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Doctoral Thesis

Many Faces of Defects in Conformal Field Theories

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Abstract

Conformal field theories (CFTs) provide a universal language for describing critical phenomena, yet real physical systems often involve impurities, boundaries, or interfaces that break e.g. translational invariance. Such extended structures, collectively referred to as defects, can fundamentally modify infrared dynamics and give rise to novel universal behavior. In this thesis, we explore the many faces of defects in conformal field theories, with emphasis on their dynamical, structural, and universal properties. We first review the general framework of defect CFTs. We then study line defects in the $O(N)$ vector model near four dimensions and demonstrate, using an axiomatic approach, that defect conformal symmetry uniquely fixes critical defect couplings and determines defect operator data. Next, we introduce a new class of conformal defects generated by quenched disorder localized on lower-dimensional subspaces. We show that such subdimensional disorder can flow to previously unexplored logarithmic conformal defects, characterized by logarithmic operator multiplets and novel infrared behavior. Finally, we develop and analyze composite defect CFTs, where defects themselves host lower-dimensional sub-defects. We construct explicit perturbative models realizing such nested conformal structures and demonstrate the emergence of new universality classes. Together, these results illustrate how defects enrich the landscape of CFTs and reveal the diverse and multifaceted roles they play in quantum field theory.

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Chapter 1

Introduction and Summary

1.1 Introduction

Quantum field theory (QFT) is one of the most successful theoretical frameworks in modern physics. A paradigmatic example is quantum electrodynamics, where the anomalous magnetic moment of the electron is predicted with astonishing precision, in remarkable agreement with experiments. In conventional formulations of QFT, the fundamental objects are point-like local operators, and this local perspective has led to many of its celebrated successes.

However, QFT admits a much richer class of observables beyond local operators. In addition to point-like excitations, there exist extended objects supported on lower-dimensional submanifolds of spacetime. In this thesis, we collectively refer to such extended operators as *defects*. Far from being peripheral, defects often play a central role in determining the dynamics of QFTs. A canonical example is provided by Wilson loop operators in gauge theories, which serve as order parameters for center symmetry and provide a diagnostic of confinement and deconfinement phases. A genuine understanding of QFT therefore requires a systematic grasp of the dynamics of defects. This can be motivated from at least the following two viewpoints.

First, from the perspective of theory space and renormalization group flows, defects are unavoidable. Consider a d -dimensional ultraviolet theory with Poincaré invariance. Standard renormalization group deformations preserve the full d -dimensional Poincaré symmetry and lead to an infrared theory that remains Poincaré invariant. More generally, however, one may consider deformations that are localized on a lower-dimensional submanifold of spacetime. Such localized perturbations explicitly break the full d -dimensional Poincaré symmetry, but they are perfectly admissible within the framework of QFT. Under renormalization group flow, these localized deformations can evolve into nontrivial infrared defects, while leaving the bulk dynamics essentially unchanged. This implies that a *complete* characterization of theory space and its renormalization group structure necessarily involves defect degrees of freedom. The presence of defects gives rise to various new conformal phases, thereby enlarging the space of universality classes beyond those realized by homogeneous bulk theories.

Second, a fundamental question in QFT concerns the spectrum of physical excitations encoded in its local operator content. In the presence of defects, this notion is naturally enlarged. In

addition to bulk local operators, one must consider operators localized on the defect itself, which describe intrinsic defect excitations and their couplings to the surrounding bulk fields. Since defects are ubiquitous in quantum field theories, a complete understanding of the operator spectrum must incorporate these defect-localized degrees of freedom on an equal footing with their bulk counterparts. From this viewpoint, defects are not merely probes inserted into a theory, but rather intrinsic constituents that enrich its dynamical structure and encode genuinely new sectors of physical excitations.

In this thesis, we focus on the properties of defects that arise in conformal field theories (CFTs), namely *conformal defects*. Before embarking on a systematic analysis of defect CFTs, however, it is instructive to revisit a paradigmatic example in which a defect plays a decisive role in determining the dynamics: the Kondo effect. By tracing the historical development of the Kondo problem, from its experimental origins to its modern field-theoretic formulations, we illustrate how a seemingly simple localized degree of freedom can drastically alter the low-energy behavior of a quantum many-body system. This historical flow not only motivates the study of defects in QFTs, but also provides valuable lessons.

History of Kondo effect/problem

One of the earliest and most influential examples of a defect problem in physics is the Kondo effect, discovered in the 1960s in the context of condensed matter physics. In an ordinary clean metal without magnetic impurities, the temperature dependence of the electrical resistivity $\rho(T)$ is well understood. At low temperatures, the dominant contribution arises from electron–phonon scattering and follows the Bloch–Grüneisen law,

$$\rho(T) \simeq \rho_0 + a T^5 \quad (T \ll \Theta_D), \quad (1.1)$$

where ρ_0 is the constant residual resistivity, Θ_D is the Debye temperature, and a is a material-dependent constant. Importantly, at low temperatures, the resistivity $\rho(T)$ is a monotonically increasing function of temperature and approaches a finite constant as $T = 0$. However, careful low-temperature experiments performed already in the 1930s revealed a striking deviation from this conventional behavior. In particular, measurements of the electrical resistivity of nominally pure noble metals, such as gold, showed that $\rho(T)$ is not a monotonically decreasing function as T is lowered, but instead exhibits a pronounced minimum at a finite temperature, below which the resistivity increases again [1–3]. The schematic picture of this behavior is drawn in Fig 1.1. This anomalous upturn at low temperatures was unexpected within the standard theory of electron–phonon scattering and remained unexplained for several decades.

As time went on, substantial experimental progress was made in clarifying the origin of this anomalous low-temperature behavior. By the early 1960s, it became increasingly clear that the resistance minimum is intimately related to the presence of dilute *impurities* in metals. More concretely, systematic measurements on dilute magnetic alloys, such as iron impurities in copper, revealed that the low-temperature behavior of the resistivity changes qualitatively as the impurity concentration is varied. These observations strongly suggested that the anomalous resistivity originates from scattering off localized magnetic moments carried by impurity atoms. Motivated by these experimental findings, Kondo constructed a minimal theoretical model to describe the interaction between conduction electrons and a localized magnetic impurity [4]. The essential

