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NUMBER OF SYLOW SUBGROUPS AND p-NILPOTENCE OF FINITE GROUPS

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1. Introduction

Let G be a finite group and $\pi(G)$ the set of primes dividing the order of G. The classical theorems of Sylow assert that there are subgroups called Sylow p-subgroups of order p^r which is the highest power of a prime p dividing the order $|G|$ of G and that the number $n_p(G)$ of Sylow p-subgroups is congruent to 1 modulo p, which is equal to the index $|G : N_G(P)|$ for some Sylow p-subgroup P. We call $n_p(G)$ the Sylow p-number of G. Some arithmetical properties of $n_p(G)$ are studied in [8] and [9].

A finite group G is said to be p-nilpotent if G has a normal p-complement, *i.e.* $G = O_{p'}(G)P$ for some $P \in Syl_p(G)$, the set of Sylow p-subgroups of G. There are several conditions equivalent to the p-nilpotence of a finite group. See [19, II, Theorem 2.27, p.155] for example.

There is an interesting conjecture concerning the Sylow p-number $n_p(G)$ of G and the p-nilpotence of G. B. Huppert made the following conjecture:

Conjecture (B. Huppert [12]**).** *Let G be a finite group. Then G is p-nilpotent if and only if for every $r \in \pi(G)$, $(p,\ n_r(G)) = 1$ holds and $N_G(R)$ is p-nilpotent for $R \in Syl_r(G)$.*

The following assertion has been made by Zhang as Theorem 2 in [20].

Claim. *Let G be a finite group. Then G is p-nilpotent if and only if $(p,\ n_r(G)) = 1$ for every $r \in \pi(G)$.*

This claim, if true, would confirm the Huppert conjecture and the paper makes other claims as applications of it. There exist, however, infinitely many counterexamples to the claim which will be mentioned in §2 of this paper. The purpose of this paper is to prove the following:

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Main Theorem. *Let G be a finite group.*

(i) *Suppose that $p \neq 3$ is a prime. Then G is p-nilpotent if and only if $(p,\ n_r(G)) = 1$ for every $r \in \pi(G)$.*

(ii) *Suppose that G does not have a composition factor isomorphic with $U_3(q)$, where $q = 2^f$, f even not divisible by 3. Then G is p-nilpotent if and only if $(p,\ n_r(G)) = 1$ for every $r \in \pi(G)$.*

Corollary 1. *The Huppert conjecture holds true.*

Our notation for simple groups and group extensions are taken from [4].

2. Counterexamples to the Claim

In this section, we will give counterexamples to the Claim in the Introduction. Before discussing them, we will state a well known lemma which plays an important role through this paper.

Lemma 1. *Let a, n be integers greater than 1. Then except in the cases $n = 2$, $a = 2^b - 1$ and $n = 6$, $a = 2$, there is a prime q with the following properties.*

(i) *q divides $a^n - 1$.*
(ii) *q does not divides $a^i - 1$ whenever $0 < i < n$.*
(iii) *q does not divide n.*

In particular, n is the order of a modulo q.

Proof. See [2, Theorem 8.3 (Zsigmondy), p.508].

Theorem 1. *Suppose that G is the projective special unitary group $U_3(q)$ where $q = u^f > 2$, u is a prime. If there exists a prime $p \in \pi(G)$ such that $(p,\ n_r(G)) = 1$ for every prime r, then $p = 3$, $q = 2^f$ and f is even not divisible by 3.*

Proof. We see that $|G| = (q^3 + 1)q^3(q^2 - 1)/\nu$, where $\nu = (q + 1,\ 3)$. Note that $(q+1,\ q-1) = 1$ or 2, $(q+1,\ q^2-q+1) = 1$ or 3 and $(q-1,\ q^2-q+1) = 1$. Suppose that q is odd. By Lemma 1, there exists a prime r such that r divides q^6-1, r does not divide q^i-1 $(1 \leq i \leq 5)$ and r does not divide 6. Then r divides q^2-q+1. By [17], we can find maximal subgroups of G. We see that $N_G(Q)$ is a maximal subgroup of order $q^3(q^2-1)/\nu$ and that $N_G(R)$ is a maximal subgroup of order $3(q^2 - q + 1)/\nu$, where $Q \in Syl_u(G)$ and $R \in Syl_r(G)$. We have $p \neq u$ since p divides $(|N_G(Q)|,\ |N_G(R)|)$ and $P \subseteq N_G(R)$ for some $P \in Syl_p(G)$. If p divides $q^2 - q + 1$, then p divides $(q^2 - q + 1,\ q^2 - 1) = 1$ or 3. It follows that $p = 3$. In this case $P \in Syl_3(G)$ is not contained in $N_G(Q)$ since $|G : N_G(Q)| = q^3 + 1 = (q + 1)(q^2 - q + 1)$ is divisible by 3, a contradiction. Hence p does not divide $q^2 - q + 1$. Since $P \subseteq N_G(R)$, we have $p = 3$. By [5, Theorem 4], we have $N_G(S) \simeq S \times Z_{k/\nu}$, where $S \in Syl_2(G)$ and k satisfies the condition $q + 1 = 2^a k$, k odd. Since $P \subseteq N_G(S)$, we have that 3 divides k which divides $q + 1$. Then $q^2 - q + 1$ is divisible by $p = 3$, a contradiction since $(p,\ q^2 - q + 1) = 1$. Hence G does not satisfy the condition if q is odd.

