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ON DOUBLY TRANSITIVE PERMUTATION GROUPS

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1. Introduction

Let G be a doubly transitive permutation group on a finite set Ω and $\alpha\in\Omega$. Using the notation of [9], we denote a normal subgroup of G_α by N^α. Then, for $\beta\in\Omega$ other, we define N^β so that $g^{-1}N^\beta g=N^\gamma$ where $\gamma=\beta^g$.

In this paper we shall prove the following:

Theorem 1. *Let G be a doubly transitive permutation group on a finite set Ω. Suppose that α is an element of Ω. If G_α has a normal simple subgroup N^α which is isomorphic to $PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ with $q=2^n$, $n\geq 2$, then one of the following holds*:

(i) $|\Omega|=6$, $G\simeq A_6$ *or* S_6 *and* $N^\alpha\simeq PSL(2, 4)$.

(ii) $|\Omega|=11$, $G\simeq PSL(2, 11)$ *and* $N^\alpha\simeq PSL(2, 4)$.

(iii) *G has a regular normal subgroup.*

We introduce some notations: Let G be a permutation group on Ω. For $X\leq G$ and $\Delta\subseteq\Omega$, we define $F(X)=\{\alpha\in\Omega\,|\,\alpha^x=\alpha$ for all $x\in X\}$, $X(\Delta)=\{x\in X\,|\,\Delta^x=\Delta\}$, $X_\Delta=\{x\in X\,|\,\alpha^x=\alpha$ for all $\alpha\in\Delta\}$ and $X^\Delta=X(\Delta)/X_\Delta$, the restriction of X on Δ. If p is a prime, we denote by $O^p(X)$, the subgroup of X generated by all p'-elements in X. Other notations are standard ([6], [16]).

2. Preliminary results

Lemma 2.1. *Let G be a doubly transitive permutation group on Ω of even degree and N^α a nonabelian simple normal subgroup of G_α with $\alpha\in\Omega$. If $C_G(N^\alpha)\neq 1$, then $N^\alpha_\beta=N^\alpha\cap N^\beta$ for $\alpha\neq\beta\in\Omega$ and $C_G(N^\alpha)$ is semi-regular on $\Omega-\{\alpha\}$.*

Proof. Set $C^\alpha=C_G(N^\alpha)$. By Corollary B3 and Lemma 2.8 of [17], C^α is semi-regular on $\Omega-\{\alpha\}$ or N^α is a T.I. set in G. Since $|\Omega|$ is even and N^α is $\frac{1}{2}$-transitive on $\Omega-\{\alpha\}$, $|N^\alpha\colon N^\alpha_\beta|$ is odd for $\alpha\neq\beta\in\Omega$. Hence N^α is not semiregular on $\Omega-\{\alpha\}$. By Theorem A of [9], N^α is not a T.I. set in G. Hence C^α is semi-regular on $\Omega-\{\alpha\}$.

Set $\Delta=F(N^\alpha_\beta)$. Since $C^\alpha\leq G(\Delta)$, $[C^\alpha, G_\Delta]\leq C^\alpha\cap G_\Delta=1$. By Corollary

B1 of [17], $N_{\alpha}^{\beta} \leq G_{\Delta}$ and so $[C^{\alpha}, N_{\alpha}^{\beta}]=1$. Let $1 \neq x \in C^{\alpha}$ and set $\beta^{x}=\gamma$. Then $N_{\alpha}^{\beta}=x^{-1}N_{\alpha}^{\beta}x=N_{\alpha}^{\gamma}$. Hence $N_{\alpha}^{\beta} \leq N_{\gamma}^{\beta}$. Since $\beta \neq \gamma$ and G is doubly transitive on Ω, $|N_{\alpha}^{\beta}|=|N_{\gamma}^{\beta}|$. Hence $N_{\alpha}^{\beta}=N_{\gamma}^{\beta}$. Similarly we have $N_{\alpha}^{\gamma}=N_{\beta}^{\gamma}$. Hence $N_{\gamma}^{\beta}=N_{\beta}^{\gamma}$ and so $N_{\gamma}^{\beta}=N^{\beta} \cap N^{\gamma}$. Since G is doubly transitive on Ω, $N_{\beta}^{\alpha}=N^{\alpha} \cap N^{\beta}$.

Lemma 2.2. *Let G be a transitive permutation group on a set Ω, H a stabilizer of a point of Ω and M a nonempty subset of G. Then*

$$|F(M)| = |N_G(M)| \times |ccl_G(M) \cap H| / |H| .$$

Here $ccl_G(M) \cap H = \{g^{-1}Mg \mid g^{-1}Mg \subseteq H, g \in G\}$.

Proof. Set $W=\{(L, \alpha) \mid L \in ccl_G(M), \alpha \in F(L)\}$ and $W_{\alpha}=\{L \mid L \in ccl_G(M), F(L) \ni \alpha\}$. By the transitivity of G, $|W_{\alpha}|=|W_{\beta}|$ holds for every $\alpha, \beta \in \Omega$. Counting the number of elements of W in two ways, we obtain $|G: N_G(M)| \times |F(M)|=|G: H| \times |ccl_G(M) \cap H|$. Thus we have Lemma 2.2.

Lemma 2.3. *Let $G \simeq PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ with $q=2^n>2$ and suppose that G is a transitive permutation group on a set Ω of odd degree. Let H be a stabilizer of a point of Ω. Then we have the following:*

(i) *H has a unique Sylow 2-subgroup S of G and $H=DS$ for a Hall $2'$-subgroup D of H where $D \leq Z_{q^2-1}$.*

(ii) *Let L be a subgroup of G such that $|L|=|H|$. Then $L \in ccl_G(H)$.*

(iii) *S is semi-regular on $\Omega - F(S)$ and $|F(S)|=|F(H)|=|N_G(S): H|$.*

(iv) *Set $D=V \times K$ where $V \leq Z_{q+1}$, $K \leq Z_{q-1}$. Then K acts semiregularly on $\Omega - F(K)$ and if $K \neq 1$, $|F(K)|=2|F(S)|$.*

Proof. Since G is generated by its two distinct Sylow 2-subgroups and $1 \neq |G: H|$ is odd, H contains a unique Sylow 2-subgroup S of G where $S=O_2(H)$. By the structure of $N_G(S)$ we have (i) (cf. § 3 of [2]).

To prove (ii) we may assume that $S \leq L$. As above $S=O_2(L)$ and $L=D_1S$ where $D_1 \leq Z_{q^2-1}$. Since $N_G(S)/S$ is cyclic and $|H|=|L|$, we get $H=L$. Thus (ii) holds.

Let $t \in I(S)$. Applying Lemma 2.2, $|F(t)|=|N_G(t)| \times |ccl_G(t) \cap H| / |H| =(|N_G(t)| \times |ccl_G(t) \cap N_G(S)| / |N_G(S)|) \times (|N_G(S)| / |H|)$. Since $N_G(S)$ is a stabilizer of the usual doubly transitive permutation representation of G, we have $|N_G(t)| \times |ccl_G(t) \cap N_G(S)| / |N_G(S)|=1$, hence $|F(t)|=|N_G(S): H|$. On the other hand, $|F(S)|=|N_G(S)| \times |ccl_G(S) \cap H| / |H|=|N_G(S): H|$. Therefore S acts semi-regularly on $\Omega - F(S)$. As $N_G(H)=N_G(S)$, similarly we have $|F(S)|=|F(H)|$. Thus (iii) holds.

Let x be a nontrivial element of K. Then we have $|F(\langle x \rangle)|=|N_G(\langle x \rangle)| \times |ccl_G(\langle x \rangle) \cap H| / |H|=(|N_G(\langle x \rangle)| \times |ccl_G(\langle x \rangle) \cap N_G(S)| / |N_G(S)|)(|N_G(S|)/|H|)$. As before we have $|N_G(\langle x \rangle)| \times |ccl_G(\langle x \rangle) \cap N_G(S)| / |N_G(S)|=2$. Hence $|F(x)| =2 \cdot |N_G(S): H|$ and this is independent of the choice of $x \in K^{\#}$. Thus (iv) holds.

Lemma 2.4. *Let $G \simeq PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ with $q=2^n>2$ and S be a Sylow 2-subgroup of G, $H=N_G(S)$, t an involution outside H, $D=H \cap H^t$, $V=C_D(t)$ and $K=\{d \in D \mid d^t=d^{-1}\}$. Then the following hold:*

(i) *$N_G(\langle k \rangle)=\langle t \rangle D$ whenever $1 \neq k \in K$.*

(ii) *If $G \simeq PSU(3, q)$ and $1 \neq U$ is a subgroup of V, then $N_G(U)=C_G(V)$ $=N \times V$ where N is a subgroup of G isomorphic to $PSL(2, q)$.*

Proof. (i) follows from the structure of $PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ (§ 3 of [2]).

We now regard $PSU(3, q)$ as a usual doubly transitive permutation group on a set Ω with q^3+1 points. Then V is semi-regular on $\Omega - F(V)$ and $G(F(U))/G_{F(U)}$ is doubly transitive on $F(U)=F(V)$. Clearly $N_G(U) \leq G(F(U))$ and $G_{F(U)}=V$. Hence $N_G(U) \leq N_G(V)$. Since V is cyclic, $N_G(V) \leq N_G(U)$ and so $N_G(U)=N_G(V)$. We now set $M=O^{2'}(N_G(V))$. Then as $[Z(S), V]=1$ and $Z(S)$ is a Sylow 2-subgroup of $N_G(V)$, M centralizes V. By the Frattini argument $N_G(V)=(N_G(V) \cap N(Z(S))M=N_H(V)M=DZ(S) \cdot M \leq C_G(V)$. Hence $N_G(V)=C_G(V)$. By the direct computation, we obtain (ii).

Lemma 2.5. *Let $G \simeq PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ with $q=2^n>2$ and let S be a Sylow 2-subgroup of G.*

(i) *If T is a maximal subgroup of S, then $N_G(T)=S$.*

(ii) *Unless $G \simeq PSU(3, q)$ where $q=2^n$ and n is odd, then by conjugation $N_G(S)$ acts regularly on the set of all maximal subgroups of S.*

Proof. Since $N_G(S)$ is strongly embedded in G, $S \leq N_G(T) \leq N_G(S)$ and so $N_G(T)=RS$ where R is a Hall $2'$-subgroup of $N_G(T)$. As $|S: T|=2$, R centralizes $S/T \simeq Z_2$ and hence there exists an element $t \in C_S(R)-T$. If $G \simeq PSL(2, q)$ or $Sz(q)$, then $R=1$ (§ 3 of [2]). If $G \simeq PSU(3, q)$ and $R \neq 1$, then by (ii) of Lemma 2.4, $t \in I(S)=\Omega_1(S) \leq T$, a contradiction. Thus (i) holds.

Let Γ be the set of all maximal subgroups of S. Then by conjugation, $N_G(S)$ acts on Γ and $(N_G(S))_T=S$ for $T \in \Gamma$ by (i). Under the assumption of (ii), we can easily verify $|\Gamma|=|N_G(S): S|$. From this (ii) follows at once.

Lemma 2.6. *Let $G \simeq PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ with $q=2^n>2$ and A be the full automorphism gruop of G. Let S be a Sylow 2-subgroup of G. Then $C_A(S)=Z(S)$. Here we identify G with the inner automorphism group of G.*

Proof. Let Ω be the set of all Sylow 2-subgroups of G. Then A acts faithfully on Ω and the action of G on Ω is the same as the usual doubly transitive permutation representation. Hence S is regular on $\Omega - \{S\}$ and so $C_A(S)$ is a 2-subgroup of A. If $G \simeq Sz(q)$, A/G is cyclic of odd order and so $C_A(S) \leq G$. Hence $C_A(S)=C_G(S)=Z(S)$. If $G \simeq PSL(2, q)$, S is abelian, so that $C_A(S)=S$

$=Z(S)$. If $G\simeq PSU(3, q)$, there exists a field automorphism such that $\langle f\rangle S$ is a Sylow 2-subgroup of $N_A(S)$. From this $C_A(S)\leq O_2(N_A(S))\leq\langle f\rangle S$. If $gs\in C_A(S)-S$ where $g\in\langle f\rangle$ and $s\in S$, then g centralizes $Z(S)$ and so g is a field automorphism of order 2 by the structural property of A. Since g centralizes s, s must be contained in $Z(S)$. Therefore g centralizes S, while g is a field automorphism of order 2. This is a contradiction. Thus $C_A(S)=S\cap C_A(S)=Z(S)$.

