  $S_1, gS_2g^{-1} \in W$ . Since  $F(S_1) = F(t) = F(S_2)$  by (ii) of (3.1),  $\langle S_1, S_2 \rangle \leq N^\gamma \cap N^\delta$  for  $\gamma, \delta \in F(t)$ . Hence  $S_1 = S_2$  by (i) of (3.12). Applying (ii) of (3.13) to  $S_1$ , we obtain  $E = O_2(C_G(S_1)) = O_2(C_G(S_2)) = E^g$ . Thus (iii) holds.

(3.15) Suppose  $N^\alpha \simeq PSL(2, q)$  and  $G \neq A_6, S_6$ . Then  $G$  has a regular normal subgroup.

Proof. We count the set  $\{(\gamma, T) \mid \gamma \in F(T), T \in ccl_G(S)\}$  in two ways and we have  $q^2 \times (q+1) = |ccl_G(S)| \times q$  by (3.12). Hence  $|ccl_G(S)| = q(q+1)$ . On the other hand we have  $|ccl_G(S)| = |G : N_G(E)| \times q$  by (i), (ii) of (3.14). From this,  $|G : N_G(E)| = q+1$ .

Set  $\Gamma = ccl_G(E)$ . We now consider the action of  $G$  on  $\Gamma$ . By definition,  $G$  is transitive on  $\Gamma$  and  $N_G(E)$  is a stabilizer of  $E \in \Gamma$ . We argue that  $S$  is regular on  $\Gamma - \{E\}$ . Suppose not and let  $1 \neq s \in S$  such that  $s^{-1}E^s s = E^s$  for some  $E^s \in \Gamma - \{E\}$ . Then  $gsg^{-1} \in N_G(E)$ . By (ii) of (3.14),  $gsg^{-1} \in E$  and hence  $gsg^{-1} \in E \cap gEg^{-1}$ . By (iii) of (3.14),  $E = gEg^{-1}$ . Hence  $E = E^s$ , a contradiction. Since  $S \leq N_G(E)$  and  $|S| = |\Gamma| - 1$ ,  $S$  is regular on  $\Gamma - \{E\}$  and  $G^\Gamma$  is doubly transitive. Since  $S$  is abelian and regular on  $\Gamma - \{E\}$ ,  $G^\Gamma \cap C(S^\Gamma) = S^\Gamma$ . From this,  $E^\Gamma = S^\Gamma$  because  $E \geq S$  and  $E$  is abelian. Therefore  $G_\Gamma \neq 1$ . Set  $M = G_\Gamma$ . Suppose  $M \cap N^\alpha \neq 1$ , then  $M \geq N^\alpha$  as  $N^\alpha$  is simple. Hence  $N^\alpha \leq N_G(E)$  and so  $N^\alpha$  normalizes  $E \cap G_\alpha = S$ , a contradiction. Thus  $M \cap N^\alpha = 1$ . Hence  $M_\alpha \leq C_G(N^\alpha) = C^\alpha$ , so that  $M_\alpha = 1$  or  $M_\alpha \neq 1$  and  $M$  is a Frobenius group on  $\Omega$  by (iii) of (3.1). In both cases,  $G$  has a regular normal subgroup.

We now consider the case that  $N^\alpha \simeq PSU(3, q)$ . By (3.7) and (3.11),  $N_\beta^\alpha = US$  where  $U$  is a Hall  $2'$ -subgroup of  $N_\beta^\alpha$  and  $U \leq Z_{q+1/\varepsilon}$  with  $\varepsilon = (q+1, 3)$ . As in the proof of (3.1)', we set  $N_{N^\alpha}(S) = DS$  and  $D = V \times K$ . Here  $V \simeq Z_{q+1/\varepsilon}$  and  $K \simeq Z_{q-1}$ . Since  $N_{N^\alpha}(S) \triangleright N_\beta^\alpha$ , we may assume  $U = V \cap N_\beta^\alpha$ .

(3.16) Suppose  $N^\alpha \simeq PSU(3, q)$ . Then  $N_\beta^\alpha = N^\alpha \cap N^\beta$  is a Sylow 2-subgroup of  $N^\alpha$ . In particular  $\mu = q^2 - 1/\varepsilon$ .

