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Coding-Theoretic Approaches to Classical and Quantum Predictive Games in Game-Theoretic Probability

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Chapter 1

Introduction

The standard mathematical framework for describing random phenomena is probability theory, founded by Kolmogorov on a measure-theoretic foundation. Within this framework, statistical properties of random variables are typically formulated as events that occur with probability one. While this formulation provides a powerful and internally consistent mathematical theory, it treats probability as a primitive concept rather than explaining how probabilistic behavior arises operationally.

An alternative foundational approach is provided by *game-theoretic probability theory*, advocated by Shafer and Vovk [1]. This framework departs fundamentally from measure-theoretic probability by avoiding any a priori assumption of an underlying probability measure. Instead, it describes statistical behaviors through an asymmetric sequential betting game involving three players: Forecaster, Skeptic, and Reality.

In each round of the game, Forecaster announces odds for bets on Reality's next move, Skeptic selects a betting strategy based on the past history of Reality, and Reality then reveals the outcome, *after* observing Skeptic's bet. At first sight, this protocol appears to place Skeptic at a severe disadvantage, since Reality moves last and can respond adversarially. Nevertheless, Shafer and Vovk established a remarkable result [1]: there exists a prudent betting strategy by which Skeptic can force unbounded capital growth unless Reality's behavior conforms to the law of large numbers. In this sense, randomness is no longer postulated as an underlying assumption but is instead imposed as a constraint on Reality's permissible behavior.

This shift in perspective yields two fundamental consequences. First, probabilistic laws such as the law of large numbers are reinterpreted as statements about the existence of explicit betting strategies for Skeptic. Rather than asserting that a property holds almost surely under a given probability measure, game-theoretic probability asserts that any persistent violation of such a law can be exploited by a suitable strategy to yield unbounded capital growth. Second, probability itself is no longer imposed a priori but emerges as an operational concept from the betting

protocol itself.

A key feature of game-theoretic probability is that Skeptic’s capital process admits a natural information-theoretic interpretation. When Forecaster announces odds for Reality’s next move, Skeptic may adopt an alternative predictive distribution based on the observed past. The resulting capital update compares these two predictive descriptions and the evolution of the capital process can be rewritten in terms of the likelihood ratio between the Skeptic’s prediction and the Forecaster’s announced odds.

From this viewpoint, the logarithm of Skeptic’s capital can be interpreted as the difference between two codelengths: the Shannon codelength associated with Forecaster’s announcement and the codelength induced by Skeptic’s alternative predictive scheme. Capital growth thus corresponds to cumulative gain in compression. In this sense, the capital process in game-theoretic probability possesses an inherently coding-theoretic aspect.

This observation reveals a close connection between game-theoretic probability and coding theory. The existence of a betting strategy where Skeptic forces Reality into statistical behavior is closely related to the existence of a coding method that describes Reality’s sequence more efficiently than the code implicit in the Forecaster’s announcement.

However, optimal compression in the sense of Kolmogorov complexity is in general not computable. This fundamental limitation motivates the use of computable universal coding schemes as practical substitutes, which in turn lead naturally to coding-theoretic betting strategies within game-theoretic probability.

This coding-theoretic interpretation raises a natural question: what consequences arise in game-theoretic probability when betting strategies derived from universal coding schemes are employed? Specifically, if Reality’s behavior deviates from the statistical properties implicit in Forecaster’s announcement, a universal coding strategy can exploit this deviation by achieving shorter descriptions of Reality’s sequence. As a result, Skeptic can force unbounded growth of capital while maintaining prudence.

In this dissertation, we develop a coding-theoretic perspective on game-theoretic probability. This perspective provides a unified way to analyze classical results in game-theoretic probability and serves as a foundation for exploring extensions to new settings. We formulate betting strategies based on explicit algorithmic constructions. Furthermore, we investigate how structural insights from coding theory can inform both classical and quantum predictive games.

Part I of this thesis aims to elucidate the coding-theoretic aspects underpinning game-theoretic probability by presenting an alternative proof of the game-theoretic law of large numbers and its generalizations. Our approach offers new insights into game-theoretic probability by employing a single, coding-theoretic (pure) strategy. This stands in contrast to Shafer and Vovk’s original proof, which relies on a diver-

sified (“mixed”) betting strategy using countably many operating accounts [1], as well as to subsequent alternative proofs based on statistical estimation [2, 3]. We also note that Feder [4] previously proposed a closely related strategy. Although the aims and scope of that work differ from ours, we cite it for completeness and to acknowledge the prior contribution.

We adopt this coding-theoretic perspective and examine strategies based on Lempel–Ziv universal coding [5]. Specifically, we show that the game-theoretic law of large numbers holds for arbitrary finite-length sequences under this strategy in a time-homogeneous Markovian predictive game, where the betting odds are determined by a k -th order Markov kernel. Furthermore, this coding-theoretic formulation reveals that entropy emerges naturally as a central quantity in the game. This observation establishes a direct conceptual link between game-theoretic probability and physics, in particular thermodynamics. As an illustration, we provide a game-theoretic formulation of Szilárd’s engine [6], a paradigmatic thought experiment in statistical mechanics. Within this framework, the second law of thermodynamics admits an alternative interpretation, highlighting aspects that are specific to the game-theoretic probabilistic viewpoint.

Part II examines possible extensions of game-theoretic probability to quantum settings. Quantum information theory, while extending classical probability theory, inherently lacks a single underlying probability space and is fundamentally non-commutative. These features pose both conceptual and technical challenges for any direct game-theoretic formulation, making quantum extensions a natural yet non-trivial research direction.

We investigate how quantum states and measurements can be incorporated into predictive games, and examine both the scope and the limitations of such extensions. We focus on two representative topics. First, we consider a predictive game in which Forecaster announces an extension from a classical probability distribution p to a quantum state ρ . Using the framework of mutually unbiased bases, we show that a law of large numbers holds in situations that remain close to the classical setting. While this result does not yet constitute a fully quantum game-theoretic framework, it suggests directions for further extensions, as well as criteria for what would qualify as genuinely quantum predictive games.

Second, we investigate a game-theoretic formulation of the CHSH inequality [7], a prominent variant of Bell’s inequality [8], which characterizes correlations compatible with local hidden variable models. Although numerous experimental tests of the CHSH inequality have been performed, several loopholes have been identified, and their closure remains both theoretically and experimentally significant. This dissertation reinterprets these loopholes within the framework of game-theoretic probability, revealing structural aspects that are less transparent in traditional measure-theoretic formulations.

This thesis is organized as follows. In Part I, we study a coding theoretic ap-

proach to time-homogeneous Markovian-predictive game. Chapter 2 introduces a fundamental result from game-theoretic probability theory and motivates the study of predictive game from coding theoretic perspective. Chapter 3 presents a betting strategy based on the incremental parsing scheme of Ziv and Lempel [5], and establishes several lemmas that lay the groundwork for proving the main result. In Chapter 4, we prove the main result by incorporating properties of Lempel-Ziv incremental parsing established in the previous chapter, thereby bringing ideas inspired by universal coding into game-theoretic probability. Chapter 5 explores applications of main theorem to thermodynamics, specifically a game-theoretic interpretation of Szilárd’s engine and a discussion of entropy in predictive games. For the reader’s convenience, additional information on stationary distributions of Markov chains and an alternative proof of Theorem 2.1 are provided in Appendices A and B, respectively.

In Part II, we explore extensions of game-theoretic probability to quantum settings and applications to quantum foundations. Chapter 6 introduces a quantum analogue of the predictive game and establishes a quantum version of the game-theoretic law of large numbers under slowly varying measurement settings. We also discuss the limitations of this framework and propose a generalized quantum predictive game in which both the predicted state and the choice of measurements are treated as strategic choices. Chapter 7 revisits the CHSH inequality from an information-structural perspective. Section 7.1 provides a brief review of the CHSH inequality, while Section 7.2 introduces the necessary preliminaries for translating the CHSH scenario into the game-theoretic probability framework. Section 7.3 presents the main results, formalizing the locality and measurement-dependence (“freedom-of-choice”) loopholes as structural constraints in sequential games and demonstrating how these conditions determine the validity or violation of the CHSH inequality.

Finally, Chapter 8 provides concluding remarks, summarizing the main contributions of the thesis and discussing open problems and directions for future research.

Part I

Coding Theoretic Study of Homogeneous Markovian Predictive Games

Chapter 2

A Coding-Theoretic View of Game-Theoretic Probability

Game-theoretic probability theory [1], proposed by Shafer and Vovk in 2001, offers a framework for studying stochastic behavior without relying on traditional concept of probability. To illustrate this approach, we begin by recalling a fundamental result from game-theoretic probability theory.

Let $\Omega := \{1, 2, \dots, A\}$ be a finite alphabet, and let Ω^n , Ω^* , and Ω^∞ denote the sets of sequences over Ω of length n , finite length, and infinite (one-sided) length, respectively. The empty string is denoted by λ . An element of Ω^n is represented symbolically as ω^n . We also introduce the notation $\omega_i^j := \omega_i \omega_{i+1} \cdots \omega_j$ for $i \leq j$, denoting the substring from the i th to the j th coordinates of a longer sequence $\omega_1 \omega_2 \cdots \omega_n \cdots$. By convention, if $i > j$, we set $\omega_i^j := \lambda$. For $x \in \Omega^*$ and $y \in \Omega^* \cup \Omega^\infty$, we write $x \sqsubset y$ to indicate that x is a prefix of y .

Let us introduce the following set:

$$\mathcal{P}(\Omega) := \left\{ p : \Omega \rightarrow (0, 1) \mid \sum_{\omega \in \Omega} p(\omega) = 1 \right\}.$$

Given a $p \in \mathcal{P}(\Omega)$, consider the following game.

Simple predictive game

Players : Skeptic and Reality.

Protocol : $K_0 = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

Skeptic announces $\beta_n \in \mathbb{R}^\Omega$.

Reality announces $\omega_n \in \Omega$.

$$K_n := K_{n-1} + \left\{ \beta_n(\omega_n) - \sum_{a \in \Omega} \beta_n(a) p(a) \right\}.$$

END FOR.

This protocol can be understood as a betting game in which Skeptic predicts Reality's "stochastic" move, regarding $p(a)$ as the "probability" of the occurrence of $a \in \Omega$. Further, β_n and K_n denote Skeptic's bet and capital at step n , respectively, with the recursion formula specifying how the capital evolves. Namely, the formula states that, at step n , after Skeptic announces $\beta_n = (\beta_n(1), \beta_n(2), \dots, \beta_n(A))$ and Reality announces $\omega_n \in \Omega$, Skeptic gains $\beta_n(\omega_n)$ and loses $\sum_{a \in \Omega} \beta_n(a) p(a)$ in assets. This loss term $\sum_{a \in \Omega} \beta_n(a) p(a)$ reflects the "expected value" of the Skeptic's stakes under the odds p and can be regarded as a participation cost that must be paid in advance, regardless of the actual outcome. Note that β_n can take negative values. Since β_n can depend on Reality's past move ω_1^{n-1} , we identify Skeptic's strategy $\{\beta_n\}_n$ with a map $\beta : \Omega^* \rightarrow \mathbb{R}^\Omega$ as $\beta_n(a) := (\beta(\omega_1^{n-1}))(a)$.

Apparently, this game is in favor of Reality because Reality announces ω_n *after* knowing Skeptic's bet β_n , preventing Skeptic from becoming rich. However, Shafer and Vovk showed the following surprising result.

Theorem 2.1 (Game-theoretic law of large numbers). *In the simple predictive game, Skeptic has a prudent strategy $\beta : \Omega^* \rightarrow \mathbb{R}^\Omega$ that ensures $\lim_{n \rightarrow \infty} K_n = \infty$ unless*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_a(\omega_i) = p(a)$$

for all $a \in \Omega$, where δ_a denotes the Kronecker delta. Here, a strategy is called prudent if $K_n > 0$ for all $n \in \mathbb{Z}_{>0}$ and every sequence $\omega_1^n \in \Omega^n$ chosen by Reality.

The theorem implies that there exists a betting strategy β_n that guarantees Skeptic becomes infinitely rich if Reality's moves deviate from the "law of large numbers," all while avoiding the risk of bankruptcy¹. Note that Skeptic's capital may diverge even while the empirical frequencies of Reality's moves converge; the two phenomena are not mutually exclusive. For more information, see Appendix B.

¹If Reality's objective is to prevent Skeptic from becoming infinitely rich, then Theorem 2.1 can be rephrased as follows: *Skeptic can force the event $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_a(\omega_i) = p(a)$, which corresponds to the original statement by Shafer and Vovk [1].*

After the publication of Shafer and Vovk's seminal book [1], game-theoretic probability theory has been widely studied across various contexts and applications. Notable developments include the game-theoretic proof of Lévy's zero-one law [9, 10], investigations into Reality's strategic choices [11, 12], and the game-theoretic formulation of Jeffrey's law [13], among others. These contributions are further extended in Shafer and Vovk's second book [14], which builds upon the foundations established in the first, deepening the analysis and exploring new frontiers in game-theoretic probability. In recent years, many studies inspired by game-theoretic probability have explored new applications, including statistical inference [15], statistical testing [16] and machine learning [17]. These works have significantly advanced the field, addressing both theoretical and practical challenges within the framework of game-theoretic probability. However, despite these developments, the connection between game-theoretic probability theory and information theory, particularly in the context of universal coding, has received limited attention; see, for example, [18, 19] for related work primarily addressing sequential testing problems.

In order to explicate our coding theoretic approach, consider the following generalized game in which a Forecaster comes into play to announce a possibly “non-i.i.d.” process.

Generalized predictive game

Players : Forecaster, Skeptic, and Reality.

Protocol : $K_0 = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

Forecaster announces $p_n \in \mathcal{P}(\Omega)$.

Skeptic announces $\beta_n \in \mathbb{R}^\Omega$.

Reality announces $\omega_n \in \Omega$.

$$K_n := K_{n-1} + \left\{ \beta_n(\omega_n) - \sum_{a \in \Omega} \beta_n(a) p_n(a) \right\}.$$

END FOR.

Let us identify Skeptic's betting strategy $\beta_n \in \mathbb{R}^\Omega$ with $\alpha_n \in \mathbb{R}^\Omega$ that satisfies

$$\beta_n(a) = K_{n-1} \cdot \alpha_n(a), \quad (a \in \Omega).$$

Then, the recursion formula for the capital is rewritten as

$$K_n = K_{n-1} \left\{ 1 + \alpha_n(\omega_n) - \sum_{a \in \Omega} \alpha_n(a) p_n(a) \right\}. \quad (2.1)$$

We shall call α_n a betting strategy as well, and call it prudent if the corresponding β_n is prudent. We also identify $\{\alpha_n\}_n$ with a map $\alpha : \Omega^* \rightarrow \mathbb{R}^\Omega$ as $\alpha_n(a) = (\alpha(\omega_1^{n-1}))(a)$.

Now, associated with a prudent strategy α_n is the following quantity:

$$Q(\omega \mid \omega_1^{n-1}) := \left\{ 1 + \alpha_n(\omega) - \sum_{a \in \Omega} \alpha_n(a) p_n(a) \right\} p_n(\omega). \quad (2.2)$$

Since α_n is prudent, we see that $Q(\omega \mid \omega_1^{n-1}) > 0$ for all $\omega \in \Omega$. Moreover,

$$\begin{aligned} \sum_{\omega \in \Omega} Q(\omega \mid \omega_1^{n-1}) &= \sum_{\omega \in \Omega} \left\{ 1 + \alpha_n(\omega) - \sum_{a \in \Omega} \alpha_n(a) p_n(a) \right\} p_n(\omega) \\ &= 1 + \sum_{\omega \in \Omega} \alpha_n(\omega) p_n(\omega) - \sum_{a \in \Omega} \alpha_n(a) p_n(a) = 1. \end{aligned}$$

Therefore, the quantity $Q(\omega \mid \omega_1^{n-1})$ defined by (2.2) can be regarded as a conditional probability. Conversely, for any conditional probability $Q(\omega \mid \omega_1^{n-1})$, there exists a prudent strategy α_n (although not unique) that satisfies (2.2): for instance, let $\alpha_n(a) := Q(a \mid \omega_1^{n-1})/p_n(a)$ for each $a \in \Omega$. Thus, the role of Skeptic in the above predictive game is regarded as announcing a conditional probability $Q(\omega \mid \omega_1^{n-1})$.

Now, suppose that Forecaster happens to have a predetermined probability measure P on $(\Omega^\infty, \mathcal{F})$, with $\mathcal{F} := \sigma(\{\Gamma_x\}_{x \in \Omega^*})$ being the σ -algebra generated by the cylinder sets $\Gamma_x := \{y \in \Omega^\infty : x \sqsubset y\}$, and announces each function p_n as the conditional probability, given the past data ω_1^{n-1} , as follows:

$$p_n(a) := P(a \mid \omega_1^{n-1}) := P(a \mid \Gamma_{\omega_1^{n-1}}), \quad (a \in \Omega).$$

Then, we have from (2.1) and (2.2) that

$$\frac{K_n}{K_{n-1}} = \left\{ 1 + \alpha_n(\omega) - \sum_{a \in \Omega} \alpha_n(a) p_n(a) \right\} = \frac{Q(\omega_n \mid \omega_1^{n-1})}{P(\omega_n \mid \omega_1^{n-1})},$$

and hence

$$K_n = K_0 \prod_{i=1}^n \frac{K_i}{K_{i-1}} = \frac{Q(\omega_1^n)}{P(\omega_1^n)}. \quad (2.3)$$

Put differently, the capital process K_n is nothing but the likelihood ratio process between P and Q . This perspective is consistent with the interpretation developed in [14, Section 10.5] and further elaborated in [20, Section 2.2 and 2.3].

Let $\mathcal{F}_n := \sigma(\{\Gamma_{x^n}\}_{x^n \in \Omega^n})$ for each $n \in \mathbb{Z}_{>0}$. Then the capital process (2.3) is a P -martingale relative to the natural filtration $\{\mathcal{F}_n\}_n$, in that

$$\mathbb{E} \left[\frac{Q(\omega_1^n)}{P(\omega_1^n)} \mid \mathcal{F}_{n-1} \right] = \sum_{\omega_n \in \Omega} \frac{Q(\omega_1^n)}{P(\omega_1^n)} P(\omega_n \mid \omega_1^{n-1}) = \sum_{\omega_n \in \Omega} \frac{Q(\omega_1^n)}{P(\omega_1^{n-1})} = \frac{Q(\omega_1^{n-1})}{P(\omega_1^{n-1})}.$$

It then follows from the martingale convergence theorem [21] that the capital process K_n converges almost surely to some nonnegative value under the probability measure P . Game-theoretic probability provides a reciprocal description of this mechanism by asserting that K_n can diverge “on a P -null set,” which, in the context of predictive games, is interpreted as “if Reality does not align with Forecaster P .”

Furthermore, due to (2.3), the logarithm of the capital process is written as

$$\log_A K_n = \{-\log_A P(\omega_1^n)\} - \{-\log_A Q(\omega_1^n)\}. \quad (2.4)$$

This expression is simply the difference between the Shannon codelengths for P and Q , evoking the notion of *randomness deficiency* in algorithmic randomness theory [22]. For a computable probability measure P on Ω^∞ , an infinite sequence $\omega_1^\infty \in \Omega^\infty$ is Martin-Löf P -random if and only if the sequence

$$-\log_A P(\omega_1^n) - \mathcal{K}(\omega_1^n) \quad (2.5)$$

of randomness deficiencies is bounded from above, where $\mathcal{K}(\omega_1^n)$ is the prefix Kolmogorov complexity of $\omega_1^n \in \Omega^*$. Obviously, (2.4) and (2.5) are similar in form, as both represent differences in codelengths. However, it is crucial to note that (2.5) contains an uncomputable quantity $\mathcal{K}(\omega_1^n)$. This observation motivates the design of the conditional probability Q in (2.2), or equivalently, a betting strategy α_n in the generalized predictive game, by employing ideas and techniques from computable universal coding schemes.

Nevertheless, due to the difficulty of addressing fully general predictive games, we focus here on the time-homogeneous Markovian predictive game² as a first step toward developing methods applicable to the general stationary ergodic case. Suppose Forecaster has a k th-order Markov kernel $M : \Omega \times \Omega^k \rightarrow (0, 1) : (a, \omega^k) \mapsto M(a \mid \omega^k)$ that satisfies

$$\sum_{a \in \Omega} M(a \mid \omega^k) = 1$$

for all $\omega^k \in \Omega^k$.