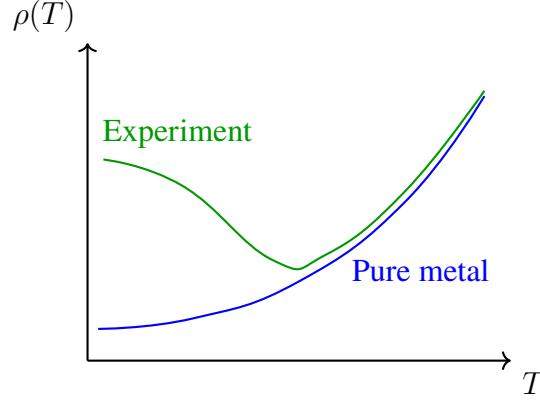


Figure 1.1: Schematic picture of the temperature dependence of electrically resistivity. Before 1930s, at low temperature, the resistivity is believed to be monotonically increasing function with respect to temperature (blue curve) as given in (1.1). However, various experiments showed that the resistivity does not behave like this way and there is a minimum at finite temperature (green curve).

ingredient is to add the antiferromagnetic exchange interaction between the electron spin $\vec{s}_e$ and the impurity spin $\vec{s}_{\text{imp}}$,

$$H_{\text{Kondo}} \sim J \vec{s}_e \cdot \vec{s}_{\text{imp}} , \quad (J < 0) , \quad (1.2)$$

to the pure Hamiltonian giving rise to the resistivity (1.1). Using perturbation theory in the above exchange coupling $J < 0$, Kondo showed that the contribution of magnetic impurity scattering to the resistivity exhibits a logarithmic temperature,

$$\rho(T) \simeq \rho_0 + a T^5 + b J \log T \quad (T \ll \Theta_D) , \quad (1.3)$$

where b is some positive constant depending on materials. This logarithmic enhancement, what is called *Kondo effect*, naturally explains the experimentally observed upturn of the resistivity at low temperatures.

However, the Kondo model contains a strong-coupling problem, and the perturbative treatment inevitably breaks down at sufficiently low temperatures. Indeed, the logarithmic term in (1.3) diverges as $T \rightarrow 0$, indicating the failure of perturbation theory and signaling the onset of genuinely nonperturbative physics.¹ In reality, the resistivity is known to remain finite at zero temperature, implying that the infrared behavior of the model must be governed by a qualitatively different mechanism. Understanding what happens in the deep infrared, in particular at zero temperature, thus constitutes a highly nontrivial problem, commonly referred to as the *Kondo problem*.

Although there are several successful approaches to this problem, such as Wilson's numerical renormalization group [5], here we focus on the boundary conformal field theory (BCFT) approach developed by Affleck and Ludwig [6–8]. When the Hamiltonian possesses rotational symmetry around the magnetic impurity, the low-energy electron–impurity scattering is dominated by the s -wave channel. As a consequence, the problem reduces to an effective one-dimensional scattering

¹One can compute the beta function of electron-impurity coupling J as a perturbative series of J . It turns out that the leading contribution is negative as the same as quantum chromo dynamics (QCD) in $3 + 1$ dimensions.

problem along the radial direction. The Kondo interaction then appears as a boundary coupling between the fermionic fields and the spin impurity. In this way, the infrared behavior of the Kondo Hamiltonian is mapped onto a boundary problem in a $(1+1)$ -dimensional CFT. From this perspective, the renormalization group flow of the Kondo model corresponds to a flow between different boundary conditions of the underlying CFT. Determining the infrared fate of the system thus amounts to classifying the possible conformally invariant boundary conditions and identifying which one is reached under the RG flow. While such boundary classification problems are highly nontrivial in general, conformal symmetry imposes powerful constraints on the allowed boundary dynamics.

What lessons, then, can be drawn from the history of Kondo effect and problem? Perhaps the most important one is that even a localized imperfection, such as a single magnetic impurity, can dramatically alter the low-energy dynamics of quantum many-body system. A second lesson is the remarkable power of the conformal field theory framework. By recasting the Kondo problem as a boundary problem in a $(1+1)$ -dimensional CFT, the infrared physics becomes amenable to exact and largely model-independent analysis.

1.2 Structure and summary of this thesis

In this thesis, we investigate various properties of defects in CFTs, with particular emphasis on their dynamical and universal features. By studying defects from multiple viewpoints, we aim to clarify their role in organizing and enriching the landscape of CFTs. In what follows, we provide an overview of the thesis and summarize the main results presented in each chapter.

In Chap. 2, we present a pedagogical review of CFTs and defects in CFTs. We begin with an elementary overview of homogeneous CFTs and then emphasize the new structures that arise in the presence of defects, such as modified symmetry constraints and additional operator sectors. This chapter provides the basic theoretical framework and notation used throughout the thesis and is essential for understanding the subsequent chapters.

In Chap. 3, we study a line defect in the $O(N)$ vector model in $d = 4 - \epsilon$ dimensions and focuses on extracting defect CFT data at criticality. We analyze the model at the Wilson–Fisher fixed point, where the presence of the line defect breaks the global symmetry to $SL(2, \mathbb{R}) \times SO(d-1) \times O(N-1)$ and gives rise to nontrivial defect local operators. Employing the axiomatic approach of Rychkov and Tan, combined with defect operator expansions, we show that defect conformal symmetry uniquely fixes the critical defect coupling. We further compute the leading anomalous dimensions of various defect local operators, including composite operators, by exploiting analyticity properties of correlation functions. Our results are in complete agreement with previous diagrammatic calculations and demonstrate that the axiomatic method provides a powerful and systematic tool for studying defect CFTs beyond conventional perturbation theory.

In Chap. 4, we introduce and analyze a new class of defects generated by quenched disorder localized on a lower-dimensional Euclidean spacetime $\mathbb{R}^d$. Instead of considering random couplings distributed throughout the entire space, we study subdimensional disorder and investigate its infrared fate after ensemble averaging. Assuming that the infrared theory is conformal, we develop a model-independent framework for *logarithmic conformal defects*, in which bulk operators remain ordinary

conformal primaries while defect local operators form logarithmic representations of the residual conformal symmetry. Using symmetry and representation theory alone, we establish general results for correlation functions, conformal block decompositions, and logarithmic defect operator expansions. As concrete realizations, we study free scalar and $O(N)$ vector models with Gaussian subdimensional disorder. We find that a nontrivial infrared logarithmic defect CFT emerges only for codimension-two defects, and that various defect operators such as the displacement and tilt operators organize into logarithmic multiplets. These results provide explicit examples of previously unexplored conformal defects induced purely by localized disorder.

In Chap. 5, we introduce and systematically study composite defect CFTs, in which a conformal defect itself hosts a lower-dimensional sub-defect. We first develop a general defect CFT framework adapted to such nested defects, clarifying the residual symmetry structures, conformal correlators, operator expansions, and conformal block decompositions. This formalism generalizes wedge CFTs to composite defects with arbitrary co-dimensions. We then construct an explicit perturbative realization of composite defect CFTs using the $O(N)$ vector model in $d = 3 - \epsilon$ dimensions, where a surface defect and an embedded line defect simultaneously flow to non-trivial infrared fixed points. Remarkably, a conformal composite defect emerges even with a free bulk theory due to the interplay between localized RG flows on the surface and the line. We analyzed the pattern of $O(N)$ symmetry breaking on the line defect and identified multiple infrared fixed points, including both symmetric and symmetry-breaking ones. Our results demonstrate that composite defects lead to new universality classes beyond conventional defect CFTs and enrich the phase structures.