Proof. Suppose not and  $U \neq 1$ . If  $U^g \leq G_{\alpha\beta}$  for  $g \in G$ ,  $U^g \leq N_\alpha^{\alpha^g} \cap N_\beta^{\beta^g} = N_\alpha^{\alpha^g} \cap N^\alpha \cap N^\beta \leq N^\alpha \cap N^\beta$ . Hence  $U$  is conjugate to  $U^g$  in  $N^\alpha \cap N^\beta \leq G_{\alpha\beta}$ . By the Witt's Theorem  $N_G(U)$  is doubly transitive on  $F(U)$ . By (ii) of Lemma 2.4,  $N_{N^\alpha}(U) = N \times V$  where  $N \simeq PSL(2, q)$ . Hence  $N_G(U)^{F(U)}$  satisfies the assumption of Theorem 1. By (i) of (3.1), the number of fixed points of  $U$  on  $\Delta_i$  is constant for each  $N^\alpha$ -orbit  $\Delta_i$  and so  $|F(U)| = |F(U) \cap \Delta_i| \times r + 1 = (|N_{N^\alpha}(U)| \times |N_\beta^\alpha : N_{N_\beta^\alpha}(U)| / |N_\beta^\alpha|) \times r + 1 = (|PSL(2, q)| \times |V| / |Z(S)| \times |U|) \times r + 1 = (q^2 - 1) \times r \times |V : U| + 1$ . Hence  $|F(U)|$  is even and  $|F(U)| \neq 6$ . Applying (3.12) to  $N_G(U)^{F(U)}$ , we obtain  $|F(U)| = q^2$ ,  $|F(U) \cap F(Z(S))| = q$ . Hence  $r = 1$ ,  $U = V$ ,  $N_\beta^\alpha = VS$  and  $|F(V)| = q^2$  and so  $\mu = |N_{N^\alpha}(S) : N_\beta^\alpha| = q - 1$ . Since by (ii) of (3.1)  $F(U) \supseteq F(S)$ ,  $|F(Z(S))| = |F(S)| = q$ . Furthermore by (3.15),  $N_G(V)^{F(V)}$  has a regular normal elementary abelian 2-subgroup, say  $E^{F(V)}$ . Clearly  $E^{F(V)} \leq C_G(V)^{F(V)}$ . Hence we may assume that  $E$  is a 2-subgroup of  $C_G(V)$ . Put  $P = E_{F(V)}$ . Then  $|E| = |P|q^2$ . By (i) of (3.4),  $|\Omega| = q^4 - q^3 + q$  and so  $2q \nmid |\Omega - F(V)|$ . Hence there exists  $\gamma \in \Omega - F(V)$  such that  $|E : E_\gamma| \leq q$ . Let  $T$  be a Sylow 2-subgroup of  $G_\gamma$  containing  $E_\gamma$ . Since  $E_\gamma/E_\gamma \cap T \cap N^\gamma$  is isomorphic to a subgroup of  $T/T \cap N^\gamma$  and  $T/T \cap N^\gamma \simeq TN^\gamma/N^\gamma \leq G_\gamma/N^\gamma$ ,  $E_\gamma/E_\gamma \cap T \cap N^\gamma$  is cyclic. If  $E_\gamma \cap T \cap N^\gamma = 1$ ,  $E_\gamma$  is cyclic and so  $|E_\gamma/E_\gamma \cap P| \leq 2$ . Then  $|E_\gamma \cap P| \geq |E_\gamma|/2 \geq |P|q/2 > |P|$ , a contradiction. Hence  $E_\gamma \cap T \cap N^\gamma \neq 1$ . Let  $z \in E_\gamma \cap T \cap N^\gamma$  with  $z \neq 1$ . Since  $|F(z)| = q < |F(P)|$ ,  $z \in E$  and  $E^{F(V)}$  is regular, we have  $F(z) \cap F(V) = \phi$ . Hence  $V$  acts semi-regularly on  $F(z)$ . From this,  $q = |F(z)| = (q+1/\varepsilon) \times k$  for some integer  $k \geq 1$ . Since  $q$  is a power

of 2,  $q+1/\varepsilon=1$ , a contradiction.

(3.17) Suppose  $N^\alpha \simeq PSU(3, q)$ . Then the following hold.

(i)  $|\Omega| = q^5 - q^3 + q^2$ ,  $|F(S)| = q^2$ .

(ii)  $N_G(S)^{F(S)}$  has a regular normal subgroup.

Proof. If  $C^\alpha \neq 1$ , (ii) follows from (v) of (3.3) and so  $|F(S)|$  is a power of 2. By (3.4) and (3.16),  $|F(S)| = (q^2 - 1)r/\varepsilon + 1$  and  $(q^2 - 1)r/\varepsilon + 1 \mid (q^3 + 1)(q^2 - 1)r/\varepsilon + 1$ , hence  $(q^2 - 1)r/\varepsilon + 1 \mid q^3$ . By calculation, we obtain  $r = \varepsilon$ . So (i) follows.

We now assume  $C^\alpha = 1$ . By (ii) of (3.4),  $q \mid |F(S)| = (q^2 - 1)r/\varepsilon + 1$ , so that  $r = qk + \varepsilon$  for an integer  $k \geq 0$ . Since  $C^\alpha = 1$ ,  $r$  is a divisor of  $|G_\alpha/N^\alpha|$ . Hence  $r \mid 2n\varepsilon$ , so that  $r \mid n\varepsilon$ . Therefore  $n\varepsilon \geq r = qk + \varepsilon = 2^n \times k + \varepsilon$ . Hence  $k = 0$  and  $r = \varepsilon$ . From this (i) follows.

Let  $f$  be a field automorphism as defined in (3.1)' and let  $T$  be a Sylow 2-subgroup of  $N_G(S)$  which contains  $\langle f \rangle S$ . Since  $|N_G(S) : N_{G_\alpha}(S)| = |F(S)| = q^2$  by (i),  $|T| = 2^m q^5$  where  $|\langle f \rangle| = 2^m$ . Since  $T \triangleright S$  and  $\Omega - F(S) = q^3(q^2 - 1)$  there exists  $\gamma \in \Omega - F(S)$  such that  $|T : T_\gamma| = q^3$ , hence  $|T_\gamma| = 2^m q^2$  and  $T = ST_\gamma$ . Set  $W = T_\gamma \cap N^\gamma$ . Then  $W$  is semi-regular on  $F(S)$  because  $\gamma \in \Omega - F(S)$ . In particular  $|W| \leq |F(S)| = q^2$ . We note that  $|T_\gamma N^\gamma / N^\gamma| \leq 2^m$ . Since  $T_\gamma / W \simeq T_\gamma N^\gamma / N^\gamma$ , we have  $|W| \geq q^2$ . Hence  $|W| = q^2$  and  $W$  is regular on  $F(S)$ . Moreover  $|T_\gamma : W| = 2^m$ .