²The Markovian predictive game introduced in this chapter is entirely distinct from the Markov game commonly used in the field of operations research [23].

Time-homogeneous k th-order Markovian predictive game

Players : Forecaster, Skeptic, and Reality.

Protocol : $K_0 = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

Forecaster announces $p_n \in \mathcal{P}(\Omega)$ such that

$p_n(a) := M(a \mid \omega_{n-k}^{n-1})$ for $n > k$.

Skeptic announces $\alpha_n \in \mathbb{R}^\Omega$.

Reality announces $\omega_n \in \Omega$.

$$K_n := K_{n-1} \left\{ 1 + \alpha_n(\omega_n) - \sum_{a \in \Omega} \alpha_n(a) p_n(a) \right\}.$$

END FOR.

This protocol can be understood as a variant of the generalized predictive game in which Forecaster announces $p_n : \Omega \rightarrow (0, 1)$ for $n > k$ according to a time-homogeneous Markov kernel M as $p_n(a) = M(a \mid \omega_{n-k}^{n-1})$ based on Reality's past moves. Note that for $n \leq k$, $p_n(a)$ can be arbitrary as long as $p_n \in \mathcal{P}(\Omega)$.

For each $\omega^\ell \in \Omega^\ell$ with $\ell \in \mathbb{Z}_{\geq k}$, let $P(\omega^\ell)$ be defined by

$$P(\omega^\ell) := \pi(\omega^k) \prod_{i=k+1}^{\ell} M(\omega_i \mid \omega_{i-k}^{i-1}),$$

where $\pi : \Omega^k \rightarrow (0, 1)$ is the stationary distribution associated with the Markov kernel M . The main result of Part I is the following:

Theorem 2.2. *In the time-homogeneous k th-order Markovian predictive game, Skeptic has a prudent strategy $\alpha : \Omega^* \rightarrow \mathbb{R}^\Omega$ that ensures $\lim_{n \rightarrow \infty} K_n = \infty$ unless*

$$\lim_{n \rightarrow \infty} \frac{S_n(a^\ell)}{n} = P(a^\ell)$$

for all $\ell \in \mathbb{Z}_{\geq k}$ and $a^\ell \in \Omega^\ell$, where $S_n(a^\ell)$ denotes the number of occurrences of a^ℓ in ω^n .

Theorem 2.2 establishes that there exists a betting strategy that guarantees Skeptic can become infinitely rich if Reality's moves do not align with the Markovian Forecaster's announcements³. Specifically, this happens when the relative frequency of occurrences of some string of length ℓ ($\geq k$) fails to converge to the stationary joint distribution associated with the k th-order Markov kernel.

³Using Shafer and Vovk's terminology [1] once again, Theorem 2.2 can be restated as follows: *Skeptic can force the event $\lim_{n \rightarrow \infty} S_n(a^\ell)/n = P(a^\ell)$ for all $\ell \in \mathbb{Z}_{\geq k}$ and $a^\ell \in \Omega^\ell$.*

Chapter 3

Lempel–Ziv strategy and its properties

In this chapter, we develop a betting strategy using the technique of incremental parsing and establish several lemmas in preparation for the proof of Theorem 2.2.

3.1 Betting strategy inspired by Lempel-Ziv coding scheme

We outline an algorithm for incremental parsing [5], which divides a string into substrings separated by slashes, with each substring being the shortest one not previously encountered. The algorithm runs as follows: Start with an initial slash. After each slash, scan the input sequence until the shortest string that has not yet been marked off is identified. Since this string is the shortest unseen string, all its prefixes must have appeared earlier in the sequence. For example, a sequence 1000011101011 of length 13 is decomposed into

$$/1/0/00/01/11/010/11.$$

Suppose a sequence ω_1^n is parsed as

$$\omega_1^n = / \omega_{n_0+1}^{n_1} / \omega_{n_1+1}^{n_2} / \cdots / \omega_{n_{T-1}+1}^{n_T} / \omega_{n_T+1}^n,$$

where $n_0 = 0$, $n_1 = 1$, and all parsed substrings except the last one, $\omega_{n_T+1}^n$, are distinct. In what follows, the substrings $\omega_{n_0+1}^{n_1}, \omega_{n_1+1}^{n_2}, \dots, \omega_{n_{T-1}+1}^{n_T}$ are referred to as parsed phrases. Note that the number T of parsed phrases depends on the sequence ω_1^n , and the last string $\omega_{n_T+1}^n$ may be empty.

We now construct a betting strategy at step n , given Reality's past moves ω_1^{n-1} ($n \geq 2$). Using incremental parsing, we decompose ω_1^{n-1} into

$$\omega_1^{n-1} = / \omega_{n_0+1}^{n_1} / \omega_{n_1+1}^{n_2} / \cdots / \omega_{n_{T-1}+1}^{n_T} / \omega_{n_T+1}^{n-1}.$$

Next, we define the set

$$V(\omega_1^{n_T}) := \left\{ \xi \in \Omega^* \left| \begin{array}{l} \xi \neq \omega_{n_{j-1}+1}^{n_j} \text{ for all } j \in \{0, 1, \dots, T\}, \text{ and} \\ \xi = \omega_{n_{j-1}+1}^{n_j} b \text{ for some } j \in \{0, 1, \dots, T\} \text{ and } b \in \Omega \end{array} \right. \right\},$$

where $\omega_{n_{-1}+1}^{n_0} = \lambda$ is the empty string. Equivalently, $V(\omega_1^{n_T})$ consists of all one-symbol extensions of already parsed phrases that have not yet appeared as phrases; it is the “frontier” of the current dictionary, whose elements are candidates to become new phrases at some later step (not necessarily the next). The size of this set is given by

$$|V(\omega_1^{n_T})| = A + T(A - 1).$$

This can be shown by induction on T : For $T = 0$, we have $n = 1$, and

$$V(\omega_1^{n-1}) = V(\lambda) = \Omega.$$

For $T \geq 1$, the set $V(\omega_1^{n_T})$ is constructed as

$$V(\omega_1^{n_T}) = \left(V(\omega_1^{n_{T-1}}) \setminus \{\omega_{n_{T-1}+1}^{n_T}\} \right) \cup \{\omega_{n_{T-1}+1}^{n_T} b \mid b \in \Omega\},$$

which yields a recursive formula $|V(\omega_1^{n_T})| = |V(\omega_1^{n_{T-1}})| - 1 + A$, ensuring the desired result.

Finally, we define the conditional probability $Q_{LZ}(\omega_n \mid \omega_1^{n-1})$, which determines the betting strategy $\alpha_n(\omega_n)$, as follows. For $n = 1$, let $Q_{LZ}(a) := 1/A$. For $n \geq 2$, we define

$$Q_{LZ}(a \mid \omega_1^{n-1}) := \frac{|\{\xi \in V(\omega_1^{n_T}) \mid \omega_{n_{T+1}}^{n-1} a \sqsubset \xi\}|}{|\{\xi \in V(\omega_1^{n_T}) \mid \omega_{n_{T+1}}^{n-1} \sqsubset \xi\}|}. \quad (3.1)$$

Note that

$$\{\xi \in V(\omega_1^{n_T}) \mid \omega_{n_{T+1}}^{n-1} \sqsubset \xi\} = \bigsqcup_{a \in \Omega} \{\xi \in V(\omega_1^{n_T}) \mid \omega_{n_{T+1}}^{n-1} a \sqsubset \xi\},$$

which follows from the definition of $V(\omega_1^{n_T})$. Moreover, the set

$$\{\xi \in V(\omega_1^{n_T}) \mid \omega_{n_{T+1}}^{n-1} a \sqsubset \xi\}$$

is nonempty for any $a \in \Omega$. Thus, we conclude that

$$\sum_{a \in \Omega} Q_{LZ}(a \mid \omega_1^{n-1}) = 1 \quad \text{and} \quad Q_{LZ}(a \mid \omega_1^{n-1}) > 0.$$

The motivation behind the definition (3.1) is now in order. When $n - 1 = n_T$ (i.e., when $\omega_{n_{T+1}}^{n-1} = \lambda$), we have

$$Q_{LZ}(a \mid \omega_1^{n_T}) = \frac{|\{\xi \in V(\omega_1^{n_T}) \mid a \sqsubset \xi\}|}{|V(\omega_1^{n_T})|}.$$

Thus, for each $\xi = \xi_1^\tau \in V(\omega_1^{n_T})$,

$$Q_{LZ}(\xi \mid \omega_1^{n_T}) := Q_{LZ}(\xi_1 \mid \omega_1^{n_T}) \prod_{t=2}^{\tau} Q_{LZ}(\xi_t \mid \omega_1^{n_T} \xi_1^{t-1}) = \frac{1}{|V(\omega_1^{n_T})|}.$$

In other words, the conditional probability $Q_{LZ}(a \mid \omega_1^{n-1})$ is designed to induce the uniform distribution over $V(\omega_1^{n_T})$. This observation also implies that

$$Q_{LZ}(\omega_{n_T+1}^n \mid \omega_1^{n_T}) > \frac{1}{|V(\omega_1^{n_T})|} \quad (3.2)$$

whenever $n > n_T$.

In what follows, we refer to the betting strategy α_n based on the conditional probability $Q_{LZ}(a \mid \omega_1^{n-1})$ as the *Lempel-Ziv betting strategy*. Since α_n is canonically constructed via $\alpha_n(a) = Q_{LZ}(a \mid \omega_1^{n-1})/p_n(a)$, we may identify α_n with $Q_{LZ}(\cdot \mid \omega_1^{n-1})$ and, for simplicity, also refer to $Q_{LZ}(\cdot \mid \omega_1^{n-1})$ itself as the Lempel-Ziv betting strategy.

Example. Let $\Omega = \{0, 1\}$, and suppose that Skeptic has observed the sequence of Reality's outcomes $\omega_1^{13} = /1/0/00/01/11/010/11$. At this stage, the current set of candidate phrases is

$$V(\omega_1^{11}) = \{10, 000, 001, 011, 110, 111, 0100, 0101\},$$

and the conditional probabilities $Q_{LZ}(a \mid \omega_1^{13})$ at step $n = 14$ are given by

$$Q_{LZ}(0 \mid \omega_1^{13}) = \frac{|\{110\}|}{|\{110, 111\}|} = \frac{1}{2}, \quad Q_{LZ}(1 \mid \omega_1^{13}) = \frac{|\{111\}|}{|\{110, 111\}|} = \frac{1}{2}.$$

Upon hearing Forecaster's announcement $p_{14}(a)$, Skeptic announces the bet

$$\alpha_{14}(a) = \frac{Q_{LZ}(a \mid \omega_1^{13})}{p_{14}(a)} = \frac{1}{2p_{14}(a)}, \quad (a \in \Omega).$$

Suppose further that Reality's announcement at step $n = 14$ is $\omega_{14} = 0$. Skeptic then updates the set of candidate phrases accordingly:

$$V(\omega_1^{14}) = \{10, 000, 001, 011, 111, 0100, 0101, 1100, 1101\},$$

and the updated strategy becomes

$$\begin{aligned} Q_{LZ}(0 \mid \omega_1^{14}) &= \frac{|\{000, 001, 011, 0100, 0101\}|}{|V(\omega_1^{14})|} = \frac{5}{9}, \\ Q_{LZ}(1 \mid \omega_1^{14}) &= \frac{|\{10, 111, 1100, 1101\}|}{|V(\omega_1^{14})|} = \frac{4}{9}. \end{aligned}$$

Remark. From the above discussion,

$$-\log Q_{LZ}(\omega_1^{n_T}) = \sum_{j=0}^{T-1} \log(A + j(A-1)), \quad (3.3)$$

where

$$Q_{LZ}(\omega_1^n) := \prod_{i=1}^n Q_{LZ}(\omega_i \mid \omega_1^{i-1}).$$

On the other hand, the Lempel-Ziv codelength ℓ_{LZ} [5] is given by

$$\ell_{LZ}(\omega_1^n) = \sum_{j=1}^{T+1} \lceil \log_A(jA) \rceil.$$

3.2 Properties of Lempel-Ziv betting strategy

In this section, we outline several fundamental properties of the Lempel-Ziv betting strategy.

Define the complexity $c(\omega_1^n)$ of a sequence $\omega_1^n \in \Omega^n$ as the total number T of parsed phrases obtained through its incremental parsing [24, p. 448]. The quantity $c(\omega_1^n) \log c(\omega_1^n)$ is known to play an essential role in Lempel-Ziv coding, and it is also important in this section as a bridge between several key quantities. The following lemma provides an asymptotic lower bound for the difference between $c(\omega_1^n) \log c(\omega_1^n)$ and $-\log Q_{LZ}(\omega_1^n)$.

Lemma 3.1. *For any $\omega_1^\infty \in \Omega^\infty$, the following inequality holds:*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \{c(\omega_1^n) \log c(\omega_1^n) - (-\log Q_{LZ}(\omega_1^n))\} \geq 0.$$

Proof. Since

$$\begin{aligned} -\log Q_{LZ}(\omega_1^n) &= -\log Q_{LZ}(\omega_1^{n_T} \omega_{n_T+1}^n) \\ &= -\log Q_{LZ}(\omega_1^{n_T}) - \log Q_{LZ}(\omega_{n_T+1}^n \mid \omega_1^{n_T}) \end{aligned}$$

for $n > n_T$, it follows from (3.2) and (3.3) that, for any n ,

$$\begin{aligned} -\log Q_{LZ}(\omega_1^n) &\leq -\log Q_{LZ}(\omega_1^{n_T}) + \log |V(\omega_1^{n_T})| = \sum_{j=0}^{c(\omega_1^n)} \log(A + j(A-1)) \\ &< \sum_{j=0}^{c(\omega_1^n)} \log(A + c(\omega_1^n)(A-1)) = (c(\omega_1^n) + 1) \log(A + c(\omega_1^n)(A-1)). \end{aligned}$$

Consequently,

$$\begin{aligned}
& \frac{1}{n} \{c(\omega_1^n) \log c(\omega_1^n) - (-\log Q_{LZ}(\omega_1^n))\} \\
& > \frac{1}{n} c(\omega_1^n) \log c(\omega_1^n) - \frac{1}{n} (c(\omega_1^n) + 1) \log(A + c(\omega_1^n)(A - 1)) \\
& = \frac{c(\omega_1^n)}{n} \log \frac{c(\omega_1^n)}{A + c(\omega_1^n)(A - 1)} - \frac{\log(A + c(\omega_1^n)(A - 1))}{n}.
\end{aligned}$$

Thus, the following Lemma 3.2 proves the claim. $\square$

Lemma 3.2. *For sufficiently large n and for all $\omega_1^n \in \Omega^n$,*

$$(0 <) c(\omega_1^n) < \frac{n}{(1 - \varepsilon_n) \log_A n},$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Specifically, $c(\omega_1^n)/n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. See Lemma 13.5.3 of [24]. $\square$

For $n, \ell \in \mathbb{Z}_{>0}$ with $n > \ell$, let $T_n(a_1^\ell)$ denote the number of occurrences of $a_1^\ell \in \Omega^\ell$ in the cyclically extended word $\omega_1^n \omega_1^{\ell-1}$ of length $n + \ell - 1$. Similarly, let $T_n(b \mid a_1^\ell)$ represent the number of occurrences of b immediately following $a_1^\ell \in \Omega^\ell$ in the extended word $\omega_1^n \omega_1^\ell$ of length $n + \ell$. In other words, $T_n(b \mid a_1^\ell) = T_n(a_1^\ell b)$.

Note that, by definition,

$$\sum_{a_1^\ell \in \Omega^\ell} T_n(a_1^\ell) = n.$$

Moreover, the quantity $T_n(a_1^\ell)$ is asymptotically equivalent to $S_n(a_1^\ell)$, the number of occurrences of a_1^ℓ in ω_1^n , in the sense that

$$\lim_{n \rightarrow \infty} \frac{1}{n} (T_n(a_1^\ell) - S_n(a_1^\ell)) = 0.$$

The following lemma states that the marginals of conditional counts $T_n(b \mid a_1^\ell)$ correspond to $T_n(a_1^\ell)$ and $T_n(a_2^\ell b)$.

Lemma 3.3. *Let $n, \ell \in \mathbb{Z}_{>0}$ with $n > \ell$. Then, for all $a_1^\ell \in \Omega^\ell$ and $b \in \Omega$, the following identities hold:*

$$\sum_{b \in \Omega} T_n(b \mid a_1^\ell) = T_n(a_1^\ell) \quad \text{and} \quad \sum_{a_1 \in \Omega} T_n(b \mid a_1^\ell) = T_n(a_2^\ell b).$$

Proof. The first identity follows from

$$\begin{aligned}
\sum_{b \in \Omega} T_n(b \mid a_1^\ell) &= \sum_{b \in \Omega} (\text{number of occurrences of } a_1^\ell b \text{ in } \omega_1^n \omega_1^\ell) \\
&= (\text{number of occurrences of } a_1^\ell \text{ in } \omega_1^n \omega_1^{\ell-1}) = T_n(a_1^\ell).
\end{aligned}$$

On the other hand, observe that

$$\begin{aligned}
\sum_{a_1 \in \Omega} T_n(b \mid a_1^\ell) &= \sum_{a_1 \in \Omega} (\text{number of occurrences of } a_1^\ell b \text{ in } \omega_1^n \omega_1^\ell) \\
&= (\text{number of occurrences of } a_2^\ell b \text{ in } \omega_2^n \omega_1^\ell) \\
&= (\text{number of occurrences of } a_2^\ell b \text{ in } \omega_1^n \omega_1^{\ell-1}) + \Delta_1 + \Delta_\ell,
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1 &:= (\text{adjustment for the effect of adding } \omega_1 \text{ to the head of } \omega_2^n \omega_1^\ell) \\
&= \begin{cases} -1 & (\omega_1^\ell = a_2^\ell b), \\ 0 & (\text{otherwise}), \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\Delta_\ell &:= (\text{adjustment for the effect of removing } \omega_\ell \text{ from the tail of } \omega_2^n \omega_1^\ell) \\
&= \begin{cases} 1 & (\omega_1^\ell = a_2^\ell b), \\ 0 & (\text{otherwise}). \end{cases}
\end{aligned}$$

Since $\Delta_1 + \Delta_\ell = 0$, the second identity holds. $\square$

Let us now fix $\ell \in \mathbb{Z}_{>0}$ arbitrarily. Given ω_1^n with $n > \ell$, we define

$$\hat{M}_n^\ell(b \mid a^\ell) := \frac{T_n(b \mid a^\ell)}{T_n(a^\ell)}$$

for $a^\ell \in \Omega^\ell$ satisfying $T_n(a^\ell) > 0$. Due to Lemma 3.3, we observe that

$$\sum_{b \in \Omega} \hat{M}_n^\ell(b \mid a^\ell) = \sum_{b \in \Omega} \frac{T_n(b \mid a^\ell)}{T_n(a^\ell)} = 1.$$

Thus, with an appropriate definition of $\hat{M}_n^\ell(b \mid a^\ell)$ for $a^\ell \in \Omega^\ell$ satisfying $T_n(a^\ell) = 0$, $\hat{M}_n^\ell$ can be regarded as an ℓ th-order Markov kernel formally associated with ω_1^n . The following lemma provides a condition that ensures $T_n(a^k)/n$ converges to the stationary distribution π of the k th-order Markov kernel M .

Lemma 3.4. *If*

$$\lim_{n \rightarrow \infty} \hat{M}_n^k(b \mid a_1^k) = M(b \mid a_1^k)$$

for all $a_1^k \in \Omega^k$ and $b \in \Omega$, then,

$$\lim_{n \rightarrow \infty} \frac{T_n(a_1^k)}{n} = \pi(a_1^k)$$

for all $a_1^k \in \Omega^k$.

Proof. The assumption $\hat{M}_n^k \rightarrow M$ ensures that for sufficiently large n , we have $\hat{M}_n^k(b \mid a_1^k) > 0$ for all $a_1^k \in \Omega^k$ and $b \in \Omega$. For $\omega^k \in \Omega^k$, define $\hat{q}_n(\omega^k) := T_n(\omega^k)/n$. Then, by Lemma 3.3, the empirical distribution $\hat{q}_n$ is stationary under the k th Markov kernel $\hat{M}_n^k$ [25]:

$$\sum_{a_1 \in \Omega} \hat{M}_n^k(b \mid a_1^k) \hat{q}_n(a_1^k) = \sum_{a_1 \in \Omega} \frac{T_n(b \mid a_1^k)}{T_n(a_1^k)} \frac{T_n(a_1^k)}{n} = \frac{1}{n} \sum_{a_1 \in \Omega} T_n(b \mid a_1^k) = \frac{T_n(a_2^k b)}{n} = \hat{q}_n(a_2^k b).$$

Thus, by Lemma A.2 in Appendix A, $\hat{q}_n$ is the unique stationary distribution of Markov matrix $\hat{M}_n^k$.