Suppose that q is even. By [11], the maximal subgroups of G are:

$$\begin{array}{ll} q^{1+2}:(q^2-1)/\nu, & \\ (q+1)/\nu \times U_2(q), & \\ ((q+1)\times(q+1)/\nu).S_3, & \\ ((q^2-q+1)/\nu):3, & \\ U_3(2^m) & \text{if } k/m>3 \text{ is odd prime,} \\ U_3(2^m).3 & \text{if } m \text{ is odd and } k=3m. \end{array}$$

Put Q_r a Sylow r-subgroup of G for $r \in \pi(G)$. Since $N_G(Q_2) = q^{1+2} : (q^2-1)/\nu$, we have that p does not divide $q^3+1 = (q+1)(q^2-q+1)$. By Lemma 1, there exists a prime r such that r divides q^6-1, r does not divide q^i-1 $(1 \le i \le 5)$ and r does not divide 6. Then r divides q^2-q+1 and therefore we have $p \neq r$ since $P \subseteq N_G(Q_2)$ for some $P \in Syl_p(G)$. Since $N_G(R) = ((q^2-q+1)/\nu):3$, we have $p=3$. Furthermore $|G|$ is not divisible by 9 since $P \subseteq N_G(R)$ for some $P \in Syl_3(G)$.

If f is odd, then $q+1$ is divisible by 3. In this case $|G|$ is divisible by 9.

Hence we may assume that f is even. Then $q-1$ is divisible by 3 and $\nu=1$. Note that $q-1$ is divisible by 9 if and only if f is divisible by 6. If f is even and f is not divisible by 3, $N_G(Q_r)$ and $n_r(G)$ are the following:

$$\begin{array}{lll} N_G(Q_2) = q^{1+2}:(q^2-1), & n_2(G) = q^3+1 & \\ N_G(Q_r) = (q+1)\times D_{2(q-1)}, & n_r(G) = q^3(q^3+1)/2 & \text{for } r \in \pi(q-1) \\ N_G(Q_r) = (q+1)^2 : S_3, & n_r(G) = q^3(q-1)/6 & \text{for } r \in \pi(q+1) \\ N_G(Q_r) = (q^2-q+1):3, & n_r(G) = q^3(q+1)^2(q-1)/3 & \text{for } r\text{: otherwise} \end{array}$$

We see that $(n_r(G),\ 3)=1$ for every $r \in \pi(G)$. This completes the proof.

This theorem yields that the groups $U_3(q)$, where $q=2^f$ and f is even not divisible by 3, are counterexamples to the Claim in the Introduction. As a corollary, we have a lot of counterexamples to the Claim. Put $\mathcal{U} = \{U_3(q) |\ q=2^f,\ f \text{ is even not divisible by } 3\}$ and $\mathcal{P}$ the set of 3-nilpotent groups.

Corollary 2. *Let $G_1 \in \mathcal{U}$ and $G_2, \dots, G_t \in \mathcal{U} \cup \mathcal{P}$. Then $G = G_1 \times G_2 \times \cdots \times G_t$ satisfies $(n_r(G),\ 3)=1$ for every $r \in \pi(G)$ and G is not 3-nilpotent.*

3. Proof of the Main Theorem

In this section, we will prove the Main Theorem. Our proof is along the same line of Zhang [20]. We will need some lemmas from [20], some of which are claimed in it without proofs.

Lemma 2. *Let G be a finite group and M a normal subgroup of G. Then both $n_p(M)$ and $n_p(G/M)$ divide $n_p(G)$, and moreover $n_p(G)$ is divisible by $n_p(M)n_p(G/M)$.*

Proof. See [20, Lemma 1].

Proposition 1. *Let p be a prime. If G is p-nilpotent, then $(n_r(G),\ p)=1$ for every $r \in \pi(G)$.*

Proof. It is trivial if $r = p$. Suppose that $r \neq p$. Since G is p-nilpotent, $G = O_{p'}(G)P$ for some $P \in Syl_p(G)$. Then r divides $|O_{p'}(G)|$. Take $R \in Syl_r(G)$. Then $R \subseteq O_{p'}(G)$. By Frattini argument, we have $G = N_G(R)O_{p'}(G)$. Since p does not divide $|O_{p'}(G)|$, we see that p divides $|N_G(R)|$ and therefore $P \subseteq N_G(R)$. Thus the result follows.

Proposition 2. *Suppose that G is a finite group of minimal possible order satisfying the following two conditions:*

(i) *There exists a prime p such that $(n_r(G),\ p) = 1$ for every $r \in \pi(G)$.*
(ii) *G is not p-nilpotent.*

Then G is a simple group.

Proof. Suppose that G has a nontrivial normal subgroup. Let N be a minimal normal subgroup of G and set $L/N = O_{p'}(G/N)$ where $N \subseteq L$. By the minimality of G, N is a p-group and $L = G$. Now $(p, n_r(G)) = 1$ for any r implies that $G = N \times O_{p'}(G)$, a contradiction.

Proposition 3. *Let G be the alternating group A_n on $n \geq 5$ letters. Then for every prime $p \in \pi(G)$, there exists a prime $r \in \pi(G)$ such that $(n_r(G),\ p) \neq 1$.*

Proof. See [20, p.113].

Lemma 3. *Let G be one of the sporadic simple groups. Put l the largest prime in $\pi(G)$. Then*

(i) *$L \in Syl_l(G)$ is cyclic of order l and L is self-centralizing subgroup in G.*
(ii) *There exists a prime $r \in \pi(G)$ such that $(n_r(G),\ l) \neq 1$.*

Proof. It is easily seen that (i) holds. (See [4]).

(ii) If $(n_r(G),\ l) = 1$ for any $r \in \pi(G)$, then RL is a Frobenius group with Frobenius complement L by (i), where $R \in Syl_r(G)$ and L is a suitable Sylow l-subgroup of G. This implies that l divides $|R| - 1$ for any $r \in \pi(G)$. This is a contradiction. (See [4]).