Lemma 2.7. *Let $G\simeq PSU(3, q)$, $q=2^n$ such that n is even. Then $Aut(G)=\langle f\rangle G$ for a field automorphism f of G (see [14]). Let B be a Borel subgroup and let D be a diagonal subgroup of G. Then $B=DS$ and $S=O_2(B)$ for some Sylow 2-subgroup S of G. Set $D=V\times K$ with $V\simeq Z_{q+1}$, $K\simeq Z_{q-1}$. Then $C_A(Z(S))=\langle\tau\rangle VS$ where $A=\langle f\rangle G$ and $\{\tau\}=I(\langle f\rangle)$.*

Proof. By the structural properties of A, $[V, Z(S)]=1$ and $C_{\langle f\rangle}(Z(S))=\langle\tau\rangle$. Since $N_A(Z(S))\rhd O_2(N_G(Z(S)))=S$, $N_A(Z(S))=\langle f\rangle N_G(S)$. Hence $C_A(Z(S))=C(Z(S))\cap\langle f\rangle DS=C_{\langle f\rangle K}(Z(S))VS$. Let $gk\in C_{\langle f\rangle K}(Z(S))$ with $g\in\langle f\rangle$, $k\in K$. Since g is a field automorphism of G, it centralizes a nontrivial element s in $Z(S)$. Then k centralizes s and so $k=1$, for otherwise $s\in C_G(k)=VK$, a contradiction. So $C_{\langle f\rangle K}(Z(S))=C_{\langle f\rangle}(Z(S))=\langle\tau\rangle$. Thus $C_A(Z(S))=\langle\tau\rangle VS$.

3. The case $|\Omega|$ is even

Let G be a doubly transitive permutation group on a finite set Ω of even degree satisfying the assumption of our theorem. Let $\alpha\in\Omega$ and $\{\alpha\}, \Delta_1, \cdots, \Delta_r$ be the set of all N^α-orbits on Ω. Since N^α is normal in G_α, $|\Delta_i|=|\Delta_j|$ for $1\leq i, j\leq r$. Hence $|\Omega|=1+|\Delta_i|r$ and so both $|\Delta_i|$ and r are odd. From this, N^α_β contains a unique Sylow 2-subgroup of N^α for $\beta\neq\alpha$ by (i) of Lemma 2.3. Set $S=O_2(N^\alpha_\beta)$.

(3.1) The following hold.

(i) For each Δ_i with $1\leq i\leq r$, there exists $\beta_i\in\Delta_i$ such that $N^\alpha_\beta=N^\alpha_{\beta_i}$.

(ii) $F(S)=F(N^\alpha_\beta)$, $|F(S)|=|N_{N^\alpha}(S): N^\alpha_\beta|\times r+1$ and S is semi-regular on $\Omega-F(S)$.

(iii) Set $C^\alpha=C_G(N^\alpha)$. Then $C^\alpha=O(G_\alpha)$ and is semi-regular on $\Omega-\{\alpha\}$.

Proof. Let $\gamma\in\Delta_i$. Since $|N^\alpha_\beta|=|N^\alpha_\gamma|$, by (ii) of Lemma 2.3, $N^\alpha_\beta=(N^\alpha_\gamma)^x$ for some $x\in N^\alpha$. Put $\gamma^x=\beta_i$. Then $\beta_i\in\Delta_i$ and $N^\alpha_\beta=N^\alpha_{\beta_i}$. Thus (i) holds.

Hence by (iii) of Lemma 2.3, for each Δ_i with $1\leq i\leq r$, $F(S)\cap\Delta_i=F(N^\alpha_\beta)\cap\Delta_i$, $|F(S)\cap\Delta_i|=|N_{N^\alpha}(S): N^\alpha_\beta|$ and S is semi-regular on $\Delta_i-(\Delta_i\cap F(S))$. Thus (ii) holds.

Since $[O(G_\alpha), N^\alpha]\leq O(G_\alpha)\cap N^\alpha$ and N^α is a non abelian simple group, $[O(G_\alpha), N^\alpha]=1$ and so $O(G_\alpha)\leq C^\alpha$. By Lemma 2.1, C^α is semi-regular on

$\Omega-\{\alpha\}$. Since $G_\alpha \triangleright C^\alpha$, C^α is $\frac{1}{2}$-transitive on $\Omega-\{\alpha\}$. Hence $|C^\alpha| \mid |\Omega|-1$. From this C^α is of odd order and hence $C^\alpha \leq O(G_\alpha)$. Thus $C^\alpha=O(G_\alpha)$.

As a Chevalley group, N^α has a Borel subgroup $N_{N^\alpha}(S)$. Let D be a diagonal subgroup of $N_{N^\alpha}(S)$. Then $N_{N^\alpha}(S)=DS$. We now denote G_α/C^α by $\bar{G}_\alpha$. By the properties of $PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ ([14]), there exists a field automorphism $\bar{f}$ such that $\langle \bar{f} \rangle \bar{N}^\alpha/\bar{N}^\alpha$ is a Sylow 2-subgroup of $\bar{G}_\alpha/\bar{N}^\alpha$. Since $C^\alpha=O(G_\alpha)$, we may assume f is a 2-element in G_α. Since $DC^\alpha \cap N^\alpha=D$ and $SC^\alpha \cap N^\alpha=S$, D and S are f-invariant. Clearly $\langle f \rangle S$ is a Sylow 2-subgroup of G_α. Since $\langle \bar{f} \rangle \cap \bar{N}^\alpha=1$, $\langle f \rangle \cap S \leq C^\alpha$ and so $\langle f \rangle \cap S=1$. Thus we have the following.

(3.1)′ There exists a 2-element f in G_α satisfying the following.
(i) f acts on N^α as a field automorphism of N^α.
(ii) D and S are f-invariant and $\langle f \rangle \cap S=1$.
(iii) $\langle f \rangle S$ is a Sylow 2-subgroup of G_α.

(3.2) $N^\alpha_\beta/N^\alpha \cap N^\beta$ is cyclic of odd order.

Proof. By Lemma 2.1 and (iii) of (3.1), we may assume that $C^\alpha=1$. First we claim that $|S: S \cap N^\beta|=1$ or 2. Since $S/S \cap N^\beta \simeq SN^\beta/N^\beta$ is isomorphic to a 2-subgroup of the outer automorphism group of N^β, $S/S \cap N^\beta$ is cyclic. But S/S' is an elementary abelian 2-group and so $S/S \cap N^\beta \simeq 1$ or Z_2 and hence $|S: S \cap N^\beta|=1$ or 2.

To prove (3.2), it suffices to show that $|S: S \cap N^\beta| \neq 2$. Assume that $|S: S \cap N^\beta|=2$. Then as S and $S \cap N^\beta$ are normal subgroups of N^α_β. Then it follows from (i) of Lemma 2.5 that $N^\alpha_\beta=S$ and $|N^\alpha_\beta: N^\alpha \cap N^\beta|=2$. Since a Sylow 2-subgroup of G_α/N^α is cyclic and $G_{\alpha\beta}/N^\alpha_\beta \simeq G_{\alpha\beta}N^\alpha/N^\alpha$, a Sylow 2-subgroup of $G_{\alpha\beta}/N^\alpha_\beta$ is cyclic. As $N^\alpha_\beta N^\beta_\alpha/N^\alpha_\beta$ is a normal subgroup of $G_{\alpha\beta}/N^\alpha_\beta$ of order 2, $I(G_{\alpha\beta}) \subseteq N^\alpha_\beta N^\beta_\alpha$. Let f be as defined in (3.1)′. Then $f \neq 1$ as $N^\alpha_\beta N^\beta_\alpha \nleq N^\alpha$. Let $\tau \in I(\langle f \rangle)$. Since $\tau \in N_{G_\alpha}(S)$, $S=N^\alpha_\beta$ and $|F(S)-\{\alpha\}|$ is odd, there exists γ such that $\gamma \in F(\tau) \cap F(N^\alpha_\beta)$ and $\gamma \neq \alpha$. Clearly $N^\alpha_\beta \leq N^\alpha_\gamma$, so that $N^\alpha_\beta=N^\alpha_\gamma$. Therefore we may assume $F(\tau) \ni \beta$ and $\tau \in G_{\alpha\beta}$. By Corollary B1 of [17] $F(N^\alpha_\beta)=F(N^\beta_\alpha)$. From this $F(\tau) \supseteq F(N^\alpha_\beta N^\beta_\alpha)=F(N^\alpha_\beta)$ because $\tau \in I(G_{\alpha\beta}) \subseteq N^\alpha_\beta N^\beta_\alpha$. So $\langle \tau \rangle N^\alpha_\beta \leq (\langle \tau \rangle N^\alpha \cap N(N^\alpha_\beta))_{F(N^\alpha_\beta)}$. Let D be as defined in (3.1)′. Then $D \leq N_{N^\alpha}(N^\alpha_\beta)$ and D is τ-invariant. Hence $[D, \tau] \leq (\langle \tau \rangle N^\alpha \cap N(N^\alpha_\beta))_{F(N^\alpha_\beta)} \cap D=1$. Therefore τ centralizes D. Since τ is a field automorphism of N^α of order 2 and D is a diagonal subgroup of N^α, this is a contradiction.

(3.3) The following hold.
(i) $N^\alpha \cap N^\beta=N^\gamma \cap N^\delta$ for, γ, $\delta \in F(N^\alpha \cap N^\beta)$ with $\gamma \neq \delta$.
(ii) $G(F(S))=N_G(N^\alpha \cap N^\beta)$.
(iii) Let M be a subgroup of $N^\alpha \cap N^\beta$ which contains S. Then $F(M)=$

$F(S)$ and $N_G(M)$ is doubly transitive on $F(S)$.

(iv) $C_{G_\alpha}(S)=Z(S)\times C^\alpha$.

(v) Let M be as defined in (iii) and suppose $C^\alpha\neq 1$. Then $O_2(C_G(M))^{F(S)}$ is a regular normal elementary abelian 2-subgroup of $N_G(M)^{F(S)}$.

Proof. Let $\gamma, \delta\in F(N^\alpha\cap N^\beta)$ with $\gamma\neq\delta$. We may assume $\alpha\neq\gamma$. Since G is doubly transitive on Ω, $|N^\alpha\cap N^\beta|=|N^\alpha\cap N^\gamma|$. By the choice of γ, $N^\alpha\cap N^\beta\leq N^\alpha_\gamma$ and $N_{N^\alpha}(S)/S$ is cyclic. Hence $N^\alpha\cap N^\beta=N^\alpha\cap N^\gamma$. Similarly $N^\gamma\cap N^\alpha=N^\gamma\cap N^\delta$. Thus (i) holds.

Since $N_G(N^\alpha\cap N^\beta)\leq N_G(S)$, $N_G(N^\alpha\cap N^\beta)\leq G(F(S))$. Let $x\in G(F(S))$. Then $\alpha^x, \beta^x\in F(S)$ and $F(S)=F(N^\alpha_\beta)$ by (ii) of (3.1). Hence $\alpha^x, \beta^x\in F(N^\alpha\cap N^\beta)$. Therefore by (i) $N^{\alpha^x}\cap N^{\beta^x}=N^\alpha\cap N^\beta$ and so $x\in N_G(N^\alpha\cap N^\beta)$. Thus (ii) holds.

Suppose $S\leq M\leq N^\alpha\cap N^\beta$. If $M^g\leq G_{\alpha\beta}$ for some $g\in G_\alpha$. Then $M^g\leq N^\alpha\cap G_{\alpha\beta}=N^\alpha_\beta$. Hence $M^g=M$ because $S\leq M$ and N^α_β/S is cyclic of odd order. By the Witt's Theorem $N_{G_\alpha}(M)$ is transitive on $F(M)-\{\alpha\}$. Similarly $N_{G_\beta}(M)$ is transitive on $F(M)-\{\beta\}$. We may assume $|F(M)|>2$. Hence $N_G(M)$ is doubly transitive on $F(M)$. By (ii) of (3.1), $F(M)=F(S)$. Thus (iii) holds.