Since  $N_{G_{\alpha\beta}}(S)/S \simeq N_{G_{\alpha\beta}}(S)N^\alpha/N^\alpha$  by (3.16),  $N_{G_{\alpha\beta}}(S)^{F(S)}$  is isomorphic to a homomorphic image of a subgroup of the outer automorphism group of  $N^\alpha$ . Hence  $N_{G_{\alpha\beta}}(S)^{F(S)}$  is abelian when  $n$  is even or  $f = 1$ . In this case by [1], (ii) holds because  $|F(S)| = q^2$ . We now assume  $n$  is odd and  $|\langle f \rangle| = 2^m = 2$ . Since  $T = ST_\gamma$  and  $|T_\gamma : W| = 2$ ,  $|T^{F(S)} : W^{F(S)}| = 2$ . Claim  $f^{F(S)} \neq 1$ . For otherwise  $f \in N_G(S)_{F(S)}$  and  $[f, D] \leq N_G(S)_{F(S)} \cap D = 1$  as  $D$  is  $f$ -invariant and  $D \leq N_G(S)$ . But since  $f \neq 1$ ,  $f$  does not centralize  $D$ . Therefore  $f^{F(S)} \neq 1$ . As  $f \in G_\alpha$ ,  $f^{F(S)} \notin W^{F(S)}$ . Hence  $T^{F(S)} = \langle f \rangle^{F(S)} W^{F(S)} \triangleright W^{F(S)}$ . Since  $W^{F(S)}$  is regular,  $f^{F(S)}$  is not conjugate to any element in  $W^{F(S)}$ . Hence  $f^{F(S)}$  is not contained in  $O^2(N_G(S)^{F(S)})$  by Lemma 2 of [3]. Since  $\langle f^{F(S)} \rangle$  is a Sylow 2-subgroup of  $(N_G(S)^{F(S)})_{\alpha\beta}$ ,  $O^2(N_G(S)^{F(S)})_{\alpha\beta}$  is of odd order. As before  $(N_G(S)^{F(S)})_{\alpha\beta}$  is isomorphic to a homomorphic image of a subgroup of the outer automorphism group of  $N^\alpha$ ,  $O^2(N_G(S)^{F(S)})_{\alpha\beta}$  is abelian. Again by [1],  $O^2(N_G(S)^{F(S)})$  has a regular normal subgroup as  $|F(S)| = q^2$ . Thus (ii) also holds in this case

(3.18)  $N^\alpha \not\simeq PSU(3, q)$ .

Proof. Let  $f$  be as in (3.1)'. By the same argument as in the proof of (ii) of (3.17), we have  $I(\langle f \rangle) \not\leq N_G(S)_{F(S)}$  and so  $S$  is a Sylow 2-subgroup of  $N_G(S)_{F(S)}$ .

By (ii) of (3.17), there is a normal subgroup  $L$  of  $N_G(S)$  such that  $L \geq N_G(S)_{F(S)}$  and  $L^{F(S)}$  is an elementary abelian 2-subgroup of  $N_G(S)^{F(S)}$ . Let  $T$  be a Sylow 2-subgroup of  $\langle f \rangle L$  which contains  $f$ . Set  $E = T \cap L$ . Then  $E$

is a Sylow 2-subgroup of  $L$ . Since  $S$  is a unique Sylow 2-subgroup of  $N_G(S)_{F(S)}$ ,  $E/S \cong L^{F(S)}$  is an elementary abelian 2-subgroup of order  $q^2$ . As  $\langle f \rangle \cap E = \langle f \rangle \cap E \cap G_\omega = \langle f \rangle \cap S = 1$ ,  $T = \langle f \rangle E \triangleright E$ .

Since  $T \triangleright S$  and  $|\Omega - F(S)| = q^3(q^2 - 1)$  by (i) of (3.17), we can choose  $\gamma \in \Omega - F(S)$  such that  $|T : T_\gamma| = q^3$ . Hence  $|T_\gamma| = 2^m q^2$  where  $2^m$  is the order of  $f$ . Since  $T_\gamma / T_\gamma \cap C^\gamma N^\gamma \cong T_\gamma N^\gamma C^\gamma / C^\gamma N^\gamma$  is cyclic of order at most  $2^m$ ,  $|T_\gamma \cap C^\gamma N^\gamma| = |T_\gamma \cap N^\gamma| \geq q^2$ . Moreover  $T_\gamma \cap N^\gamma / T_\gamma \cap N^\gamma \cap E \cong (T_\gamma \cap N^\gamma)E/E$  is cyclic of order at most  $2^m$ , we have  $|T_\gamma \cap N^\gamma \cap E| \geq q^2/2^m$ . Since the order of  $f$  is a divisor of  $2n$ , we have  $|T_\gamma \cap N^\gamma \cap E| \geq q(2^n/2^m) \geq q$ .