Consider a convergent subsequence $\{\hat{q}_{n_i} - \pi\}_i$ of $\{\hat{q}_n - \pi\}_n$, which converges to some $r \in [-1, 1]^{\Omega^k}$. By the assumption $\hat{M}_n^k \rightarrow M$, we obtain

$$\begin{aligned} r &= \lim_{i \rightarrow \infty} (\hat{q}_{n_i} - \pi) = \lim_{i \rightarrow \infty} (\hat{M}_{n_i}^k \hat{q}_{n_i} - M\pi) \\ &= \lim_{i \rightarrow \infty} \left\{ \hat{M}_{n_i}^k (\hat{q}_{n_i} - \pi) + (\hat{M}_{n_i}^k - M)\pi \right\} = Mr. \end{aligned}$$

By the Perron-Frobenius theorem (Section 4.4 of [54]), there exists a constant $c \in \mathbb{R}$ satisfying $r = c\pi$. Furthermore,

$$\sum_{a_1^k \in \Omega^k} r(a_1^k) = \lim_{i \rightarrow \infty} \sum_{a_1^k \in \Omega^k} (\hat{q}_{n_i}(a_1^k) - \pi(a_1^k)) = 0.$$

Thus, we must have $c = 0$, completing the proof. $\square$

Next, for $n, \ell \in \mathbb{Z}_{>0}$ with $n > \ell$, we define

$$\hat{R}_n^\ell(\omega_1^n) := \left\{ \prod_{i=1}^{\ell} \hat{M}_n^\ell(\omega_i \mid \omega_{n-\ell+i}^n \omega_1^{i-1}) \right\} \left\{ \prod_{i=\ell+1}^n \hat{M}_n^\ell(\omega_i \mid \omega_{i-\ell}^{i-1}) \right\}.$$

In the first factor, which represents $\hat{R}_n^\ell(\omega_1^\ell)$, we formally introduce concatenated strings $\omega_{n-\ell+i}^n \omega_1^{i-1}$ of length ℓ for $i = 1, \dots, \ell$, to facilitate the application of the ℓ th Markov kernel $\hat{M}_n^\ell$. The following lemma presents an important inequality relating $c(\omega_1^n) \log c(\omega_1^n)$ to $-\log \hat{R}_n^\ell(\omega_1^n)$. While its proof resembles Ziv's inequality and the asymptotic optimality of the Lempel-Ziv algorithm [24, Section 13.5.2], our setting differs in that $\hat{M}_n^\ell$ depends solely on Reality's moves and does not assume any underlying stochastic process.

Lemma 3.5. *For any $n, \ell \in \mathbb{Z}_{>0}$ with $n > \ell$ and $\omega_1^n \in \Omega^n$, the following inequality holds:*

$$\frac{1}{n} c(\omega_1^n) \log c(\omega_1^n) \leq -\frac{1}{n} \log \hat{R}_n^\ell(\omega_1^n) + \delta_\ell(n),$$

where $\delta_\ell(n) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Suppose the sequence ω_1^n is parsed into C_n distinct substrings as

$$\omega_1^n = / \omega_{n_0+1}^{n_1} / \omega_{n_1+1}^{n_2} / \cdots / \omega_{n_{C_n-1}+1}^{n_{C_n}},$$

where $n_0 = 0$ and $n_{C_n} = n$. For example, if ω_1^n is parsed using the incremental parsing algorithm as

$$\omega_1^n = / \omega_{n_0+1}^{n_1} / \omega_{n_1+1}^{n_2} / \cdots / \omega_{n_{T-1}+1}^{n_T} / \omega_{n_T+1}^n,$$

we define $C_n = T$ and set $\omega_{n_{C_n-1}+1}^{n_{C_n}} := \omega_{n_{T-1}+1}^{n_T} \omega_{n_T+1}^n$.

Now, define $s_i := \omega_{i-\ell}^{i-1}$ for $\ell+1 \leq i \leq n$, and extend them cyclically for $1 \leq i \leq \ell$ as follows:

$$s_1 := \omega_{n-\ell+1}^n, \quad s_2 := \omega_{n-\ell+2}^n \omega_1, \quad \dots, \quad s_\ell := \omega_n \omega_1^{\ell-1}.$$

For $m \in \mathbb{Z}_{>0}$ and $s \in \Omega^\ell$, let $c_{m,s}$ denote the number of occurrences of the word $\omega_{n_{j-1}+1}^{n_j}$ of length m such that $s_{n_{j-1}+1} = \omega_{n_{j-1}-\ell+1}^{n_j-1} = s$ among the C_n substrings $\omega_{n_0+1}^{n_1}, \omega_{n_1+1}^{n_2}, \dots, \omega_{n_{C_n-1}+1}^{n_{C_n}}$, i.e.,

$$c_{m,s} := \left| \left\{ j \in \{1, 2, \dots, C_n\} \mid |\omega_{n_{j-1}+1}^{n_j}| = m, s_{n_{j-1}+1} = s \right\} \right|.$$

Letting $\mathcal{U} := \{(m, s) \in \mathbb{Z}_{>0} \times \Omega^\ell \mid c_{m,s} > 0\}$, we have

$$\sum_{(m,s) \in \mathcal{U}} c_{m,s} = C_n \quad \text{and} \quad \sum_{(m,s) \in \mathcal{U}} m \cdot c_{m,s} = n.$$

For each $(m, s) \in \mathcal{U}$, let

$$\mathcal{J}_{m,s} := \{j \in \{1, 2, \dots, C_n\} \mid n_j - n_{j-1} = m, s_{n_{j-1}+1} = s\}.$$

With a slight abuse of notation, we define, for any $m \in \mathbb{Z}_{>\ell}$ and $a_1^m \in \Omega^m$,

$$\hat{M}_n^\ell(a_{\ell+1}^m \mid a_1^\ell) := \prod_{i=\ell+1}^m \hat{M}_n^\ell(a_i \mid a_{i-\ell}^{i-1}).$$

Then, we can evaluate $\log \hat{R}_n^\ell(\omega_1^n)$ as follows:

$$\begin{aligned} \log \hat{R}_n^\ell(\omega_1^n) &= \sum_{j=1}^{C_n} \log \hat{M}_n^\ell(\omega_{n_{j-1}+1}^{n_j} \mid s_{n_{j-1}+1}) \\ &= \sum_{(m,s) \in \mathcal{U}} \sum_{j \in \mathcal{J}_{m,s}} \log \hat{M}_n^\ell(\omega_{n_{j-1}+1}^{n_j} \mid s_{n_{j-1}+1}) \\ &= \sum_{(m,s) \in \mathcal{U}} c_{m,s} \sum_{j \in \mathcal{J}_{m,s}} \frac{1}{c_{m,s}} \log \hat{M}_n^\ell(\omega_{n_{j-1}+1}^{n_j} \mid s_{n_{j-1}+1}) \\ &\leq \sum_{(m,s) \in \mathcal{U}} c_{m,s} \log \left(\frac{1}{c_{m,s}} \sum_{j \in \mathcal{J}_{m,s}} \hat{M}_n^\ell(\omega_{n_{j-1}+1}^{n_j} \mid s_{n_{j-1}+1}) \right). \end{aligned}$$

In the last inequality, we used Jensen's inequality. Since the parsed substrings $\{\omega_{n_{j-1}+1}^{n_j}\}_{1 \leq j \leq C_n}$ are distinct, we have

$$\sum_{j \in \mathcal{J}_{m,s}} \hat{M}_n^\ell(\omega_{n_{j-1}+1}^{n_j} \mid s_{n_{j-1}+1}) \leq 1$$

for all $(m, s) \in \mathcal{U}$. As a consequence,

$$\begin{aligned} \log \hat{R}_n^\ell(\omega_1^n) &\leq - \sum_{(m,s) \in \mathcal{U}} c_{m,s} \log c_{m,s} \\ &= -c(\omega_1^n) \log c(\omega_1^n) - c(\omega_1^n) \sum_{(m,s) \in \mathcal{U}} \frac{c_{m,s}}{c(\omega_1^n)} \log \frac{c_{m,s}}{c(\omega_1^n)}, \end{aligned}$$

where $c(\omega_1^n) = C_n$. Writing $\pi_{m,s} := c_{m,s}/c(\omega_1^n)$, we have

$$\sum_{(m,s) \in \mathcal{U}} \pi_{m,s} = 1 \quad \text{and} \quad \sum_{(m,s) \in \mathcal{U}} m \cdot \pi_{m,s} = \frac{n}{c(\omega_1^n)}.$$

We now define the random variables U and V as follows:

$$\Pr(U = m, V = s) := \pi_{m,s}.$$

From the above bound on $\log \hat{R}_n^\ell(\omega_1^n)$, it follows that

$$-\frac{1}{n} \log \hat{R}_n^\ell(\omega_1^n) \geq \frac{c(\omega_1^n)}{n} \log c(\omega_1^n) - \frac{c(\omega_1^n)}{n} H(U, V),$$

where

$$H(U, V) := - \sum_{(m,s) \in \mathcal{U}} \frac{c_{m,s}}{c(\omega_1^n)} \log \frac{c_{m,s}}{c(\omega_1^n)}.$$

By the subadditivity of entropy, we have

$$H(U, V) \leq H(U) + H(V).$$

Since the expectation of U is given by

$$\mathbb{E}[U] = \frac{n}{c(\omega_1^n)},$$

applying Lemma 3.6 below, we can bound $H(U)$ as

$$\begin{aligned} H(U) &\leq (\mathbb{E}[U] + 1) \log(\mathbb{E}[U] + 1) - (\mathbb{E}[U]) \log(\mathbb{E}[U]) \\ &= \log \frac{n}{c(\omega_1^n)} + \left(\frac{n}{c(\omega_1^n)} + 1 \right) \log \left(\frac{c(\omega_1^n)}{n} + 1 \right). \end{aligned}$$

On the other hand, since $H(V) \leq \log |\Omega|^\ell = \ell \log A$, we obtain

$$\begin{aligned} \delta_\ell(n) &:= \frac{c(\omega_1^n)}{n} H(U, V) \\ &\leq \frac{c(\omega_1^n)}{n} \log \frac{n}{c(\omega_1^n)} + \left(1 + \frac{c(\omega_1^n)}{n}\right) \log \left(\frac{c(\omega_1^n)}{n} + 1\right) + \frac{c(\omega_1^n)}{n} \ell \log A. \end{aligned}$$

Since $c(\omega_1^n)/n \rightarrow 0$ as $n \rightarrow \infty$ by Lemma 3.2, it follows that $\delta_\ell(n) \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. $\square$

Lemma 3.6. *Let Z be a nonnegative integer-valued random variable with mean μ . Then the entropy $H(Z)$ is bounded by*

$$H(Z) \leq (\mu + 1) \log(\mu + 1) - \mu \log \mu.$$

Proof. See Lemma 13.5.4 of [24]. $\square$

Finally, for $n \in \mathbb{Z}_{\geq k}$, let us introduce

$$\tilde{P}(\omega_1^n) := \left\{ \prod_{i=1}^k p_i(\omega_i) \right\} \cdot \left\{ \prod_{i=k+1}^n M(\omega_i \mid \omega_{i-k}^{i-1}) \right\},$$

which represents Forecaster's announcements. The next lemma establishes the relationship between $\tilde{P}$ and $\hat{R}_n^\ell$.

Lemma 3.7. *For any $n, \ell \in \mathbb{Z}_{\geq k}$ with $n > \ell$, the following identity holds:*

$$\begin{aligned} -\log \tilde{P}(\omega_1^n) + \log \hat{R}_n^\ell(\omega_1^n) &= -\log \tilde{P}(\omega_1^\ell) + \log \prod_{i=1}^\ell M(\omega_i \mid \omega_{n-k+i}^n \omega_1^{i-1}) \\ &\quad + \sum_{a_1^\ell \in \Omega^\ell} T_n(a_1^\ell) \cdot D\left(\hat{M}_n^\ell(\cdot \mid a_1^\ell) \parallel M(\cdot \mid a_{\ell-k+1}^\ell)\right), \end{aligned}$$

where $D(\cdot \parallel \cdot)$ is the Kullback-Leibler divergence.

Proof. For $n, \ell \in \mathbb{Z}_{>0}$ satisfying $n > \ell \geq k$,

$$\begin{aligned}
& -\log \tilde{P}(\omega_1^n) + \log \hat{R}_n^\ell(\omega_1^n) \\
&= -\log \tilde{P}(\omega_1^\ell) - \log \tilde{P}(\omega_{\ell+1}^n \mid \omega_1^\ell) + \log \hat{R}_n^\ell(\omega_1^n) \\
&= -\log \tilde{P}(\omega_1^\ell) - \log \prod_{i=\ell+1}^n M(\omega_i \mid \omega_{i-k}^{i-1}) + \log \hat{R}_n^\ell(\omega_1^n) \\
&= -\log \tilde{P}(\omega_1^\ell) + \log \prod_{i=1}^\ell M(\omega_i \mid \omega_{n-k+i}^n \omega_1^{i-1}) - \log \prod_{i=1}^\ell M(\omega_i \mid \omega_{n-k+i}^n \omega_1^{i-1}) \\
&\quad - \log \prod_{i=\ell+1}^n M(\omega_i \mid \omega_{i-k}^{i-1}) + \log \hat{R}_n^\ell(\omega_1^n) \\
&= -\log \tilde{P}(\omega_1^\ell) + \log \prod_{i=1}^\ell M(\omega_i \mid \omega_{n-k+i}^n \omega_1^{i-1}) - \sum_{a_1^\ell \in \Omega^\ell} \sum_{b \in \Omega} T_n(b \mid a_1^\ell) \log M(b \mid a_{\ell-k+1}^\ell) + \log \hat{R}_n^\ell(\omega_1^n).
\end{aligned}$$

In the last equality, we used the fact that

$$\begin{aligned}
M(\omega_i \mid \omega_{n-k+i}^n \omega_1^{i-1}) &= M(\omega_i \mid \omega_{n-\ell+i}^n \omega_1^{i-1}) \quad \text{and} \\
M(\omega_i \mid \omega_{i-k}^{i-1}) &= M(\omega_i \mid \omega_{i-\ell}^{i-1}),
\end{aligned}$$

since $\ell \geq k$ and M is the k th-order Markov kernel.

Substituting the definition of $\hat{R}_n^\ell(\omega_1^n)$, the computation follows as:

$$\begin{aligned}
& -\log \tilde{P}(\omega_1^n) + \log \hat{R}_n^\ell(\omega_1^n) \\
&= -\log \tilde{P}(\omega_1^\ell) + \log \prod_{i=1}^\ell M(\omega_i \mid \omega_{n-k+i}^n \omega_1^{i-1}) \\
&\quad + \sum_{a_1^\ell \in \Omega^\ell} \sum_{b \in \Omega} T_n(b \mid a_1^\ell) \cdot \left(-\log M(b \mid a_{\ell-k+1}^\ell) + \log \hat{M}_n^\ell(b \mid a_1^\ell) \right) \\
&= -\log \tilde{P}(\omega_1^\ell) + \log \prod_{i=1}^\ell M(\omega_i \mid \omega_{n-k+i}^n \omega_1^{i-1}) \\
&\quad + \sum_{a_1^\ell \in \Omega^\ell} T_n(a_1^\ell) \cdot D \left(\hat{M}_n^\ell(\cdot \mid a_1^{\ell-1}) \parallel M(\cdot \mid a_{\ell-k+1}^\ell) \right).
\end{aligned}$$

This completes the proof. □

Chapter 4

Proof of Theorem 2.2

Let $K^\alpha : \Omega^* \rightarrow \mathbb{R}$ represent the capital process associated with a betting strategy α , so that $K_n = K^\alpha(\omega_1^n)$ for $\omega_1^n \in \Omega^*$. The following lemma demonstrates that the limit supremum for the capital process can be replaced by a limit.

Lemma 4.1. *Suppose a strategy α satisfies*

$$\limsup_{n \rightarrow \infty} K^\alpha(\omega_1^n) = \infty$$

for a specific sequence of Reality's moves $\{\omega_1^n\}_{n \in \mathbb{Z}_{>0}}$. Then there exists another strategy α_ such that*

$$\lim_{n \rightarrow \infty} K^{\alpha_*}(\omega_1^n) = \infty$$

for the same sequence $\{\omega_1^n\}_{n \in \mathbb{Z}_{>0}}$.

Proof. The required strategy α_* can be defined as follows: Strategy α_* uses α as long as the capital K_n retains below 2. Once K_n reaches or exceeds 2, α_* transfers the net gain $\Delta K := K_n - 1$ (≥ 1) into an external storage. It then restart the game with a capital of 1, employing the strategy α again. $\square$

Lemma 4.2. *In the time-homogeneous k th-order Markovian predictive game, the Lempel-Ziv betting strategy Q_{LZ} ensures $\lim_{n \rightarrow \infty} K_n = \infty$ unless*

$$\lim_{n \rightarrow \infty} \hat{M}_n^k(b \mid a_1^k) = M(b \mid a_1^k)$$

for all $a_1^k \in \Omega^k$ and $b \in \Omega$.

Proof. Using (2.3) and applying Lemma 3.5, we obtain

$$\begin{aligned} \frac{\log K_n}{n} &= \frac{1}{n} \left\{ -\log \tilde{P}(\omega_1^n) - (-\log Q_{LZ}(\omega_1^n)) \right\} \\ &= \frac{1}{n} \left\{ -\log \tilde{P}(\omega_1^n) - c(\omega_1^n) \log c(\omega_1^n) \right\} + \frac{1}{n} \left\{ c(\omega_1^n) \log c(\omega_1^n) - (-\log Q_{LZ}(\omega_1^n)) \right\} \\ &\geq \frac{1}{n} \left\{ -\log \tilde{P}(\omega_1^n) + \log \hat{R}_n^k(\omega_1^n) - n\delta_k(n) \right\} + \frac{1}{n} \left\{ c(\omega_1^n) \log c(\omega_1^n) - (-\log Q_{LZ}(\omega_1^n)) \right\}. \end{aligned}$$

Applying Lemma 3.7 with $\ell = k$, we further evaluate K_n as

$$\begin{aligned} \frac{\log K_n}{n} &\geq \sum_{a_1^k \in \Omega^k} \frac{T_n(a_1^k)}{n} \cdot D\left(\hat{M}_n^k(\cdot | a_1^k) \parallel M(\cdot | a_1^k)\right) \\ &\quad + \frac{1}{n} \left\{ -\log \tilde{P}(\omega_1^k) + \log \prod_{i=1}^k M(\omega_i | \omega_{n-k+i}^n \omega_1^{i-1}) \right\} \\ &\quad - \delta_k(n) + \frac{1}{n} \left\{ c(\omega_1^n) \log c(\omega_1^n) - (-\log Q_{LZ}(\omega_1^n)) \right\}. \end{aligned}$$

From the definition of $\tilde{P}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ -\log \tilde{P}(\omega_1^k) + \log \prod_{i=1}^k M(\omega_i | \omega_{n-k+i}^n \omega_1^{i-1}) \right\} = 0.$$

Furthermore, by Lemma 3.5, we know that $\delta_k(n) \rightarrow 0$ as $n \rightarrow \infty$, and by Lemma 3.1,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \{ c(\omega_1^n) \log c(\omega_1^n) - (-\log Q_{LZ}(\omega_1^n)) \} \geq 0.$$

In light of Lemma 4.1, therefore, it now suffices to prove the following claim: For a given $\omega_1^\infty \in \Omega^\infty$, if there exists $b_1^{k+1} \in \Omega^{k+1}$ such that $\hat{M}_n^k(b_{k+1} | b_1^k)$ does not converge to $M(b_{k+1} | b_1^k)$ as $n \rightarrow \infty$, then

$$\limsup_{n \rightarrow \infty} \sum_{a_1^k \in \Omega^k} \frac{T_n(a_1^k)}{n} \cdot D\left(\hat{M}_n^k(\cdot | a_1^k) \parallel M(\cdot | a_1^k)\right) > 0.$$