Proposition 4. *Let G be one of the sporadic simple groups. Then for every prime $p \in \pi(G)$, there exists a prime $r \in \pi(G)$ such that $(n_r(G),\ p) \neq 1$.*

Proof. Suppose that there exists a prime p such that $(n_r(G),\ p) = 1$ for any $r \in \pi(G)$. Lemma 3 yields that $p \neq l$, the largest prime in $\pi(G)$. Since $L \in Syl_l(G)$ is self-centralizing by Lemma 3 and $P \subset N_G(L)$ for some $P \in Syl_p(G)$, LP is a Frobenius group and $|P|$ divides $l - 1$. For J_2, Suz, He, Fi_{22}, Fi'_{24}, HN, Th, M, $O'N$, J_3, l and $l - 1$ are as follows:

$$\begin{array}{lllll}
J_2 & : & l=7 & \text{and} & l-1=6=2\cdot 3,\\
Suz & : & l=13 & \text{and} & l-1=12=2^2\cdot 3,\\
He & : & l=17 & \text{and} & l-1=16=2^4,\\
Fi_{22} & : & l=13 & \text{and} & l-1=12=2^2\cdot 3,\\
Fi'_{24} & : & l=29 & \text{and} & l-1=28=2^2\cdot 7,\\
HN & : & l=19 & \text{and} & l-1=18=2\cdot 3^2,\\
Th & : & l=31 & \text{and} & l-1=30=2\cdot 3\cdot 5,\\
M & : & l=71 & \text{and} & l-1=70=2\cdot 5\cdot 7,\\
O'N & : & l=31 & \text{and} & l-1=30=2\cdot 3\cdot 5,\\
J_3 & : & l=19 & \text{and} & l-1=18=2\cdot 3^2,
\end{array}$$

We get a contradiction for each cases since $|P|$ does not divide $l-1$.

Suppose that there exists a prime $s \neq l$ in $\pi(G)$ such that $S \in Syl_s(G)$ is cyclic of order s, S is self-centralizing in G and s does not divide $l-1$. Then $s \neq p$ and the order of Sylow p-subgroup of G divides $(l-1,\ s-1)$. For J_1, M_{22}, HS, Ly, Fi_{23}, J_4, B, there exists such a prime s.

$$\begin{array}{llllll}
J_1 & : & s=11, & l=19 & \text{and} & (l-1,s-1)=2,\\
M_{22} & : & s=7, & l=11 & \text{and} & (l-1,s-1)=2,\\
HS & : & s=7, & l=11 & \text{and} & (l-1,s-1)=2,\\
Ly & : & s=37, & l=67 & \text{and} & (l-1,s-1)=6,\\
Fi_{23} & : & s=17, & l=23 & \text{and} & (l-1,s-1)=2,\\
J_4 & : & s=37, & l=43 & \text{and} & (l-1,s-1)=6,\\
B & : & s=31, & l=47 & \text{and} & (l-1,s-1)=2.
\end{array}$$

In any case, we have a contradiction.

The rest of the sporadic simple groups are M_{11}, M_{12}, M_{23}, M_{24}, McL, Ru, Co_3, Co_2 and Co_1. The Sylow 2-subgroup Q of these groups is self-normalizing. (See [4]). Hence $p=2$ and $|Q|$ must divide $l-1$, a contradiction. This completes the proof.

Let G be a connected reductive algebraic group over the algebraic closure of the finite field $\mathbb{F}_q$ of $q=u^f$ elements, with a Frobenius map $F: G \longrightarrow G$, where u is a prime. The fixed points in G under F then form a finite group G^F, and all finite simple groups of Lie type occur as the nonabelian composition factors of such G^F.

Let Σ be an irreducible root system. For $a \in \Sigma$, set $a^* = 2a/(a,a)$, the coroot of a. Then $\Sigma^* = \{a^* | a \in \Sigma\}$ is also a root system, the dual of Σ. A prime p is called a torsion prime if $L(\Sigma^*)/L(\Sigma_1^*)$ has a p-torsion for some closed subsystem Σ_1 of Σ, where $L(\Sigma)$ denotes the lattice generated by Σ.

Lemma 4. *For various root systems, the torsion primes are as follows:*

(i) *For type A_l C_l: none.*
(ii) *For type B_l, D_l, G_2: 2.*
(iii) *For type E_6, E_7, F_4: 2, 3.*
(iv) *For type E_8: 2, 3, 5.*

Proof. See [1, I.4.4, p.178].

To study the case of simple groups of Lie type, we will need the following:

Lemma 5. *Let G be a connected semisimple algebraic group over the algebraic closure of the field $\mathbb{F}_q$ and F a Frobenius map of G.*

(i) *A direct product $E = Y_1 \times \cdots \times Y_m$ of cyclic semisimple subgroups Y_i's of G^F can be embedded into a maximal torus T^F of G^F, if the number of $|Y_i|$ not prime to all torsion primes of G is at most two. In particular, we then have $N_{G^F}(E)/C_{G^F}(E) \leq W(G)$ the Weyl group of G.*

(ii) *A Sylow r-subgroup for r prime to the characteristic and the order of the Weyl group can be embedded into a maximal torus T^F of G^F; in particular, it is abelian.*

Proof. See [15, Lemma 1.7].

Lemma 6. *Suppose that G and H are finite groups satisfying $G/Z(G) \simeq H$. Let r be a prime. Then $n_r(G) = n_r(H)$. In particular, if $(n_r(H),\ p) = 1$ for some prime p, then $(n_r(G),\ p) = 1$.*

Proof. Note that $N_G(\tilde{R})/Z(G) = N_{G/Z(G)}(\tilde{R}Z(G)/Z(G))$ for $\tilde{R} \in Syl_r(G)$. Since $\tilde{R}Z(G)/Z(G) \in Syl_r(G/Z(G))$, we have

$$\begin{aligned}|G : N_G(\tilde{R})| &= |G/Z(G) : N_G(\tilde{R})/Z(G)| \\ &= |G/Z(G) : N_{G/Z(G)}(\tilde{R}Z(G)/Z(G))| \\ &= |H : N_H(R)|\end{aligned}$$

for some $R \in Syl_r(H)$. Thus the result follows.