If  $T_\gamma \cap N^\gamma \cap E$  contains no element of order 4, then  $T_\gamma \cap N^\gamma \cap E$  is an elementary abelian 2-subgroup of  $N^\gamma$  of order  $q$  and hence  $T_\gamma \cap N^\gamma / T_\gamma \cap N^\gamma \cap E$  is an elementary abelian 2-group. Therefore  $|(T_\gamma \cap N^\gamma)E/E| \leq 2$  and so  $|T_\gamma \cap N^\gamma \cap E| \geq q^2/2 > q$ , a contradiction.

If  $T_\gamma \cap N^\gamma \cap E$  contains an element  $x$  of order 4, then  $1 \neq x^2 \in S$  because  $E/S$  is an elementary abelian 2-group. Since  $\gamma \in F(x^2)$ , by (ii) of (3.1) we have  $\gamma \in F(S)$ , which is contrary to  $\gamma \in \Omega - F(S)$ . Thus (3.18) holds.

In this section we have proved the following:

**Theorem 2.** *Suppose  $G^\Omega$  satisfies the hypothesis of Theorem 1 and  $|\Omega|$  is even. Then  $N^\omega \neq Sz(q)$ ,  $PSU(3, q)$ ,  $N^\omega \cong PSL(2, q)$  and either*

- (i)  $G^\Omega \cong A_6$  or  $S_6$  or
- (ii)  $|\Omega| = q^2$ ,  $|N_\beta^\omega| = |N^\omega \cap N^\beta| = q$  and  $G$  has a regular normal subgroup.

#### 4. The case $|\Omega|$ is odd

Let  $G$  be a doubly transitive permutation group on  $\Omega$  of odd degree satisfying the assumption of Theorem 1. By Theorem A of [10] and Theorem B of [11], we may assume  $C_G(N^\omega) = 1$ . Hence  $G_\omega/N^\omega$  is isomorphic to a subgroup of the outer automorphism group of  $N^\omega$ . Let  $\{\alpha\}, \Delta_1, \Delta_2, \dots, \Delta_r$  be the set of all  $N^\omega$ -orbits on  $\Omega$ . Clearly  $r$  is a divisor of  $|G_\omega/N^\omega|$ .

From now on we assume that  $G$  has no regular normal subgroup and prove that  $G \cong PSL(2, 11)$ . Let  $M$  be a minimal normal subgroup of  $G$ . Then by assumption,  $M_\omega \neq 1$ .

(4.1)  $M$  is simple and  $N^\omega \leq M$ .

*Proof.* Since  $G$  is doubly transitive and  $M_\omega \neq 1$ ,  $M$  is a simple group (cf. Exercise 12.4 of [16]). If  $N^\omega \not\leq M$ , then  $M_\omega \cap N^\omega = 1$  as  $N^\omega$  is simple and hence  $M_\omega \leq C_G(N^\omega) = 1$ , a contradiction. Thus  $N^\omega \leq M$ .

As in (3.1)', there is a 2-element  $f$  of  $M_\omega$  such that  $f$  acts on  $N^\omega$  as a field automorphism,  $\langle f \rangle S \triangleright S$ ,  $\langle f \rangle \cap S = 1$  and  $\langle f \rangle S$  is a Sylow 2-subgroup of  $M_\omega$ , where  $N_{N^\omega}(S) = DS$  is a Borel subgroup of  $N^\omega$ ,  $S$  is a unipotent subgroup of  $N^\omega$ , and  $D$  is a diagonal subgroup of  $N^\omega$ .



(4.2) If  $f \neq 1$ , then  $I(N_\beta^\alpha) \not\subseteq N^\alpha \cap N^\beta$  for  $\beta \neq \alpha$ .

Proof. Suppose  $f \neq 1$  and  $\tau \in I(\langle f \rangle)$ . Since  $M$  is a simple group with a Sylow 2-subgroup  $\langle f \rangle S$ ,  $\tau^g \in S$  for some  $g \in M_\alpha$  by Lemma 2 of [3]. Set  $\gamma = \alpha^{g^{-1}}$ . Then  $\tau \in N_\gamma^\alpha$  and clearly  $\tau \notin N^\gamma \cap N^\alpha$ , so that  $I(N_\gamma^\alpha) \not\subseteq N^\gamma \cap N^\alpha$ . By the transitivity of  $G$ , we obtain  $I(N_\beta^\alpha) \not\subseteq N^\alpha \cap N^\beta$  for any  $\beta \neq \alpha$ .

(4.3) Suppose  $f \neq 1$ . Then  $N^\alpha \neq Sz(q)$ ,  $PSU(3, q)$ .

Proof. If  $N^\alpha \simeq Sz(q)$ ,  $|G_\alpha/N^\alpha|$  is odd and hence  $f=1$ , a contradiction. Therefore  $N^\alpha \neq Sz(q)$ .