We prove this claim by contradiction. Assume that for a given $\omega_1^\infty \in \Omega^\infty$, there exists $b_1^{k+1} \in \Omega^{k+1}$ such that $\hat{M}_n^k(b_{k+1} | b_1^k)$ does not converge to $M(b_{k+1} | b_1^k)$ as $n \rightarrow \infty$, and yet

$$\lim_{n \rightarrow \infty} \sum_{a_1^k \in \Omega^k} \frac{T_n(a_1^k)}{n} \cdot D\left(\hat{M}_n^k(\cdot | a_1^k) \parallel M(\cdot | a_1^k)\right) = 0. \quad (4.1)$$

From this assumption, there exists a subsequence $(n_i)_i \subset (n)$ such that

$$\lim_{i \rightarrow \infty} D\left(\hat{M}_{n_i}^k(\cdot | b_1^k) \parallel M(\cdot | b_1^k)\right) > 0.$$

Combining this with (4.1), we obtain

$$\lim_{i \rightarrow \infty} \frac{T_{n_i}(b_1^k)}{n_i} = 0. \quad (4.2)$$

Further, using (4.1) and (4.2), we can deduce that for all $d_k \in \Omega$,

$$\lim_{i \rightarrow \infty} \frac{T_{n_i}(d_k b_1^{k-1})}{n_i} = 0. \quad (4.3)$$

To establish (4.3), we first recall the second identity in Lemma 3.3, which yields

$$T_n(da_2^\ell b) \leq \sum_{a_1 \in \Omega} T_n(a_1 a_2^\ell b) = T_n(a_2^\ell b)$$

for any $d \in \Omega$ and $a_2^\ell b \in \Omega^\ell$. Thus, for any $d_k \in \Omega$,

$$0 \leq \frac{T_{n_i}(d_k b_1^k)}{n_i} \leq \frac{T_{n_i}(b_1^k)}{n_i}.$$

From (4.2), it follows that

$$\lim_{i \rightarrow \infty} \frac{T_{n_i}(d_k b_1^k)}{n_i} = 0. \quad (4.4)$$

Let us prove (4.3) by contradiction. Suppose there exists $d_k \in \Omega$ such that

$$\limsup_{i \rightarrow \infty} \frac{T_{n_i}(d_k b_1^{k-1})}{n_i} > 0.$$

Then, there exists a subsequence $(n_{i_j})_j \subset (n_i)_i$ such that

$$\lim_{j \rightarrow \infty} \frac{T_{n_{i_j}}(d_k b_1^{k-1})}{n_{i_j}} > 0. \quad (4.5)$$

Consequently, for sufficiently large j , we have

$$\begin{aligned} \frac{T_{n_{i_j}}(d_k b_1^k)}{n_{i_j}} &= \frac{T_{n_{i_j}}(b_k \mid d_k b_1^{k-1})}{n_{i_j}} = \frac{T_{n_{i_j}}(d_k b_1^{k-1})}{n_{i_j}} \frac{T_{n_{i_j}}(b_k \mid d_k b_1^{k-1})}{T_{n_{i_j}}(d_k b_1^{k-1})} \\ &= \frac{T_{n_{i_j}}(d_k b_1^{k-1})}{n_{i_j}} \hat{M}_{n_{i_j}}^k(b_k \mid d_k b_1^{k-1}), \end{aligned}$$

and thus, combining (4.4) and (4.5), we find $\lim_{j \rightarrow \infty} \hat{M}_{n_{i_j}}^k(b_k \mid d_k b_1^{k-1}) = 0$. Recalling $M(b_k \mid d_k b_1^{k-1}) > 0$, we find that

$$\lim_{j \rightarrow \infty} \hat{M}_{n_{i_j}}^k(b_k \mid d_k b_1^{k-1}) \neq M(b_k \mid d_k b_1^{k-1}). \quad (4.6)$$

Since (4.5) and (4.6) contradict (4.1), we have (4.3).

By repeatedly applying this reasoning, adding d_{k-i+1} to the front and removing b_{k-i+1} from the rear of the word b_1^k in (4.2) for $i = 1, \dots, k$, we conclude that

$$\lim_{i \rightarrow \infty} \frac{T_{n_i}(d_1^k)}{n_i} = 0$$

for all $d_1^k \in \Omega^k$. This contradicts the fact that

$$\sum_{d_1^k \in \Omega} T_n(d_1^k) = n$$

for all $n (\geq k)$. Hence, the claim is proved. $\square$

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. We prove the assertion by induction in $\ell (\geq k)$. For $\ell = k$, the assertion holds by Lemmas 3.4 and 4.2.

Assume that the assertion holds for some $\ell (\geq k)$, namely,

$$\lim_{n \rightarrow \infty} \frac{T_n(a_1^\ell)}{n} = P(a_1^\ell) = \pi(a_1^k) \prod_{i=k+1}^{\ell} M(a_i | a_{i-k}^{i-1}) (> 0)$$

for all $a_1^\ell \in \Omega^\ell$. Since

$$\frac{T_n(a_1^{\ell+1})}{n} = \frac{T_n(a_1^\ell)}{n} \hat{M}_n^\ell(a_{\ell+1} | a_1^\ell),$$

we see that

$$\lim_{n \rightarrow \infty} \frac{T_n(a_1^{\ell+1})}{n} = P(a_1^{\ell+1})$$

holds if and only if

$$\lim_{n \rightarrow \infty} \hat{M}_n^\ell(a_{\ell+1} | a_1^\ell) = M(a_{\ell+1} | a_{\ell-k+1}^\ell).$$

By a similar evaluation in the proof of Lemma 4.2, we see that

$$\begin{aligned} \frac{\log K_n}{n} &\geq \sum_{a_1^\ell \in \Omega^\ell} \frac{T_n(a_1^\ell)}{n} \cdot D \left(\hat{M}_n^\ell(\cdot | a_1^\ell) \parallel M(\cdot | a_{\ell-k+1}^\ell) \right) \\ &\quad + \frac{1}{n} \left\{ -\log \tilde{P}(\omega_1^\ell) + \log \prod_{i=1}^{\ell} M(\omega_i | \omega_{n-k+i}^n \omega_1^{i-1}) \right\} \\ &\quad - \delta_\ell(n) + \frac{1}{n} \left\{ c(\omega_1^n) \log c(\omega_1^n) - (-\log Q_{LZ}(\omega_1^n)) \right\}. \end{aligned}$$

By the assumption of induction, $T_n(a_1^\ell)/n$ converges to a positive number $P(a_1^\ell)$ for all $a_1^\ell \in \Omega^\ell$ as $n \rightarrow \infty$. Therefore, if $\hat{M}_n^\ell(a_{\ell+1} | a_1^\ell)$ does not converge to $M(a_{\ell+1} | a_{\ell-k+1}^\ell)$ as $n \rightarrow \infty$, we have $\limsup_{n \rightarrow \infty} K_n = \infty$. This completes the proof. $\square$

Chapter 5

Applications

In the previous chapter, we demonstrated that adopting a coding-theoretic idea leads to a new way of understanding game-theoretic probability. This methodological innovation motivates further investigation into its broader implications. Here we present two applications of our main result. The first concerns Szilárd’s engine, which bridges thermodynamics and information theory, while the second raises questions related to stationary ergodic games and the role of entropy within game-theoretic contexts.

5.1 Szilárd’s engine game

We begin by applying the framework of predictive games to thermodynamics, conceptualizing a thermodynamic cyclic as a betting game between Scientist and Nature¹. As a fundamental prototype, we consider a work-extracting game inspired by Leó Szilárd’s thought experiment [6]², which has been widely discussed in the context of the second law of thermodynamics and its relationship to information theory.

Consider the following work-extracting game played on a hypothetical engine illustrated in Figure 5.1:

- (i) A partition, connected to two containers by inextensible strings, is placed at a specific position within a cylinder and fixed in place.
- (ii) Scientist places a weight $m(0)$ on the left container and another weight $m(1)$ on the right container.

¹A related perspective is discussed in [26], which aims to give a game-theoretic characterization of Gibbs’ distribution. Our approach differs by emphasizing the coding theoretic aspect of Szilárd’s engine while adhering closely to the original formalism of the Shafer-Vovk theorem.

²The original work [6] is in German; an English translation is available [?].

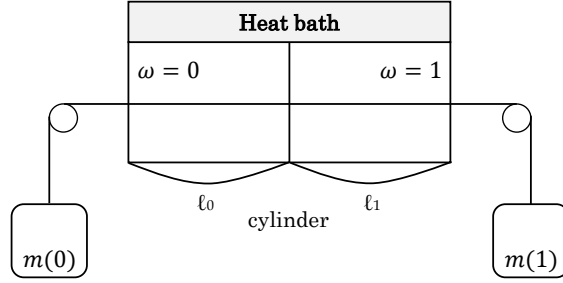


Figure 5.1: Szilárd's engine game.

- (iii) Nature inserts a single molecule into one side of the partition, announces whether the molecule is in the left chamber ($\omega = 0$) or the right chamber ($\omega = 1$), and then releases the partition.
- (iv) If $\omega = 0$, the molecule pushes the partition to the right, and Scientist gains potential energy $m(0)g\ell_1 - m(1)g\ell_1$, where g is the gravitational acceleration and ℓ_1 is the displacement of the weights. If $\omega = 1$, on the other hand, Scientist instead gains potential energy $m(1)g\ell_0 - m(0)g\ell_0$.
- (v) Once the partition reaches the end of the cylinder, it is reset to its original position as in step (i).

Letting $\Omega := \{0, 1\}$ and

$$r := \frac{\ell_1}{\ell_0 + \ell_1} \in (0, 1),$$

the above procedure can be formulated as a game-theoretic process:

Szilárd's engine game

Players : Scientist and Nature.

Protocol : $W_0 = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

Scientist announces $m_n = (m_n(0), m_n(1)) \in \mathbb{R}^\Omega$.

Nature announces $\omega_n \in \Omega$.

$W_n := W_{n-1} + (m_n(1) - m_n(0))g(\ell_0 + \ell_1)(\omega_n - r)$.

END FOR.

At first glance, this game appears to favor Nature, as Nature announces ω_n after Scientist has set the weights. However, we can prove the following.

Theorem 5.1. *In Szilárd’s engine game, Scientist has a prudent strategy $\{m_n\}_n$ that ensures $\lim_{n \rightarrow \infty} W_n = \infty$ unless*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_1(\omega_i) = r.$$

Proof. Observe that the recurrence relation is rewritten as

$$W_n := W_{n-1} \left[1 + \sum_{a \in \Omega} \alpha_n(a) (\delta_{\omega_n}(a) - p(a)) \right],$$

where $p := (p(0), p(1)) := (1 - r, r)$ and

$$\alpha_n(a) := \frac{m_n(a)g(\ell_0 + \ell_1)}{W_{n-1}}.$$

Thus, Theorem 5.1 is an immediate consequence of Theorem 2.1. $\square$

The implication of Theorem 5.1 is as follows: If Nature does not behave in accordance with the expected statistical law, Scientist can extract an infinite amount of work from the engine. A distinctive feature of this finding is that it does not require invoking Maxwell’s demon [27] or employing any measurement scheme to determine a molecule’s position before setting weights. Instead, Scientist only needs to detect deviations in Nature’s behavior from the law of large numbers.

This result closely resembles Kelvin’s formulation of the second law of thermodynamics, which asserts that it is impossible to extract any net amount of work from a thermodynamic system while leaving the system in the same state. To clarify the position of our framework within existing theory, it is helpful to compare it with the traditional information-theoretic interpretation of Maxwell’s demon. In that framework, the paradox is resolved by introducing the costs of information acquisition and erasure, where “information” serves as a compensating quantity [28, 29]. In contrast, our formulation is more flexible: it does not forbid *any* net conversion of heat into work from a single heat bath by an arbitrary cyclic process, but rather forbids the *unbounded* extraction of work from the system.

Extending the previous argument to the case when the outcome space Ω is an arbitrary finite set is straightforward. Consider a device illustrated in Figure 5.2, corresponding to the case when $\Omega = \{1, 2, 3\}$. The cylinder contains two partitions, dividing it into three chambers labeled by $\omega = 1, 2, 3$. Each partition is connected to two containers by inextensible strings and negligibly small pulleys that can move horizontally. Weights can be placed on these containers. The containers correspond one-to-one with the chambers and are labeled accordingly.

A generalized Szilard’s engine game for $\Omega = \{1, 2, 3\}$ runs as follows:

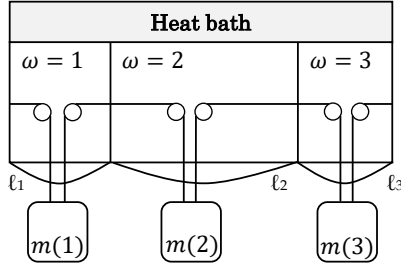


Figure 5.2: Generalized Szilárd's engine game having three chambers. The pulleys can move horizontally and are assumed to be negligibly small.

- (i) Each of two partitions is placed at a specific position within the cylinder and fixed in place.
- (ii) Scientist places a weight $m(a)$ on each container a for $a = 1, 2, 3$.
- (iii) Nature places a single molecule in one of the three chambers, announces its label ω , and releases the partitions.
- (iv) The molecule pushes the partitions at the boundaries of chamber ω , causing the chamber to expand until all the partitions are pressed against the end(s) of the cylinder, and Scientist gains potential energy as follows: If $\omega = 1$, the work extracted is

$$m(1)g\frac{\ell_2 + \ell_3}{2} - m(2)g\frac{\ell_2}{2} - m(3)g\frac{\ell_3}{2}.$$

If $\omega = 2$, the work extracted is

$$m(2)g\frac{\ell_3 + \ell_1}{2} - m(3)g\frac{\ell_3}{2} - m(1)g\frac{\ell_1}{2}.$$

If $\omega = 3$, the work extracted is

$$m(3)g\frac{\ell_1 + \ell_2}{2} - m(1)g\frac{\ell_1}{2} - m(2)g\frac{\ell_2}{2}.$$

- (v) Once the partitions come to rest, they return to their original positions as in step (i).

In a single round of the game, Scientist extracts the following amount of work:

$$\sum_{a=1}^3 \frac{m(a)g}{2} (\ell_1 + \ell_2 + \ell_3) \left(\delta_\omega(a) - \frac{\ell_a}{\ell_1 + \ell_2 + \ell_3} \right).$$

Generalizing Szilárd's engine game to an arbitrary finite set $\Omega = \{1, 2, \dots, A\}$ is straightforward: one simply increases the number of chambers illustrated in Figure 5.2. Defining

$$p(a) := \frac{\ell_a}{\ell_1 + \dots + \ell_A}, \quad (a \in \Omega),$$

we can formulate the generalized work-extracting protocol as follows.

Generalized Szilárd's engine game

Players : Scientist and Nature.

Protocol : $W_0 = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

 Scientist announces $m_n \in \mathbb{R}^\Omega$.

 Nature announces $\omega_n \in \Omega$.

$W_n := W_{n-1}$

$$+ \sum_{a \in \Omega} \frac{m_n(a)g}{2} (\ell_1 + \dots + \ell_A) (\delta_{\omega_n}(a) - p(a)).$$

END FOR.

Now, we extend Theorem 5.1 to this generalized setting:

Theorem 5.2. *In generalized Szilárd's engine game, Scientist has a prudent strategy $\{m_n\}_n$ that ensures $\lim_{n \rightarrow \infty} W_n = \infty$ unless*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_a(\omega_i) = p(a), \quad (\forall a \in \Omega).$$

We can further generalize the game described above by allowing the chamber size ratios $p(a)$ to vary in each round n , introducing a Forecaster who announces these ratios. In this extended protocol, Theorem 2.2 provides a generalization of Theorem 5.2, incorporating a time-homogeneous finite-order Markovian Forecaster.

5.2 Entropy

Given the pivotal role of universal coding schemes in establishing game-theoretic law of large numbers, it is natural to expect that the protocol of a predictive game is also intertwined with the concept of entropy. The next proposition formalizes this connection, where we continue to assume that M is a k th-order Markov kernel with strictly positive entries and π is the stationary distribution of M .

Proposition 5.3. *In the time-homogeneous k th-order Markovian predictive game, Skeptic has a prudent strategy $\alpha : \Omega^* \rightarrow \mathbb{R}^\Omega$ that ensures $\lim_{n \rightarrow \infty} K_n = \infty$ unless*

$$\lim_{n \rightarrow \infty} \frac{-\log Q_{LZ}(\omega_1^n)}{n} = H(M), \quad (5.1)$$

where

$$H(M) := - \sum_{a_1^k \in \Omega^k} \pi(a_1^k) \sum_{b \in \Omega} M(b | a_1^k) \log M(b | a_1^k)$$

is the entropy rate.

Proof. We observed in the proof of Theorem 2.2 that, under the Lempel-Ziv betting strategy Q_{LZ} , the boundedness of the capital process, i.e., $\limsup_{n \rightarrow \infty} K_n < \infty$, guarantees not only that

$$\frac{\log K_n}{n} = \frac{1}{n} \left\{ -\log \tilde{P}(\omega_1^n) - (-\log Q_{LZ}(\omega_1^n)) \right\} \longrightarrow 0$$

but also that

$$\hat{M}_n^k(\cdot | a_1^k) \longrightarrow M(\cdot | a_1^k) \quad \text{and} \quad \frac{T_n(a_1^k)}{n} \longrightarrow \pi(a_1^k)$$

for all $a_1^k \in \Omega^k$ as $n \rightarrow \infty$.

As a consequence, using a similar computation as in the proof of Lemma 3.7, we obtain

$$\begin{aligned} -\frac{1}{n} \log \tilde{P}(\omega_1^n) &= - \sum_{a_1^k \in \Omega^k} \frac{T_n(a_1^k)}{n} \sum_{b \in \Omega} \hat{M}_n^k(b | a_1^k) \log M(b | a_1^k) \\ &\quad + \frac{1}{n} \left\{ -\log \tilde{P}(\omega_1^k) + \log \prod_{i=1}^k M(\omega_i | \omega_{n-k+i}^n \omega_1^{i-1}) \right\} \\ &\longrightarrow H(M). \end{aligned}$$

Combining these asymptotic properties, (5.1) follows immediately. $\square$

The implication of Proposition 5.3 is as follows: To prevent Skeptic from becoming infinitely rich, Reality must ensure that the asymptotic compression rate of its moves coincides with the entropy rate.

Note that Proposition 5.3 bears a close resemblance to Lempel-Ziv's theorem [5]

$$\lim_{n \rightarrow \infty} \frac{\ell_{LZ}(\omega_1^n)}{n} = H_A(P), \quad P\text{-a.s.}$$

as well as Brudno's theorem [30]

$$\lim_{n \rightarrow \infty} \frac{\mathcal{K}(\omega_1^n)}{n} = H_A(P), \quad P\text{-a.s.}$$

for the prefix Kolmogorov complexity $\mathcal{K}(\omega_1^n)$ when data ω_1^n are drawn according to a stationary ergodic probability measure P on Ω^∞ , where

$$H_A(P) := \lim_{n \rightarrow \infty} \mathbb{E} \left[-\frac{1}{n} \log_A P(\omega_1^n) \right]$$

is the entropy rate to the base A . In addition, the last asymptotic property in the proof of Proposition 5.3 corresponds to the Shannon-McMillan-Breiman theorem [24]

$$\lim_{n \rightarrow \infty} \left\{ -\frac{1}{n} \log P(\omega_1^n) \right\} = H(P), \quad P\text{-a.s.}$$

which also has a counterpart in algorithmic randomness theory [31, 32, 33]. These observations prompt us to call a Forecaster *stationary ergodic* if they announce predictions according to the prescription

$$p_n(\omega_n) := P(\omega_n \mid \omega_1^{n-1}), \quad (\omega_1^n \in \Omega^n),$$

where P represents a predetermined stationary ergodic probability measure.

If we were to discover a compression algorithm capable of efficiently compressing Reality's moves within the game-theoretic context, we could define the entropy of a game as the asymptotic data compression rate, assuming that Reality faithfully follows the predictions of a stationary ergodic Forecaster and thereby prevents Skeptic from becoming infinitely rich.

However, this definition, which relies on the existence of a stationary ergodic Forecaster, may not be fully satisfactory from the perspective of Dawid's prequential principle [34], as the prequential framework does not impose any structural assumptions on the underlying data-generating mechanisms but instead evaluates predictive performance based solely on the observed data sequence. The validation and further exploration of the concepts of game entropy and stationary ergodic Forecasters remain topics for future investigation.