Proposition 5. *Let G be a finite simple group of Lie type. If G is not isomorphic with ${}^2A_2(q)$, then for every prime $p \in \pi(G)$, there exists a prime $r \in \pi(G)$ such that $(n_r(G),\ p) \neq 1$.*

Proof. Let G be a finite simple group of Lie type over a field $\mathbb{F}_q$ of q elements, where $q = u^f$, u is a prime and f is a positive integer. Suppose that there exists a prime p such that $(n_r(G),\ p) = 1$ for any $r \in \pi(G)$. Since G has a subgroup $B = UH$ so called Borel subgroup, where $U \in Syl_u(G)$ and $N_G(U) = B$, it follows that p divides $|B|$. By [6, Theorem 4.253, p.313], U has a trivial signalizer. Hence we have $p \neq u$. This yields that p divides $|H|$ and $P_0 \subseteq H$ for some $P_0 \in Syl_p(G)$. Since H is abelian for any type, P_0 is abelian. Also G has a subgroup $N \rhd H$ such that $N/H \simeq W(G)$, the Weyl group of G. Since $P_0 \subseteq H$, p does not divide $|W(G)|$. In particular, $p \neq 2$ since the Weyl group has even order for any type.

There exists a simply connected algebraic group $\tilde{G}$ with Frobenius map F such that $\tilde{G}^F/Z(\tilde{G}^F) \simeq G$ except for the case where $G \simeq {}^2F_4(2)'$. By Lemma 6, we have $(n_r(\tilde{G}^F),\ p) = 1$ for any $r \in \pi(G) = \pi(\tilde{G}^F)$. We will consider $\tilde{G}^F$ except for some cases. Note that $\tilde{G}^F$ has a Borel subgroup $\tilde{B}^F = \tilde{U}^F \tilde{H}^F$, where $\tilde{U}^F \in Syl_u(\tilde{G}^F)$ and $N_{\tilde{G}^F}(\tilde{U}^F) = \tilde{B}^F$. We see that $P \subseteq \tilde{H}^F$ for some $P \in Syl_p(\tilde{G}^F)$, P is abelian and p does not divide $|W(\tilde{G}^F)|$.

$G \simeq A_l(q)$: $|\tilde{G}^F| = q^{l(l+1)/2} \prod_{i=2}^{l+1}(q^i - 1)$ and $\tilde{H}^F \simeq Z_{q-1} \times \cdots \times Z_{q-1}$ (l times). Since $P \subseteq \tilde{H}^F$ for some $P \in Syl_p(\tilde{G}^F)$, p divides $q-1$ and therefore $q > 2$. By [1, G] or [3], $\tilde{G}^F$ has a maximal torus T of type $T(A_l)$ whose order is $(q^{l+1}-1)/(q-1)$. By Lemma 1, there exists a prime r such that r divides $q^{l+1}-1$, r does not divide $q^i - 1$ $(1 \leq i \leq l)$ and r does not divide $l+1$ except for the cases where $l+1 = 2$ and $q = 2^b - 1$ or $l+1 = 6$ and $q = 2$. Then we see that $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since there exist no torsion primes for type A_l by Lemma 4, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. Since p does not divide $|W(\tilde{G}^F)| = |W(\tilde{G})| = (l+1)!$, P is contained in $C_{\tilde{G}^F}(R)$. Then $PR = P \times R$ is abelian and $P \times R$ is contained in some maximal torus T_1 by Lemma 5. Since $T(A_l)$ is the only type of maximal torus whose order is divisible by r, we have $|T_1| = (q^{l+1} - 1)/(q - 1)$. This yields that p divides $((q^{l+1}-1)/(q-1),\ q-1)$. Since $((q^{l+1}-1)/(q-1),\ q-1) = (l+1, q-1)$, p divides $l+1$ which divides $|W(\tilde{G}^F)|$, a contradiction.

Suppose that $l+1 = 2$ and $q = 2^b - 1$. By [19, I, Excercise 7, p.417], we can find maximal subgroups of $A_1(q) \simeq L_2(q)$. We have $N_G(S) = S$ for $S \in Syl_2(G)$. This implies that $p = 2$, a contradiction.

$G \simeq B_l(q)$ $(l \geq 2)$: $|\tilde{G}^F| = q^{l^2} \prod_{i=1}^{l}(q^{2i} - 1)$ and $\tilde{H}^F \simeq Z_{q-1} \times \cdots \times Z_{q-1}$ (l times). Then p divides $q-1$ and therefore $q > 2$. By [1, G] or [3], $\tilde{G}^F$ has a maximal torus T of type $T(B_l)$ whose order is $q^l + 1$. By Lemma 1, there exists a prime r such that r divides $q^{2l} - 1$, r does not divide $q^i - 1$ $(1 \leq i \leq 2l-1)$ and r does not divide $2l$ except for the cases where $2l = 2$ and $q = 2^b - 1$ or $2l = 6$ and $q = 2$. Then we see that $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since $q(q^2-1)$ is divisible by 2, we have $r \neq 2$. Since the torsion prime for type B_l is 2, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. Since p does not divide $|W(\tilde{G}^F)| = |W(\tilde{G})| = 2^l \cdot l!$, P is contained in $C_{\tilde{G}^F}(R)$ for some $P \in Syl_p(\tilde{G}^F)$. Then $PR = P \times R$ is abelian and $P \times R$ is contained in some maximal torus T_1 by Lemma 5. Since $T(B_l)$ is the only type of maximal torus whose order is divisible by r, we have $|T_1| = q^l + 1$. This yields that p divides $(q^l+1,\ q-1)$. Since $q-1$ divides $q^l - 1$, $(q^l+1,\ q-1) = 1$ or 2. This is a contradiction.