We end up with giving conclusion and future directions related to the contents of this thesis. The appendices provided at the end of Chap. 3, Chap. 4 and Chap. 5 are devoted to technical details that supplement the main text. They contain explicit derivations, intermediate steps of calculations, and additional discussions that would otherwise interrupt the flow of the presentation.

Chapter 2

Fundamental Aspects of Defects in Conformal Field Theories

2.1 Brief review on CFT

In relativistic quantum field theory (QFT), it is customary to impose Poincaré invariance, consisting of spacetime translations and Lorentz transformations. A natural question is whether one can extend the Poincaré symmetry of QFTs. However, possible extensions are severely constrained by the famous *Coleman-Mandula* and *Haag-Lopuszanski-Sohnius* theorems. Their statements are as follows.²

Coleman–Mandula Theorem [9]

In quantum field theories containing *massive* particles in $d \geq 2 + 1$ dimensions, any (bosonic) symmetry of the S -matrix must be a direct product of the Poincaré group with an internal symmetry group.

Haag-Lopuszanski-Sohnius Theorem [10]

In quantum field theories containing *only massless* particles in $d \geq 2 + 1$ dimensions, any (bosonic) symmetry of the S -matrix must be a direct product of the *conformal* group with an internal symmetry group.

These theorems together assert that the largest possible bosonic extension of Poincaré symmetry in spacetime is the conformal group.^{3,4} Also, as mentioned in the Introduction, conformal symmetry

²There are several assumptions underlying these theorems. For instance, one assumes that the S -matrix is invariant under the Poincaré group and that the elastic two-body scattering amplitude is an analytic function of the scattering angle. We refer the reader to [9, 10] for a complete list of assumptions and a detailed discussion.

³Here, we restrict our attention to bosonic extensions of the spacetime symmetry. If one allows fermionic generators, the maximal extension is instead the superconformal symmetry [10]. In this thesis, however, we will not consider supersymmetry and will confine ourselves entirely to non-supersymmetric CFTs.

⁴We stress that the proof for these no-go theorems rely crucially on the existence of an S -matrix and thus on a particle description of the theory. Defining an S -matrix requires well-defined single-particle asymptotic states, which

frequently emerges at renormalization-group fixed points and therefore plays a central role in all QFTs. Before turning to defect CFT, let us begin with recalling the structures of CFTs, which will be the relevant setting for most of this thesis. We refer the reader to [11, 12] for excellent and comprehensive reviews on this subject. Additional details will be incorporated as needed.

2.1.1 Conformal algebra

Conformal Killing vector equation

In this subsection, we define conformal symmetry on a flat spacetime of dimension $d \geq 3$. The Poincaré symmetry consists of those coordinate transformations that leave the Minkowski metric invariant. Conformal field theories (CFTs) generalize this by allowing transformations that preserve the metric only up to a local rescaling:

$$\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \eta_{\alpha\beta} = \Omega(x) \eta_{\mu\nu}. \quad (2.1)$$

To obtain the conformal Lie algebra, we consider an infinitesimal transformation of the form

$$x^\mu \mapsto x'^\mu = x^\mu + \varepsilon^\mu(x), \quad \Omega(x) = 1 - f(x), \quad |\varepsilon| \ll 1, \quad |f| \ll 1, \quad (2.2)$$

and determine the general structure of $\varepsilon^\mu(x)$.

Substituting (2.2) into (2.1), we find

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x) \eta_{\mu\nu}. \quad (2.3)$$

Taking the trace of this equation determines $f(x)$ as

$$f(x) = \frac{2}{d} \partial \cdot \varepsilon, \quad \partial \cdot \varepsilon \equiv \partial_\mu \varepsilon^\mu. \quad (2.4)$$

Plugging (2.4) back into (2.3) gives the conformal Killing vector (CKV) equation:

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d} (\partial \cdot \varepsilon) \eta_{\mu\nu}. \quad (2.5)$$

Solving this equation determines the generators of the conformal Lie algebra. Acting with ∂^μ on both sides, we obtain

$$\partial^2 \varepsilon_\nu = \frac{2-d}{d} \partial_\nu (\partial \cdot \varepsilon). \quad (2.6)$$

Note that the right-hand side of (2.6) does not vanish since we assume $d \geq 3$ in this thesis.⁵ Next, applying ∂_μ to both sides of (2.6) and symmetrizing indices,⁶ we obtain

$$\partial^2 \partial_{\{\mu} \varepsilon_{\nu\}} = \frac{2-d}{d} \partial_\mu \partial_\nu (\partial \cdot \varepsilon). \quad (2.7)$$

generally fail to exist in theories with massless particles due to infrared issues. More broadly, many CFTs do not admit a particle interpretation at all, and therefore lie outside the scope of these no-go theorems.

⁵When $d = 2$, the right-hand side identically vanishes, leading to an infinite-dimensional symmetry algebra [13].

⁶The symmetrization symbol $\{ \}$ is defined by

$$A_{\{\mu_1 \mu_2 \dots \mu_n\}} \equiv \frac{1}{n!} (A_{\mu_1 \mu_2 \dots \mu_n} + (\text{symmetrized})).$$

Using (2.5) on the left-hand side and then taking a trace yields

$$\partial^2(\partial \cdot \epsilon) = 0. \quad (2.8)$$

Substituting (2.8) into (2.7) then gives

$$\partial_\mu \partial_\nu (\partial \cdot \epsilon) = 0, \quad (2.9)$$

which implies that $\partial \cdot \epsilon$ is at most linear in the coordinates:

$$\partial \cdot \epsilon = A + B \cdot x \quad (A, B_\mu : \text{constants}). \quad (2.10)$$

Next, acting with ∂_ρ on (2.3) and cyclically permuting indices, we obtain

$$\partial_\rho \partial_\mu \epsilon_\nu + \partial_\rho \partial_\nu \epsilon_\mu = \eta_{\mu\nu} \partial_\rho f, \quad (2.11)$$

$$\partial_\nu \partial_\rho \epsilon_\mu + \partial_\nu \partial_\mu \epsilon_\rho = \eta_{\rho\mu} \partial_\nu f, \quad (2.12)$$

$$\partial_\mu \partial_\nu \epsilon_\rho + \partial_\mu \partial_\rho \epsilon_\nu = \eta_{\nu\rho} \partial_\mu f. \quad (2.13)$$

From (2.11)+(2.12)-(2.13), together with (2.10), we find

$$\partial_\rho \partial_\nu \epsilon_\mu = \text{constant}. \quad (2.14)$$

Thus a general conformal Killing vector $\epsilon_\mu(x)$ is at most quadratic in coordinates:

$$\epsilon_\mu(x) = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho, \quad c_{\mu\nu\rho} = c_{\mu\rho\nu}, \quad (2.15)$$

where a_μ , $b_{\mu\nu}$, and $c_{\mu\nu\rho}$ are constant coefficients. Their independent components correspond precisely to the generators of the conformal Lie algebra.