Part II

Quantum Information Theoretic Approach to Game-Theoretic Probability Theory

Chapter 6

Quantum Game with Mutually Unbiased Bases

Adaptive (or online) quantum state estimation [35, 36] refers to the task of estimating an unknown quantum state while sequentially updating measurement settings based on past measurement outcomes. It has become increasingly important in quantum information processing, including adaptive tomography, quantum control, and quantum computing. A quantum-specific difficulty, distinct from the classical case, is the intrinsic coupling between the choice of measurements and the information extractable from the system. Achieving optimal convergence rates often necessitates adaptive design of measurement schemes. Most existing approaches rely on probabilistic analyses assuming underlying models, focusing on variance, expected error, or convergence in probability.

In contrast, game-theoretic probability theory allows one to interpret the construction of betting strategies as a form of sequential inference driven solely by Reality's data sequence, without assuming an underlying probabilistic model. From this perspective, this chapter provides a first step toward a game-theoretic framework for quantum state estimation.

We introduce a predictive quantum game in which Skeptic sequentially predicts quantum measurement outcomes, while Reality adversarially selects the outcomes. Our setting models online quantum state estimation in a scenario where the available POVMs may vary from round to round. The main result establishes a quantum analogue of the game-theoretic law of large numbers for sequential quantum estimation under adversarially varying measurements.

Let $\mathcal{H}$ be a d -dimensional Hilbert space, and let

$$\mathcal{S}(\mathcal{H}) := \{\rho \in \mathcal{L}(\mathcal{H}) \mid \rho > 0, \text{Tr}\rho = 1\}$$

denote the set of quantum states on $\mathcal{H}$. We note that $\rho \in \mathcal{S}(\mathcal{H})$ is a full-rank

quantum state by definition. We say that k orthonormal bases of $\mathcal{H}$

$$\{|\phi^{(1)}\rangle\}_{\phi \in \{1,2,\dots,d\}}, \{|\psi^{(2)}\rangle\}_{\psi \in \{1,2,\dots,d\}}, \dots, \{|\chi^{(k)}\rangle\}_{\chi \in \{1,2,\dots,d\}}$$

are mutually unbiased [37] if they satisfy, for all $s, t \in \{1, 2, \dots, k\}$ with $s \neq t$, and for all $\psi, \phi \in \{1, 2, \dots, d\}$,

$$|\langle \phi^{(s)} | \psi^{(t)} \rangle|^2 = \frac{1}{d}.$$

When we can take $k = d + 1$ mutually unbiased bases (MUBs), $\mathcal{H}$ is said to admit a full set of MUBs. The existence of a full set of MUBs is known in the case where d is a prime power [38]. We suppose that $\mathcal{H}$ admits a full set of MUBs throughout this chapter. An important property of MUBs is that every density operator $\rho \in \mathcal{S}(\mathcal{H})$ admits a unique representation of the form

$$\rho = \sum_{s=1}^{d+1} \sum_{\psi=1}^d \xi_{\psi}^{(s)} M^{(s)}(\psi^{(s)}) - I, \quad (6.1)$$

where $M^{(s)}(\psi^{(s)}) := |\psi^{(s)}\rangle \langle \psi^{(s)}|$ and coefficients $\xi_{\psi}^{(s)}$ are nonnegative and satisfy $\sum_{\psi=1}^d \xi_{\psi}^{(s)} = 1$ for each s [39].

In this chapter, we define $\Omega := \{1, 2, \dots, d\}$. For a fixed $\rho \in \mathcal{S}(\mathcal{H})$, consider the following game.

Quantum MUB-type predictive game

Players : Forecaster, Skeptic and Reality.

Protocol : $K_0 = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

Forecaster announces a set of MUBs $\{ \{ |\psi_n^{(s)}\rangle \}_{\psi=1}^d \}_{s=1}^{d+1}$

Skeptic announces $\{q_n^{(s)} \in \mathcal{P}(\Omega)\}_{s=1}^{d+1}$.

Reality announces $\omega_n = (\omega_n^{(1)}, \omega_n^{(2)}, \dots, \omega_n^{(d+1)}) \in \Omega^{d+1}$.

$$K_n := K_{n-1} \cdot \prod_{s=1}^{d+1} \frac{q_n^{(s)}(\omega_n^{(s)})}{\text{Tr} \rho M_n^{(s)}(\omega_n^{(s)})}.$$

END FOR.

In this game, Forecaster determines Skeptic's measurement protocol by declaring a set of mutually unbiased bases. Physically, each round may be interpreted as follows. Forecaster prepares $d + 1$ independent copies of the same quantum state ρ . Each copy is measured using a different mutually unbiased basis announced by Forecaster, yielding outcomes $\omega_n^{(1)}, \dots, \omega_n^{(d+1)}$. Skeptic's capital process is then defined as a likelihood-ratio betting scheme against the Born probabilities induced by ρ .

We associate the announced MUBs $\{ \{ |\psi_n^{(s)}\rangle \}_{\psi=1}^d \}_{s=1}^{d+1}$ with the corresponding projective measurement $M_n = \{ \{ M_n^{(s)}(a) \}_{a \in \Omega} \}_{s=1}^{d+1}$. We define $\hat{\rho}_n$ by

$$\hat{\rho}_n := \left[\frac{1}{n} \sum_{i=1}^n \left(\sum_{s=1}^{d+1} M_i^{(s)}(\omega_i^{(s)}) - I \right) \right].$$

This estimator is motivated by the linear reconstruction formula (6.1) for quantum states based on a full set of MUBs. We note that $\hat{\rho}_n$ is not necessarily a quantum state. We call the sequence $(M_n, q_n)_n$ of measurements M_n constructed from MUBs and a family of $q_n = \{ q_n^{(s)} \in \mathcal{P}(\Omega) \}_{s=1}^{d+1}$ a strategy of Skeptic.

With the above notations, here is the main result of this chapter.

Theorem 6.1. *If Forecaster announces measurements M_n satisfying*

$$\| M_{n+1}^{(s)}(a) - M_n^{(s)}(a) \| = o(1/n) \quad \text{uniformly in } s \in \{1, 2, \dots, d+1\} \text{ and } a \in \Omega,$$

with respect to the norm induced from the Hilbert-Schmidt inner product, then Skeptic has a strategy $(M_n, q_n)_n$ that ensures $\lim_{n \rightarrow \infty} K_n = \infty$ unless

$$\lim_{n \rightarrow \infty} \hat{\rho}_n = \rho.$$

Throughout this chapter, $\| \cdot \|$ denotes the Hilbert-Schmidt norm, unless stated otherwise. Informally, Theorem 6.1 states that if measurement settings do not change too quickly, then Skeptic can enforce convergence of the empirical estimator to the true state, in analogy with the classical game-theoretic law of large numbers without assuming any probability model.

In other words, even in a fully adversarial setting with no probabilistic assumptions and non-commuting measurements, the behavior of the observed outcomes is forced to be asymptotically consistent with ρ .

A key technical challenge in proving Theorem 6.1, which has no classical analogue, arises from the presence of measurement drift under non-commuting observation operators. This necessitates a regret-type analysis adapted to the quantum setting.

This prevents a direct application of classical arguments based on estimation theory or coding theory, and instead requires a careful analysis that explicitly accounts for quantum-specific sources of regret.

6.1 Regret analysis and proof of the main theorem

In this section, we construct Skeptic's betting strategy and analyze the associated regrets. This regret analysis provides the key quantitative control needed to prove the main theorem.

We construct Skeptic's betting strategy $(q_n)_n$ using the Laplace estimator, which plays a fundamental role in universal coding and in classical proofs of the game-theoretic law of large numbers (see Appendix B). For all $s \in \{1, \dots, d+1\}$, define

$$q_n^{(s)}(a) := \frac{n-1}{n-1+d} \cdot \frac{S_{n-1}^{(s)}(a)}{n-1} + \frac{d}{n-1+d} \cdot \frac{1}{d} = \frac{S_{n-1}^{(s)}(a) + 1}{n-1+d},$$

where $S_n^{(s)}(a)$ denotes the number of occurrences of the symbol a in the sequence $\omega_1^{(s)} \dots \omega_n^{(s)}$. By construction, it is easy to see that $q_n^{(s)} \in \mathcal{P}(\Omega)$ for all n and s .

6.1.1 Regret analysis

The following lemma evaluates the log-loss regret between the sequential estimation and maximum likelihood estimation. Equivalently, Skeptic's cumulative log-loss using $(q_n)_n$ is within $O(\log n)$ of the best fixed distribution in hindsight. The proof of this lemma relies only on classical arguments.

Lemma 6.2. *Define the regret R_n^1 as the difference between the cumulative log-loss of the sequential estimator $(q_n)_n$ and that of the maximum likelihood estimator in hindsight as follows:*

$$R_n^1 := \sum_{i=1}^n \sum_{s=1}^{d+1} \left[\log q_i^{(s)}(\omega_i^{(s)}) \right] - \sum_{i=1}^n \sum_{s=1}^{d+1} \left[\log \hat{q}_n^{(s)}(\omega_i^{(s)}) \right],$$

where $\hat{q}_n^{(s)}(a) := S_n^{(s)}(a)/n$. Then

$$R_n^1 = O(\log n).$$

Proof. For fixed $s \in \{1, \dots, d+1\}$,

$$\begin{aligned} \sum_{i=1}^n \log q_i^{(s)}(\omega_i^{(s)}) &= \sum_{i=1}^n \log \frac{S_{i-1}^{(s)}(\omega_i^{(s)}) + 1}{i-1+d} = \log \frac{S_n(1)^{(s)}! S_n(2)^{(s)}! \dots S_n(d)^{(s)}!}{(n-1+d)!} \\ &= \sum_{a \in \Omega} (S_n^{(s)}(a) \log S_n^{(s)}(a) - S_n^{(s)}(a)) \\ &\quad - (n-1+d) \log(n-1+d) + n + O(\log n) \\ &= n \sum_{a \in \Omega} \hat{q}_n^{(s)}(a) \log \hat{q}_n^{(s)}(a) + O(\log n), \end{aligned}$$

where we used Stirling's formula $\log n! = n \log n - n + O(\log n)$ in the third equation. Hence

$$R_n^1 = \sum_{s=1}^{d+1} \left\{ n \sum_{a \in \Omega} \hat{q}_n^{(s)}(a) \log \hat{q}_n^{(s)}(a) + O(\log n) - n \sum_{a \in \Omega} \hat{q}_n^{(s)}(a) \log \hat{q}_n^{(s)}(a) \right\} = O(\log n)$$

□

The following lemma bounds the contribution of measurement drift to the cumulative log-loss. Under the assumption that the measurement settings vary sufficiently slowly, this contribution grows only sublinearly in the number of rounds. Together with Lemma 6.2, which controls the estimation regret of Skeptic's strategy, this result completes the control of all error terms except for the divergence term that drives the proof of the main theorem.

Lemma 6.3. *For any $\sigma \in \mathcal{S}(\mathcal{H})$, define*

$$R_n^2(\sigma) := \sum_{i=1}^n \sum_{s=1}^{d+1} \left[-\log \text{Tr} \sigma M_i^{(s)}(\omega_i^{(s)}) \right] + \sum_{i=1}^n \sum_{s=1}^{d+1} \left[\log \text{Tr} \sigma M_n^{(s)}(\omega_i^{(s)}) \right].$$

Assume that the measurement operators satisfy

$$\|M_{t+1}^{(s)}(a) - M_t^{(s)}(a)\| = o(1/t), \quad \text{uniformly in } s \in \{1, \dots, d+1\} \text{ and } a \in \Omega.$$

Then

$$\lim_{n \rightarrow \infty} \frac{R_n^2(\sigma)}{n} = 0.$$

Proof. For each s, a , define

$$x_i^{(s)} := \text{Tr} \sigma M_i^{(s)}(\omega_i^{(s)}), \quad y_i^{(s)} := \text{Tr} \sigma M_n^{(s)}(\omega_i^{(s)}).$$

Then R_n^2 can be written as

$$R_n^2(\sigma) = \sum_{i=1}^n \sum_{s=1}^{d+1} \log \frac{y_i^{(s)}}{x_i^{(s)}}.$$

Since σ is a full-rank quantum state, all $x_i^{(s)}$ and $y_i^{(s)}$ are bounded below by the minimal eigenvalue of σ , $\lambda_{\min}(\sigma) > 0$. Using the inequality $|\log x - \log y| \leq |x - y|/(\min(x, y))$ for $x, y > 0$, we have

$$\left| \log \frac{y_i^{(s)}}{x_i^{(s)}} \right| \leq \frac{|y_i^{(s)} - x_i^{(s)}|}{\lambda_{\min}(\sigma)}.$$

Using the Cauchy–Schwarz inequality with respect to the Hilbert–Schmidt inner product, we obtain

$$|\text{Tr} \sigma A| \leq \|\sigma\| \cdot \|A\| \leq \|A\|,$$

since $\|\sigma\| \leq 1$ for any density operator σ . This is because, since σ is a density operator, we have $\|\sigma\|^2 = \text{Tr}(\sigma^2) \leq \text{Tr}(\sigma) = 1$. Hence we get that

$$\left| y_i^{(s)} - x_i^{(s)} \right| = |\text{Tr} \sigma (M_n^{(s)}(\omega_i) - M_i^{(s)}(\omega_i))| \leq \|M_n^{(s)} - M_i^{(s)}\|.$$

Thus

$$|R_n^2(\sigma)| \leq \frac{(d+1)}{\lambda_{\min}(\sigma)} \sum_{i=1}^n \|M_n - M_i\|,$$

where M_n represents the whole PVM indexed by (s, a) . Moreover, by the assumption, we get

$$\|M_n - M_i\| \leq \sum_{t=i}^{n-1} \|M_{t+1} - M_t\| = \sum_{t=i}^{n-1} o(1/t).$$

By the uniformity of the $o(1/t)$ bound, there exists a sequence $\varepsilon_t \rightarrow 0$ such that $\|M_{t+1} - M_t\| \leq \varepsilon_t/t$ for all t . Then, interchanging sums gives

$$\sum_{i=1}^n \sum_{t=i}^{n-1} o(1/t) = \sum_{t=1}^{n-1} t \cdot o(1/t) = \sum_{t=1}^{n-1} o(1) = o(n).$$

Thus $R_n^2(\sigma) = o(n)$ and hence $\lim_{n \rightarrow \infty} R_n^2(\sigma)/n = 0$. $\square$

The next lemma shows that, under the slow-variation assumption on the measurement sequence, the empirical average of measurement outcomes from earlier rounds is close, in operator norm, to the operator constructed from the measurements in the most recent round. This bound is crucial for controlling the regret term arising from measurement drift.

Lemma 6.4. *Let*

$$X_n := \left(\sum_{s=1}^{d+1} \sum_{a=1}^d \hat{q}_n^{(s)}(a) M_n^{(s)}(a) \right) - I, \quad \hat{\rho}_n := \frac{1}{n} \sum_{i=1}^n \left(\sum_s M_i^{(s)}(\omega_i^{(s)}) - I \right).$$

Assume that the measurement operators satisfy

$$\|M_{t+1}^{(s)}(a) - M_t^{(s)}(a)\| = o(1/t), \quad \text{uniformly in } s \text{ and } a.$$

Then

$$\|X_n - \hat{\rho}_n\| = o(1),$$

Proof. Rewrite $\hat{\rho}_n$ and X_n as

$$\hat{\rho}_n = \sum_{s,a} \frac{S_n^{(s)}(a)}{n} \cdot \frac{1}{S_n^{(s)}(a)} \sum_{i: \omega_i^{(s)}=a} M_i^{(s)}(a) - I, \quad \text{and } X_n = \sum_{s,a} \frac{S_n^{(s)}(a)}{n} M_n^{(s)}(a) - I.$$

We note that, for those (s, a) such that $\hat{q}_n^{(s)}(a) \rightarrow 0$, the corresponding summands are multiplied by $\hat{q}_n^{(s)}(a)$ and hence vanish in the limit $n \rightarrow \infty$. Thus it suffices to

consider indices satisfying $\liminf_n \hat{q}_n^{(s)}(a) > 0$. Then

$$X_n - \hat{\rho}_n = \sum_{s,a} \hat{q}_n^{(s)}(a) \left(M_n^{(s)}(a) - \frac{1}{S_n^{(s)}(a)} \sum_{i:\omega_i^{(s)}=a} M_i^{(s)}(a) \right).$$

Taking the Hilbert-Schmidt norm, and using the triangle inequality, we obtain

$$\begin{aligned} \|X_n - \hat{\rho}_n\| &\leq \sum_{s,a} \hat{q}_n^{(s)}(a) \frac{1}{S_n^{(s)}(a)} \sum_{i:\omega_i^{(s)}=a} \|M_n^{(s)}(a) - M_i^{(s)}(a)\| \\ &\leq \sum_{s,a} \hat{q}_n^{(s)}(a) \frac{n}{S_n^{(s)}(a)} \frac{1}{n} \sum_{i=1}^n \|M_n^{(s)}(a) - M_i^{(s)}(a)\|. \end{aligned}$$

Since $\hat{q}_n^{(s)}(a) = S_n^{(s)}(a)/n$, we obtain

$$\|X_n - \hat{\rho}_n\| \leq \sum_{s,a} \frac{1}{n} \sum_{i=1}^n \|M_n^{(s)}(a) - M_i^{(s)}(a)\|.$$

The inner term can be evaluated by

$$\|M_n^{(s)}(a) - M_i^{(s)}(a)\| \leq \sum_{t=i}^{n-1} \|M_{t+1} - M_t\| = \sum_{t=i}^{n-1} o(1/t).$$

Using the same summation as in Lemma 6.3,

$$\frac{1}{n} \sum_{i=1}^n \|M_n^{(s)}(a) - M_i^{(s)}(a)\| \leq o(1).$$

Since the sum is over finitely many indices, we conclude that

$$\|X_n - \hat{\rho}_n\| = o(1).$$

□

6.1.2 Proof of Theorem 6.1

In this subsection, we complete the proof of Theorem 6.1. The key idea is to decompose the capital process into three contributions: (i) the classical regret term arising from the Laplace estimator (Lemma 6.2), (ii) the drift term due to changing measurements (Lemma 6.3), and (iii) the divergence between the empirical frequencies and the Born probabilities associated with the measurement at time n . The first two terms are shown to be negligible compared to n . Therefore, Reality can prevent Skeptic from becoming infinitely rich only if the divergence term vanishes asymptotically. This yields the quantum version of the game-theoretic law of large numbers.

Proof of Theorem 6.1. We evaluate the asymptotic behavior of the capital process $\log K_n$. Using the decomposition introduced above, we obtain

$$\begin{aligned}
\frac{\log K_n}{n} &= \frac{1}{n} \left\{ \sum_{i=1}^n \sum_{s=1}^{d+1} \left[\log q_i^{(s)}(\omega_i^{(s)}) \right] - \sum_{i=1}^n \sum_{s=1}^{d+1} \left[\log \text{Tr} \rho M_i^{(s)}(\omega_i^{(s)}) \right] \right\} \\
&= \frac{R_n^1}{n} + \frac{1}{n} \sum_{i=1}^n \sum_{s=1}^{d+1} \left[\log \hat{q}_n^{(s)}(\omega_i) - \log \text{Tr} \rho M_i^{(s)}(\omega_i^{(s)}) \right] \\
&= \frac{R_n^1}{n} + \frac{1}{n} \sum_{i=1}^n \sum_{s=1}^{d+1} \left[\log \hat{q}_n^{(s)}(\omega_i^{(s)}) - \log \text{Tr} \rho M_n^{(s)}(\omega_i^{(s)}) \right] + \frac{R_n^2(\rho)}{n} \\
&= \frac{R_n^1}{n} + \sum_{s=1}^{d+1} D(\hat{q}_n^{(s)} \| \text{Tr} \rho M_n^{(s)}) + \frac{R_n^2(\rho)}{n}.
\end{aligned}$$

By Lemmas 6.2, 6.3,

$$\frac{R_n^1}{n} \rightarrow 0, \quad \frac{R_n^2(\rho)}{n} \rightarrow 0.$$

Due to the property of the divergence, to prevent Skeptic becoming infinitely rich, Reality should satisfy, for all $s \in \{1, \dots, d+1\}$ and $a \in \Omega$,

$$\lim_{n \rightarrow \infty} \{ \hat{q}_n^{(s)}(a) - \text{Tr} \rho M_n^{(s)}(a) \} = 0.$$

Since measurements are chosen from a complete set of MUBs, the linear reconstruction formula (6.1) implies that the limiting empirical frequencies uniquely determine the state, that is,

$$\lim_{n \rightarrow \infty} X_n = \lim_{n \rightarrow \infty} \sum_{s=1}^{d+1} \sum_{a=1}^d \hat{q}_n^{(s)}(a) M_n^{(s)}(a) - I = \rho.$$

Finally, Lemma 6.4 ensures that

$$\hat{\rho}_n \rightarrow \rho.$$

□

6.2 Discussions

The quantum predictive game studied in this chapter provides a first step toward a game-theoretic formulation of quantum probability. However, it remains close to the classical setting in several important respects. In particular, the use of mutually unbiased bases and the assumption that measurement settings vary sufficiently slowly ensure that the effective behavior is asymptotically almost commutative.