$G \simeq C_l(q)$ $(l \geq 2)$: By an argument similar to that in the case where $G \simeq B_l(q)$, there is no prime p for the groups $G \simeq C_l(q)$ satisfying $(n_r(G),\ p) = 1$ for any $r \in \pi(G)$.

$G \simeq D_l(q)$ $(l \geq 4)$: $|\tilde{G}^F| = q^{l(l-1)}(q^l - 1)\prod_{i=1}^{l-1}(q^{2i} - 1)$ and $\tilde{H}^F \simeq Z_{q-1} \times \cdots \times Z_{q-1}$ (l times). Then p divides $q-1$ and $q > 2$. By [1, G] or [3], $\tilde{G}^F$ has a maximal torus T of type $T(D_l)$ whose order is $(q^{l-1}+1)(q+1)$. By Lemma 1, there exists a prime r such that r divides $q^{2(l-1)} - 1$, r does not divide $q^i - 1$ $(1 \leq i \leq 2l-3)$ and r does not divide $2(l-1)$ except for the cases where $2(l-1) = 2$ and $q = 2^b - 1$

or $2(l-1)=6$ and $q=2$. We see that $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since $q(q^2-1)$ is divisible by 2, we have $r \neq 2$. Since the torsion prime for type D_l is 2, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. Since p does not divide $|W(\tilde{G}^F)| = |W(\tilde{G})| = 2^{l-1} \cdot l!$, P is contained in $C_{\tilde{G}^F}(R)$ for some $P \in Syl_p(\tilde{G}^F)$. Then $PR = P \times R$ is abelian and $P \times R$ is contained in some maximal torus T_1 by Lemma 5. Since $T(D_l)$ is the only type of maximal torus whose order is divisible by r, we have $|T_1| = (q^{l-1}+1)(q+1)$. This yields that p divides $((q^{l-1}+1)(q+1),\ q-1) = 1$, 2 or 4, a contradiction.

$G \simeq E_6(q)$: $|\tilde{G}^F| = q^{36}(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)$ and $\tilde{H}^F \simeq Z_{q-1} \times \cdots \times Z_{q-1}$ (6 times). Then p divides $q-1$. By [1, G] or [3], $\tilde{G}^F$ has a maximal torus T of type $T(E_6)$ whose order is $(q^4-q^2+1)(q^2+q+1)$. By Lemma 1, there exists a prime r such that r divides $q^{12}-1$, r does not divide q^i-1 $(1 \le i \le 11)$ and r does not divide 12. Since $q^{12}-1 = (q^6-1)(q^2+1)(q^4-q^2+1)$, r divides q^4-q^2+1 and $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since the torsion primes for type E_6 are 2 and 3, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. Since p does not divide $|W(\tilde{G}^F)| = |W(\tilde{G})| = 2^7 \cdot 3^4 \cdot 5$, P is contained in $C_{\tilde{G}^F}(R)$. Then $PR = P \times R$ is abelian and $P \times R$ is contained in some maximal torus T_1 by Lemma 5. Since $T(E_6)$ is the only type of maximal torus whose order is divisible by r, we have $|T_1| = (q^4-q^2+1)(q^2+q+1)$. This yields that p divides $((q^4-q^2+1)(q^2+q+1),\ q-1) = 1$ or 3, a contradiction.

$G \simeq E_7(q)$: $|\tilde{G}^F| = q^{63}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)$ and $\tilde{H}^F \simeq Z_{q-1} \times \cdots \times Z_{q-1}$ (7 times). Then p divides $q-1$. By [1, G] or [3], $\tilde{G}^F$ has a maximal torus T of type $T(E_7)$ whose order is $(q^6-q^3+1)(q+1)$. By Lemma 1, there exists a prime r such that r divides $q^{18}-1$, r does not divide q^i-1 $(1 \le i \le 17)$ and r does not divide 18. Since $q^{18}-1 = (q^9-1)(q^3+1)(q^6-q^3+1)$, r divides q^6-q^3+1 and $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since the torsion primes for type E_7 are 2 and 3, we have $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. Since p does not divide $|W(\tilde{G}^F)| = |W(\tilde{G})| = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$, P is contained in $C_{\tilde{G}^F}(R)$. Then $PR = P \times R$ is abelian and $P \times R$ is contained in some maximal torus T_1 by Lemma 5. Since $T(E_7)$ is the only type of maximal torus whose order is divisible by r, we have $|T_1| = (q^6-q^3+1)(q+1)$. This yields that p divides $((q^6-q^3+1)(q+1),\ q-1) = 1$ or 2, a contradiction.

$G \simeq E_8(q)$: $|\tilde{G}^F| = q^{120}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)$ and $\tilde{H}^F \simeq Z_{q-1} \times \cdots \times Z_{q-1}$ (8 times). Then p divides $q-1$. By [1, G] or [3], $\tilde{G}^F$ has a maximal torus T of type $T(E_8)$ whose order is $q^8+q^7-q^5-q^4-q^3+q+1$. By Lemma 1, there exists a prime r such that r divides $q^{30}-1$, r does not divide q^i-1 $(1 \le i \le 29)$ and r does not divide 30. Since $q^{30}-1 = (q^{15}-1)(q^5+1)(q^2+q+1)(q^8+q^7-q^5-q^4-q^3+q+1)$, r divides $q^8+q^7-q^5-q^4-q^3+q+1$ and $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since the torsion primes for type E_8 are 2, 3 and 5 and $q(q^8-1)$ is divisible by 30, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma

5. Since p does not divide $|W(\tilde{G}^F)| = |W(\tilde{G})| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$, P is contained in $C_{\tilde{G}^F}(R)$. Then $PR = P \times R$ is abelian and $P \times R$ is contained in some maximal torus T_1 by Lemma 5. Since $T(E_8)$ is the only type of maximal torus whose order is divisible by r, we have $|T_1| = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$. This yields that p divides $(q^8 + q^7 - q^5 - q^4 - q^3 + q + 1,\ q - 1) = 1$, a contradiction.