- a_μ : Since $\epsilon_\mu = a_\mu$ trivially satisfies (2.5) without any constraints, there are d independent parameters corresponding to spacetime translations.
- $b_{\mu\nu}$: Substituting (2.15) into (2.5), we obtain

$$b_{\{\mu\nu\}} - \frac{1}{d} b^\rho{}_\rho \eta_{\mu\nu} = 0, \quad (2.16)$$

$$c_{\{\mu\nu\}\tau} - \frac{1}{d} c^\rho{}_{\rho\tau} \eta_{\mu\nu} = 0. \quad (2.17)$$

A rank-two tensor decomposes into symmetric traceless, antisymmetric, and trace parts; the condition (2.16) removes the symmetric traceless part. Hence the independent components of $b_{\mu\nu}$ are $\frac{d(d-1)}{2} + 1$ and can be parametrized as

$$b_{\mu\nu} = \omega_{\mu\nu} + \lambda \eta_{\mu\nu}, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}. \quad (2.18)$$

Here $\omega_{\mu\nu}$ and λ correspond to Lorentz transformations and dilations, respectively.

For example, a rank-2 / rank-3 tensor satisfies

$$A_{\{\mu\nu\}} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}),$$

$$B_{\{\mu\nu\rho\}} = \frac{1}{6}(B_{\mu\nu\rho} + B_{\nu\rho\mu} + B_{\rho\mu\nu} + B_{\nu\mu\rho} + B_{\mu\rho\nu} + B_{\rho\nu\mu}).$$

- $c_{\mu\nu\rho}$: It is convenient to decompose the first two indices of $c_{\mu\nu\rho}$ as follows:

$$c_{\mu\nu\rho} = \frac{1}{d} c^\tau_{\tau\rho} \eta_{\mu\nu} + c_{[\mu\nu]\rho} , \quad (2.19)$$

where $[\]$ denotes antisymmetrization.⁷ From (2.19), one finds

$$c_{\mu\nu\rho} + c_{\nu\mu\rho} = \frac{2}{d} c^\tau_{\tau\rho} \eta_{\mu\nu} , \quad (2.20)$$

$$c_{\nu\rho\mu} + c_{\rho\nu\mu} = \frac{2}{d} c^\tau_{\tau\mu} \eta_{\nu\rho} , \quad (2.21)$$

$$c_{\rho\mu\nu} + c_{\mu\rho\nu} = \frac{2}{d} c^\tau_{\tau\nu} \eta_{\rho\mu} . \quad (2.22)$$

Combining (2.20) + (2.21) – (2.22), one obtains

$$c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu , \quad (2.23)$$

where $b_\mu \equiv c^\tau_{\tau\mu}/d$. Thus $c_{\mu\nu\rho}$ contains d independent parameters corresponding to special conformal transformations (SCTs).

Combining all pieces, the action of the conformal transformations on coordinates is

$$\delta x^\mu = a^\mu + \omega^{\mu\nu} x_\nu + \lambda x^\mu + 2(b \cdot x) x^\mu - x^2 b^\mu , \quad (2.24)$$

and the number of independent parameters is

$$d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+2)(d+1)}{2} . \quad (2.25)$$

Finally, we summarize the finite conformal transformations and the associated scale factor $\Omega(x)$ defined in (2.1):

$$\begin{aligned} \text{Translations: } x'^\mu &= x^\mu + a^\mu , & \Omega(x) &= 1 , \\ \text{Lorentz transformations: } x'^\mu &= \Lambda^\mu_\nu x^\nu , & \Omega(x) &= 1 , \\ \text{Dilations: } x'^\mu &= \lambda x^\mu , & \Omega(x) &= \lambda^{-2} , \\ \text{SCT: } x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2} , & \Omega(x) &= (1 - 2b \cdot x + b^2 x^2)^2 . \end{aligned} \quad (2.26)$$

Conformal algebra

So far, we introduced conformal symmetry and derived the action of conformal transformations on spacetime coordinates (2.24) by solving the conformal Killing equation. Our next goal is first to

⁷The antisymmetrization symbol $[\]$ is defined by

$$A_{[\mu_1 \mu_2 \dots \mu_n]} \equiv \frac{1}{n!} (A_{\mu_1 \mu_2 \dots \mu_n} \pm \{\text{permutations}\}) .$$

derive the conformal (Lie) algebra generating this conformal transformations. We then show that the resulting algebra is isomorphic to $\mathfrak{so}(2, d)$.

In general, the action of a conformal transformation on the coordinates x^μ can be written as

$$\delta x^\mu = i a^\nu \widehat{P}_\nu(x^\mu) + \frac{i}{2} \omega^{\nu\lambda} \widehat{M}_{\nu\lambda}(x^\mu) + i \lambda \widehat{D}(x^\mu) + i b^\nu \widehat{K}_\nu(x^\mu). \quad (2.27)$$

Here $\widehat{P}_\nu$, $\widehat{M}_{\nu\lambda}$, $\widehat{D}$, and $\widehat{K}_\nu$ denote the differential representations of generators for translation, Lorentz transformation, dilatation, and SCT, respectively. By comparing this with the expression obtained in the previous section, Eq. (2.24), we read off the explicit differential operators:

$$\begin{aligned} \widehat{P}_\mu &= -i \partial_\mu, & \widehat{M}_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu), \\ \widehat{D} &= -i x^\mu \partial_\mu, & \widehat{K}_\mu &= -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu). \end{aligned} \quad (2.28)$$