We denote G_α/C^α by $\bar{G}_\alpha$. Clearly $C_{\bar{G}_\alpha}(\bar{N}^\alpha)=\bar{1}$. Applying Lemma 2.6, $C_{\bar{G}_\alpha}(\bar{S})=Z(\bar{S})$, hence $C_{G_\alpha}(S)\leq Z(S)\times C^\alpha$. The converse implication is obvious. Thus (iv) holds.

Suppose $C^\alpha\neq 1$. Then since C^α is semi-regular on $\Omega-\{\alpha\}$, $C_G(M)^{F(S)}\geq (C^\alpha)^{F(S)}\neq 1$. As $N_G(M)^{F(S)}$ is doubly transitive by (iii), $C_G(M)^{F(S)}$ is transitive. By (iv), $(C^\alpha)^{F(S)}\leq C_{G_\alpha}(M)^{F(S)}\leq (Z(S)\times C^\alpha)^{F(S)}$ and so $C_{G_\alpha}(M)^{F(S)}=(C^\alpha)^{F(S)}$. Hence $C_G(M)^{F(S)}$ is a Frobenius group and so $O_2(C_G(M)^{F(S)})\neq 1$ because $|F(S)|$ is even. Since $C_G(M)_{F(S)}\leq (Z(S)\times C^\alpha)_{F(S)}=Z(S)$, $O_2(C_F(M)^{F(S)})=O_2(C_G(M))^{F(S)}$ and this is regular on $F(S)$. As $N_G(M)^{F(S)}\triangleright O_2(C_G(M))^{F(S)}$, $O_2(C_G(M))^{F(S)}$ must be a regular normal elementary abelian 2-subgroup of $N_G(M)^{F(S)}$. Thus (v) holds.

(3.4) There exists an involution t such that $ccl_G(t)\cap S\neq\phi$, $\alpha^t=\beta$ and $F(t)\cap F(S)=\phi$. Set $\mu=|N_{N^\alpha}(S)\colon N^\alpha_\beta|$ and $|S|=q^i$. Then we have

(i) $|\Omega|=(q^i+1)\mu r+1$.

(ii) $|C_S(t)|\geq\sqrt{q}$, $\sqrt{2q}$ or q according as $N^\alpha\simeq PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$, respectively. Furthermore $|C_S(t)|\mid |F(S)|=\mu r+1$.

(iii) If $\mu=1$, then $|\Omega|=6$ and $G\simeq A_6$ or S_6.

(iv) $|\Omega|_2=|F(S)|_2\cdot|G\colon N_G(S)|_2$.

Proof. Since $|\Delta_i|=|N^\alpha\colon N^\alpha_\beta|=|N^\alpha\colon N_{N^\alpha}(S)|\times|N_{N^\alpha}(S)\colon N^\alpha_\beta|=(q^i+1)\mu$ and $|\Omega|=|\Delta_i|r+1$. Hence (i) holds.

Since G is doubly transitive on Ω, there exists an involution t such that $ccl_G(t)\cap S\neq\phi$ and $\alpha^t=\beta$. Then t normalizes $O_2(N^\alpha\cap N^\beta)=S$. Claim $F(t)\cap F(S)=\phi$. Suppose not and let $\gamma\in F(t)\cap F(S)$. As $S\leq N^\alpha_\gamma$, $S\leq N^\alpha\cap N^\gamma$ by (i) of (3.3). Let g be such that $t^g\in S$. Then $t\in N^\delta\cap G_\gamma=N^\delta_\gamma$ where $\delta=\alpha^{g^{-1}}$ and hence $t\in N^{\gamma}$. Since t normalizes S and $\langle t\rangle S\leq N^{\gamma}$, t must be contained in S, a contradiction. Hence $F(t)\cap F(S)=\phi$. From this $C_S(t)$ acts semi-regularly on $F(t)$ and so $|F(t)|$ is divisibly by $|C_S(t)|$. Since $t^g\in S$, $|F(t)|=|F(t^g)|=|F(S)|$, hence $|C_S(t)|\,|\,|F(S)|$.

If $N^{\alpha}\simeq PSL(2, q)$, then $|\Omega_1(S/S')|=|S|=q$ and by Lemma 1 of [7], $|C_S(t)|\geq\sqrt{q}$. If $N^{\alpha}\simeq Sz(q)$, then $|\Omega_1(S/S')|=q$. Since q is an odd power of 2 in this case, similarly $|C_S(t)|\geq\sqrt{2q}$. If $N^{\alpha}\simeq PSU(3,q)$, then $|\Omega_1(S/S')|=q^2$ and so similarly $|C_S(t)|\geq q$. Thus we have (ii).

Suppose $\mu=1$. Then N^{α} is doubly transitive on each N^{α}-orbit $\neq\{\alpha\}$. Applying Theorem D of [10], $r=1$. Therefore, $|F(S)|=\mu r+1=2$ and so by (i) and (ii), $q=4$, $N^{\alpha}\simeq PSL(2, 4)$ and $|\Omega|=6$. Thus (iii) holds.

Since $|\Omega|=|G:N_G(S)|\times|N_G(S):N_{G_\alpha}(S)|/|G_\alpha:N_{G_\alpha}(S)|$ and $|G_\alpha:N_{G_\alpha}(S)|$ is odd, (iv) holds.

(3.5) Let π be the set of primes which divides $q-1$ and K a Hall π-subgroup of $N^{\alpha}\cap N^{\beta}$. If $K\neq 1$, then $C^{\alpha}=1$.

Proof. Suppose $K\neq 1$ and $C^{\alpha}\neq 1$. Set $\Gamma_i=\Delta_i\cap F(S)$ and $\Lambda_i=\Delta_i\cap F(K)$. Then by (i) of (3.1) and Lemma 2.3, for each i with $1\leq i\leq r$ $|\Lambda_i|=2|\Gamma_i|=2|N_{N^\alpha}(S)\colon N^{\alpha}_{\beta_i}|=2|N_{N^\alpha}(S)\colon N^{\alpha}_{\beta}|$ and K is semi-regular on $\Delta_i-\Lambda_i$. By (v) of (3.3), $O_2(C_G(KS))^{F(S)}$ is a regular normal elementary abelian 2-subgroup of $N_G(KS)^{F(S)}$. Set $E=O_2(C_G(KS))$. It follows from (iv) of (3.3) that $E_{F(S)}\leq(Z(S)\times C^{\alpha})_{F(S)}$. Since $F(Z(S))=F(S)$ by (ii) of (3.1) and $(C^{\alpha})_{F(S)}=1$ by (iii) of (3.1), $(Z(S)\times C^{\alpha})_{F(S)}=Z(S)$. On the other hand $Z(S)\cap C(K)=1$ (cf. § 3 of [2]) and so $E_{F(S)}=1$. Hence $E\simeq E^{F(S)}$. Since E is regular on $F(S)$, $|F(S)|=|E^{F(S)}|$ and so we have $|F(S)|=|E|$. Since KS is a subgroup of N^{α}_{β} which contains S, by (ii) of (3.1) we have $F(S)=F(KS)$. From this $F(S)$ is a subset of $F(K)$. Hence $|F(K)-F(S)|=|F(K)-\{\alpha\}|-|F(S)-\{\alpha\}|=\sum_{i=1}^{r}|\Lambda_i|-\sum_{i=1}^{r}|\Gamma_i|=r\times|N_{N^\alpha}(S)\colon N^{\alpha}_{\beta}|$. Since r is odd, $|F(K)-F(S)|$ is odd. On the other hand E fixes $F(K)-F(S)$ setwise because E centralizes S and K. Therefore E fixes an element $\gamma\in F(K)-F(S)$ as E is a 2-subgroup of G. Since $N^{\alpha}_{\gamma}/O_2(N^{\alpha}_{\gamma})$ is cyclic of odd order, $K\leq N^{\alpha}_{\gamma}$ and $|K\cdot O_2(N^{\alpha}_{\gamma})|\,|\,|N^{\alpha}\cap N^{\gamma}|$, we have $K\cdot O_2(N^{\alpha}_{\gamma})\leq N^{\alpha}\cap N^{\gamma}$. Hence $K\leq N^{\gamma}$ and so $|C_{N^\gamma}(K)|$ is odd by (i) of Lemma 2.4. Since $C_{G_\gamma}(K)/C_{N^\gamma}(K)C^{\gamma}\simeq C_{G_\gamma}(K)N^{\gamma}C^{\gamma}/N^{\gamma}C^{\gamma}$, a Sylow 2-subgroup of $C_{G_\gamma}(K)$ is cyclic. But $E\leq C_{G_\gamma}(K)$ and hence $|E|=|F(S)|=2=\mu r+1$. From this $\mu=r=1$. By (iii) of (3.4) $C^{\alpha}=1$, which is contrary to the assumption $C^{\alpha}\neq 1$. So (3.5) holds.

(3.6) Suppose $K\neq 1$ and let S_1 be a subgroup of S. If $S_1{}^g\leq N_G(S)$ and $S_1{}^g\not\leq S$ for some $g\in G$, then $S_1\leq Z_2\times Z_4$ and $|S_1|\,|\,2|G_\alpha/N^{\alpha}|$.

Proof. Set $S_1{}^g=T$. By (ii) of (3.1), T is semi-regular on $\Omega-F(T)$. Claim

$F(T)\cap F(S)=\phi$. Suppose not and let $\gamma\in F(T)\cap F(S)$. Then $T\leq N_\gamma^{\alpha^g}$ and $S\leq N_\gamma^\alpha$. By (3.2) $T\leq N^{\alpha^g}\cap N^\gamma$ and $S\leq N^\alpha\cap N^\gamma$ and so $TS\leq N^\gamma$. Since S is a Sylow 2-subgroup of N^γ, $TS=S$. Hence $T\leq S$, a contradiction. Thus $F(T)\cap F(S)=\phi$. From this T acts semi-regularly on $F(S)$. By (ii) of (3.3), T normalizes $N^\alpha\cap N^\beta$ and so $T\leq N_G(S)\cap N_G(KS)$. By the Frattini argument $KST=N_{KST}(K)\cdot KS=N_{ST}(K)\cdot KS$, so that $N_{ST}(K)^{F(S)}=T^{F(S)}$ as $F(S)=F(KS)$. For an arbitrary $\gamma\in F(S)$, $N_{ST}(K)_\gamma=N_S(K)=C_S(K)=1$, whence $N_{ST}(K)\simeq N_{ST}(K)^{F(S)}$. Hence $T\simeq N_{ST}(K)$. Now $N_{ST}(K)$ acts on $F(K)-F(S)$ and $|F(K)-F(S)|$ is odd. Hence $N_{ST}(K)$ fixes some $\delta\in F(K)-F(S)$. Since $K\leq N_\delta^\alpha$ and $|K\cdot O_2(N_\delta^\alpha)|\,|\,|N^\alpha\cap N^\delta|$, we have $K\leq N^\alpha\cap N^\delta$ as in the proof of (3.5). By (i) of Lemma 2.4, $N_{N^\delta}(K)=D\langle u\rangle\triangleright D$ where u is an involution and D is a cyclic subgroup of N^δ of odd order. Since $N_{G_\delta}(K)/N_{N^\delta}(K)\simeq N_{G_\delta}(K)N^\delta/N^\delta$ and a Sylow 2-subgroup of G_δ/N^δ is cyclic, a Sylow 2-subgroup of $N_{G_\delta}(K)$ is isomorphic to a subgroup of $Z_2\times Z_m$ for some integer m. Since $T\leq S^g$ and S is of exponent at most 4, (3.6) follows immediately.

(3.7) One of the following holds.
(i) $|\Omega|=6$ and $G\simeq A_6$ or S_6.
(ii) $N^\alpha\cap N^\beta$ is a π'-group.