While this allows us to establish a quantum analogue of the game-theoretic law of large numbers, it also highlights the intrinsic limitations of the present framework. Genuinely quantum effects, such as noncommutativity in measurement sequences or in sequences of quantum states declared by agents, are largely suppressed. Thus, the present results should be seen as a conservative extension of classical game-theoretic probability rather than a fully quantum theory.

A natural question is how to formulate predictive games in which noncommutativity plays an essential role. As a step toward this direction, we propose the following generalized quantum predictive game:

Generalized quantum predictive game

Players : Forecaster, Skeptic, and Reality.

Protocol : $K_0 = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

Forecaster announces a quantum state $\rho_n \in \mathcal{S}(\mathcal{H})$.

Skeptic (or Forecaster) announces an Ω -valued POVM $\{M_n(a)\}_{a \in \Omega}$.

Skeptic announces a quantum state $\sigma_n \in \mathcal{S}(\mathcal{H})$.

Reality announces $\omega_n \in \Omega$.

$$K_n := K_{n-1} \cdot \frac{\text{Tr} \sigma_n M_n(\omega_n)}{\text{Tr} \rho_n M_n(\omega_n)}.$$

END FOR.

This game extends the classical predictive protocol in two ways. First, the prediction target is a quantum state rather than a classical probability distribution; second, the choice of measurements is an explicit strategic choices, intertwining predictions and observations in a fully quantum manner. The capital update is defined via a likelihood ratio based on the Born rule, comparing Skeptic's hypothetical state σ_n with the Forecaster's announced state ρ_n . This formulation captures features absent in the classical theory. These include the noncommutativity of observables, the dependence of outcomes on measurement choices, and a loss function that explicitly depends on the measurements.

The generalized quantum predictive game also opens connections to other quantum information concepts. From the viewpoint of online learning, the game suggests a fully quantum notion of regret, where both predicted states and measurements contribute to the loss. From an information-theoretic perspective, one may expect analogues of the typical subspaces, the quantum source coding, and the Holevo bound to play a role.

Moreover, the game raises questions about quantum analogues of martingales and algorithmic randomness. In classical game-theoretic probability, capital processes form martingales and underpin notions of sequence randomness. In the quantum setting, however, capital evolution depends on both outcome probabilities and

measurement choices, complicating direct generalizations.

Finally, the choice of logarithmic loss itself merits careful consideration. In classical game-theoretic probability, the logarithm converts multiplicative capital growth into an additive quantity, naturally leading to the Kullback–Leibler divergence and operational interpretations in terms of coding and thermodynamics. In the quantum case, however, the logarithm is applied only to classical probabilities obtained after measurements, so the capital process and regret remain defined after reduction to classical data. Whether a more intrinsic quantum formulation, for example using quantum relative entropy or other operator-valued divergences, can serve as an appropriate notion of loss remains an open question. Clarifying the operational meaning of such quantum loss functions, and their relation to capital growth, coding theorems, or thermodynamic interpretations, is an important direction for future research.

In summary, while the results of this chapter establish a quantum analogue of the classical game-theoretic law of large numbers under slowly varying measurement conditions, they are inherently limited in capturing fully quantum phenomena. The generalized quantum predictive game provides a natural framework to explore these limitations and to guide future investigations into genuinely quantum game-theoretic probability.

Chapter 7

A game-theoretic probability approach to loopholes in CHSH experiments

Bell’s inequality was formulated in the context of the EPR paradox [40], and it characterizes correlations that are compatible with local hidden-variable models (see, e.g. Bell’s seminal work [8] and comprehensive reviews [41]). Among its variants, the CHSH inequality [7] has played a central role. It has been extensively studied in foundations of quantum mechanics and subjected to thorough experimental tests [42, 43, 44].

In certain experimental configurations, however, it cannot be ruled out that a system may appear to violate the CHSH inequality even though it follows a hidden-variable model. Such possibilities are known as “loopholes” and the identification and the control of loopholes in CHSH-type experiments have been a topic of sustained theoretical and experimental interest [45, 46].

This chapter focuses on two representative loopholes, the locality loophole and the measurement-dependence loophole, also known as the freedom-of-choice loophole. The locality loophole refers to the possibility that the outcomes at spatially separated measurement sites may influence one another. This loophole is addressed experimentally by performing measurements at sufficiently distant locations, where the measurement results do not exert an immediate influence. The measurement-dependence loophole refers to the possibility that the choice of measurement settings is statistically correlated with a hidden variable. Experimentally, this loophole is considered closed by selecting the measurement settings “freely” or “randomly,” for example using physical random number generators after particle generation [47]. Modern “loophole-free” Bell tests that simultaneously aim to close these loopholes have reported violations of the CHSH inequality consistent with quantum-mechanical predictions [48, 49].

While most analyses of the CHSH inequality rely on probabilistic or measure-theoretic formulations, comparatively little attention has been paid to how the availability and timing of information themselves constrain possible correlations. The objective of this chapter is to shed new light on the conditions under which the CHSH inequality holds or is violated, from this informational perspective. To this end, we employ the framework of game-theoretic probability theory [1].

Game-theoretic probability theory models statistical behavior as arising from a betting game, without introducing an underlying probability space. Within this framework, players act sequentially, so that one player may choose a move after observing the other's. This asymmetry in information and timing plays a central role in the emergence of statistical behavior. This explicit representation of these structural features enables us to reconsider the assumptions underlying the CHSH inequality from an alternative perspective.

In this chapter, we formalize the locality loophole and the measurement-dependence loophole as structural conditions in games between Scientists and Nature inspired by hidden-variable scenarios. We then introduce a loopholes-closed game equipped with two capital processes. One is designed to test whether empirical conditional frequencies of outcomes are consistent with the CHSH correlations. The other probes whether, across trials, the chosen measurement settings exhibit any systematic dependence on Nature's outcome assignments. Importantly, this second test is formulated in terms of empirical frequencies, providing an operational non-probabilistic analogue of the usual measurement-independence assumption.

The main result shows that these two requirements cannot be satisfied simultaneously within this game-theoretic framework. This incompatibility leads to a formulation in which violations of the CHSH inequality can be understood operationally, in terms of sequential betting strategies, once the relevant structural conditions are enforced.

7.1 CHSH inequality and its assumption

This section summarizes the CHSH inequality and related topics for the reader's convenience and clarifies terminology usage. First, we give the mathematical formulation and proof of the CHSH inequality. Next, we explain how this assumption is interpreted in physical implementations. We finally recall how quantum mechanics gives rise to correlations that violate the inequality.

We first introduce the CHSH inequality in a measure-theoretic setting. Let X_s , for $s \in \{1, 2, 3, 4\}$, be a real-valued random variables on a probability space $(\Lambda, \mathcal{F}, P)$.

For $s, t \in \{1, 2, 3, 4\}$, define the correlations

$$C(s, t) := \mathbb{E}[X_s \cdot X_t] = \int_{\Lambda} X_s(\lambda) X_t(\lambda) dP(\lambda),$$

then the following inequality holds.

Theorem 7.1 (CHSH inequality). *Assume that, for any $s \in \{1, 2, 3, 4\}$, $|X_s| \leq 1$ almost surely. Then*

$$|S| \leq 2,$$

where

$$S := C(1, 3) - C(1, 4) + C(2, 3) + C(2, 4).$$

Proof. By the definition,

$$\begin{aligned} C(1, 3) - C(1, 4) &= \mathbb{E}[X_1 X_3] - \mathbb{E}[X_1 X_4] \\ &= \mathbb{E}[X_1 X_3 \cdot (1 \pm X_2 X_4)] - \mathbb{E}[X_1 X_4 \cdot (1 \pm X_2 X_3)], \end{aligned}$$

where the same choice of sign is taken throughout. Due to the assumptions that $1 \pm X_2 X_4, 1 \pm X_2 X_3 \geq 0$ a.s. P ,

$$\begin{aligned} |C(1, 3) - C(1, 4)| &\leq \mathbb{E}[|X_1 X_3| \cdot (1 \pm X_2 X_4)] + \mathbb{E}[|X_1 X_4| \cdot (1 \pm X_2 X_3)] \\ &\leq 2 \pm \{C(2, 4) + C(2, 3)\}. \end{aligned}$$

Hence we get

$$\begin{aligned} C(1, 3) - C(1, 4) &\leq 2 - \{C(2, 4) + C(2, 3)\}, \text{ and} \\ C(1, 3) - C(1, 4) &\geq -2 - \{C(2, 4) + C(2, 3)\}. \end{aligned}$$

Thus we conclude that

$$|S| \leq 2.$$

□

In the measure-theoretic formulation above, the fact that the random variables $X_1, \dots, X_4$ are defined on a single probability space is simply a structural assumption on the model. When these variables are interpreted as outcomes of measurements labeled by $s \in \{1, 2, 3, 4\}$, this structural assumption is reinterpreted as the requirement that all four outcome values are simultaneously defined when λ is fixed, including those for settings that are not actually chosen in a given run of the experiment. These hypothetically assigned values are often referred to as counterfactual outcomes, in the sense that they correspond to outcomes of measurements that could have been performed but were not. In this sense, a hidden-variable model assumes that every experimental run carries predetermined values for all possible measurement settings, even though only one pair of them is observed.

On the other hand, within quantum mechanics, one can design experiments that violate the CHSH inequality. Let $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ and let ρ be a maximally entangled pure state on $\mathcal{H}_A \otimes \mathcal{H}_B$. For a standard choice of four local dichotomic observables

corresponding to measurement settings on $\mathcal{H}_A$ and $\mathcal{H}_B$ that maximize the CHSH violation, one obtains

$$C(1, 3) - C(1, 4) + C(2, 3) + C(2, 4) = -2\sqrt{2}.$$

The resulting joint outcome probabilities are consistent with the CHSH table used in the main text. For further details on Bell's theorem and the CHSH inequality in the quantum-mechanical setting, see [53].

7.2 From CHSH scenario to a game-theoretic framework

In this section, we explain the physical and game-theoretic situations we study. Scientist A and Scientist B are located at spatially separated sites. Each receives a particle, chooses one of two measurement settings, and performs the corresponding measurement; Nature then produces an outcome in $\Omega := \{+1, -1\}$ for each party.

In preparation for a game-theoretic interpretation of the CHSH inequality, we introduce a hidden-variable-style representation in which outcomes for all four measurement settings are preassigned. Let Λ be a set, and for each $s \in \{1, 2, 3, 4\}$, let $X_s : \Lambda \rightarrow \Omega$ be a function. We assume that Nature fixes the functions $X_1, \dots, X_4$ prior to the game protocol and that they are disclosed to all participants before the game begins. This differs from the usual hidden-variable narrative, where the hidden variable $\lambda \in \Lambda$ is not accessible to the experimenters; however, this difference is inessential for the arguments below, since we will not exploit any informational advantage from knowing λ itself.

In the standard CHSH formulation, Λ is equipped with a probability measure and each X_s is a random variable. Here, by contrast, we impose no probabilistic structure: Λ is merely a set and each X_s is simply a function. Our only structural assumption is that the map

$$\lambda \mapsto (X_1(\lambda), X_2(\lambda), X_3(\lambda), X_4(\lambda))$$

is surjective onto Ω^4 ; equivalently, for every $(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4$, there exists $\lambda \in \Lambda$ such that $X_s(\lambda) = \omega^s$ for all $s \in \{1, 2, 3, 4\}$.

Let

$$\Theta := \Theta^A \times \Theta^B := \{1, 2\} \times \{3, 4\}.$$

Here, $\Theta^A = \{1, 2\}$ and $\Theta^B = \{3, 4\}$ denote the sets of measurement settings available to Scientist A and Scientist B, respectively. The product $\Theta = \Theta^A \times \Theta^B$ therefore represents the set of all joint measurement settings (θ^A, θ^B) that can occur in each measurement.

$(\omega^A, \omega^B) \backslash (\theta^A, \theta^B)$	(1, 3)	(1, 4)	(2, 3)	(2, 4)
(+1, +1)	μ	ν	μ	μ
(-1, -1)	μ	ν	μ	μ
(+1, -1)	ν	μ	ν	ν
(-1, +1)	ν	μ	ν	ν

Table 7.1: CHSH table specifying the odds $p(\omega^A, \omega^B \mid \theta^A, \theta^B)$ for each setting pair (θ^A, θ^B) and outcome (ω^A, ω^B) , where $\mu := (2 - \sqrt{2})/8$ and $\nu := (2 + \sqrt{2})/8$.

We model Scientists and Nature as players in a game-theoretic setting, who place bets on the outcomes produced by Nature under these measurement choices. In this setting, the betting odds depends on the measurement settings and the outcomes generated by Nature. Table 7.1 specifies the betting odds assigned to each experimental outcome for each pair of measurement settings; we call it the *CHSH table*. The parameters $\mu := (2 - \sqrt{2})/8$ and $\nu := (2 + \sqrt{2})/8$ are chosen so as to match the quantum-mechanical prediction for the standard CHSH configuration achieving the maximal violation (the Tsirelson bound; see [50]).

For example, suppose Scientists A and B choose the measurement setting pair $(\theta^A, \theta^B) = (1, 4)$. Then the odds assigned to the outcome $(\omega^A, \omega^B) = (+1, -1)$ are μ ; we write $p(+1, -1 \mid 1, 4) = \mu$.

Although the CHSH table may be read as a table of “conditional probabilities,” this probabilistic interpretation plays no role in our game-theoretic framework. What matters for the game is only that Scientists agree on these odds before betting.

The key observation is that for each fixed measurement pair $(s, t) \in \Theta$, the distribution $p(\cdot, \cdot \mid s, t)$ is the element of $\mathcal{P}(\Omega^2)$, that is,

$$\sum_{(a,b) \in \Omega^2} p(a, b \mid s, t) = 2\mu + 2\nu = 1.$$

Thus, at the level of each fixed measurement pair (s, t) , the situation admits a fully classical probabilistic description and no inconsistency appears.

The difficulty arises only when one asks whether these four distributions can be realized simultaneously within a single hidden-variable model, that is, whether there exists a probability space Λ and random variables $X_1, \dots, X_4 : \Lambda \rightarrow \Omega$ whose pairwise marginals reproduce $p(\cdot, \cdot \mid s, t)$ for all $(s, t) \in \Theta$. Indeed, the existence of such a single joint distribution would imply the CHSH inequality

$$|C(1, 3) - C(1, 4) + C(2, 3) + C(2, 4)| \leq 2,$$

where,

$$C(s, t) := \sum_{(a,b) \in \Omega^2} ab \cdot p(a, b \mid s, t)$$

denotes the correlation (see Appendix 7.1 for details). However, the parameters (μ, ν) specified in Table 7.1 do not satisfy CHSH inequality. In fact, the CHSH table yields

$$C(1, 3) = -C(1, 4) = C(2, 3) = C(2, 4) = -\frac{1}{\sqrt{2}},$$

and consequently

$$C(1, 3) - C(1, 4) + C(2, 3) + C(2, 4) = -2\sqrt{2},$$

which violates the CHSH bound. Equivalently, there is no single underlying probability space (i.e., no joint distribution) whose marginals reproduce the CHSH table for all $(s, t) \in \Theta$ simultaneously.

7.3 Game-theoretic analysis of CHSH experiments with and without loopholes

In this section we recast two major loopholes in CHSH-type experiments, the locality loophole and the measurement-dependence loophole, as *structural* constraints in a hidden-variable game. We start from a baseline game in which Nature is explicitly allowed to exploit the locality loophole, in the sense that Nature may choose the hidden variable after obtaining partial information from one wing of the experiment. We then identify what additional restrictions on the timing and availability of information are needed to exclude both loopholes, and show that imposing these restrictions leads naturally to the loopholes-closed game studied in the main theorem.

7.3.1 A hidden-variable game with the locality loophole

We first address the locality loophole. Our aim is to model a situation in which, when Nature selects the hidden variable λ , information about A's measurement setting s and outcome ω^s , and possibly also B's setting, may already be available. In such a scenario Nature can correlate λ with the settings and/or outcomes, thereby simulating correlations that would otherwise be ruled out by locality-based arguments. Operationally, this corresponds to experimental situations in which the two measurements are not space-like separated (e.g., they are not effectively simultaneous), so that setting or outcome information can be transmitted between the two wings before λ is fixed.

To formalize this situation into a game-theoretic framework, we consider the following game. We define, for any $a, b \in \Omega$, $s \in \Theta^A$, and $t \in \Theta^B$,

$$p(a \mid s) := \sum_{b' \in \Omega} p(a, b' \mid s, 3) = \sum_{b' \in \Omega} p(a, b' \mid s, 4) = \frac{1}{2},$$

$$p(b \mid a, s, t) := \frac{p(a, b \mid s, t)}{p(a \mid s)} = 2p(a, b \mid s, t).$$

Hidden-variable game with the locality loophole

Players : Scientist A, Scientist B, Nature A, and Nature B.

Protocol : $K_0^A = K_0^B = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

Scientist A announces $\{q_n^A(\cdot \mid s) \in \mathcal{P}(\Omega)\}_{s \in \Theta^A}$ and $s_n \in \Theta^A$.

Nature A announces $\omega_n^{s_n} \in \Omega$.

Scientist B announces $\{q_n^B(\cdot \mid \omega_n^{s_n}, s_n, t) \in \mathcal{P}(\Omega)\}_{t \in \Theta^B}$ and $t_n \in \Theta^B$.

Nature B chooses $\lambda_n \in X_{s_n}^{-1}(\{\omega_n^{s_n}\})$ and announces $\omega_n^{t_n} := X_{t_n}(\lambda_n)$.

$$K_n^A := K_{n-1}^A \cdot \frac{q_n^A(\omega_n^{s_n} \mid s_n)}{p(\omega_n^{s_n} \mid s_n)},$$

$$K_n^B := K_{n-1}^B \cdot \frac{q_n^B(\omega_n^{t_n} \mid \omega_n^{s_n}, s_n, t_n)}{p(\omega_n^{t_n} \mid \omega_n^{s_n}, s_n, t_n)}.$$

END FOR.

This game is intended to model the locality loophole as follows. Scientist A first specifies a betting distribution on Ω for each setting s , and then chooses the setting s_n . After observing A's move, Nature A produces the corresponding outcome $\omega_n^{s_n}$. Next, Scientist B specifies a betting distribution that may depend on the pair $(s_n, \omega_n^{s_n})$ and then chooses the setting t_n . Finally, Nature B is allowed to use the information available from both wings—in particular, the setting and outcome on A's side—to select a hidden variable λ_n consistent with A's realized outcome, and to generate B's outcome via $\omega_n^{t_n} = X_{t_n}(\lambda_n)$.

The key feature is that Nature B may choose λ_n *after* learning $(s_n, \omega_n^{s_n})$. This explicitly captures the operational content of the locality loophole: information can propagate between the two wings before the hidden variable is effectively fixed, allowing correlations that would be excluded under strict locality (space-like separation).

The following theorem holds.

Theorem 7.2. *In the hidden-variable game with the locality loophole, Scientists A and B have a joint strategy that ensures $\lim_{n \rightarrow \infty} K_n^A = \infty$ or $\lim_{n \rightarrow \infty} K_n^B = \infty$*

unless, for any $s \in \Theta^A$, $t \in \Theta^B$ and $a, b \in \Omega$,

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \delta_{(a,b,s,t)}(\omega_i^{s_i}, \omega_i^{t_i}, s_i, t_i)}{\sum_{i=1}^n \delta_{(s,t)}(s_i, t_i)} = p(a, b \mid s, t).$$

Proof. Using Theorem 2.1, Scientist A can force Nature A to realize the event

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \delta_{(a,s)}(\omega_i^{s_i}, s_i)}{\sum_{i=1}^n \delta_s(s_i)} = p(a \mid s),$$

for any $(a, s) \in \Omega \times \Theta^A$. Similarly, for fixed $(a, s) \in \Omega \times \Theta^A$, again by Theorem 2.1, Scientist B can force Nature B to realize the event

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \delta_{(a,b,s,t)}(\omega_i^{s_i}, \omega_i^{t_i}, s_i, t_i)}{\sum_{i=1}^n \delta_{(a,s,t)}(\omega_i^{s_i}, s_i, t_i)} = p(b \mid a, s, t),$$

for any $(b, t) \in \Omega \times \Theta^B$.