Using these differential representations, one can compute the commutation relations among the generators. The result is

$$\begin{aligned} [\widehat{P}_\mu, \widehat{P}_\nu] &= 0, & [\widehat{P}_\mu, \widehat{M}_{\nu\lambda}] &= i \eta_{\mu\nu} \widehat{P}_\lambda - i \eta_{\mu\lambda} \widehat{P}_\nu, \\ [\widehat{M}_{\mu\nu}, \widehat{M}_{\rho\lambda}] &= -i \eta_{\mu\rho} \widehat{M}_{\nu\lambda} + i \eta_{\nu\rho} \widehat{M}_{\mu\lambda} + i \eta_{\mu\lambda} \widehat{M}_{\nu\rho} - i \eta_{\nu\lambda} \widehat{M}_{\mu\rho}, \\ [\widehat{D}, \widehat{P}_\mu] &= i \widehat{P}_\mu, & [\widehat{D}, \widehat{K}_\mu] &= -i \widehat{K}_\mu, & [\widehat{D}, \widehat{M}_{\mu\nu}] &= 0, & [\widehat{K}_\mu, \widehat{K}_\nu] &= 0, \\ [\widehat{M}_{\mu\nu}, \widehat{K}_\rho] &= i(\eta_{\nu\rho} \widehat{K}_\mu - \eta_{\mu\rho} \widehat{K}_\nu), & [\widehat{P}_\mu, \widehat{K}_\nu] &= -2i(\eta_{\mu\nu} \widehat{D} + \widehat{M}_{\mu\nu}). \end{aligned} \quad (2.29)$$

Thus, by lifting these differential operators into the abstract generators, we obtain the so-called conformal algebra which generates the conformal transformation:

Conformal algebra on $\mathbb{R}^{1,d-1}$ ($\mu, \nu = 0, 1, \dots, d-1$)

$$[P_\mu, P_\nu] = 0, \quad (2.30)$$

$$[P_\mu, M_{\nu\lambda}] = i \eta_{\mu\nu} P_\lambda - i \eta_{\mu\lambda} P_\nu, \quad (2.31)$$

$$[M_{\mu\nu}, M_{\rho\lambda}] = -i \eta_{\mu\rho} M_{\nu\lambda} + i \eta_{\nu\rho} M_{\mu\lambda} + i \eta_{\mu\lambda} M_{\nu\rho} - i \eta_{\nu\lambda} M_{\mu\rho}, \quad (2.32)$$

$$[D, P_\mu] = i P_\mu, \quad (2.33)$$

$$[D, K_\mu] = -i K_\mu, \quad (2.34)$$

$$[D, M_{\mu\nu}] = 0, \quad (2.35)$$

$$[K_\mu, K_\nu] = 0, \quad (2.36)$$

$$[M_{\mu\nu}, K_\rho] = i(\eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu), \quad (2.37)$$

$$[P_\mu, K_\nu] = -2i(\eta_{\mu\nu} D + M_{\mu\nu}). \quad (2.38)$$

The conformal algebra in d -dimensional Minkowski spacetime is isomorphic to $\mathfrak{so}(2, d)$. This can be seen by embedding the conformal generators into the $(d+2) \times (d+2)$ matrices J_{mn}

$(m, n = -2, -1, 0, \dots, d-1)$:

$$J_{mn} = \begin{matrix} m \backslash n \rightarrow \\ \downarrow \end{matrix} \begin{matrix} -2 & -1 & \nu \\ 0 & D & \frac{1}{2}(P_\nu - K_\nu) \\ -D & 0 & \frac{1}{2}(P_\nu + K_\nu) \\ -\frac{1}{2}(P_\mu - K_\mu) & -\frac{1}{2}(P_\mu + K_\mu) & M_{\mu\nu} \end{matrix} \quad (2.39)$$

One then finds that J_{mn} obey the commutation relations

$$\begin{aligned} [J_{mn}, J_{rs}] &= -i\eta_{mr} J_{ns} + i\eta_{nr} J_{ms} + i\eta_{ms} J_{nr} - i\eta_{ns} J_{mr} , \\ \eta_{mn} &\equiv \text{diag}(-1, 1, -1, 1, 1, \dots, 1) , \end{aligned} \quad (2.40)$$

which coincide with the Lie algebra $\mathfrak{so}(2, d)$. This identification underlies, for instance, the embedding-space formalism used later to construct correlation functions efficiently as demonstrated in Chapter 5.

The conformal algebra in d -dimensional Euclidean spacetime $\mathbb{R}^d$ can be obtained from its Minkowski counterpart simply by replacing the Minkowski metric $\eta_{\mu\nu}$ with the Euclidean metric $\delta_{\mu\nu}$. With this replacement, the metric in (2.40) becomes $\eta_{MN} = \text{diag}(-1, 1, +1, \dots, 1)$ for $M, N = -1, 0, 1, \dots, d$, and the resulting algebra is isomorphic to $\mathfrak{so}(1, d+1)$. Since much of our analysis of this thesis takes place on $\mathbb{R}^d$, we record the Euclidean conformal algebra below.

Conformal algebra on $\mathbb{R}^d$ ($\mu, \nu = 1, \dots, d$)

$$i[\mathbf{P}_\mu, \mathbf{P}_\nu] = 0 , \quad (2.41)$$

$$i[\mathbf{P}_\mu, \mathbf{M}_{\nu\lambda}] = \delta_{\mu\nu} \mathbf{P}_\lambda - \delta_{\mu\lambda} \mathbf{P}_\nu , \quad (2.42)$$

$$i[\mathbf{M}_{\mu\nu}, \mathbf{M}_{\rho\lambda}] = -\delta_{\mu\rho} \mathbf{M}_{\nu\lambda} + \delta_{\nu\rho} \mathbf{M}_{\mu\lambda} + \delta_{\mu\lambda} \mathbf{M}_{\nu\rho} - \delta_{\nu\lambda} \mathbf{M}_{\mu\rho} , \quad (2.43)$$

$$i[\mathbf{D}, \mathbf{P}_\mu] = \mathbf{P}_\mu , \quad (2.44)$$

$$i[\mathbf{D}, \mathbf{K}_\mu] = -\mathbf{K}_\mu , \quad (2.45)$$

$$i[\mathbf{D}, \mathbf{M}_{\mu\nu}] = 0 , \quad (2.46)$$

$$i[\mathbf{K}_\mu, \mathbf{K}_\nu] = 0 , \quad (2.47)$$

$$i[\mathbf{M}_{\mu\nu}, \mathbf{K}_\rho] = \delta_{\nu\rho} \mathbf{K}_\mu - \delta_{\mu\rho} \mathbf{K}_\nu , \quad (2.48)$$

$$i[\mathbf{P}_\mu, \mathbf{K}_\nu] = -2(\delta_{\mu\nu} \mathbf{D} + \mathbf{M}_{\mu\nu}) . \quad (2.49)$$

$$\mathbf{J}_{MN} = \begin{matrix} M \backslash N \rightarrow \\ \downarrow \end{matrix} \begin{matrix} -1 & 0 & \nu \\ 0 & \mathbf{D} & \frac{1}{2}(\mathbf{P}_\nu - \mathbf{K}_\nu) \\ -\mathbf{D} & 0 & \frac{1}{2}(\mathbf{P}_\nu + \mathbf{K}_\nu) \\ -\frac{1}{2}(\mathbf{P}_\mu - \mathbf{K}_\mu) & -\frac{1}{2}(\mathbf{P}_\mu + \mathbf{K}_\mu) & \mathbf{M}_{\mu\nu} \end{matrix} . \quad (2.50)$$

$$\begin{aligned} i[\mathbf{J}_{MN}, \mathbf{J}_{RS}] &= -\eta_{MR} \mathbf{J}_{NS} + \eta_{NR} \mathbf{J}_{MS} + \eta_{MS} \mathbf{J}_{NR} - \eta_{NS} \mathbf{J}_{MR} , \\ \eta_{MN} &\equiv \text{diag}(-1, 1, 1, \dots, 1) . \end{aligned} \quad (2.51)$$