Suppose  $N^\alpha \simeq PSU(3, q)$  and let  $\tau \in I(\langle f \rangle)$ . Let  $s \in Z(\langle f \rangle S) \cap I(S)$ . As in the proof of (4.2),  $ccl_M(\tau) \cap S \neq \emptyset$ . Then since  $s$  is an extremal element there is  $g \in M$  such that  $\tau^g = s$  and  $(C_{\langle f \rangle S}(\tau))^g \leq \langle f \rangle S$ . Since  $\tau$  is a field automorphism of order 2,  $Z(S) \leq C_{\langle f \rangle S}(\tau)$ . Put  $\beta = \alpha^{g^{-1}}$ . Then  $\tau \in N_\beta^\alpha$  and  $Z(S) \leq N_\beta^\alpha$ . By (4.2)  $Z(S) \not\subseteq N^\alpha \cap N^\beta$  and so  $|Z(S) : Z(S) \cap N^\alpha \cap N^\beta| = 2$  because  $Z(S)/Z(S) \cap N^\alpha \cap N^\beta \simeq Z(S)(N^\alpha \cap N^\beta)/N^\alpha \cap N^\beta \leq N_\beta^\alpha/N^\alpha \cap N^\beta \simeq N_\beta^\alpha N^\beta/N^\beta \leq G_\beta/N^\beta$ .

Claim  $N_\beta^\alpha \leq N_{N^\alpha}(S)$ . Suppose not. Then  $N_\beta^\alpha \cap N_{N^\alpha}(S)$  is a strongly embedded subgroup of  $N_\beta^\alpha$ . Since  $|N_\beta^\alpha/N^\alpha \cap N^\beta|$  is even and  $N_\beta^\alpha \geq Z(S) \geq Z_2 \times Z_2$ , by Bender's Theorem ([2]),  $N_\beta^\alpha/N^\alpha \cap N^\beta$  is not solvable, while  $N_\beta^\alpha/N^\beta \cap N^\beta \simeq N_\beta^\alpha N^\beta/N^\beta$  is solvable, a contradiction.

Let  $V_1$  be a  $\tau$ -invariant Hall 2'-subgroup of  $N_\beta^\alpha$ . Then since  $V_1$  normalizes  $\Omega_1(O_2(N_\beta^\alpha)) = Z(S)$ ,  $V_1$  centralizes  $Z(S)/Z(S) \cap N^\alpha \cap N^\beta \simeq Z_2$ . Hence by (i) of Lemma 2.4,  $V_1 \leq Z_{q+1}$  and so  $[V_1, Z(S)] = 1$  by (ii) of Lemma 2.4. Therefore  $I(N_\beta^\alpha) \subseteq Z(N_\beta^\alpha)$ . Similarly  $I(N_\alpha^\beta) \subseteq Z(N_\alpha^\beta)$ . Since  $\tau \in I(N_\alpha^\beta)$ , we have  $N^\alpha \cap N^\beta \leq C(\tau) \cap N_{N^\alpha}(S)$ . Since  $\tau$  is a field automorphism of  $N^\alpha$  of order 2,  $C(\tau) \cap N_{N^\alpha}(S) = KZ(S)$  where  $K$  is a cyclic subgroup of  $N_{N^\alpha}(S)$  of order  $q-1$ . Hence  $N^\alpha \cap N^\beta \leq KZ(S) \cap N_\beta^\alpha = Z(S)(K \cap V_1 O_2(N_\beta^\alpha)) = Z(S)$  and so  $|Z(S) : N^\alpha \cap N^\beta| = 2$ .

We claim that  $F(z) = F(Z(S))$  for  $z \in I(N_\beta^\alpha)$ . Let  $\Delta_i$  be an arbitrary  $N^\alpha$ -orbit on  $\Omega - \{\alpha\}$ . Since all elementary abelian 2-subgroups of  $N^\alpha$  of order  $q$  are conjugate in  $N^\alpha$ , there exists  $\gamma \in \Delta_i$  with  $Z(S) \leq N_\gamma^\alpha$ . Hence by Lemma 2.2,  $|F(z) \cap \Delta_i| = |C_{N^\alpha}(z)| \times |Z(S)^*|/|N_\gamma^\alpha| = (q+1/\varepsilon) \times q^3(q-1)/|N_\gamma^\alpha|$  for  $z \in I(N_\beta^\alpha)$ . On the other hand  $|F(Z(S)) \cap \Delta_i| = |N_{N^\alpha}(Z(S))|/|N_\gamma^\alpha| = (q^2-1/\varepsilon) \times q^3/|N_\beta^\alpha|$ . Hence  $F(z) \cap \Delta_i = F(Z(S)) \cap \Delta_i$  and so  $F(z) = F(Z(S))$ . In particular  $F(\tau) = F(Z(S))$  because  $\tau \in I(N_\alpha^\beta)$  and  $N^\alpha \cap N^\beta \neq 1$ .

We claim that  $(V_1)_{F(Z(S))} = 1$ . Set  $S_1 = O_2(N_\beta^\alpha)$ . Let  $d \in V_1$  with  $d \neq 1$ ,  $\Delta_i$  be a  $N^\alpha$ -orbit which contains  $\beta$  and let  $D_1$  be a  $\tau$ -invariant Hall 2'-subgroup of  $N_{N^\alpha}(S)$  which contains  $V_1$ . Put  $X = \langle d \rangle Z(S)$ . Then by Lemma 2.2,  $|F(X) \cap \Delta_i| = |N_{N^\alpha}(X)|/|N_\beta^\alpha : N_{N_\beta^\alpha}(X)|/|N_\beta^\alpha| = |D_1 Z(S)|/|N_\beta^\alpha : V_1 Z(S)|/|N_\beta^\alpha| = (q^2-1/\varepsilon)|S_1|/|N_\beta^\alpha| = |F(Z(S)) \cap \Delta_i|/|S : S_1|$ . Since  $S_1/N^\alpha \cap N^\beta$  is cyclic and  $N^\alpha \cap N^\beta \leq Z(S)$ ,  $S \neq S_1$ . Therefore  $F(X) \neq F(Z(S))$  and so  $(V_1)_{F(Z(S))} = 1$ .