$G \simeq F_4(q)$: $|\tilde{G}^F| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$ and $\tilde{H}^F \simeq Z_{q-1} \times \cdots \times Z_{q-1}$ (4 times). Then p divides $q-1$. By [1, G] or [3], $\tilde{G}^F$ has a maximal torus T of type $T(F_4)$ whose order is $q^4 - q^2 + 1$. By Lemma 1, there exists a prime r such that r divides $q^{12} - 1$, r does not divide $q^i - 1$ $(1 \leq i \leq 11)$ and r does not divide 12. Since $q^{12} - 1 = (q^6 - 1)(q^2 + 1)(q^4 - q^2 + 1)$, r divides $q^4 - q^2 + 1$ and $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since the torsion primes for type F_4 are 2 and 3, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. Since p does not divide $|W(\tilde{G}^F)| = |W(\tilde{G})| = 2^7 \cdot 3^2$, P is contained in $C_{\tilde{G}^F}(R)$. Then $PR = P \times R$ is abelian and $P \times R$ is contained in some maximal torus T_1 by Lemma 5. Since $T(F_4)$ is the only type of maximal torus whose order is divisible by r, we have $|T_1| = q^4 - q^2 + 1$. This yields that p divides $(q^4 - q^2 + 1,\ q - 1) = 1$, a contradiction.

$G \simeq G_2(q)$: We see that $|\tilde{G}^F| = q^6(q^2-1)(q^6-1)$ and $\tilde{H}^F \simeq Z_{q-1} \times Z_{q-1}$. Then p divides $q-1$. By [1, G] or [3], $\tilde{G}^F$ has a maximal torus T of type $T(G_2)$ whose order is $q^2 - q + 1$. By Lemma 1, there exists a prime r such that r divides $q^6 - 1$, r does not divide $q^i - 1$ $(1 \leq i \leq 5)$ and r does not divide 6 except for the case where $q = 2$. Note that $G_2(2)$ is not simple. We see that $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since the torsion prime for type G_2 is 2, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. Since p does not divide $|W(\tilde{G}^F)| = |W(\tilde{G})| = 12$, P is contained in $C_{\tilde{G}^F}(R)$. Then $PR = P \times R$ is abelian and $P \times R$ is contained in some maximal torus T_1 by Lemma 5. Since $T(G_2)$ is the only type of maximal torus whose order is divisible by r, we have $|T_1| = q^2 - q + 1$. This yields that p divides $(q^2 - q + 1,\ q - 1) = 1$, a contradiction.

$G \simeq {}^2A_l(q)$ $(l \geq 3)$: $|\tilde{G}^F| = q^{l(l+1)/2} \prod_{i=2}^{l+1}(q^i - (-1)^i)$.

If l is even, then $|\tilde{H}^F| = (q-1)^{l/2}(q+1)^{l/2}$. Since $(q+1)^l$ divides $|\tilde{G}^F|$, p divides $q-1$. By [1, E.II.1.10, p.188–190], we have the orders of maximal tori of $\tilde{G}^F$. There is a maximal torus T of order $q^{l+1} + 1/q + 1$. By Lemma 1, there exists a prime r such that r divides $q^{2(l+1)} - 1$, r does not divide $q^i - 1$ $(1 \leq i \leq 2(l+1) - 1)$ and r does not divide $2(l+1)$ except for the cases where $2(l+1) = 2$ and $q = 2^b - 1$ or $2(l+1) = 6$ and $q = 2$. We see that $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since there exist no torsion primes for type A_l by Lemma 4, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. Note that $W(\tilde{G}^F) \neq W(\tilde{G})$. Since $l \geq 4$, there exist $x \in P$ and $y \in R$ such that $[x,\ y] = 1$. Put $z = xy$. By Lemma 5, there is a maximal torus T_1 containing z. Since there is only a type of maximal torus whose order is divisible by r, we have $|T_1| = (q^{l+1})/(q+1)$. This yields that p divides $((q^{l+1} + 1)/(q+1),\ q - 1) = 1$ or 2, a contradiction.

If l is odd, then $|\tilde{H}^F| = (q-1)^{(l+1)/2}(q+1)^{(l-1)/2}$. Since $(q+1)^l$ divides $|\tilde{G}^F|$, p divides $q-1$. There is a maximal torus T of order q^l+1. By Lemma 1, there exists a prime r such that r divides $q^{2l}-1$, r does not divide q^i-1 $(1 \le i \le 2l-1)$ and r does not divide $2l$ except for the cases where $2l = 2$ and $q = 2^b - 1$ or $2l = 6$ and $q = 2$. Since r does not divide $q^{l+1}-1$ we see that $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since there exist no torsion primes for type A_l by Lemma 4, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. Since $l \ge 3$, there exist $x \in P$ and $y \in R$ such that $[x,\ y] = 1$. Put $z = xy$. By Lemma 5, there is a maximal torus T_1 containing z. Since there is only a type of maximal torus whose order is divisible by r, we have $|T_1| = q^l + 1$. This yields that p divides $(q^l+1,\ q-1) = 1$ or 2, a contradiction.

If $2l = 6$ and $q = 2$, then $G \simeq {}^2A_3(2) \simeq C_2(3)$. It is already discussed.