We remark that the denominators are determined by the declarations of Scientists A and B. Hence, without loss of generality, we may assume that they are eventually nonzero and diverge. In particular, Scientists can ensure that $\sum_{i=1}^n \delta_{(a,s,t)}(\omega_i^{s_i}, s_i, t_i) \rightarrow \infty$.

Combining these results, it remains to show that

$$\frac{\sum_{i=1}^n \delta_{(a,s)}(\omega_i^{s_i}, s_i)}{\sum_{i=1}^n \delta_s(s_i)} \cdot \frac{\sum_{i=1}^n \delta_{(a,b,s,t)}(\omega_i^{s_i}, \omega_i^{t_i}, s_i, t_i)}{\sum_{i=1}^n \delta_{(a,s,t)}(\omega_i^{s_i}, s_i, t_i)} - \frac{\sum_{i=1}^n \delta_{(a,b,s,t)}(\omega_i^{s_i}, \omega_i^{t_i}, s_i, t_i)}{\sum_{i=1}^n \delta_{(s,t)}(s_i, t_i)} = o(1). \quad (7.1)$$

Since Scientist B announces $t_n \in \Theta^B$, Scientist B can adopt a strategy (t_i) ensuring that

$$\frac{\sum_{i=1}^n \delta_{(a,s)}(\omega_i^{s_i}, s_i)}{\sum_{i=1}^n \delta_s(s_i)} - \frac{\sum_{i=1}^n \delta_{(a,s,t)}(\omega_i^{s_i}, s_i, t_i)}{\sum_{i=1}^n \delta_{(s,t)}(s_i, t_i)} = o(1).$$

Thus, under this strategy, (7.1) holds in the limit. $\square$

This theorem implies that Scientists have a joint betting strategy that allows them to make their capital diverge if Nature's move, namely, the empirical frequencies of outcomes, fails to converge to the CHSH table. In the sense of Shafer and Vovk [1], Theorem 7.2 states that if Nature, following the hidden-variable model, is allowed to choose λ after observing both the measurement setting and the outcome on A's side, then Scientists can force Nature to reproduce empirical correlations consistent with the CHSH table. From this perspective, the locality loophole is precisely the structural feature that prevents the CHSH inequality from emerging as a game-theoretic law of large numbers. Similarly, one may construct a game where Nature can exploit the measurement-dependence loophole. In that case, an analogous theorem holds for the corresponding hidden-variable game.

7.3.2 A hidden-variable game with loopholes closed

We next consider the game inspired by the CHSH experiment with loopholes closed. By closing the currently considered loopholes, is Nature, following a hidden variable model, forced to exhibit behavior consistent with the CHSH table?

To address this question, we examine structural requirements to close the locality loophole and measurement-dependence loophole. One naive game-theoretic method for closing the locality loophole is to adopt a protocol where, *after* Nature declares λ , Scientists determine the measurement setting (s, t) . Experimentally, this can be viewed as analogous to a Bell-type delayed-choice experiment [51, 52], that is, a scheme in which the choice of measurement directions is delayed until after the particle is generated and enters the observation apparatus. Thus, in this formulation, Nature is no longer the last player to act. This seems somewhat conceptually incompatible with the usual role of Nature in game-theoretic probability, where Nature is modeled as the last player to move. Furthermore, this formulation imposes additional requirements to close the measurement-dependence loophole, since the measurement settings might have correlations with λ . This is also unsatisfactory from a game-theoretic probabilistic perspective.

To address these conceptual tensions, we reinterpret the formulation as “the locality loophole does not occur if Nature behaves *as if* it does not anticipate the experimental outcome” and embeds this into the game. Likewise, the measurement-dependence loophole is resolved by having Nature behave “*as if* it does not anticipate the measurement direction at all.” In other words, rather than postulating a probabilistic independence like constraint between the hidden variable and the settings, we introduce an additional capital process into betting game, in which any deviation from such “independence-like” behavior allows the Scientists’ capital to diverge. Thus, Nature must act as though it cannot exploit information about outcomes or settings when declaring λ in order to prevent Scientists from becoming infinitely rich. Formalizing this operational constraint leads us to the “loopholes-closed game.”

Before discussing the loopholes-closed game, we introduce several definitions that allow us to formulate an analogue of measurement independence in a game-theoretic, frequency-based manner. Suppose that we are given sequences $\{(\omega_k^1, \omega_k^2, \omega_k^3, \omega_k^4) \in$

$\Omega^4\}_{k=1}^n$ and $\{(s_k, t_k) \in \Theta\}_{k=1}^n$. For $(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4$ and $(s, t) \in \Theta$, define

$$\begin{aligned} T_n(\omega^1, \omega^2, \omega^3, \omega^4; s, t) &:= \left| \left\{ k \mid \begin{array}{l} 1 \leq k \leq n, s_k = s, t_k = t, \\ (X_1(\lambda_k), X_2(\lambda_k), X_3(\lambda_k), X_4(\lambda_k)) = (\omega^1, \omega^2, \omega^3, \omega^4) \end{array} \right\} \right|, \\ T_n(\omega^1, \omega^2, \omega^3, \omega^4) &:= |\{k \mid 1 \leq k \leq n, (X_1(\lambda_k), X_2(\lambda_k), X_3(\lambda_k), X_4(\lambda_k)) = (\omega^1, \omega^2, \omega^3, \omega^4)\}| \\ &= \sum_{(s,t) \in \Theta} T_n(\omega^1, \omega^2, \omega^3, \omega^4; s, t), \\ T_n(s, t) &:= |\{k \mid 1 \leq k \leq n, s_k = s, t_k = t\}| \\ &= \sum_{(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4} T_n(\omega^1, \omega^2, \omega^3, \omega^4; s, t). \end{aligned}$$

The quantity $T_n(\omega^1, \omega^2, \omega^3, \omega^4; s, t)$ counts how often the hidden variable assignments produce the outcome quadruple $(\omega^1, \omega^2, \omega^3, \omega^4)$ under the measurement setting (s, t) . The quantities $T_n(\omega^1, \omega^2, \omega^3, \omega^4)$ and $T_n(s, t)$ are the corresponding marginal counts. We then define empirical distributions on $\Omega^4 \times \Theta$, Ω^4 , and Θ by

$$\begin{aligned} \hat{P}_n(\omega^1, \omega^2, \omega^3, \omega^4; s, t) &:= \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4; s, t)}{n}, \\ \hat{Q}_n(\omega^1, \omega^2, \omega^3, \omega^4) &:= \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n}, \\ \hat{R}_n(s, t) &:= \frac{T_n(s, t)}{n}. \end{aligned}$$

In the usual probabilistic formulation of the CHSH inequality, measurement-independence assumption asserts that the distributions on Λ and on Θ are independent in the measure-theoretic sense. Under this assumption, one expects at least that,

$$\lim_{n \rightarrow \infty} (\hat{P}_n - \hat{Q}_n \cdot \hat{R}_n) = 0 \quad \text{almost surely.}$$

This observation motivates the construction of an additional capital process that tests, in an operational and sequential manner, whether such independence holds in the empirical data.

Then the following game provides a structural expression of closing the two loopholes within our framework. Since there is no need to distinguish between Scientists A and B in this game, these two are combined into a single player, Scientist AB. The same applies to Nature AB.

Hidden-variable game with loopholes closed

Players : Scientist AB, Nature AB.

Protocol : $K_0^{AB} = 1$, $W_0^{AB} = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

Scientist AB announces $\{(q_n^{AB}(\cdot, \cdot \mid s, t)) \in \mathcal{P}(\Omega^2)\}_{(s,t) \in \Theta}$ and $(s_n, t_n) \in \Theta$.

Nature AB announces $\lambda_n \in \Lambda$ and $(\omega_n^{s_n}, \omega_n^{t_n}) := (X_{s_n}(\lambda_n), X_{t_n}(\lambda_n))$.

$$K_n^{AB} := K_{n-1}^{AB} \cdot \frac{q_n^{AB}(\omega_n^{s_n}, \omega_n^{t_n} \mid s_n, t_n)}{p(\omega_n^{s_n}, \omega_n^{t_n} \mid s_n, t_n)}.$$

$$W_n^{AB} := W_{n-1}^{AB} \cdot \frac{\hat{P}_n(\omega_n^1, \omega_n^2, \omega_n^3, \omega_n^4; s_n, t_n)}{\hat{Q}_n(\omega_n^1, \omega_n^2, \omega_n^3, \omega_n^4) \cdot \hat{R}_n(s_n, t_n)}.$$

END FOR.

The second capital process W_n^{AB} is designed to test, in a sequential manner, whether the empirical joint distribution on $\Omega^4 \times \Theta$ factorizes asymptotically into the product of its marginals. It is defined recursively via a likelihood-ratio-type update between $\hat{P}_n$ and $\hat{Q}_n \cdot \hat{R}_n$, so that any systematic dependence between Nature's hidden-variable declarations and the chosen measurement settings leads to the divergence of W_n^{AB} .

Structurally, the present game differs from the locality-loophole game in only two respects. First, Scientist AB introduces the additional capital process W_n^{AB} in order to operationally enforce a measurement-independence-like constraint. Second, by changing the timing of the announcements (relative to the locality-loophole game), we reformulate the dependence between the hidden variable λ_n and the measurement settings (s_n, t_n) from a purely temporal dependence into a dependence that is empirically testable through frequencies. Although the timing of moves differs from that in the locality-loophole game, this does not restrict Nature's ability to generate outcome pairs $(\omega_n^{s_n}, \omega_n^{t_n})$ as functions of (s_n, t_n) . The essential difference is that the loophole-closed game makes such dependence detectable within the game, which ultimately leads to the following incompatibility result.

Theorem 7.3. *In the hidden-variable game with loopholes closed, Scientist AB has a strategy $q : \Omega^* \rightarrow \mathcal{P}(\Omega^2)$, that ensures $\lim_{n \rightarrow \infty} K_n^{AB} = \infty$ unless, for all $(s, t) \in \Theta$ and $a, b \in \Omega$,*

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \delta_{(a,b,s,t)}(\omega_i^{s_i}, \omega_i^{t_i}, s_i, t_i)}{\sum_{i=1}^n \delta_{(s,t)}(s_i, t_i)} = p(a, b \mid s, t). \quad (7.2)$$

In addition, Scientist AB's second capital process W_n^{AB} diverges, $\lim_{n \rightarrow \infty} W_n^{AB} = \infty$, unless

$$\lim_{n \rightarrow \infty} \left\{ \hat{P}_n(\omega^1, \omega^2, \omega^3, \omega^4; s, t) - \hat{Q}_n(\omega^1, \omega^2, \omega^3, \omega^4) \cdot \hat{R}_n(s, t) \right\} = 0. \quad (7.3)$$

Furthermore, Nature AB cannot satisfy (7.2) and (7.3) simultaneously. Thus in this game, either $\lim_{n \rightarrow \infty} K_n^{AB} = \infty$ or $\lim_{n \rightarrow \infty} W_n^{AB} = \infty$ must occur.

Proof. We divide the proof into three parts. First, as a direct consequence of Theorem 2.1, Scientist AB has a strategy $q : \Omega^* \rightarrow \mathcal{P}(\Omega)$, that ensures $\lim_{n \rightarrow \infty} K_n^{AB} = \infty$ unless, for all $(s, t) \in \Theta$ and $a, b \in \Omega$,

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \delta_{(a,b,s,t)}(\omega_i^{s_i}, \omega_i^{t_i}, s_i, t_i)}{\sum_{i=1}^n \delta_{(s,t)}(s_i, t_i)} = p(a, b \mid s, t).$$

We note that as in Theorem 7.2, this step is merely an application of Theorem 2.1, which forces Nature to conform to the CHSH table at the level of empirical frequencies. The novelty of Theorem 7.3 lies not in this enforcement itself, but in the fact that its consequence is incompatible with the boundedness of the capital process W_n^{AB} .

Second, we evaluate the capital process W_n^{AB} . To simplify the symbols, we write $\tau = (\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4$ and $u = (s, t) \in \Theta$. Then by the recursion formula of W_n^{AB} ,

$$W_n^{AB} = \frac{n! \cdot \prod_{\tau \in \Omega^4} \prod_{u \in \Theta} T_n(\tau; u)!}{\prod_{\tau \in \Omega^4} T_n(\tau)! \cdot \prod_{u \in \Theta} T_n(u)!}.$$

Then, taking the logarithm, and applying Stirling's formula in the second equation,

$$\begin{aligned} \frac{\log W_n^{AB}}{n} &= \frac{1}{n} \left\{ \log n! + \sum_{\tau \in \Omega^4} \sum_{u \in \Theta} \log T_n(\tau; u)! - \sum_{\tau \in \Omega^4} \log T_n(\tau)! - \sum_{u \in \Theta} \log T_n(u)! \right\} \\ &= \log n + \sum_{\tau, u} \frac{T_n(\tau; u)}{n} \log T_n(\tau; u) \\ &\quad - \sum_{\tau} \frac{T_n(\tau)}{n} \log T_n(\tau) - \sum_u \frac{T_n(u)}{n} \log T_n(u) + O\left(\frac{\log n}{n}\right) \\ &= \sum_{\tau, u} \frac{T_n(\tau; u)}{n} \left(\log \frac{T_n(\tau; u)}{n} - \log \frac{T_n(\tau)}{n} - \log \frac{T_n(u)}{n} \right) + O\left(\frac{\log n}{n}\right) \\ &= D\left(\hat{P}_n \parallel \hat{Q}_n \cdot \hat{R}_n\right) + O\left(\frac{\log n}{n}\right), \end{aligned}$$

where $D(\cdot \parallel \cdot)$ denotes the Kullback-Leibler divergence between probability distributions on $\Omega^4 \times \Theta$. Hence, we get that if

$$\hat{P}_n - \hat{Q}_n \cdot \hat{R}_n \not\rightarrow 0,$$

then $\limsup W_n^{AB} = \infty$.

Finally, we show that Nature AB cannot satisfy (7.2) and (7.3) simultaneously. Indeed, under the condition (7.3), the asymptotic behavior of Nature AB necessarily

satisfies the CHSH inequality, which contradicts (7.2), since the CHSH table violates the CHSH inequality. We now make this implication explicit by deriving an asymptotic version of the CHSH inequality from (7.3).

The condition (7.3) can be written by

$$\hat{P}_n - \hat{Q}_n \cdot \hat{R}_n = o(1).$$

Hence, by adopting a strategy satisfying $\liminf \hat{R}_n > 0$, the asymptotic relation (7.3) is equivalent to

$$\begin{aligned} & \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4; s, t)}{T_n(s, t)} - \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n} \\ &= \frac{\hat{P}_n(\omega^1, \omega^2, \omega^3, \omega^4; s, t)}{\hat{R}_n(s, t)} - \hat{Q}_n(\omega^1, \omega^2, \omega^3, \omega^4) = o(1). \end{aligned}$$

Fixing (s, t) and (ω^s, ω^t) , and summing over the remaining indices, we obtain

$$\frac{S_n(\omega^s, \omega^t)}{n} - \frac{S_n(\omega^s, \omega^t; s, t)}{T_n(s, t)} = o(1), \quad (7.4)$$

where

$$\begin{aligned} S_n(\omega^s, \omega^t) &:= |\{k \mid 1 \leq k \leq n, X_s(\lambda_k) = \omega^s, X_t(\lambda_k) = \omega^t\}|, \\ S_n(\omega^s, \omega^t; s, t) &:= |\{k \mid 1 \leq k \leq n, X_s(\lambda_k) = \omega^s, X_t(\lambda_k) = \omega^t, s_k = s, t_k = t\}|. \end{aligned}$$

By using (7.4), we prove that the behavior of Nature AB asymptotically satisfies the CHSH inequality. The following computation parallels the standard derivation of the CHSH inequality, with empirical frequencies replacing probabilities. Define the empirical correlation function by

$$C_n(s, t) := \sum_{(\omega^s, \omega^t) \in \Omega^2} \omega^s \omega^t \cdot \frac{S_n(\omega^s, \omega^t; s, t)}{T_n(s, t)}.$$

Then

$$\begin{aligned}
& C_n(1, 3) - C_n(1, 4) \\
&= \sum_{(\omega^1, \omega^3) \in \Omega^2} \omega^1 \omega^3 \cdot \frac{S_n(\omega^1, \omega^3; 1, 3)}{T_n(1, 3)} - \sum_{(\omega^1, \omega^4) \in \Omega^2} \omega^1 \omega^4 \cdot \frac{S_n(\omega^1, \omega^4; 1, 4)}{T_n(1, 4)} \\
&= \sum_{(\omega^1, \omega^3) \in \Omega^2} \omega^1 \omega^3 \cdot \frac{S_n(\omega^1, \omega^3)}{n} - \sum_{(\omega^1, \omega^4) \in \Omega^2} \omega^1 \omega^4 \cdot \frac{S_n(\omega^1, \omega^4)}{n} + o(1) \\
&= \sum_{(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4} \omega^1 \omega^3 \cdot \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n} - \sum_{(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4} \omega^1 \omega^4 \cdot \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n} + o(1) \\
&= \sum_{(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4} \omega^1 \omega^3 (1 \pm \omega^2 \omega^4) \cdot \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n} \\
&\quad - \sum_{(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4} \omega^1 \omega^4 (1 \pm \omega^2 \omega^3) \cdot \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n} + o(1),
\end{aligned}$$

where the same choice of sign is taken throughout. Hence

$$\begin{aligned}
& |C_n(1, 3) - C_n(1, 4)| \\
&\leq \sum_{(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4} |\omega^1 \omega^3| (1 \pm \omega^2 \omega^4) \cdot \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n} \\
&\quad + \sum_{(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4} |\omega^1 \omega^4| (1 \pm \omega^2 \omega^3) \cdot \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n} + o(1) \\
&= \sum_{(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4} (1 \pm \omega^2 \omega^4) \cdot \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n} \\
&\quad + \sum_{(\omega^1, \omega^2, \omega^3, \omega^4) \in \Omega^4} (1 \pm \omega^2 \omega^3) \cdot \frac{T_n(\omega^1, \omega^2, \omega^3, \omega^4)}{n} + o(1) \\
&= \sum_{(\omega^2, \omega^4) \in \Omega^2} (1 \pm \omega^2 \omega^4) \cdot \frac{S_n(\omega^2, \omega^4)}{n} + \sum_{(\omega^2, \omega^3) \in \Omega^2} (1 \pm \omega^2 \omega^3) \cdot \frac{S_n(\omega^2, \omega^3)}{n} + o(1) \\
&= \sum_{(\omega^2, \omega^4) \in \Omega^2} (1 \pm \omega^2 \omega^4) \cdot \frac{S_n(\omega^2, \omega^4; 2, 4)}{T_n(2, 4)} + \sum_{(\omega^2, \omega^3) \in \Omega^2} (1 \pm \omega^2 \omega^3) \cdot \frac{S_n(\omega^2, \omega^3; 2, 3)}{T_n(2, 3)} + o(1) \\
&= 2 \pm \{C_n(2, 4) + C_n(2, 3)\} + o(1).
\end{aligned}$$

Thus we conclude that

$$|C_n(1, 3) - C_n(1, 4) + C_n(2, 3) + C_n(2, 4)| \leq 2 + o(1).$$

This completes the proof. $\square$

Theorem 7.3 states, in game-theoretic terms, that Scientist AB possesses a *winning strategy* in the hidden-variable game with loopholes closed. More precisely, Scientist AB can choose a strategy $q : \Omega^* \rightarrow \mathcal{P}(\Omega)$ such that Nature AB cannot keep both capital processes K_n^{AB} and W_n^{AB} bounded: Nature must either reproduce the CHSH table at the level of empirical conditional frequencies (7.2) *or* exhibit setting-independence-like behavior, in the frequency sense (7.3), but it cannot achieve both simultaneously. Thus, as $n \rightarrow \infty$, at least one of K_n^{AB} or W_n^{AB} necessarily diverges, meaning that Nature inevitably loses in the sense of game-theoretic probability.

At first sight this may seem at odds with the empirical situation: in actual CHSH experiments, Bell-type inequalities are violated in accordance with quantum-mechanical predictions. The point, however, is that the theorem is not merely saying that “a violation occurs.” Rather, it clarifies what must *give way* within a local hidden-variable (LHV) picture once the relevant loopholes are controlled.

The message of Theorem 7.3 is that one cannot simultaneously reproduce the CHSH table (including quantum correlations) and maintain measurement independence (free choice of settings) within an LHV model. While this is often discussed within a probabilistic framework, our theorem recasts it as a sequential, game-theoretic validation that does not assume an underlying probability space.