Proof. Let K be a Hall π-subgroup of $N^\alpha\cap N^\beta$ and suppose $G\not\simeq A_6$, S_6 and $K\neq 1$. Let t be an involution as in (3.4) and Q a Sylow 2-subgroup of G containing $\langle t\rangle S$. Then $Q\triangleright S$. For otherwise, let $x\in N_Q(N_Q(S))-N_Q(S)$, then $S^x\neq S$ and S^xnormalizes S. Applying (3.6) to S^x, $S\simeq Z_2\times Z_2$ and $N^\alpha\simeq PSL(2, 4)$. But since $K\neq 1$, $|N^\alpha\cap N^\beta|=12$ and hence $\mu=1$. It follows from (iii) of (3.4) that $G\simeq A_6$ or S_6, which is contrary to the assumption.

Since $Q\triangleright S$ and all involutions in S are conjugate in G, t is conjugate to s for an involution $s\in Z(Q)\cap S$. As s is an extremal element in Q, there is an element $g\in G$ such that $t^g=s$ and $(C_Q(t))^g\leq Q$. Set $T=(C_S(t))^g$. If $T\leq S$, as S is semi-regular on $\Omega-F(S)$, $F(S)^g=F(S)$. Hence $F(t)=F(s)^{g^{-1}}=F(S)$, contrary to the choice of t. Therefore $T\not\leq S$. Applying (3.6) again, $C_S(t)\leq Z_2\times Z_4$, $|C_S(t)|\,|\,2\cdot|G_\alpha/N^\alpha|$.

If $N^\alpha\simeq PSL(2, q)$, by (ii) of (3.4), $\sqrt{q}\leq|C_S(t)|\,|\,2\cdot|G_\alpha/N^\alpha|$ and so $q=2^2$ or 2^4. As before, $q\neq 2^2$, hence $q=2^4$, $N^\alpha\simeq PSL(2, 2^4)$. Then $r=1$ because the outer automorphism group of $PSL(2, 2^4)$ is cyclic of order 4. Since $\mu\neq 1$ and $K\neq 1$, $(\mu, |K|, |F(K)|, |\Omega|)$ is (3, 5, 7, 52) or (5, 3, 11, 86) by (iv) of Lemma 2.3 and (i) of (3.4). By the Witt's Theorem, $N_G(K)$ is doubly transitive on $F(K)$. Hence $|G|$ is divisible by $|F(K)|$. Since $C^\alpha=1$ by (3.5), we have $|G|\,|\,|\Omega|\cdot|\mathrm{Aut}(PSL(2, 2^4))|$. Hence we can verify $|F(K)|\nmid|G|$ in both cases. This is a contradiction.

If $N^\alpha\simeq Sz(q)$, similarly we obtain $\sqrt{2q}<|C_S(t)|\,|\,2|G_\alpha/N^\alpha|$. But in this case since the outer automorphism group of N^α is cyclic of odd order, $|G_\alpha/N^\alpha|$

is odd and so $\sqrt{2q}\leq 2$. Hence $q\leq 2$, a contracdiction.

If $N^{\alpha}\simeq PSU(3, q)$, similarly $q\leq |C_S(t)|\,|\,2|G_{\alpha}/N^{\alpha}|$. Hence $q=2^2$, $N^{\alpha}\simeq PSU(3, 2^2)$. As in the first case, $r=1$ and $(\mu, |K|, |F(K)|, |\Omega|)=(5, 3, 11, 326)$ and so $11=|F(K)|\,|\,|\Omega|\cdot|\mathrm{Aut}(PSU(3, 2^2))|$, a contradiction.

In (3.8)–(3.11), we shall prove that $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$. First we note the following.

(3.8) If $C^{\alpha}\neq 1$, $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$.

Proof. Since N^{α} is a nonabelian simple group, (3.8) follows immediately form Lemma 2.1.

(3.9) Let p be a prime with $p\,|\,|N^{\alpha}_{\beta}: N^{\alpha}\cap N^{\beta}|$ and assume the following:

(*) $\qquad p\neq 3$ if $N^{\alpha}\simeq PSU/(3, 2^n)$ and n is odd.

Then $\mu=p$.

Proof. It follows from (3.8) that $C^{\alpha}=1$. Hence G_{α}/N^{α} is isomorphic to a subgroup of the outer automorphism group of N^{α} and so under the hypothesis (*), a Sylow p-subgroup of G_{α}/N^{α} is normal and cyclic ([14]). Set $=N_G(S)_{F(S)}$. Since $W/N^{\alpha}_{\beta}\leq G_{\alpha\beta}/N^{\alpha}_{\beta}\simeq G_{\alpha\beta}N^{\alpha}/N^{\alpha}$, a Sylow p-subgroup of W/N^{α}_{β} is normal and cyclic. Hence all elements in W of order p is contained in $N^{\alpha}_{\beta}N^{\beta}_{\alpha}$ because $|N^{\beta}_{\alpha}N^{\alpha}_{\beta}/N^{\alpha}_{\beta}|=|N^{\beta}_{\alpha}: N^{\beta}\cap N^{\alpha}|=|N^{\alpha}_{\beta}: N^{\alpha}\cap N^{\beta}|$ and $p\,|\,|N^{\alpha}_{\beta}: N^{\alpha}\cap N^{\beta}|$. Let P be a Sylow p-subgroup of W. Then $\Omega_1(P)\leq N^{\alpha}_{\beta}N^{\beta}_{\alpha}$. Set $Q=\Omega_1(P)$. Since $N^{\alpha}_{\beta}N^{\beta}_{\alpha}/N^{\alpha}_{\beta}\simeq N^{\beta}_{\alpha}/N^{\alpha}\cap N^{\beta}$, by (3.2) $N^{\alpha}_{\beta}N^{\beta}_{\alpha}/N^{\alpha}_{\beta}$ is cyclic and so Q' is a cyclic subgroup of N^{α}_{β}, similarly $Q'\leq N^{\beta}_{\alpha}$. Hence $Q'\leq N^{\alpha}\cap N^{\beta}$ and the p-rank of Q/Q' is at most 2.

By the Frattini argument, $N_G(S)=(N_G(S)\cap N(P))W$. Let M be a normal subgroup of $N_G(S)\cap N(P)$ such that $M^{F(S)}$ is a minimal normal subgroup of $N_G(S)^{F(S)}$. We choose M so that its order is minimal. Since $N_G(S)^{F(S)}$ is doubly transitive, $M^{F(S)}$ is an elementary abelian 2-subgroup or a direct product of isomorphic non abelian simple groups. As Q' is cyclic, $M/C_M(Q')$ is abelian and its Sylow 2-subgroup is cyclic. Hence by the minimality of M, $M=C_M(Q')$.

Set $\bar{Q}=Q/Q'$. We argue that $C_M(\bar{Q})\leq W$. To prove this, it suffices to show that $M\neq C_M(\bar{Q})$. If $M=C_M(\bar{Q})$, M stabilizes the normal series $Q\triangleright Q'\triangleright 1$ and hence $O^p(M)$ centralizes P by Theorem 5.3.2 and Theorem 5.3.1 of [6]. Obviously $O^p(M)\not\leq W$ and so $O^p(M)=M$ by the minimality of M. Therefore M centralizes P. Let x be an element of M such that $\alpha^x=\beta$, then $P\cap N^{\alpha}_{\beta}\leq N^{\alpha}\cap N^{\alpha^x}=N^{\alpha}\cap N^{\beta}$. But since $P\cap N^{\alpha}_{\beta}$ is a Sylow p-subgroup of N^{α}_{β}, $p\not|\,|N^{\alpha}_{\beta}: N^{\alpha}\cap N^{\beta}|$, a contradiction.

Set $C=C_M(\Omega_1(\bar{Q}))$. Then $M/C\leq GL(2, p)$ because the p-rank of $\bar{Q}$ is at most 2. By the minimality of M, $M/C\leq SL(2, p)$. On the other hand $O^p(C)\leq C_M(\bar{Q})\leq W$. Therefore $C^{F(S)}$ is a normal p-subgroup of $N_G(S)^{F(S)}$. Since

$p\neq 2$, $C^{F(S)}=1$ and so $C\leq W$. Hence $M^{F(S)}$ is isomorphic to a homomorphic image of a subgroup of $SL(2, p)$.

Hence if $M^{F(S)}$ is an elementary abelian 2-group, we have $M^{F(S)}\simeq Z_2\times Z_2$ and $|F(S)|=4$. From (ii) and (iii) of (3.4), $\mu=3$ and $r=1$. By (ii) of (3.4), $N^{\alpha}\simeq PSL(2, 4)$, $PSL(2, 16)$ or $PSU(3, 4)$ and hence $|G_{\alpha}: N^{\alpha}|=1$, 2 or 4, which is contrary to $p\,|\,|N^{\beta}_{\alpha}: N^{\beta}\cap N^{\alpha}|=|N^{\beta}_{\alpha}N^{\alpha}/N^{\alpha}|$.

If $M^{F(S)}$ is a direct product of isomorphic non abelian simple groups by Dickson's Theorem (Hauptsatz 8.27 [8]) $M^{F(S)}\simeq PSL(2, p)$ with $p>5$ or A_5. Claim $M^{F(S)}\not\simeq A_5$. Suppose $M^{F(S)}\simeq A_5$, then $N_G(S)^{F(S)}\simeq A_5$ or S_5 and so $|F(S)|=6$, $\mu=5$ and $r=1$. By (ii) of (3.4), we obtain $q=2^2$ and $N^{\alpha}\simeq PSL(2, 4)$. Hence $5\not|\,|N_{N^{\alpha}}(S): N^{\alpha}_{\beta}|=\mu=5$, a contradiction. Thus $M^{F(S)}\simeq PSL(2, p)$ with $p>5$. Hence $|N_G(S)^{F(S)}: M^{F(S)}|=1$ or 2. From this as $|F(S)|$ is even, $M^{F(S)}$ is also doubly transitive. Again by Dickson's Theorem, we know all maximal subgroups of $PSL(2, p)$ with $p>5$ and hence $PSL(2, p)$ with $p>5$ has a unique doubly transitive permutation representation of even degree, which is the known one. From this $|F(S)|=p+1$. Since $|F(S)|=\mu r+1=\mu+1$, we obtain $\mu=p$.

(3.10) If $N^{\alpha}\simeq PSU(3, q)$ and n is odd, then $3\not|\,|N^{\alpha}_{\beta}: N^{\alpha}\cap N^{\beta}|$.

Proof. By (3.8), we may assume $C^{\alpha}=1$. Set $W=N_G(S)_{F(S)}$ and let P be a Sylow 3-subgroup of W. As $G_{\alpha\beta}/N^{\alpha}_{\beta}\simeq G_{\alpha\beta}N^{\alpha}/N^{\alpha}\leq G_{\alpha}/N^{\alpha}$, a Sylow 3-subgroup of W/N^{α}_{β} is an abelian 3-group of rank at most 2, so that $P'\leq N^{\alpha}_{\beta}$ and similarly $P'\leq N^{\beta}_{\alpha}$. Hence $P'\leq N^{\alpha}\cap N^{\beta}$ and P' is cyclic.

Similarly as in the proof of (3.9) we can choose a normal subgroup M of $N_G(S)\cap N(P)$. Denote P/P' by $\bar{P}$. Then $\Omega_1(\bar{P})$ is an elementary abelian 3-subgroup of rank at most 3. Then as in the proof of (3.9), M centralizes P' and $C_M(\Omega_1(\bar{P}))$ is contained in W. Hence $M/C\leq SL(3, 3)$ where $C=C_M(\Omega^1(\bar{P}))$.

If $M^{F(S)}$ is an elementary abelian 2-group, by the structure of $SL(3, 3)$, $M^{F(S)}\simeq Z_2\times Z_2$ and so $|F(S)|=4$, $\mu=3$ and $r=1$. Let $p_1\in\pi$. Since n is odd, $3\notin\pi$. Therefore $p_1\neq 3$. By (3.7), $p_1\not|\,|N^{\alpha}\cap N^{\beta}|$. Hence $p_1\,|\,|N^{\alpha}_{\beta}: N^{\alpha}\cap N^{\beta}|$ and applying (3.9) to p_1, we have $\mu=p_1=3$, a contradiction.