Since  $D_1 \leq N_{N^\alpha}(Z(S))$  and  $\tau \in N_{G_\alpha}(Z(S))_{F(Z(S))}$ ,  $[\tau, D_1] \leq N_G(Z(S))_{F(Z(S))} \cap D_1$

$= (V_1)_{F\langle Z(S) \rangle} = 1$ . Hence  $D_1 \leq C(\tau) \cap N_{N^\alpha}(S) = KZ(S)$  with  $K \simeq Z_{q-1}$ , which is contrary to  $|D_1| = (q^2 - 1)/\varepsilon$ . So (4.3) is proved.

(4.4) Suppose  $N^\alpha \simeq PSL(2, q)$  and  $f \neq 1$ . Then the following hold.

(i)  $N_\beta^\alpha$  is a 2-subgroup of  $N^\alpha$  and  $|N_\beta^\alpha : N^\alpha \cap N^\beta| = 2$ .

(ii) Let  $\tau \in I(\langle f \rangle)$ . Then for some  $\beta \neq \alpha$ ,  $\tau \in N_\alpha^\beta - N_\beta^\alpha$ ,  $|C_S(\tau)| = \sqrt{q}$  and  $N^\alpha \cap N^\beta \leq C_S(\tau) \leq N_\beta^\alpha$ .

Proof. As in the proof of (4.3), there exist  $s \in I(S)$  and  $g \in M$  such that  $\tau^g = s$  and  $(C_{\langle f \rangle S}(\tau))^g \leq \langle f \rangle S$ . Put  $\beta = \alpha^{g^{-1}}$ . Then  $\tau \in N_\alpha^\beta - N_\beta^\alpha$  and  $C_S(\tau) \leq N_\beta^\alpha$ . Since  $\tau$  is a field automorphism of  $N^\alpha$  of order 2,  $|C_S(\tau)| = \sqrt{q}$ . Claim  $N_\beta^\alpha \leq N_{N^\alpha}(S)$ . If  $q \neq 2^2$ , as  $C_S(\tau) \leq N_\beta^\alpha$ , a Sylow 2-subgroup of  $N^\alpha$  is non cyclic. Hence as in the proof of (4.3),  $N_\beta^\alpha \leq N_{N^\alpha}(S)$ . If  $q = 2^2$ ,  $N^\alpha \simeq A_5$  and so  $\langle \tau \rangle N^\alpha = M_\alpha = G_\alpha \simeq S_5$ . In particular  $r = 1$ . Hence  $N_\beta^\alpha \leq N_{N^\alpha}(S)$ . For otherwise  $|N_\beta^\alpha| = 6$  or 10 and  $|\Omega| = 11$  or 7, respectively. By [13], such groups do not exist. Thus in both cases  $N_\beta^\alpha \leq N_{N^\alpha}(S)$ . On the other hand  $N_\beta^\alpha / N^\alpha \cap N^\beta$  is cyclic of even order. By (i) of Lemma 2.4,  $N_\beta^\alpha$  must be an abelian 2-subgroup of  $N^\alpha$  and  $|N_\beta^\alpha : N^\alpha \cap N^\beta| = 2$ . Since  $N_\alpha^\beta \simeq N_\beta^\alpha$  and  $\tau \in N_\alpha^\beta$ , we obtain  $N^\alpha \cap N^\beta \leq C_S(\tau)$ . Thus (i) and (ii) hold.

(4.5) Suppose  $N^\alpha \simeq PSL(2, q)$  and  $f \neq 1$ . Let  $T = N_\beta^\alpha N_\alpha^\beta$ . Then

(i)  $N_G(T)$  is doubly transitive on  $F(T)$ .

(ii)  $N_{N^\alpha}(T) = S$  and  $S_\gamma = N_\beta^\alpha$  for every  $\gamma \in F(T)$ .

Proof. Since  $G_{\alpha\beta} / N_\beta^\alpha$  is cyclic and by (i) of (4.4)  $T / N_\beta^\alpha \simeq Z_2$ ,  $I(G_{\alpha\beta}) \leq T$ . Clearly  $\langle I(G_{\alpha\beta}) \rangle = T$ . Hence by the Witt's Theorem, we have (i).

Let  $K_1$  be a Hall  $2'$ -subgroup of  $N_{N^\alpha}(T)$ . Then  $K_1$  normalizes  $T \cap N^\alpha = N_\beta^\alpha$ . Since  $T / N_\beta^\alpha \simeq Z_2$ ,  $[K_1, T / N_\beta^\alpha] = 1$  and so  $T = C_T(K_1) N_\beta^\alpha$ . If  $K_1 \neq 1$ , by (i) of Lemma 2.4  $C_T(K_1) = 1$ . Hence  $K_1 = 1$  and  $N_{N^\alpha}(T) = S$ .