2.1.2 Local operators in CFT

Primary and descendant operators

A distinctive feature of CFTs is the *state–operator correspondence*. In radial quantization, inserting a local operator at the origin of $\mathbb{R}^d$ produces a normalizable state on the sphere $\mathbb{S}^{d-1}$. More precisely, if $\mathcal{O}(0)$ is a local operator, the corresponding state is obtained by acting on the conformally invariant vacuum $|0\rangle$:

$$|\mathcal{O}\rangle \equiv \lim_{x \rightarrow 0} \mathcal{O}(x) |0\rangle . \quad (2.52)$$

Conversely, every state $|\mathcal{O}\rangle$ on $\mathbb{S}^{d-1}$ determines a unique local operator $\mathcal{O}(0)$ at the origin. This establishes a one-to-one correspondence between local operators and states, which is quite useful in the CFT analysis.⁸

Just as in the case of the Poincaré group, it is natural to classify fields in a CFT by quantum numbers associated with the conformal symmetry. Recall that a CFT appears as a fixed point of the renormalization group. At such a fixed point, scale transformations (combined with an appropriate rescaling of fields) leave the Hamiltonian and correlation functions invariant. Therefore, local operators in a CFT can be characterized by their scaling dimensions or conformal dimensions, namely the eigenvalues of the dilatation generator $\mathbf{D}$:⁹

$$\mathbf{D} |\mathcal{O}_\Delta\rangle = -i\Delta |\mathcal{O}_\Delta\rangle . \quad (2.53)$$

Here the state $|\mathcal{O}_\Delta\rangle$ is related to the operator $\mathcal{O}_\Delta(0)$ via the state–operator correspondence (2.52). The commutation relations

$$i[\mathbf{D}, \mathbf{P}_\mu] = +\mathbf{P}_\mu , \quad i[\mathbf{D}, \mathbf{K}_\mu] = -\mathbf{K}_\mu \quad (2.54)$$

show that $\mathbf{P}_\mu$ acts as a raising operator for the conformal dimension, while $\mathbf{K}_\mu$ acts as a lowering operator. A state annihilated by all special conformal generators $\mathbf{K}_\mu$ is called a *primary state*:

$$|\phi\rangle \text{ is primary} \iff \mathbf{K}_\mu |\phi\rangle = 0 . \quad (2.55)$$

Physically, on the cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$, the dilatation operator $\mathbf{D}$ becomes the Hamiltonian, whose spectrum must be bounded from below; primary states correspond precisely to the lowest-energy states in each conformal multiplet.¹⁰ Via the state–operator correspondence, a primary operator is

⁸See also footnote 10.

⁹Here, we assume that the operator spectrum is *diagonal*. One can also consider the non-diagonal operator spectrum, which leads to the logarithmic dependence of correlation functions. The non-diagonal examples are discussed in Chap. 4.

¹⁰The conformal dimension Δ of a primary operator ϕ_Δ is directly related to the energy E of the corresponding state on the cylinder spacetime $\mathbb{R} \times \mathbb{S}^{d-1}$. Under the state–operator correspondence, inserting $\phi_\Delta(0)$ at the origin of flat space prepares a state on the cylinder with energy

$$E = \frac{\Delta}{R} , \quad (2.56)$$

where R is the radius of the spatial sphere $\mathbb{S}^{d-1}$. Consequently, computing the conformal dimension of a local

defined by $|\phi\rangle = \phi(0)|0\rangle$. Since the only generators commuting with $\mathbf{D}$ are the rotation generators $\mathbf{M}_{\mu\nu}$, each primary operator is characterized by its conformal dimension and by $\text{SO}(d)$ spin.

Once a primary state is given, acting with the raising operators $\mathbf{P}_\mu$ produces an infinite tower of states called *descendants*:

$$\text{descendant states: } \quad \mathbf{P}_{\mu_1} \mathbf{P}_{\mu_2} \cdots \mathbf{P}_{\mu_n} |\phi\rangle, \quad \Delta_{\text{desc}} = \Delta_\phi + n. \quad (2.57)$$

Similarly, descendant operators are obtained from the corresponding descendant states through the state–operator correspondence. In a unitary CFT, it is known that any states can be decomposed into a linear combination of primary and descendant states. Thus, since every conformal multiplet is generated from a single primary by the action of $\mathbf{P}_\mu$, understanding primary operators plays a central role in the operator content of the theory.

Transformation laws of primary fields

As emphasized in the previous subsection, the analysis of primary operators plays a central role in CFT. In this subsection we collect the transformation laws of primary fields under the generators of the conformal group. Let $\phi_{\Delta,I}$ be a primary operator with conformal dimension Δ and transforming in an irreducible representation of $\text{SO}(d)$ labeled by the collective index I . Using the definition of primary states (2.55), the state–operator correspondence (2.52), and the conformal invariance of the vacuum, one finds the commutation relations

$$\begin{aligned} i[\mathbf{D}, \phi_{\Delta,I}(0)] &= \Delta \phi_{\Delta,I}(0), \\ i[\mathbf{M}_{\mu\nu}, \phi_{\Delta,I}(0)] &= (\mathbf{M}_{\mu\nu})_I^J \phi_{\Delta,J}(0), \\ i[\mathbf{K}_\mu, \phi_{\Delta,I}(0)] &= 0. \end{aligned} \quad (2.58)$$