If $M^{F(S)}$ is a direct product of isomorphic non abelian simple groups, we have $M^{F(S)}\simeq SL(3, 3)$ because every proper subgroup of $SL(3, 3)$ is solvable. Hence $|N_G(S)^{F(S)}: M^{F(S)}|=1$ or 2 and so $M^{F(S)}$ is also doubly transitive. By (ii) of (3.1), $N_{N^{\alpha}}(S)_{F(S)}=N^{\alpha}_{\beta}$. Therefore, $N_{N^{\alpha}}(S)^{F(S)}$ is cyclic of order μ. Since $|SL(3, 3)|=2^4 3^3 13$, $\mu=3$ or 13. If $\mu=3$, applying (3.7) and (3.9), π is empty, a contradiction. If $\mu=13$, then $(M_{\alpha})^{F(S)}\triangleright N_{N^{\alpha}}(S)^{F(S)}\simeq Z_{13}$. Hence $(M_{\alpha})^{F(S)}$ is isomorphic to the normalizer of a Sylow 13-subgroup in $SL(3, 3)$, while this permutation representation of $SL(3, 3)$ is not doubly transitive. Thus (3.10) is proved.

(3.11) $N^{\alpha}_{\beta}=N^{\alpha}\cap N^{\beta}$.

Proof. Suppose not and let p be a prime with $p \mid |N^{\alpha}_{\beta}: N^{\alpha} \cap N^{\beta}|$. Then it follows from (3.7), (3.9) and (3.10) that $q-1=p^{e}$ for some integer $e \geq 2$. If e is even, $p^{e} \equiv 1 \pmod 4$, while $q-1 \equiv -1 \pmod 4$, a contradiction. If e is odd, $2^{n}=q=c(p+1)$ where $c=p^{e-1}-p^{e-2}+\cdots-p+1$. We note that $e \geq 3$. Since c is odd, $c=1$, a contradiction. Thus $N^{\alpha}_{\beta}=N^{\alpha} \cap N^{\beta}$.

(3.12) Suppose $N^{\alpha} \simeq PSL(2, q)$ or $Sz(q)$ and $G \not\simeq A_6, S_6$. Then

(i) $N^{\alpha}_{\beta}=N^{\alpha} \cap N^{\beta}$ is a Sylow 2-subgroup of N^{α}.

(ii) If $N^{\alpha} \simeq PSL(2, q)$, then $|F(S)|=q$ and $|\Omega|=q^{2}$.

(iii) If $N^{\alpha} \simeq Sz(q)$, then $|F(S)|=q^{2}$ and $|\Omega|=q^{4}$.

(iv) There is an element x in G such that $S \neq S^{x}$, $[S, S^{x}]=1$ and $F(S) \cap F(S^{x})=\phi$.

Proof. By assumption, $N_{N^{\alpha}}(S)=(q-1)q^{i}$ where $|S|=q^{i}$. Hence (i) follows immediately from (3.7) and (3.11).

We now argue that $|F(S)|$ is a power of 2. By (v) of (3.3), it suffices to consider the case $C^{\alpha}=1$. Applying (ii) of (3.4), $q \mid |F(S)|^{2}$. By (i), $\mu= |N_{N^{\alpha}}(S): N^{\alpha}_{\beta}|=q-1$ and so $|F(S)|=\mu r+1=(q-1)r+1$. Hence $q \mid (r-1)^{2}$, while r is a divisor of n where $2^{n}=q$ because $C^{\alpha}=1$ and G_{α}/N^{α} is isomorphic to a subgroup of the outer automorphism group of N^{α}. Therefore $r=1$ and $|F(S)|=q$, a power of 2.

Hence by (iv) of (3.4), $|F(S)|=(q-1)r+1 \mid |\Omega|=(q^{i}+1)(q-1)r+1$ and so $q \mid (q-1)r+1$ and $(q-1)r+1 \mid q^{i}$. From this, $(i, r)=(1, 1)$, $(2, 1)$ or $(2, q+1)$. If $(i, r)=(1, 1)$ or $(2, q+1)$, we obtain (ii) or (iii), respectively. We argue $(i, r) \neq (2, 1)$. Suppose $(i, r)=(2, 1)$. Then $N^{\alpha} \simeq Sz(q)$, $|F(S)|=q$ and $|\Omega|= q(q^{2}-q+1)$. In this case, since $|G_{\alpha}/C^{\alpha}N^{\alpha}|$ is odd, we have $I(G_{\alpha\beta})=I(N^{\alpha} \cap N^{\beta})$. From this, all involutions in a fixed Sylow 2-subgroup of $G_{\alpha\beta}$ have a common fixed point set. By [12], G has a regular normal subgroup and so $q^{2}-q+1=1$, a contradiction.

Since by (iv) of (3.4) $|\Omega|=|F(S)| \times |G: N_{G}(S)|_{2}$, $|G: N_{G}(S)|_{2}$ is divisible by 2. Let S_1 be a Sylow 2-subgroup of $N_{G}(S)$ and S_2 a Sylow 2-subgroup of $N_{G}(S_1)$. Since $2 \mid |G: N_{G}(S)|$, $S_1 \neq S_2$. Let $x \in S_2 - S_1$, then $S \neq S^{x}$ and $S_1 \triangleright S, S^{x}$. Suppose $\gamma \in F(S) \cap F(S^{x})$. Then by (i), $SS^{x} \leq N^{\gamma}$ and so $S=S^{x}$, a contradiction. Therefore $F(S) \cap F(S^{x})=\phi$ and hence $[S, S^{x}]=1$ by (ii) of (3.1). Thus (iii) holds.

(3.13) The following hold.

(i) $N^{\alpha} \not\simeq Sz(q)$.

(ii) Suppose $N^{\alpha} \simeq PSL(2, q)$ and let S^{x} be as defined in (3.12). Then $O_2(C_G(S))$ is a Sylow 2-subgroup of $C_G(S)$ and $O_2(C_G(S))=S \times S^{x}$.

Proof. Suppose $N^{\alpha} \simeq PSL(2, q)$ or $Sz(q)$. If $C^{\alpha} \neq 1$, $O_2(C_G(S))^{F(S)}$ is a regular normal subgroup of $N_G(S)^{F(S)}$ by (v) of (3.3). If $C^{\alpha}=1$, by (iv) of (3.3)

$C_{G_\alpha}(S)=Z(S)$ and so $C_G(S)_{F(S)}=Z(S)$. By (3.12), $C_G(S)^{F(S)}\geq(S^x)^{F(S)}\neq 1$, and $|F(S)|=q^i=|S|$ and so $C_G(S)=Z(S)\times S^x$. Hence in both cases $O_2(C_G(S))$ is regular on $F(S)$.

Since by (iv) of (3.3) $C_G(S)_{F(S)}=C_{G_{\alpha\beta}}(S)=Z(S)$ and by (ii), (iii) of (3.12) $q^i=|S^x|=F|(S)|=|C_G(S)\colon C_{G_\alpha}(S)|$, we have $O_2(C_G(S))=Z(S)\times S^x$ and this is a Sylow 2-subgroup of $C_G(S)$. Since $Z(O_2(C_G(S)))^{F(S)}=Z(S^x)^{F(S)}$, $N_G(S)\triangleright Z(O_2(C_G(S)))$ and $|F(S)|=|S|$, $|Z(S^x)^{F(S)}|=|S|$. Hence $|Z(S)|=|S|$ and S is abelian. So (3.13) follows.

(3.14) Suppose $N^\alpha\simeq PSL(2, q)$ and $G\not\simeq A_6$, S_6. Put $E=O_2(C_G(S))=S\times S^x$, $W=\{T\mid T\in ccl_G(S),\ T\leq E\}$. Then we have the following:

(i) $|W|=q$ and $\Omega=\bigcup_T F(T)$ where T runs over every element of W.

(ii) $N_G(E)\cap ccl_G(s)\subseteq E$ for all $s\in I(S)$.

(iii) If $E\cap E^g\cap ccl_G(s)\neq\phi$ for some $g\in G$, then $g\in N_G(E)$.

Proof. Let D be a Hall $2'$-subgroup of $N_{N^\alpha}(S)$. Then $D\simeq Z_{q-1}$ and by (i) of (3.12) D is semi-regular on $\Omega-\{\alpha\}$. If $d\in N_D(S^x)$, $\langle d\rangle$ acts semi-regularly on $F(S^x)$ since $\alpha\notin F(S^x)$. Hence the order of d divides $|F(S)|$. But $|F(S)|=q$ by (ii) of (3.12), hence $|\langle d\rangle|\,|(q, q-1)=1$ and so $d=1$. Therefore $N_D(S^x)=1$. Hence $|\{S^{xd}\mid d\in D\}|=q-1$ and $\{S^{xd}\mid d\in D\}\subseteq W$ as D normalizes E. If $S=S^{xd}$ for some $d\in D$, $S^x=S^{d^{-1}}=S$, a contradiction. Hence $|W|\geq q$. If there exist S_1, $S_2\in W$ such that $S_1\neq S_2$ and $F(S_1)\cap F(S_2).\neq\phi$ Let $\gamma\in F(S_1)\cap F(S_2)$. Then S_1, $S_2\leq N^\gamma$ by (i) of (3.12) and so $\langle S_1, S_2\rangle=N^\gamma$, which is contrary to $\langle S_1, S_2\rangle\leq E$. Hence $F(S_1)\cap F(S_2)=\phi$ for S_1, $S_2\in W$ such that $S_1\neq S_2$. Since $|F(S)|=q$ and $|\Omega|=q^2$ by (ii) of (3.12), we have $|W|\leq q$. Thus (i) holds.

Let $s\in I(S)$ and suppose $s^g\in N_G(E)-E$ for some $g\in G$. Then $s^g\in N^\gamma$ where $\gamma=\alpha^g$. By (i) we choose $T\in W$ so that $\gamma\in F(T)$. Then $\langle s^g, T\rangle=N^\gamma$ as $s^g\notin T$ and T is a Sylow 2-subgroup of N^γ. On the other hand $\langle s^g, T\rangle\leq\langle s^g\rangle E$, which is a 2-subgroup of $N_G(E)$, a contradiction. Thus (ii) holds.

Let $1\neq t\in E\cap E^g\cap ccl_G(s)$ for $g\in G$ and $s\in I(S)$. Then there are $S_1\leq E$ and $S_2\leq E^g$ such that $t\in S_1\cap S_2$ and S_1, $gS_2g^{-1}\in W$. Since $F(S_1)=F(t)=F(S_2)$ by (ii) of (3.1), $\langle S_1, S_2\rangle\leq N^\gamma\cap N^\delta$ for γ, $\delta\in F(t)$. Hence $S_1=S_2$ by (i) of (3.12). Applying (ii) of (3.13) to S_1, we obtain $E=O_2(C_G(S_1))=O_2(C_G(S_2))=E^g$. Thus (iii) holds.

(3.15) Suppose $N^\alpha\simeq PSL(2, q)$ and $G\not\simeq A_6$, S_6. Then G has a regular normal subgroup.

Proof. We count the set $\{(\gamma, T)\mid\gamma\in F(T),\ T\in ccl_G(S)\}$ in two ways and we have $q^2\times(q+1)=|ccl_G(S)|\times q$ by (3.12). Hence $|ccl_G(S)|=q(q+1)$. On the other hand we have $|ccl_G(S)|=|G\colon N_G(E)|\times q$ by (i), (ii) of (3.14). From this, $|G\colon N_G(E)|=q+1$.