Let  $\gamma \in F(T) - \{\alpha\}$ . Then obviously  $N_\beta^\alpha \leq S_\gamma \leq N_\gamma^\alpha$ . Since  $G$  is doubly transitive on  $\Omega$ ,  $|N_\beta^\alpha| = |N_\gamma^\alpha|$ , so that  $N_\beta^\alpha = S_\gamma = N_\gamma^\alpha$ . Thus (ii) holds.

(4.6) Suppose  $N^\alpha \simeq PSL(2, q)$  and  $f \neq 1$ . Put  $q = 2^n$ . Then

(i)  $(n, |N_\beta^\alpha|) = (2, 2), (2, 2^2), (4, 2^3)$  or  $(6, 2^4)$ .

(ii) If  $(n, |N_\beta^\alpha|) = (6, 2^4)$ ,  $N_G(T)^{F(T)} \simeq A_5$ .

Proof.  $|G_\alpha / N^\alpha| \mid n$  and  $f \neq 1$ ,  $n$  is even and so we set  $n = 2m$ . By (ii) of (4.4),  $|N_\beta^\alpha| = 2^{m+\varepsilon}$  where  $\varepsilon = 0$  or 1. Since  $N_{G_{\alpha\beta}}(T) / T \leq G_{\alpha\beta} / T \simeq (G_{\alpha\beta} / N_\beta^\alpha) / (T / N_\beta^\alpha)$  and  $G_{\alpha\beta} / N_\beta^\alpha \simeq G_{\alpha\beta} N^\alpha / N^\alpha \leq G_\alpha / N^\alpha$ ,  $N_{G_{\alpha\beta}}(T)^{F(T)}$  is cyclic and  $|N_{G_{\alpha\beta}}(T)^{F(T)}| \mid m$ . By (4.5),  $N_G(T)^{F(T)}$  is doubly transitive and  $S^{F(T)} \simeq S / N_\beta^\alpha$  is semi-regular on  $F(T) - \{\alpha\}$ . Since  $N_{G_{\alpha\beta}}(T)^{F(T)}$  is cyclic, by [1]  $N_G(T)^{F(T)} \simeq PSL(2, q_1)$  where  $q_1$  is a power of 2 or  $N_G(T)^{F(T)}$  has a regular normal subgroup. If  $(n, |N_\beta^\alpha|) \neq (2, 2), (2, 2^2)$  and  $(4, 2^3)$ ,  $S^{F(T)}$  contains a four-group, which is semi-regular on  $F(T) - \{\alpha\}$ . Hence  $N_G(T)^{F(T)}$  contains no regular normal subgroup and so

$N_G(T)^{F(T)} \simeq PSL(2, q_1)$ . Since  $N_{N^\alpha}(T)^{F(T)} = S^{F(T)} \simeq S/N_\beta^\alpha$  and  $N_{G_\alpha}(T)^{F(T)} \triangleright N_{N^\alpha}(T)^{F(T)}$ ,  $q_1 = 2^{m-\varepsilon} > 2$ . Hence  $2^{m-\varepsilon} - 1 = |N_{G_\alpha}(T)^{F(T)}|$ , so that  $2^{m-\varepsilon} - 1 \mid m$ . From this,  $\varepsilon = 1$ ,  $m = 3$  and  $N_G(T)^{F(T)} \simeq A_5$ . Thus (4.6) holds.

$$(4.7) \quad f = 1.$$

Proof. Suppose  $f \neq 1$ . Then by (4.3) and (4.6), it suffices to consider the case (i) of (4.6).

If  $N^\alpha \simeq PSL(2, 2^2)$  and  $|N_\beta^\alpha| = 2$ ,  $G_\alpha = N_\alpha^\beta N^\alpha \simeq \text{Aut}(N^\alpha) \simeq S_6$ . Hence  $r = 1$ . Therefore  $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 31$  and  $|G| = |\Omega| |G_\alpha| = 2^3 \cdot 3 \cdot 5 \cdot 31$ . By the Sylow's theorem,  $G$  has a regular normal subgroup of order 31. But this is a contradiction as  $G \geq N^\alpha$ .

If  $N^\alpha \simeq PSL(2, 2^2)$  and  $|N_\beta^\alpha| = 2^4$ , as above  $G_\alpha = N_\alpha^\beta N^\alpha$  and hence  $r = 1$ . From this  $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 16$ , a contradiction.

If  $N^\alpha \simeq PSL(2, 2^4)$  and  $|N_\beta^\alpha| = 2^3$ ,  $|\text{Aut}(N^\alpha) : N^\alpha| = 4$  and so  $|G_\alpha : N_\alpha^\beta N^\alpha| \leq 2$ . Hence  $r = 1$  or 2 and  $|\Omega| = 511$  or 1021 respectively. By Lemma 2.2, for  $s \in N_\beta^\alpha - \{1\}$   $|F(s) - \{\alpha\}| = 14$  or 28 respectively. Let  $\tau$  be a field automorphism of  $N^\alpha$  of order 2 as in (4.4). Then  $C_{N^\alpha}(\tau) \simeq PSL(2, 2^2)$  and  $|F(\tau) - \{\alpha\}| = 14$  or 28 since  $\tau$  is conjugate to  $s$ . From this an element  $x$  of  $C_{N^\alpha}(\tau)$  of order 5 fixes at least four points in  $\Omega$ . Since  $5 \nmid |\Omega|$ ,  $\langle x \rangle$  is a Sylow 5-subgroup of  $G$  and so  $x^g \in N^\alpha$  for some  $g \in G$ . But  $F(x^g) = \{\alpha\}$  because  $|N_\gamma^\alpha| = |N_\beta^\alpha| = 2^3$  for all  $\gamma \neq \alpha$ . Therefore  $|F(x)| = 1$ , which is contrary to  $|F(x)| \geq 4$ .