Here $(\mathbf{M}_{\mu\nu})_I^J$ denotes the $\mathfrak{so}(d)$ representation matrices acting on the spin indices of the operator. For a general insertion point x , we use

$$\phi_{\Delta,I}(x) = e^{ix \cdot \mathbf{P}} \phi_{\Delta,I}(0) e^{-ix \cdot \mathbf{P}}, \quad (2.59)$$

together with

$$i[\mathbf{P}_\mu, \phi_{\Delta,I}(x)] = \partial_\mu \phi_{\Delta,I}(x), \quad (2.60)$$

to obtain the action of $\mathbf{D}$, $\mathbf{M}_{\mu\nu}$ and $\mathbf{K}_\mu$ at finite separation:

$$i[\mathbf{D}, \phi_{\Delta,I}(x)] = (x \cdot \partial + \Delta) \phi_{\Delta,I}(x), \quad (2.61)$$

$$i[\mathbf{M}_{\mu\nu}, \phi_{\Delta,I}(x)] = (x_\nu \partial_\mu - x_\mu \partial_\nu) \phi_{\Delta,I}(x) + (\mathbf{M}_{\mu\nu})_I^J \phi_{\Delta,J}(x), \quad (2.62)$$

$$i[\mathbf{K}_\mu, \phi_{\Delta,I}(x)] = (2x_\mu(\Delta + x \cdot \partial) - x^2 \partial_\mu) \phi_{\Delta,I}(x) + 2x^\nu (\mathbf{M}_{\nu\mu})_I^J \phi_{\Delta,J}(x). \quad (2.63)$$

operator reduces to the calculation of the ground state energy with corresponding state. A notable application appears in three-dimensional QED with N_f flavors. For sufficiently large N_f , the infrared fixed point is expected to be an interacting CFT. In this theory, monopole operators [14] play a central role, yet evaluating their conformal dimensions is notoriously difficult in flat space. Using the state–operator correspondence, the problem becomes the computation of the vacuum energy on $\mathbb{R} \times \mathbb{S}^2$ in the presence of a fixed magnetic flux through $\mathbb{S}^2$ [14, 15]. This reformulation turns the determination of monopole dimensions into a standard quantum field theory calculation on the cylinder, illustrating the practical power of the conformal framework.

Following the standard procedure of QFT, one can exponentiate the infinitesimal transformations (2.60)–(2.63) to obtain the finite conformal transformation law. Under a finite conformal map $x \mapsto x'(x)$ with scale factor $\Omega(x)$ defined in (2.26), a primary field transforms as

$$\phi'_{\Delta,I}(x') = \Omega(x)^{\Delta/2} R[\mathbf{M}_{\mu\nu}]_I^J \phi_{\Delta,J}(x), \quad (2.64)$$

where $R[\mathbf{M}_{\mu\nu}]$ is the $\text{SO}(d)$ representation matrix acting on the spin indices. In this thesis, we encounter both of the scalar and tensor primaries, for which the transformation rules simplify to

$$\phi'_\Delta(x') = \Omega(x)^{\Delta/2} \phi_\Delta(x), \quad (2.65)$$

$$T'^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_m}(x') = \Omega(x)^{(\Delta+n-m)/2} \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \dots \frac{\partial x'^{\nu_n}}{\partial x^{\beta_n}} \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \dots \frac{\partial x^{\alpha_m}}{\partial x'^{\mu_m}} T^{\beta_1 \dots \beta_n}_{\alpha_1 \dots \alpha_m}(x). \quad (2.66)$$

2.1.3 Conjugate state

Here, we define the conjugate bra state $\langle \mathcal{O} |$ associated with a primary state $|\mathcal{O}\rangle$. In radial quantization the dilatation operator $\mathbf{D}$ plays the role of the Hamiltonian, so the origin $x = 0$ corresponds to the infinite past. To define the conjugate state, we therefore need a transformation that maps the infinite past to the infinite future. The inversion

$$x^\mu \mapsto x'^\mu \equiv \frac{x^\mu}{x^2} \quad (2.67)$$

precisely plays this role. This makes it natural to define conjugation through inversion:

$$\langle \mathcal{O} | \equiv \lim_{x \rightarrow 0} \langle 0 | [\mathcal{O}(x)]^\dagger = \lim_{x \rightarrow 0} \langle 0 | \mathcal{O}'(x), \quad (2.68)$$

where $\mathcal{O}'$ denotes the operator obtained from $\mathcal{O}$ by applying an inversion. Its explicit form follows from the general transformation rules such as Eqs. (2.65) and (2.66). For example, for a scalar primary operator ϕ_Δ ,

$$\phi'_\Delta(x) = \left[\Omega\left(\frac{x^\mu}{x^2}\right) \right]^{\Delta/2} \phi_\Delta\left(\frac{x^\mu}{x^2}\right) = |x|^{-2\Delta} \phi_\Delta\left(\frac{x^\mu}{x^2}\right) \quad (2.69)$$

where the conformal factor $\Omega(x)$ for inversion can be obtained from Eq. (2.1). Consequently, the conjugate state $\langle \phi_\Delta |$ takes the form

$$\langle \phi_\Delta | = \lim_{y \rightarrow \infty} |y|^{2\Delta} \langle 0 | \phi_\Delta(y), \quad (2.70)$$

For tensor primary operators $\mathcal{O} = T^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_m}$, inversion acts as

$$T'^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_m}(x) = |x|^{-2\Delta} I^{\nu_1}_{\beta_1} \dots I^{\nu_n}_{\beta_n} I^{\alpha_1}_{\mu_1} \dots I^{\alpha_m}_{\mu_m} T^{\beta_1 \dots \beta_n}_{\alpha_1 \dots \alpha_m}\left(\frac{x^\mu}{x^2}\right) \quad (2.71)$$

where the inversion tensor

$$I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}, \quad I_{\mu\rho} I^\rho_\nu = \delta_{\mu\nu} \quad (2.72)$$

appears frequently in CFT, for instance as a building block of tensor structures in correlation functions. Using (2.71), the conjugate of a tensor primary state is

$$\langle T_{\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n} | \equiv \lim_{y \rightarrow \infty} |y|^{2\Delta} I_{\beta_1}^{\nu_1}(y) \dots I_{\beta_n}^{\nu_n}(y) I_{\mu_1}^{\alpha_1}(y) \dots I_{\mu_m}^{\alpha_m}(y) \langle 0 | T_{\alpha_1 \dots \alpha_m}^{\beta_1 \dots \beta_n}(y) . \quad (2.73)$$

Finally, let us determine the conjugation rule for the generators of the conformal algebra. The special conformal transformation can be realized as the combination of inversion $\rightarrow$ translation $\rightarrow$ inversion. Since conjugation is defined via inversion, it is natural to set

$$\mathbf{P}_\mu^\dagger = \mathbf{K}_\mu, \quad \mathbf{K}_\mu^\dagger = \mathbf{P}_\mu. \quad (2.74)$$

Requiring that the conjugation be consistent with the Lie algebra fixes the remaining relations to

$$\mathbf{D}^\dagger = -\mathbf{D}, \quad \mathbf{M}_{\mu\nu}^\dagger = -\mathbf{M}_{\mu\nu}. \quad (2.75)$$

2.1.4 Correlation functions

In the previous section, we introduced the primary and descendant operators, and discussed the transformation properties of primary operators under conformal transformations. In this section, we derive the *conformal Ward-Takahashi identities* (CWTIs), which encode the constraints imposed by conformal symmetry on one of the most important physical quantities, namely correlation functions. For simplicity, we restrict ourselves to scalar primary fields and denote $\phi_{\Delta_i}(x_i)$ simply by $\phi_i(x_i)$. The CWTIs for an n -point scalar correlation function are written as follows.