Set $\Gamma = ccl_G(E)$. We now consider the action of G on Γ. By definition, G is transitive on Γ and $N_G(E)$ is a stabilizer of $E \in \Gamma$. We argue that S is regular on $\Gamma - \{E\}$. Suppose not and let $1 \neq s \in S$ such that $s^{-1}E^g s = E^g$ for some $E^g \in \Gamma - \{E\}$. Then $gsg^{-1} \in N_G(E)$. By (ii) of (3.14), $gsg^{-1} \in E$ and hence $gsg^{-1} \in E \cap gEg^{-1}$. By (iii) of (3.14), $E = gEg^{-1}$. Hence $E = E^g$, a contradiction. Since $S \leq N_G(E))$ and $|S| = |\Gamma| - 1$, S is regular on $\Gamma - \{E\}$ and G^Γ is doubly transitive. Since S is abelian and regular on $\Gamma - \{E\}$, $G^\Gamma \cap C(S^\Gamma) = S^\Gamma$. From this, $E^\Gamma = S^\Gamma$ because $E \geq S$ and E is abelian. Therefore $G_\Gamma \neq 1$. Set $M = G_\Gamma$. Suppose $M \cap N^\alpha \neq 1$, then $M \geq N^\alpha$ as N^α is simple. Hence $N^\alpha \leq N_G(E)$ and so N^α normalizes $E \cap G_\alpha = S$, a contradiction. Thus $M \cap N^\alpha = 1$. Hence $M_\alpha \leq C_G(N^\alpha) = C^\alpha$, so that $M_\alpha = 1$ or $M_\alpha \neq 1$ and M is a Frobenius group on Ω by (iii) of (3.1). In both cases, G has a regular normal subgroup.

We now consider the case that $N^\alpha \simeq PSU(3, q)$. By (3.7) and (3.11), $N^\alpha_\beta = US$ where U is a Hall 2′-subgroup of N^α_β and $U \leq Z_{q+1/\varepsilon}$ with $\varepsilon = (q+1, 3)$. As in the proof of (3.1)′, we set $N_{N^\alpha}(S) = DS$ and $D = V \times K$. Here $V \simeq Z_{q+1/\varepsilon}$ and $K \simeq Z_{q-1}$. Since $N_{N^\alpha}(S) \triangleright N^\alpha_\beta$, we may assume $U = V \cap N^\alpha_\beta$.

(3.16) Suppose $N^\alpha \simeq PSU(3, q)$. Then $N^\alpha_\beta = N^\alpha \cap N^\beta$ is a Sylow 2-subgroup of N^α. In particular $\mu = q^2 - 1/\varepsilon$.

Proof. Suppose not and $U \neq 1$. If $U^g \leq G_{\alpha\beta}$ for $g \in G$, $U^g \leq N^{\alpha^g}_\alpha \cap N^{\beta^g}_\beta = N^{\alpha^g} \cap N^\alpha \cap N^{\beta^g} \cap N^\beta \leq N^\alpha \cap N^\beta$. Hence U is conjugate to U^g in $N^\alpha \cap N^\beta \leq G_{\alpha\beta}$. By the Witt's Theorem $N_G(U)$ is doubly transitive on $F(U)$. By (ii) of Lemma 2.4, $N_{N^\alpha}(U) = N \times V$ where $N \simeq PSL(2, q)$. Hence $N_G(U)^{F(U)}$ satisfies the assumption of Theorem 1. By (i) of (3.1), the number of fixed points of U on Δ_i is constant for each N^α-orbit Δ_i and so $|F(U)| = |F(U) \cap \Delta_i| \times r + 1 = (|N_{N^\alpha}(U)| \times |N^\alpha_\beta : N_{N^\alpha_\beta}(U)| / |N^\alpha_\beta|) \times r + 1 = (|PSL(2,q)| \times |V| / |Z(S)| \times |U|) \times r + 1 = (q^2-1) \times r \times |V : U| + 1$. Hence $|F(U)|$ is even and $|F(U)| \neq 6$. Applying (3.12) to $N_G(U)^{F(U)}$, we obtain $|F(U)| = q^2$, $|F(U) \cap F(Z(S))| = q$. Hence $r = 1$, $U = V$, $N^\alpha_\beta = VS$ and $|F(V)| = q^2$ and so $\mu = |N_{N^\alpha}(S) : N^\alpha_\beta| = q - 1$. Since by (ii) of (3.1) $F(U) \supseteq F(S)$, $|F(Z(S))| = |F(S)| = q$. Furthermore by (3.15), $N_G(V)^{F(V)}$ has a regular normal elementary abelian 2-subgroup, say $E^{F(V)}$. Clearly $E^{F(V)} \leq C_G(V)^{F(V)}$. Hence we may assume that E is a 2-subgroup of $C_G(V)$. Put $P = E_{F(V)}$. Then $|E| = |P| q^2$. By (i) of (3.4), $|\Omega| = q^4 - q^3 + q$ and so $2q \nmid |\Omega - F(V)|$. Hence there exists $\gamma \in \Omega - F(V)$ such that $|E : E_\gamma| \leq q$. Let T be a Sylow 2-subgroup of G_γ containing E_γ. Since $E_\gamma / E_\gamma \cap T \cap N^\gamma$ is isomorphic to a subgroup of $T / T \cap N^\gamma$ and $T / T \cap N^\gamma \simeq TN^\gamma / N^\gamma \leq G_\gamma / N^\gamma$, $E_\gamma / E_\gamma \cap T \cap N^\gamma$ is cyclic. If $E_\gamma \cap T \cap N^\gamma = 1$, E_γ is cyclic and so $|E_\gamma / E_\gamma \cap P| \leq 2$. Then $|E_\gamma \cap P| \geq |E_\gamma| / 2 \geq |P| q / 2 > |P|$, a contradiction. Hence $E_\gamma \cap T \cap N^\gamma \neq 1$. Let $z \in E_\gamma \cap T \cap N^\gamma$ with $z \neq 1$. Since $|F(z)| = q < |F(P)|$, $z \in E$ and $E^{F(V)}$ is regular, we have $F(z) \cap F(V) = \phi$. Hence V acts semi-regularly on $F(z)$. From this, $q = |F(z)| = (q + 1/\varepsilon) \times k$ for some integer $k \geq 1$. Since q is a power of 2, $q+1/\varepsilon=1$, a contradiction.

(3.17) Suppose $N^{\alpha}\simeq PSU(3, q)$. Then the following hold.

(i) $|\Omega|=q^5-q^3+q^2$, $|F(S)|=q^2$.

(ii) $N_G(S)^{F(S)}$ has a regular normal subgroup.

Proof. If $C^{\alpha}\neq 1$, (ii) follows from (v) of (3.3) and so $|F(S)|$ is a power of 2. By (3.4) and (3.16), $|F(S)|=(q^2-1)r/\varepsilon+1$ and $(q^2-1)r/\varepsilon+1\,|\,(q^3+1)(q^2-1)r/\varepsilon+1$, hence $(q^2-1)r/\varepsilon+1\,|\,q^3$. By calculation, we obtain $r=\varepsilon$. So (i) follows.

We now assume $C^{\alpha}=1$. By (ii) of (3.4), $q\,|\,|F(S)|=(q^2-1)r/\varepsilon+1$, so that $r=qk+\varepsilon$ for an integer $k\geq 0$. Since $C^{\alpha}=1$, r is a divisor of $|G_{\alpha}/N^{\alpha}|$. Hence $r\,|\,2n\varepsilon$, so that $r\,|\,n\varepsilon$. Therefore $n\varepsilon\geq r=qk+\varepsilon=2^n\times k+\varepsilon$. Hence $k=0$ and $r=\varepsilon$. From this (i) follows.

Let f be a field automorphism as defined in (3.1)$'$ and let T be a Sylow 2-subgroup of $N_G(S)$ which contains $\langle f\rangle S$. Since $|N_G(S)\colon N_{G_{\alpha}}(S)|=|F(S)|=q^2$ by (i), $|T|=2^m q^5$ where $|\langle f\rangle|=2^m$. Since $T\triangleright S$ and $\Omega-F(S)=q^3(q^2-1)$ there exists $\gamma\in\Omega-F(S)$ such that $|T\colon T_{\gamma}|=q^3$, hence $|T_{\gamma}|=2^m q^2$ and $T=ST_{\gamma}$. Set $W=T_{\gamma}\cap N^{\gamma}$. Then W is semi-regular on $F(S)$ because $\gamma\in\Omega-F(S)$. In particular $|W|\leq|F(S)|=q^2$. We note that $|T_{\gamma}N^{\gamma}/N^{\gamma}|\leq 2^m$. Since $T_{\gamma}/W\simeq T_{\gamma}N^{\gamma}/N^{\gamma}$, we have $|W|\geq q^2$. Hence $|W|=q^2$ and W is regular on $F(S)$. Moreover $|T_{\gamma}\colon W|=2^m$.

Since $N_{G_{\alpha\beta}}(S)/S\simeq N_{G_{\alpha\beta}}(S)N^{\alpha}/N^{\alpha}$ by (3.16), $N_{G_{\alpha\beta}}(S)^{F(S)}$ is isomorphic to a homomorphic image of a subgroup of the outer automorphism group of N^{α}. Hence $N_{G_{\alpha\beta}}(S)^{F(S)}$ is abelian when n is even or $f=1$. In this case by [1], (ii) holds because $|F(S)|=q^2$. We now assume n is odd and $|\langle f\rangle|=2^m=2$. Since $T=ST_{\gamma}$ and $|T_{\gamma}\colon W|=2$, $|T^{F(S)}\colon W^{F(S)}|=2$. Claim $f^{F(S)}\neq 1$. For otherwise $f\in N_G(S)_{F(S)}$ and $[f, D]\leq N_G(S)_{F(S)}\cap D=1$ as D is f-invariant and $D\leq N_G(S)$. But since $f\neq 1$, f does not centralize D. Therefore $f^{F(S)}\neq 1$. As $f\in G_{\alpha}$, $f^{F(S)}\notin W^{F(S)}$. Hence $T^{F(S)}=\langle f\rangle^{F(S)}W^{F(S)}\triangleright W^{F(S)}$. Since $W^{F(S)}$ is regular, $f^{F(S)}$ is not conjugate to any element in $W^{F(S)}$. Hence $f^{F(S)}$ is not contained in $O^2(N_G(S)^{F(S)})$ by Lemma 2 of [3]. Since $\langle f^{F(S)}\rangle$ is a Sylow 2-subgroup of $(N_G(S)^{F(S)})_{\alpha\beta}$, $O^2(N_G(S)^{F(S)})_{\alpha\beta}$ is of odd order. As before $(N_G(S)^{F(S)})_{\alpha\beta}$ is isomorphic to a homomorphic image of a subgroup of the outer automorphism group of N^{α}, $O^2(N_G(S)^{F(S)})_{\alpha\beta}$ is abelian. Again by [1], $O^2(N_G(S)^{F(S)})$ has a regular normal subgroup as $|F(S)|=q^2$. Thus (ii) also holds in this case

(3.18) $N^{\alpha}\not\simeq PSU(3, q)$.

Proof. Let f be as in (3.1)$'$. By the same argument as in the proof of (ii) of (3.17), we have $I(\langle f\rangle)\not\subseteq N_G(S)_{F(S)}$ and so S is a Sylow 2-subgroup of $N_G(S)_{F(S)}$.

By (ii) of (3.17), there is a normal subgroup L of $N_G(S)$ such that $L\geq N_G(S)_{F(S)}$ and $L^{F(S)}$ is an elementary abelian 2-subgroup of $N_G(S)^{F(S)}$. Let T be a Sylow 2-subgroup of $\langle f\rangle L$ which contains f. Set $E=T\cap L$. Then E

is a Sylow 2-subgroup of L. Since S is a unique Sylow 2-subgroup of $N_G(S)_{F(S)}$, $E/S \simeq L^{F(S)}$ is an elementary abelian 2-subgroup of order q^2. As $\langle f\rangle \cap E = \langle f\rangle \cap E \cap G_\alpha = \langle f\rangle \cap S = 1$, $T = \langle f\rangle E \triangleright E$.