$G \simeq {}^2B_2(q)$: $|{}^2B_2(q)| = q^2(q^2+1)(q-1)$. By [18], we have all maximal subgroups of ${}^2B_2(q)$. Since $|N_G(Q)| = q^2(q-1)$ for some $Q \in Syl_2(G)$, p divides $q-1$. By Lemma 1, there exists a prime r such that r divides q^4-1, r does not divide q^i-1 $(1 \le i \le 3)$ and r does not divide 4. Then r divides either $q+\sqrt{2q}+1$ or $q-\sqrt{2q}+1$. We see that $R \in Syl_r(G)$ is contained in either a cyclic group A_1 of order $q+\sqrt{2q}+1$ or a cyclic group A_2 of order $q-\sqrt{2q}+1$ and $N_G(R)$ is contained in either $N_G(A_1)$ of order $4(q+\sqrt{2q}+1)$ or $N_G(A_2)$ of order $4(q-\sqrt{2q}+1)$. In both cases, p divides q^2+1. This yields that p divides $(q^2+1,\ q-1) = 1$ or 2, a contradiction.

$G \simeq {}^2D_l(q)$ $(l \ge 4)$: $|\tilde{G}^F| = q^{l(l-1)}(q^l+1)\prod_{i=1}^{l-1}(q^{2i}-1)$ and $|\tilde{H}^F| = (q-1)^{l-1}(q+1)$. Then p divides $p-1$. There is a maximal torus T of order q^l+1. By Lemma 1, there exists a prime r such that r divides $q^{2l}-1$, r does not divide q^i-1 $(1 \le i \le 2l-1)$ and r does not divide $2l$ except for the cases where $2l = 2$ and $q = 2^b-1$ or $2l = 6$ and $q = 2$. Since $l \ge 4$, we do not have to consider the exceptional cases. We see that $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since the torsion prime for type D_l is 2 by Lemma 4, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. There exist $x \in P$ and $y \in R$ such that $[x,\ y] = 1$. Put $z = xy$. By Lemma 5, there is a maximal torus T_1 containing z. Since there is only a type of maximal torus whose order is divisible by r, we have $|T_1| = q^l+1$. This yields that p divides $(q^l+1,\ q-1) = 1$ or 2, a contradiction.

$G \simeq {}^3D_4(q)$: We have that $|{}^3D_4(q)| = q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$. By [13], we have all maximal subgroups of ${}^3D_4(q)$. Since $|N_G(Q)| = q^{12}(q^3-1)(q-1)$ for some $Q \in Syl_u(G)$ and q^2+q+1 divides q^8+q^4+1, p divides $q-1$. There is a maximal torus T of order q^4-q^2+1. By Lemma 1, there exists a prime r such that r divides $q^{12}-1$, r does not divide q^i-1 $(1 \le i \le 11)$ and r does not divide 12. Since r divides q^4-q^2+1, we see that $R \subseteq T$ for some $R \in Syl_r(G)$ and $N_G(R) \subseteq N_G(T)$, whose order is $4(q^4-q^2+1)$. This yields that p divides $(q^4-q^2+1,\ q-1) = 1$, a contradiction.

$G \simeq {}^2E_6(q)$: $|\tilde{G}^F| = q^{36}(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1$ and $|\tilde{H}^F| = (q-1)^4(q+1)^2$. We see that P is abelian and p divides $p-1$. There is a maximal torus T of order $(q^4-q^2+1)(q^2-q+1)$. By Lemma 1, there exists a prime r such that r divides $q^{12}-1$, r does not divide q^i-1 $(1 \leq i \leq 11)$ and r does not divide 12. We see that r divides q^4-q^2+1 and $R \subseteq T$ for some $R \in Syl_r(\tilde{G}^F)$. Since the torsion primes for type E_6 are 2 and 3, it follows that $N_{\tilde{G}^F}(R)/C_{\tilde{G}^F}(R) \subseteq W(\tilde{G})$ by Lemma 5. There exist $x \in P$ and $y \in R$ such that $[x, y] = 1$. Put $z = xy$. By Lemma 5, there is a maximal torus T_1 containing z. Since there is only a type of maximal torus whose order is divisible by r, we have $|T_1| = (q^4-q^2+1)(q^2-q+1)$. This yields that p divides $(q^4-q^2+1)(q^2-q+1)$, $q-1) = 1, 2$ or 4, a contradiction.

$G \simeq {}^2F_4(q)$ $(q > 2)$: $|{}^2F_4(q)| = q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1)$. By [16], we have all maximal subgroups of ${}^2F_4(q)$. Since $|N_G(Q)| = q^{12}(q-1)^2$ for some $Q \in Syl_2(G)$, p divides $q-1$. By Lemma 1, there exists a prime r such that r divides $q^{12}-1$, r does not divide q^i-1 $(1 \leq i \leq 11)$ and r does not divide 12. Then r divides either $q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$ or $q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$ since $q^{12}-1 = (q^6-1)(q^2+1)(q^4-q^2+1) = (q^6-1)(q^2+1)(q^2+\sqrt{2q^3}+q+\sqrt{2q}+1)(q^2-\sqrt{2q^3}+q-\sqrt{2q}+1)$. We see that $R \in Syl_r(G)$ is contained in either a cyclic group of order $q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$ or a cyclic group of order $q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$ and $N_G(R)$ is contained in either a maximal subgroup of order $12(q^2+\sqrt{2q^3}+q+\sqrt{2q}+1)$ or a maximal subgroup of order $12(q^2-\sqrt{2q^3}+q-\sqrt{2q}+1)$. In both cases, p divides q^6+1 since $|{}^2F_4(q)|$ is divisible by 9. This yields that p divides $(q^6+1,\ q-1) = 1$ or 2, a contradiction.