If  $N^\alpha \simeq PSL(2, 2^6)$  and  $|N_\beta^\alpha| = 2^4$ , by (ii) of (4.6),  $|N_{G_\alpha}(T)^{F(T)}| = 3$ . Hence  $3 \mid |G_\alpha : N_\alpha^\beta|$ . Since  $|G_\alpha : N_\alpha^\beta| = |G_\alpha N^\alpha : N^\alpha|$  and  $|N_\alpha^\beta N^\alpha : N^\alpha| = 2$  by (i) of (4.4), we have  $G_\alpha N^\alpha = G_\alpha \simeq \text{Aut}(N^\alpha)$ . In particular  $r = 1$  and  $|\Omega| = 16381$ . Moreover  $|F(s) - \{\alpha\}| = 60$ . As before  $|F(\tau) - \{\alpha\}| = 60$ ,  $C_{N^\alpha}(\tau) \simeq PSL(2, 2^3)$  and an element of  $C_{N^\alpha}(\tau)$  of order 7 fixes at least five points. But since  $7 \nmid |\Omega|$  and  $7 \nmid |N_\beta^\alpha|$ , every element of order 7 fixes exactly one point, a contradiction.

$$(4.8) \quad G^\alpha \simeq PSL(2, 11), |\Omega| = 11.$$

Proof. By (4.7),  $|M_\alpha : N^\alpha|$  is odd and so a Sylow 2-subgroup of  $N^\alpha$  is also that of  $M$ . By [4], [5] and [15], it suffices to consider the following cases:

- (i)  $N^\alpha \simeq PSL(2, 2^2)$ ,  $M \simeq PSL(2, q_1)$ ,  $q_1 \equiv 3$  or  $5 \pmod{8}$ ,  $q_1 > 3$ .
- (ii)  $N^\alpha \simeq PSL(2, 2^3)$ ,  $C_M(t) \simeq Z_2 \times PSL(2, 3^{2m+1})$ ,  $t \in I(M)$  ( $m \geq 1$ ).
- (iii)  $N^\alpha \simeq PSL(2, 2^3)$ ,  $M \simeq J_1$ , the smallest Janko group.

First we consider the case (i). If  $|N_\beta^\alpha|$  is odd, every involution in  $M$  has a unique fixed point and so  $M \simeq PSL(2, 5)$  by [2]. But then  $M = N^\alpha$ , a contradiction. Hence  $|N_\beta^\alpha| = 2, 4, 6, 10$  or 12. On the other hand  $r = 1$  or 2 because  $|\text{Aut}(N^\alpha) : N^\alpha| = 2$ . From this  $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| r = 7, 11, 13, 21, 31$  or 61. Since  $M \simeq PSL(2, q_1)$  and  $|M| = |\Omega| |N^\alpha|$ , we get  $|\Omega| = 11$ ,  $|N_\beta^\alpha| = 6$  and  $M \simeq PSL(2, 11)$ . Thus  $|\Omega| = 11$  and  $G \simeq PSL(2, 11)$ .

Next we consider the case (ii). As in the case (i),  $|N_\beta^\alpha|$  is even. Let  $t \in I(N_\beta^\alpha)$ . Since  $|M_\alpha: N^\alpha| = 1$  or  $3$ ,  $I(M_\alpha) = \{t^g \mid g \in M_\alpha\}$  and so  $C_M(t)$  is transitive on  $F(t)$ . Hence  $|F(t)| = |C_M(t): C_{M_\alpha}(t)|$ . Since  $|C_{M_\alpha}(t)| = |C_{M_\alpha}(t)N^\alpha: N^\alpha| \cdot |C_{N^\alpha}(t)|$ ,  $|F(t)| \geq (3^{2m+1} - 1)3^{2m+1}(3^{2m+1} + 1)/24$ . Since  $|M_\alpha: N^\alpha| = 1$  or  $3$ ,  $r = 1$  or  $3$ . Therefore  $|F(t)| = 1 + (|C_{N^\alpha}(t)| \cdot |I(N_\beta^\alpha)| / |N_\beta^\alpha|) \cdot r < 1 + 8 \times 3 = 25$ . Hence  $25 > (3^{2m+1} - 1)^3/24$  and so  $3^{2m+1} < 11$ , a contradiction.

Finally we consider the case (iii). Since  $N^\alpha \simeq PSL(2, 2^3)$ ,  $3^2 \mid |N^\alpha|$ . But  $3^2 \nmid |M| = |J_1| = 2^3 \cdot 3 \cdot 7 \cdot 11 \cdot 19$ , a contradiction.

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