Since $T \triangleright S$ and $|\Omega - F(S)| = q^3(q^2-1)$ by (i) of (3.17), we can choose $\gamma \in \Omega - F(S)$ such that $|T: T_\gamma| = q^3$. Hence $|T_\gamma| = 2^m q^2$ where 2^m is the order of f. Since $T_\gamma/T_\gamma \cap C^\gamma N^\gamma \simeq T_\gamma N^\gamma C^\gamma/C^\gamma N^\gamma$ is cyclic of order at most 2^m, $|T_\gamma \cap C^\gamma N^\gamma| = |T_\gamma \cap N^\gamma| \geq q^2$. Moreover $T_\gamma \cap N^\gamma/T_\gamma \cap N^\gamma \cap E \simeq (T_\gamma \cap N^\gamma)E/E$ is cyclic of order at most 2^m, we have $|T_\gamma \cap N^\gamma \cap E| \geq q^2/2^m$. Since the order of f is a divisor of $2n$, we have $|T_\gamma \cap N^\gamma \cap E| \geq q(2^n/2^m) \geq q$.

If $T_\gamma \cap N^\gamma \cap E$ contains no element of order 4, then $T_\gamma \cap N^\gamma \cap E$ is an elementary abelian 2-subgroup of N^γ of order q and hence $T_\gamma \cap N^\gamma/T_\gamma \cap N^\gamma \cap E$ is an elementary abelian 2-group. Therefore $|(T_\gamma \cap N^\gamma)E/E| \leq 2$ and so $|T_\gamma \cap N^\gamma \cap E| \geq q^2/2 > q$, a contradiction.

If $T_\gamma \cap N^\gamma \cap E$ contains an element x of order 4, then $1 \neq x^2 \in S$ because E/S is an elementary abelian 2-group. Since $\gamma \in F(x^2)$, by (ii) of (3.1) we have $\gamma \in F(S)$, which is contrary to $\gamma \in \Omega - F(S)$. Thus (3.18) holds.

In this section we have proved the following:

Theorem 2. *Suppose G^Ω satisfies the hypothesis of Theorem* 1 *and $|\Omega|$ is even. Then $N^\alpha \not\simeq Sz(q)$, $PSU(3, q)$, $N^\alpha \simeq PSL(2, q)$ and either*

(i) $G^\Omega \simeq A_6$ *or* S_6 *or*

(ii) $|\Omega| = q^2$, $|N^\alpha_\beta| = |N^\alpha \cap N^\beta| = q$ *and G has a regular normal subgroup.*

4. The case $|\Omega|$ is odd

Let G be a doubly transitive permutation group on Ω of odd degree satisfying the assumption of Theorem 1. By Theorem A of [10] and Theorem B of [11], we may assume $C_G(N^\alpha) = 1$. Hence G_α/N^α is isomorphic to a subgroup of the outer automorphism group of N^α. Let $\{\alpha\}$, $\Delta_1, \Delta_2, \cdots, \Delta_r$ be the set of all N^α-orbits on Ω. Clearly r is a divisor of $|G_\alpha/N^\alpha|$.

From now on we assume that G has no regular normal subgroup and prove that $G \simeq PSL(2, 11)$. Let M be a minimal normal subgroup of G. Then by assumption, $M_\alpha \neq 1$.

(4.1) *M is simple and $N^\alpha \leq M$.*

Proof. Since G is doubly transitive and $M_\alpha \neq 1$, M is a simple group (cf. Exercise 12.4 of [16]). If $N^\alpha \not\leq M$, then $M_\alpha \cap N^\alpha = 1$ as N^α is simple and hence $M_\alpha \leq C_G(N^\alpha) = 1$, a contradiction. Thus $N^\alpha \leq M$.

As in (3.1)′, there is a 2-element f of M_α such that f acts on N^α as a field automorphism, $\langle f\rangle S \triangleright S$, $\langle f\rangle \cap S = 1$ and $\langle f\rangle S$ is a Sylow 2-subgroup of M_α, where $N_{N^\alpha}(S) = DS$ is a Borel subgroup of N^α, S is a unipotent subgroup of N^α, and D is a diagonal subgroup of N^α.

(4.2) If $f\neq 1$, then $I(N^\alpha_\beta)\nsubseteq N^\alpha\cap N^\beta$ for $\beta\neq\alpha$.

Proof. Suppose $f\neq 1$ and $\tau\in I(\langle f\rangle)$. Since M is a simple group with a Sylow 2-subgroup $\langle f\rangle S$, $\tau^g\in S$ for some $g\in M_\alpha$ by Lemma 2 of [3]. Set $\gamma=\alpha^{g^{-1}}$. Then $\tau\in N^\gamma_\alpha$ and clearly $\tau\notin N^\gamma\cap N^\alpha$, so that $I(N^\gamma_\alpha)\nsubseteq N^\gamma\cap N^\alpha$. By the transitivity of G, we obtain $I(N^\alpha_\beta)\nsubseteq N^\alpha\cap N^\beta$ for any $\beta\neq\alpha$.

(4.3) Suppose $f\neq 1$. Then $N^\alpha\not\simeq Sz(q)$, $PSU(3, q)$.

Proof. If $N^\alpha\simeq Sz(q)$, $|G_\alpha/N^\alpha|$ is odd and hence $f=1$, a contradiction. Therefore $N^\alpha\not\simeq Sz(q)$.

Suppose $N^\alpha\simeq PSU(3, q)$ and let $\tau\in I(\langle f\rangle)$. Let $s\in Z(\langle f\rangle S)\cap I(S)$. As in the proof of (4.2), $ccl_M(\tau)\cap S\neq\phi$. Then since s is an extremal element there is $g\in M$ such that $\tau^g=s$ and $(C_{\langle f\rangle S}(\tau))^g\leq\langle f\rangle S$. Since τ is a field automorphism of order 2, $Z(S)\leq C_{\langle f\rangle S}(\tau)$. Put $\beta=\alpha^{g^{-1}}$. Then $\tau\in N^\beta_\alpha$ and $Z(S)\leq N^\alpha_\beta$. By (4.2) $Z(S)\nleq N^\alpha\cap N^\beta$ and so $|Z(S)\colon Z(S)\cap N^\alpha\cap N^\beta|=2$ because $Z(S)/Z(S)\cap N^\alpha\cap N^\beta\simeq Z(S)(N^\alpha\cap N^\beta)/N^\alpha\cap N^\beta\leq N^\alpha_\beta/N^\alpha\cap N^\beta\simeq N^\alpha_\beta N^\beta/N^\beta\leq G_\beta/N^\beta$.

Claim $N^\alpha_\beta\leq N_{N^\alpha}(S)$. Suppose not. Then $N^\alpha_\beta\cap N_{N^\alpha}(S)$ is a strongly embedded subgroup of N^α_β. Since $|N^\alpha_\beta/N^\alpha\cap N^\beta|$ is even and $N^\alpha_\beta\geq Z(S)\geq Z_2\times Z_2$, by Bender's Theorem ([2]), $N^\alpha_\beta/N^\alpha\cap N^\beta$ is not solvable, while $N^\alpha_\beta/N^\beta\cap N^\beta\simeq N^\alpha_\beta N^\beta/N^\beta$ is solvable, a contradiction.

Let V_1 be a τ-invariant Hall $2'$-subgroup of N^α_β. Then since V_1 normalizes $\Omega_1(O_2(N^\alpha_\beta))=Z(S)$, V_1 centralizes $Z(S)/Z(S)\cap N^\alpha\cap N^\beta\simeq Z_2$. Hence by (i) of Lemma 2.4, $V_1\leq Z_{q+1}$ and so $[V_1, Z(S)]=1$ by (ii) of Lemma 2.4. Therefore $I(N^\alpha_\beta)\subseteq Z(N^\alpha_\beta)$. Similarly $I(N^\beta_\alpha)\subseteq Z(N^\beta_\alpha)$. Since $\tau\in I(N^\beta_\alpha)$, we have $N^\alpha\cap N^\beta\leq C(\tau)\cap N_{N^\alpha}(S)$. Since τ is a field automorphism of N^α of order 2, $C(\tau)\cap N_{N^\alpha}(S)=KZ(S)$ where K is a cyclic subgroup of $N_{N^\alpha}(S)$ of order $q-1$. Hence $N^\alpha\cap N^\beta\leq KZ(S)\cap N^\alpha_\beta=Z(S)(K\cap V_1O_2(N^\alpha_\beta))=Z(S)$ and so $|Z(S)\colon N^\alpha\cap N^\beta|=2$.

We claim that $F(z)=F(Z(S))$ for $z\in I(N^\alpha_\beta)$. Let Δ_i be an arbitrary N^α-orbit on $\Omega-\{\alpha\}$. Since all elementary abelian 2-subgroups of N^α of order q are conjugate in N^α, there exists $\gamma\in\Delta_i$ with $Z(S)\leq N^\alpha_\gamma$. Hence by Lemma 2.2, $|F(z)\cap\Delta_i|=|C_{N^\alpha}(z)|\times|Z(S)^\sharp|/|N^\alpha_\gamma|=(q+1/\varepsilon)\times q^3(q-1)/|N^\alpha_\gamma|$ for $z\in I(N^\alpha_\beta)$. On the other hand $|F(Z(S))\cap\Delta_i|=|N_{N^\alpha}(Z(S))|/|N^\alpha_\gamma|=(q^2-1/\varepsilon)\times q^3/|N^\alpha_\beta|$. Hence $F(z)\cap\Delta_i=F(Z(S))\cap\Delta_i$ and so $F(z)=F(Z(S))$. In particular $F(\tau)=F(Z(S))$ because $\tau\in I(N^\beta_\alpha)$ and $N^\alpha\cap N^\beta\neq 1$.

We claim that $(V_1)_{F(Z(S))}=1$. Set $S_1=O_2(N^\alpha_\beta)$. Let $d\in V_1$ with $d\neq 1$, Δ_i be a N^α-orbit which contains β and let D_1 be a τ-invariant Hall $2'$-subgroup of $N_{N^\alpha}(S)$ which contains V_1. Put $X=\langle d\rangle Z(S)$. Then by Lemma 2.2, $|F(X)\cap\Delta_i|=|N_{N^\alpha}(X)|\,|N^\alpha_\beta\colon N_{N^\alpha_\beta}(X)|/|N^\alpha_\beta|=|D_1Z(S)|\,|N^\alpha_\beta\colon V_1Z(S)|/|N^\alpha_\beta|$ $=(q^2-1/\varepsilon)|S_1|/|N^\alpha_\beta|=|F(Z(S))\cap\Delta_i|/|S\colon S_1|$. Since $S_1/N^\alpha\cap N^\beta$ is cyclic and $N^\alpha\cap N^\beta\leq Z(S)$, $S\neq S_1$. Therefore $F(X)\neq F(Z(S))$ and so $(V_1)_{F(Z(S))}=1$.

Since $D_1\leq N_{N^\alpha}(Z(S))$ and $\tau\in N_{G_\alpha}(Z(S))_{F(Z(S))}$, $[\tau, D_1]\leq N_G(Z(S))_{F(Z(S))}\cap D_1$

$=(V_1)_{F(Z(S))}=1$. Hence $D_1 \leq C(\tau) \cap N_{N^\alpha}(S)=KZ(S)$ with $K \simeq Z_{q-1}$, which is contrary to $|D_1|=(q^2-1)/\varepsilon$. So (4.3) is proved.

(4.4) Suppose $N^\alpha \simeq PSL(2, q)$ and $f \neq 1$. Then the following hold.

(i) N^α_β is a 2-subgroup of N^α and $|N^\alpha_\beta: N^\alpha \cap N^\beta|=2$.

(ii) Let $\tau \in I(\langle f \rangle)$. Then for some $\beta \neq \alpha$, $\tau \in N^\beta_\alpha - N^\alpha_\beta$, $|C_S(\tau)|=\sqrt{q}$ and $N^\alpha \cap N^\beta \leq C_S(\tau) \leq N^\alpha_\beta$.