$G \simeq {}^2F_4(2)'$: There is no prime p satisfying $(n_r(G),\ p) = 1$ for any $r \in \pi(G)$ since $N_G(S) = S$ for $S \in Syl_2(G)$.

$G \simeq {}^2G_2(q)$: $|{}^2G_2(q)| = q^3(q^3+1)(q-1)$. By [14], we have all maximal subgroups of ${}^2G_2(q)$. Since $|N_G(Q)| = q^3(q^2-1)$ for some $Q \in Syl_3(G)$, p divides $q-1$. By Lemma 1, there exists a prime r such that r divides q^6-1, r does not divide q^i-1 $(1 \leq i \leq 5)$ and r does not divide 6 since q is a power of 3. Then r divides either $q+\sqrt{3q}+1$ or $q-\sqrt{3q}+1$. We see that $R \in Syl_r(G)$ is contained in either a cyclic group A_1 of order $q+\sqrt{3q}+1$ or a cyclic group A_2 of order $q-\sqrt{3q}+1$ and $N_G(R)$ is contained in either $N_G(A_1)$ of order $6(q+\sqrt{3q}+1)$ or $N_G(A_2)$ of order $6(q-\sqrt{3q}+1)$. In both cases, p divides q^3+1. This yields that p divides $(q^3+1,\ q-1) = 2$, a contradiction.

Proof of the Main Theorem. (i) It follows from Theorem 1, Propositions 1, 2, 3, 4 and 5.

(ii) It follows from Propositions 1, 2, 3, 4 and 5.

To prove Corollary 1, we need the following:

Proposition 6. *Let G be a finite group. Suppose that for every $r \in \pi(G)$, $(p,\ n_r(G)) = 1$ holds and $N_G(R)$ is p-nilpotent for $R \in Syl_r(G)$. Then G is p-nilpotent.*

Proof. Suppose that the assertion is false. Take G a counterexample of minimal possible order. Then G is simple by Proposition 2. Propositions 3, 4 and 5 yields that $G \simeq U_3(q)$ $q = 2^f$, f is even not divisible by 3 and $p = 3$. In the proof of

Theorem 1, we have that $N_G(Q_3) = (q+1) \times D_{2(q-1)}$, which is not 3-nilpotent. This completes the proof.

Proof of Corollary 1. It follows from Propositions 1 and 6.

4. Sylow Graphs of Finite Groups

As applications of the Claim in the Introduction, a couple of claims are made in [20], about which we will discuss below. Note that the Claim is not necessary for the proofs of Theorem 2 and of Theorem 3.

The Sylow graph $\Gamma_s(G)$ of a finite group G is defined as follows. The vertices $V(\Gamma_s(G))$ of $\Gamma_s(G)$ are the set of prime divisors of any Sylow numbers of G and two vertices p and q are joined by an edge if pq divides some Sylow Sylow numbers of G.

Note that $V(\Gamma_s(G))$ is not always equal to $\pi(G)$. In fact, let $G = U_3(q)$, $q = 2^f$, f even not divisible by 3. Then we see that $V(\Gamma_s(G)) \subsetneq \pi(G)$ by Theorem 1.

Theorem 2 ([20, Theorem 3]). *If G is not a finite group with a non-connected Sylow graph then G is not simple.*

Proof. It is easily seen that $\Gamma_s(G)$ is connected for any sporadic simple group G. See [4].

The Sylow graphs of the alternating groups A_5 is connected. See [4]. Assume that $n \geq 6$. Take l the largest prime in $\pi(A_n)$. Then $|N_{A_n}(L)| = l \cdot (l-1) \cdot (n-l)!/2$. This yields that $n_l(A_n)$ is divisible by 2. If there exists an odd prime $r \in \pi(A_n)$ such that r does not divide $n_l(G)$, then there exists Hall $\{l, r\}$-subgroup of A_n. This contradicts [10, Theorem A4]. This yields that any odd prime in $\pi(A_n)$ not equal to l divides $n_l(A_n)$. By [7], $n_p(G) \neq l$ for any $p \in \pi(A_n)$ since $n \geq 5$. Hence $\Gamma_s(A_n)$ is connected for $n \geq 6$.

Suppose that G is a simple group of Lie type over $GF(q)$, $q = u^f$, u a prime. By [6, Theorem 4.253, p.313], $U \in Syl_u(G)$ has a trivial signalizer. This yields that $n_p(G)$ is divisible by u if $p \neq u$. Suppose that any r dividing $n_u(G)$ does not divide $n_p(G)$ for any $p \neq u$. Then $\{r \in \pi(G) | r \text{ divides } n_p(G),\ p \neq u\} \subseteq \pi(B)$, where $B = UH$ is the Borel subgroup of G. Suppose that $\pi(H) \neq \varnothing$. Take $r \in \pi(H)$. Then $N_G(R) \supseteq H$ since H is abelian. Since $|G : N_G(R)|$ is not a prime power by [7], there exists a prime $t \neq u$ which divides $n_r(G)$. This is a contradiction since $t \notin \pi(B)$. This yields that $\pi(B) = \{u\}$. Then $n_p(G)$ is a power of u for any $p \neq u$. This contradicts [7]. This implies that there exists a prime r dividing $n_u(G)$ such that r divides $n_p(G)$ for some $p \neq u$. Hence $\Gamma_s(G)$ is connected. Suppose that $\pi(H) = \varnothing$. Then $q = 2$. If $\Gamma_s(G)$ is disconnected, $n_r(G)$ must be a power of 2 for any odd prime $r \in V(\Gamma_s(G))$. This contradicts [7]. This completes the proof.

This implies the following:

Theorem 3 ([20, Theorem 4]). *Let G be a finite group such that all Sylow numbers of G are prime power. Then G is solvable.*

Proof. It follows from Lemma 2 and Theorem 2.

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