Scientists run two capital processes in parallel.

- K_n^{AB} tests whether the empirical conditional frequencies converge to the CHSH table.
- W_n^{AB} tests whether the empirical distribution on $\Omega^4 \times \Theta$ factorizes asymptotically into the product of its marginal distributions on Ω^4 and on Θ , providing a frequency-based analogue of measurement independence.

Note that (7.3) is weaker than the probabilistic independence assumptions usually imposed in the context of the CHSH inequality. Theorem 7.3 asserts that Nature (the LHV side) cannot satisfy these two convergence requirements simultaneously. Consequently, as $n \rightarrow \infty$, at least one of K_n^{AB} or W_n^{AB} must diverge.

Physically, an LHV model can survive asymptotically only by abandoning one of the two: either it fails to reproduce the CHSH table, or it gives up measurement independence by correlating the setting choices with the hidden variable. Operationally, the two capital processes function as sequential test statistics that diagnose compatibility with each requirement along the data stream. Thus, in a loophole-closed setting, Scientists obtain an operational winning strategy: they inevitably detect a breakdown, either in reproducing the CHSH table at the level of empirical frequencies or in setting-independence-like behaviour, thereby excluding LHV explanations in a data-driven sense.

Chapter 8

Concluding Remarks

This thesis has explored several facets of game-theoretic probability and its extensions to classical, quantum, and operationally relevant settings.

In Part I, we generalized the classical game-theoretic law of large numbers to a time-homogeneous k th-order Markovian predictive game. By constructing a Lempel-Ziv-inspired strategy based on incremental parsing and the martingale properties of the game, we provided new insights into the relationship between game-theoretic randomness and coding theory. Applications to thermodynamics were explored via a game-theoretic formulation of Szilard's engine, demonstrating that Nature must behave stochastically to satisfy the law of large numbers and avoid violating the second law. We further introduced the concept of entropy in predictive games, linking it to the codelength of universal coding. Despite these advances, integrating additional thermodynamic concepts such as temperature, thermal contact, and free energies, or extending the framework to non-Markovian processes, remains an important open problem.

In Part II, we explored the possibilities of extension of game-theoretic probability theory into the quantum realm. The first topic of this direction (Chapter 6) focused on the quantum predictive game. Here, we established a quantum analogue of the game-theoretic law of large numbers under the assumption of slowly varying measurement settings. While this provides a first step toward a quantum theory of game-theoretic probability, it is inherently limited: asymptotic commutativity suppresses genuinely quantum effects such as noncommutativity between measurements or declared quantum states. To address these limitations, we proposed a generalized quantum predictive game in which both the prediction target and the choice of measurement are explicit strategic variables, and discussed some open questions that should be addressed in future research.

Finally, in the second topic (Chapter 7), we revisited the CHSH inequality from an explicitly information-structural perspective, emphasizing that the validity of Bell-type constraints depends not only on algebraic relations among correlations

but also on who can access which information, and when. To make this timing and availability explicit, we worked in the framework of game-theoretic probability, in which statistical claims are expressed through the existence of betting strategies rather than by positing an underlying probability space.

Our main contribution was to formalize two representative loopholes, the locality loophole and the measurement-dependence (“freedom-of-choice”) loophole, as structural constraints in sequential games inspired by hidden-variable scenarios. Within this framework, we designed a loopholes-closed game and constructed two capital processes. One monitors whether empirical conditional frequencies converge to the CHSH table, while the other monitors whether, across trials, the association between Nature’s internally fixed outcome assignments and the chosen measurement settings remains free of systematic dependence. This second test provides an operational, frequency-based surrogate for the usual measurement-independence assumption. The central theorem shows that Nature cannot keep both processes bounded. Equivalently, at least one capital process must diverge, so that Nature inevitably loses in the sense of game-theoretic probability. This provides an operational reading of loophole-free CHSH violations: when the structural conditions corresponding to locality and freedom of choice are suitably controlled, Scientists possess a winning strategy against any hidden-variable explanation attempting to satisfy both requirements simultaneously.

Several directions for future work appear natural. First, in this thesis, we tested independence only at the level of single-trial empirical frequencies, which yields a deliberately weak, operational notion of measurement independence. It would be interesting to investigate how the CHSH constraint changes when one imposes stronger regularity conditions, for example by monitoring correlations across multiple trials or higher-order configuration frequencies. Such extensions may correspond, in game-theoretic terms, to strengthening the independence requirement beyond simple factorization of single-trial frequencies. Second, it would be interesting to reinterpret various variants of Bell inequalities within the framework of game-theoretic probability. Such reinterpretations may reveal structural or operational aspects that are less transparent in standard probabilistic formulations. Finally, the present framework suggests a systematic way to compare different notions of “loophole closure” by translating them into game constraints; clarifying the relations among such constraints may help sharpen the interpretation of experimental protocols.

Taken together, the contributions of this thesis highlight both the power and the limitations of game-theoretic probability as a unifying framework. Classical, quantum, and operationally motivated applications illustrate how sequential betting games can capture stochasticity, measurement constraints, and information-theoretic structure. At the same time, the limitations, for example, suppression of noncommutative effects, restricted notions of measurement independence, and the classical reduction implicit in quantum log-loss, point to rich avenues for further

theoretical development. We hope that the generalized quantum predictive game and the information-structural approach to CHSH-type scenarios will serve as a foundation for future work in genuinely quantum game-theoretic probability and operational quantum foundations.

Appendix

A Stationary distributions of Markov chains

Given a Markov matrix $M : \Omega \times \Omega \rightarrow (0, 1) : (a, b) \mapsto M(a \mid b)$ satisfying

$$\sum_{a \in \Omega} M(a \mid b) = 1$$

for all $b \in \Omega$, let $M^{(m)}$ be the m th power of M , in that $M^{(1)} := M$ and

$$M^{(m)}(a \mid b) := \sum_{c \in \Omega} M(a \mid c) M^{(m-1)}(c \mid b).$$

We recall the following well-known fact.

Lemma A.1. *There exists a unique probability distribution μ on Ω such that for any $a, b \in \Omega$,*

$$\lim_{m \rightarrow \infty} M^{(m)}(a \mid b) = \mu(a),$$

and μ is the stationary distribution of M .

Proof. See Theorem 6 in Chapter 4 of [54]. □

Now, consider a k th-order Markov kernel $M : \Omega \times \Omega^k \rightarrow (0, 1) : (a, \omega_1^k) \mapsto M(a \mid \omega_1^k)$ that satisfies

$$\sum_{a \in \Omega} M(a \mid \omega_1^k) = 1 \quad (\forall \omega_1^k \in \Omega^k).$$

Lemma A.2. *For the k th-order Markov kernel M , there is a unique stationary distribution $\pi : \Omega^k \rightarrow (0, 1)$ satisfying*

$$\pi(a_2^{k+1}) = \sum_{a_1 \in \Omega} M(a_{k+1} \mid a_1^k) \pi(a_1^k).$$

Proof. Define $\tilde{M} : \Omega^k \times \Omega^k \rightarrow [0, 1]$ by

$$\tilde{M}(a_1^k | b_1^k) := \begin{cases} M(a_k | b_1^k) & (a_1^k = b_2^k a_k), \\ 0 & (\text{otherwise}). \end{cases}$$

Since

$$\sum_{a_1^k \in \Omega^k} \tilde{M}(a_1^k | b_1^k) = \sum_{a_k \in \Omega} \sum_{a_1^{k-1} \in \Omega^{k-1}} \tilde{M}(a_1^k | b_1^k) = \sum_{a_k \in \Omega} M(a_k | b_1^k) = 1,$$

we can regard $\tilde{M}$ as a first-order Markov kernel on Ω^k . Moreover it is straightforward to verify that

$$\tilde{M}^{(k)}(a_1^k | b_1^k) = \prod_{i=1}^k M(a_i | b_i^k a_1^{i-1}).$$

This expression ensures that $\tilde{M}^{(k)}(a_1^k | b_1^k) > 0$ for all $a_1^k, b_1^k \in \Omega^k$. Thus, by applying Lemma A.1 to the Markov matrix $\tilde{M}^{(k)}$, we conclude that there exists a unique distribution π on Ω^k satisfying

$$\tilde{M}^{(k)}\pi = \pi.$$

Furthermore, since

$$\tilde{M}^{(k)}(M\pi) = M(\tilde{M}^{(k)}\pi) = M\pi,$$

the uniqueness of the stationary distribution for $\tilde{M}^{(k)}$ implies that $M\pi = \pi$. $\square$

B Lynch-Davisson betting strategy for simple predictive game

In this appendix, we present an alternative proof of Theorem 2.1 using one of the simplest universal data compression schemes [55, 56]. As a by-product, we also analyze the convergence rate of the empirical distribution.

We begin with a binary case and consider describing a binary sequence $x^n = 11001$ of length $n = 5$. For $a \in \{0, 1\}$, let $S_n(a)$ denote the number of occurrences of a in x^n . The sequence can be identified by first specifying its type (also known as the empirical distribution):

$$\hat{P}_{x^n} = \left(\frac{S_n(0)}{n}, \frac{S_n(1)}{n} \right) = \left(\frac{2}{5}, \frac{3}{5} \right)$$

and then specifying the index of this sequence among all sequences of length $n = 5$ that share this type. Thus, the given binary sequence x^n can be described by another

binary sequence as follows:

$$\underbrace{\boxed{\text{specify the type}}}_{\lceil \log_2(n+1) \rceil \text{ bits}} + \underbrace{\boxed{\text{specify the sequence in the type class}}}_{\left\lceil \log_2 \binom{n}{S_n(0), S_n(1)} \right\rceil \text{ bits}}$$

This scheme is called the *Lynch-Davisson code*, and its codelength is given by

$$\begin{aligned} \ell_{LD}(x^n) &= \lceil \log_2(n+1) \rceil + \left\lceil \log_2 \frac{n!}{S_n(0)! S_n(1)!} \right\rceil \\ &= \log_2 \frac{(n+1)!}{S_n(0)! S_n(1)!} + O(1). \end{aligned}$$

Generalizing to a generic alphabet $\Omega = \{1, 2, \dots, A\}$ is straightforward, and the corresponding Lynch-Davisson codelength is

$$\begin{aligned} \ell_{LD}(x^n) &= \left\lceil \log_A \frac{(n+A-1)!}{n!(A-1)!} \right\rceil + \left\lceil \log_A \frac{n!}{S_n(1)! S_n(2)! \dots S_n(A)!} \right\rceil \\ &= \log_A \frac{(n+A-1)!}{(A-1)! S_n(1)! S_n(2)! \dots S_n(A)!} + O(1). \end{aligned}$$

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let us introduce the reference probability measure P on Ω^* defined by

$$P(\omega_1^n) := \prod_{i=1}^n p(\omega_i),$$

and consider the “randomness deficiency” function $\mathcal{L}_{LD}(\omega_1^n)$ for the Lynch-Davisson codelength $\ell_{LD}(\omega_1^n)$ relative to the Shannon codelength $-\log_A P(\omega_1^n)$ defined by

$$\mathcal{L}_{LD}(\omega_1^n) := -\log_A P(\omega_1^n) - \ell_{LD}(\omega_1^n).$$

A crucial observation is that

$$\mathcal{L}_{LD}(\omega_1^n) = -\log_A \prod_{i=1}^n p(\omega_i) + \log_A \frac{(A-1)! S_n(1)! S_n(2)! \dots S_n(A)!}{(n+A-1)!} + O(1) \quad (1)$$

$$= -\sum_{a \in \Omega} S_n(a) \log_A p(a) + \log_A \frac{(A-1)! S_n(1)! S_n(2)! \dots S_n(A)!}{(n+A-1)!} + O(1) \quad (2)$$

$$\begin{aligned} &= (\log_A e) \left\{ \sum_{a \in \Omega} S_n(a) \log \frac{S_n(a)}{p(a)} - n \log n - O(\log n) \right\} \\ &= n(\log_A e) \left\{ D(\hat{P}_{\omega_1^n} \| p) - O\left(\frac{\log n}{n}\right) \right\}, \end{aligned} \quad (3)$$

where Stirling's formula was used in the third equality.

The relation (3) shows that $\limsup_{n \rightarrow \infty} \mathcal{L}_{LD}(\omega_1^n) = \infty$ if $\hat{P}_{\omega_1^n}$ does not converge to p . It then suffices to show that there exists a prudent betting strategy α_n that realizes

$$K_n \propto A^{\mathcal{L}_{LD}(\omega_1^n)} = \frac{(A-1)! S_n(1)! S_n(2)! \cdots S_n(A)!}{\prod_{i=1}^n p(\omega_i) \cdot (n+A-1)!}. \quad (4)$$

If this were the case, then

$$\begin{aligned} \frac{K_n}{K_{n-1}} &= \frac{1}{p(\omega_n)(n+A-1)} \cdot \frac{S_n(1)! S_n(2)! \cdots S_n(A)!}{S_{n-1}(1)! S_{n-1}(2)! \cdots S_{n-1}(A)!} \\ &= \frac{1}{p(\omega_n)(n+A-1)} \cdot (S_{n-1}(\omega_n) + 1). \end{aligned} \quad (5)$$

Comparing this with the recursion formula

$$K_n := K_{n-1} \left\{ 1 + \alpha_n(\omega_n) - \sum_{a \in \Omega} \alpha_n(a) p(a) \right\},$$

we find that

$$\alpha_n(a) := \frac{S_{n-1}(a) + 1}{p(a)(n+A-1)} \quad (6)$$

gives a desired prudent betting strategy that satisfies (4). In fact, since

$$\sum_{a \in \Omega} \alpha_n(a) p(a) = \frac{1}{n+A-1} \sum_{a \in \Omega} \{S_{n-1}(a) + 1\} = \frac{1}{n+A-1} \{(n-1) + A\} = 1,$$

we have

$$\left\{ 1 + \alpha_n(\omega_n) - \sum_{a \in \Omega} \alpha_n(a) p(a) \right\} = \alpha_n(\omega_n),$$

which is identical to the right-hand side of (5).

In summary, the prudent betting strategy (6) ensures that

$$\limsup_{n \rightarrow \infty} \log_A K_n = \limsup_{n \rightarrow \infty} \mathcal{L}_{LD}(\omega_1^n) = \infty$$

if $\hat{P}_{\omega_1^n}$ does not converge to p as $n \rightarrow \infty$. The proof is complete. $\square$

Remark. In Theorem 2.1, the two events $K_n \rightarrow \infty$ and $\hat{P}_{\omega_1^n} \rightarrow p$ are not necessarily mutually exclusive, and both may occur simultaneously². For example, suppose that $\hat{P}_{\omega_1^n}$ converges to p at the rate

$$\|\hat{P}_{\omega_1^n} - p\| = O(\sqrt{\log n/n})$$

²A similar argument is found in [2].

and satisfies

$$\limsup_{n \rightarrow \infty} \frac{n}{\log n} \sum_{a \in \Omega} \frac{(\hat{P}_{\omega_1^n}(a) - p(a))^2}{p(a)} > A - 1.$$

Then, we have

$$\limsup_{n \rightarrow \infty} K_n = \infty.$$

Proof. Applying Stirling's formula

$$\log n! = \left(n + \frac{1}{2}\right) \log n - n + O(1),$$

we get

$$\begin{aligned} \log \frac{S_n(1)! S_n(2)! \cdots S_n(A)!}{(n + A - 1)!} &= \sum_{a \in \Omega} \left\{ \left(S_n(a) + \frac{1}{2}\right) \log S_n(a) - S_n(a) \right\} \\ &\quad - \left\{ \left(n + A - 1 + \frac{1}{2}\right) \log(n + A - 1) - (n + A - 1) \right\} \\ &\quad + O(1) \\ &= \sum_{a \in \Omega} S_n(a) \log S_n(a) + \frac{1}{2} \sum_{a \in \Omega} \log S_n(a) \\ &\quad - \left(n + A - \frac{1}{2}\right) \log(n + 1) + O(1). \end{aligned}$$

Thus, the randomness deficiency function $\mathcal{L}_{LD}(\omega_1^n)$ is evaluated using (2) as

$$\begin{aligned} \frac{\mathcal{L}_{LD}(\omega_1^n)}{n \log_A e} &= - \sum_{a \in \Omega} \frac{S_n(a)}{n} \log p(a) + \frac{1}{n} \log \frac{S_n(1)! S_n(2)! \cdots S_n(A)!}{(n + A - 1)!} + O\left(\frac{1}{n}\right) \\ &= \sum_{a \in \Omega} \frac{S_n(a)}{n} (-\log p(a) + \log S_n(a)) + \frac{1}{2n} \sum_{a \in \Omega} \log S_n(a) \\ &\quad - \frac{1}{n} \left(n + A - \frac{1}{2}\right) \log(n + A - 1) + O\left(\frac{1}{n}\right) \\ &= \sum_{a \in \Omega} \hat{P}_{\omega_1^n}(a) \left(\log \frac{\hat{P}_{\omega_1^n}(a)}{p(a)} + \log n \right) + \frac{1}{2n} \sum_{a \in \Omega} (\log \hat{P}_{\omega_1^n}(a) + \log n) \\ &\quad - \left(1 + \frac{A}{n} - \frac{1}{2n}\right) \log(n + A - 1) + O\left(\frac{1}{n}\right) \\ &= \sum_{a \in \Omega} \hat{P}_{\omega_1^n}(a) \log \frac{\hat{P}_{\omega_1^n}(a)}{p(a)} + \frac{1}{2n} \sum_{a \in \Omega} \log \hat{P}_{\omega_1^n}(a) - \frac{A - 1}{2n} \log n + O\left(\frac{1}{n}\right). \end{aligned} \tag{7}$$

Letting $Q_n(a) := \hat{P}_{\omega_1^n}(a) - p(a)$, we evaluate the first term of (7) as

$$\begin{aligned}
\sum_{a \in \Omega} \hat{P}_{\omega_1^n}(a) \log \frac{\hat{P}_{\omega_1^n}(a)}{p(a)} &= \sum_{a \in \Omega} p(a) \left(1 + \frac{Q_n(a)}{p(a)} \right) \log \left(1 + \frac{Q_n(a)}{p(a)} \right) \\
&= \sum_{a \in \Omega} p(a) \left\{ \frac{Q_n(a)}{p(a)} + \frac{1}{2} \left(\frac{Q_n(a)}{p(a)} \right)^2 - O \left(\frac{Q_n(a)}{p(a)} \right)^3 \right\} \\
&= 0 + \frac{1}{2} \sum_{a \in \Omega} \frac{Q_n(a)^2}{p(a)} + O(|Q_n|^3). \tag{8}
\end{aligned}$$

Combining (7) and (8), we have

$$\begin{aligned}
\frac{\mathcal{L}_{LD}(\omega_1^n)}{n \log_A e} &= \frac{1}{2} \sum_{a \in \Omega} \frac{Q_n(a)^2}{p(a)} - \frac{A-1}{2n} \log n \\
&\quad + \frac{1}{2n} \sum_{a \in \Omega} \log \hat{P}_{\omega_1^n}(a) + O(|Q_n|^3) + O\left(\frac{1}{n}\right),
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\mathcal{L}_{LD}(\omega_1^n)}{\log_A e} &= \frac{\log n}{2} \left[\frac{n}{\log n} \sum_{a \in \Omega} \frac{Q_n(a)^2}{p(a)} - (A-1) \right] \\
&\quad + \sum_{a \in \Omega} \log \hat{P}_{\omega_1^n}(a) + nO(|Q_n|^3) + O(1). \tag{9}
\end{aligned}$$

Now, by the assumption that $|Q_n| = |\hat{P}_{\omega_1^n} - p| = O(\sqrt{\log n/n})$, we have

$$\sum_{a \in \Omega} \log \hat{P}_{\omega_1^n}(a) = O(1) \quad \text{and} \quad nO(|Q_n|^3) \rightarrow 0.$$

It then follows from (9) that

$$\limsup_{n \rightarrow \infty} \frac{n}{\log n} \sum_{a \in \Omega} \frac{(\hat{P}_{\omega_1^n}(a) - p(a))^2}{p(a)} > A-1,$$

implies $\limsup_{n \rightarrow \infty} \mathcal{L}_{LD}(\omega_1^n) = \infty$. This completes the proof. $\square$

Note that the quantity

$$\sum_{a \in \Omega} \frac{(\hat{P}_{\omega_1^n}(a) - p(a))^2}{p(a)}$$

corresponds to the Fisher information.

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