Proof. As in the proof of (4.3), there exist $s \in I(S)$ and $g \in M$ such that $\tau^g = s$ and $(C_{\langle f \rangle S}(\tau))^g \leq \langle f \rangle S$. Put $\beta = \alpha^{g^{-1}}$. Then $\tau \in N^\beta_\alpha - N^\alpha_\beta$ and $C_S(\tau) \leq N^\alpha_\beta$. Since τ is a field automorphism of N^α of order 2, $|C_S(\tau)|=\sqrt{q}$. Claim $N^\alpha_\beta \leq N_{N^\alpha}(S)$. If $q \neq 2^2$, as $C_S(\tau) \leq N^\alpha_\beta$, a Sylow 2-subgroup of N^α is non cyclic. Hence as in the proof of (4.3), $N^\alpha_\beta \leq N_{N^\alpha}(S)$. If $q=2^2$, $N^\alpha \simeq A_5$ and so $\langle \tau \rangle N^\alpha = M_\alpha = G_\alpha \simeq S_5$. In particular $r=1$. Hence $N^\alpha_\beta \leq N_{N^\alpha}(S)$. For otherwise $|N^\alpha_\beta|=6$ or 10 and $|\Omega|=11$ or 7, respectively. By [13], such groups do not exist. Thus in both cases $N^\alpha_\beta \leq N_{N^\alpha}(S)$. On the other hand $N^\alpha_\beta / N^\alpha \cap N^\beta$ is cyclic of even order. By (i) of Lemma 2.4, N^α_β must be an abelian 2-subgroup of N^α and $|N^\alpha_\beta: N^\alpha \cap N^\beta|=2$. Since $N^\beta_\alpha \simeq N^\alpha_\beta$ and $\tau \in N^\beta_\alpha$, we obtain $N^\alpha \cap N^\beta \leq C_S(\tau)$. Thus (i) and (ii) hold.

(4.5) Suppose $N^\alpha \simeq PSL(2, q)$ and $f \neq 1$. Let $T = N^\alpha_\beta N^\beta_\alpha$. Then

(i) $N_G(T)$ is doubly transitive on $F(T)$.

(ii) $N_{N^\alpha}(T)=S$ and $S_\gamma = N^\alpha_\beta$ for every $\gamma \in F(T)$.

Proof. Since $G_{\alpha\beta}/N^\alpha_\beta$ is cyclic and by (i) of (4.4) $T/N^\alpha_\beta \simeq Z_2$, $I(G_{\alpha\beta}) \subseteq T$. Clearly $\langle I(G_{\alpha\beta}) \rangle = T$. Hence by the Witt's Theorem, we have (i).

Let K_1 be a Hall $2'$-subgroup of $N_{N^\alpha}(T)$. Then K_1 normalizes $T \cap N^\alpha = N^\alpha_\beta$. Since $T/N^\alpha_\beta \simeq Z_2$, $[K_1, T/N^\alpha_\beta]=1$ and so $T=C_T(K_1)N^\alpha_\beta$. If $K_1 \neq 1$, by (i) of Lemma 2.4 $C_T(K_1)=1$. Hence $K_1=1$ and $N_N\alpha(T)=S$.

Let $\gamma \in F(T)-\{\alpha\}$. Then obviously $N^\alpha_\beta \leq S_\gamma \leq N^\alpha_\gamma$. Since G is doubly transitive on Ω, $|N^\alpha_\beta|=|N^\alpha_\gamma|$, so that $N^\alpha_\beta = S_\gamma = N^\alpha_\gamma$. Thus (ii) holds.

(4.6) Suppose $N^\alpha \simeq PSL(2, q)$ and $f \neq 1$. Put $q=2^n$. Then

(i) $(n, |N^\alpha_\beta|)=(2, 2), (2, 2^2), (4, 2^3)$ or $(6, 2^4)$.

(ii) If $(n, |N^\alpha_\beta|)=(6, 2^4)$, $N_G(T)^{F(T)} \simeq A_5$.

Proof. $|G_\alpha/N^\alpha| \mid n$ and $f \neq 1$, n is even and so we set $n=2m$. By (ii) of (4.4), $|N^\alpha_\beta|=2^{m+\varepsilon}$ where $\varepsilon=0$ or 1. Since $N_{G_{\alpha\beta}}(T)/T \leq G_{\alpha\beta}/T \simeq (G_{\alpha\beta}/N^\alpha_\beta)/(T/N^\alpha_\beta)$ and $G_{\alpha\beta}/N^\alpha_\beta \simeq G_{\alpha\beta}N^\alpha/N^\alpha \leq G_\alpha/N^\alpha$, $N_{G_{\alpha\beta}}(T)^{F(T)}$ is cyclic and $|N_{G_{\alpha\beta}}(T)^{F(T)}| \mid m$. By (4.5), $N_G(T)^{F(T)}$ is doubly transitive and $S^{F(T)} \simeq S/N^\alpha_\beta$ is semi-regular on $F(T)-\{\alpha\}$. Since $N_{G_{\alpha\beta}}(T)^{F(T)}$ is cyclic, by [1] $N_G(T)^{F(T)} \simeq PSL(2, q_1)$ where q_1 is a power of 2 or $N_G(T)^{F(T)}$ has a regular normal subgroup. If $(n, |N^\alpha_\beta|) \neq (2, 2), (2, 2^2)$ and $(4, 2^3)$, $S^{F(T)}$ contains a four-group, which is semi-regular on $F(T)-\{\alpha\}$. Hence $N_G(T)^{F(T)}$ contains no regular normal subgroup and so

$N_G(T)^{F(T)} \simeq PSL(2, q_1)$. Since $N_{N^\alpha}(T)^{F(T)} = S^{F(T)} \simeq S/N_\beta^\alpha$ and $N_{G_\alpha}(T)^{F(T)} \triangleright N_{N^\alpha}(T)^{F(T)}$, $q_1 = 2^{m-\varepsilon} > 2$. Hence $2^{m-\varepsilon} - 1 = |N_{G_{\alpha\beta}}(T)^{F(T)}|$, so that $2^{m-\varepsilon} - 1 | m$. From this, $\varepsilon = 1$, $m = 3$ and $N_G(T)^{F(T)} \simeq A_5$. Thus (4.6) holds.

(4.7) $f = 1$.

Proof. Suppose $f \neq 1$. Then by (4.3) and (4.6), it suffices to consider the case (i) of (4.6).

If $N^\alpha \simeq PSL(2, 2^2)$ and $|N_\beta^\alpha| = 2$, $G_\alpha = N_\alpha^\beta N^\alpha \simeq \mathrm{Aut}(N^\alpha) \simeq S_6$. Hence $r = 1$. Therefore $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 31$ and $|G| = |\Omega||G_\alpha| = 2^3 \cdot 3 \cdot 5 \cdot 31$. By the Sylow's theorem, G has a regular normal subgroup of order 31. But this is a contradiction as $G \geq N^\alpha$.

If $N^\alpha \simeq PSL(2, 2^2)$ and $|N_\beta^\alpha| = 2^4$, as above $G_\alpha = N_\alpha^\beta N^\alpha$ and hence $r = 1$. From this $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| = 16$, a contradiction.

If $N^\alpha \simeq PSL(2, 2^4)$ and $|N_\beta^\alpha| = 2^3$, $|\mathrm{Aut}(N^\alpha) : N^\alpha| = 4$ and so $|G_\alpha : N_\alpha^\beta N^\alpha| \leq 2$. Hence $r = 1$ or 2 and $|\Omega| = 511$ or 1021 respectively. By Lemma 2.2, for $s \in N_\beta^\alpha - \{1\}$ $|F(s) - \{\alpha\}| = 14$ or 28 respectively. Let τ be a field automorphism of N^α of order 2 as in (4.4). Then $C_{N^\alpha}(\tau) \simeq PSL(2, 2^2)$ and $|F(\tau) - \{\alpha\}| = 14$ or 28 since τ is conjugate to s. From this an element x of $C_{N^\alpha}(\tau)$ of order 5 fixes at least four points in Ω. Since $5 \nmid |\Omega|$, $\langle x \rangle$ is a Sylow 5-subgroup of G and so $x^g \in N^\alpha$ for some $g \in G$. But $F(x^g) = \{\alpha\}$ because $|N_\gamma^\alpha| = |N_\beta^\alpha| = 2^3$ for all $\gamma \neq \alpha$. Therefore $|F(x)| = 1$, which is contrary to $|F(x)| \geq 4$.

If $N^\alpha \simeq PSL(2, 2^6)$ and $|N_\beta^\alpha| = 2^4$, by (ii) of (4.6), $|N_{G_{\alpha\beta}}(T)^{F(T)}| = 3$. Hence $3 \mid |G_{\alpha\beta} : N_\beta^\alpha|$. Since $|G_{\alpha\beta} : N_\beta^\alpha| = |G_{\alpha\beta} N^\alpha : N^\alpha|$ and $|N_\alpha^\beta N^\alpha : N^\alpha| = 2$ by (i) of (4.4), we have $G_{\alpha\beta} N^\alpha = G_\alpha \simeq \mathrm{Aut}(N^\alpha)$. In particular $r = 1$ and $|\Omega| = 16381$. Moreover $|F(s) - \{\alpha\}| = 60$. As before $|F(\tau) - \{\alpha\}| = 60$, $C_{N^\alpha}(\tau) \simeq PSL(2, 2^3)$ and an element of $C_{N^\alpha}(\tau)$ of order 7 fixes at least five points. But since $7 \nmid |\Omega|$ and $7 \nmid |N_\beta^\alpha|$, every element of order 7 fixes exactly one point, a contradiction.

(4.8) $G^\Omega \simeq PSL(2, 11)$, $|\Omega| = 11$.

Proof. By (4.7), $|M_\alpha : N^\alpha|$ is odd and so a Sylow 2-subgroup of N^α is also that of M. By [4], [5] and [15], it suffices to consider the following cases:

(i) $N^\alpha \simeq PSL(2, 2^2)$, $M \simeq PSL(2, q_1)$, $q_1 \equiv 3$ or 5 (mod 8), $q_1 > 3$.

(ii) $N^\alpha \simeq PSL(2, 2^3)$, $C_M(t) \simeq Z_2 \times PSL(2, 3^{2m+1})$, $t \in I(M)$ $(m \geq 1)$.

(iii) $N^\alpha \simeq PSL(2, 2^3)$, $M \simeq J_1$, the smallest Janko group.

First we consider the case (i). If $|N_\beta^\alpha|$ is odd, every involution in M has a unique fixed point and so $M \simeq PSL(2, 5)$ by [2]. But then $M = N^\alpha$, a contradiction. Hence $|N_\beta^\alpha| = 2$, 4, 6, 10 or 12. On the other hand $r = 1$ or 2 because $|\mathrm{Aut}(N^\alpha) : N^\alpha| = 2$. From this $|\Omega| = 1 + |N^\alpha : N_\beta^\alpha| r = 7$, 11, 13, 21, 31 or 61. Since $M \simeq PSL(2, q_1)$ and $|M| = |\Omega||N^\alpha|$, we get $|\Omega| = 11$, $|N_\beta^\alpha| = 6$ and $M \simeq PSL(2, 11)$. Thus $|\Omega| = 11$ and $G \simeq PSL(2, 11)$.

Next we consider the case (ii). As in the case (i), $|N^{\alpha}_{\beta}|$ is even. Let $t\in I(N^{\alpha}_{\beta})$. Since $|M_{\alpha}:N^{\alpha}|=1$ or 3, $I(M_{\alpha})=\{t^g|g\in M_{\alpha}\}$ and so $C_M(t)$ is transitive on $F(t)$. Hence $|F(t)|=|C_M(t)\colon C_{M_{\alpha}}(t)|$. Since $|C_{M_{\alpha}}(t)|=|C_{M_{\alpha}}(t)N^{\alpha}:N^{\alpha}|\,|C_{N^{\alpha}}(t)|$, $|F(t)|\geq(3^{2m+1}-1)3^{2m+1}(3^{2m+1}+1)/24$. Since $|M_{\alpha}:N^{\alpha}|=1$ or 3, $r=1$ or 3. Therefore $|F(t)|=1+(|C_{N^{\alpha}}(t)|\,|I(N^{\alpha}_{\beta})|/|N^{\alpha}_{\beta}|)\cdot r<1+8\times 3=25$. Hence $25>(3^{2m+1}-1)^3/24$ and so $3^{2m+1}<11$, a contradiction.

Finally we consider the case (iii). Since $N^{\alpha}\simeq PSL(2,2^3)$, $3^2\,|\,|N^{\alpha}|$. But $3^2\not|\,|M|=|J_1|=2^3\cdot 3\cdot 7\cdot 11\cdot 19$, a contradiction.

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