◇ Conformal Ward-Takahashi identity under finite conformal transformations

$$\langle \phi_1(x'_1) \phi_2(x'_2) \dots \phi_n(x'_n) \rangle = \left(\prod_{i=1}^n \Omega(x_i)^{\Delta_i/2} \right) \langle \phi_1(x_1) \phi_2(x_2) \dots \phi_n(x_n) \rangle . \quad (2.76)$$

◇ Conformal Ward-Takahashi identity under infinitesimal conformal transformations

$$\langle [\mathbf{J}_{MN}, \phi_1(x_1) \phi_2(x_2) \dots \phi_n(x_n)] \rangle = 0 . \quad (2.77)$$

Next, using these constraints, we illustrate how correlation functions are fixed, taking the simplest case of scalar primary operators as an example.

One-point function The one-point function $\langle \phi(x) \rangle$ vanishes except for the identity operator **1**. This is because, first, translation invariance, $\langle \phi(x+a) \rangle = \langle \phi(x) \rangle$, implies that $\langle \phi(x) \rangle$ is constant. Furthermore, the CWTI for scale transformations $x' = \lambda x$,

$$\langle \phi(\lambda x) \rangle = \lambda^{-\Delta} \langle \phi(x) \rangle, \quad (2.78)$$

implies that,

$$\langle \phi(x) \rangle = 0, \quad \text{unless } \Delta = 0 . \quad (2.79)$$

Two-point function Next, we consider how the form of the two-point function $\langle \phi_1(x_1) \phi_2(x_2) \rangle$ is constrained by conformal symmetry. From the CWTI associated with Poincaré symmetry, the two-point function depends only on the distance between the points:

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = f(|x_{12}|) , \quad (2.80)$$

where $x_{ij} \equiv x_i - x_j$. Moreover, the CWTI for scale transformations implies

$$f(|x_{12}|) = \lambda^{\Delta_1 + \Delta_2} f(\lambda |x_{12}|) , \quad (2.81)$$

from which we conclude

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{C_{12}}{|x_{12}|^{\Delta_1 + \Delta_2}} . \quad (2.82)$$

Finally, considering the CWTI for the SCT :

$$\begin{aligned} \langle \phi_1(x'_1) \phi_2(x'_2) \rangle &= \frac{C_{12}}{|x'_{12}|^{\Delta_1 + \Delta_2}} \\ &= (\gamma_1 \gamma_2)^{\frac{\Delta_1 + \Delta_2}{2}} \frac{C_{12}}{|x_{12}|^{\Delta_1 + \Delta_2}} \\ &\stackrel{\text{CWTI}}{=} \gamma_1^{\Delta_1} \gamma_2^{\Delta_2} \langle \phi_1(x_1) \phi_2(x_2) \rangle \\ &= \gamma_1^{\Delta_1} \gamma_2^{\Delta_2} \frac{C_{12}}{|x_{12}|^{\Delta_1 + \Delta_2}} . \end{aligned} \quad (2.83)$$

Here, during the manipulations, we used the following relations¹¹.

$$\gamma_i \equiv 1 - 2b \cdot x_i + b^2 x_i^2 , \quad |x'_{12}| = \frac{|x_{12}|}{\gamma_1^{1/2} \gamma_2^{1/2}} . \quad (2.85)$$

By comparing the second and fourth lines of (2.83), we see that a nontrivial two-point function exists only if $\Delta_1 = \Delta_2$. Therefore, the two-point correlation function is finally given by

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{C_{12} \delta_{\Delta_1 \Delta_2}}{|x_{12}|^{2\Delta_1}} . \quad (2.86)$$

We note that the structure of correlation functions can be determined either using the finite CWTI (2.76) or the infinitesimal CWTI (2.77). Indeed, it is straightforward to verify that (2.86) satisfies the differential equation (2.77).

¹¹Not only for SCTs, but for all finite conformal transformations (see (2.26)), the following holds:

$$|x'_{12}|^2 = \frac{|x_{12}|^2}{\Omega(x_1)^{1/2} \Omega(x_2)^{1/2}} . \quad (2.84)$$

Three-point function Similarly, conformal symmetry completely fixes the form of the three-point function of scalar primary operators:

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} . \quad (2.87)$$

Regarding the overall coefficients C_{12} and C_{123} , in many references C_{12} is absorbed into a redefinition of the scalar primary fields, setting $C_{12} = 1$. In that case, the overall coefficient C_{123} of the three-point function cannot generally be removed by field redefinitions and carries important physical information. As we will see in the following sections, C_{123} is related to the OPE coefficients, which are part of the CFT data.

Four-point function Unlike the previous cases, for four-point functions there exist two dimensionless quantities, called *cross ratios*, that are invariant under conformal transformations:

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \quad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} . \quad (2.88)$$

The invariance of the cross ratios under conformal transformations follows immediately from (2.84). Thanks to these invariants, the four-point function can be written as

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = \frac{F(u, v)}{\prod_{i < j}^4 |x_{ij}|^{\Delta_i + \Delta_j - \frac{\bar{\Delta}}{3}}} , \quad \bar{\Delta} \equiv \sum_{i=1}^4 \Delta_i . \quad (2.89)$$

Here, the denominator is determined by conformal symmetry, as in the two- and three-point functions, while $F(u, v)$ is the part not fixed by the CWTIs. In particular, for four identical scalar primaries ($\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = \Delta$), the correlation function can be written in the compact form:

$$\begin{aligned} \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle &= \frac{F(u, v)}{|x_{12}|^{\frac{2\Delta}{3}} |x_{34}|^{\frac{2\Delta}{3}} |x_{13}|^{\frac{2\Delta}{3}} |x_{24}|^{\frac{2\Delta}{3}} |x_{14}|^{\frac{2\Delta}{3}} |x_{23}|^{\frac{2\Delta}{3}}} \\ &= \frac{u^{\frac{2\Delta}{3}} v^{-\frac{\Delta}{3}} F(u, v)}{|x_{12}|^{2\Delta} |x_{34}|^{2\Delta}} \\ &\equiv \frac{G(u, v)}{|x_{12}|^{2\Delta} |x_{34}|^{2\Delta}} . \end{aligned} \quad (2.90)$$

The function $F(u, v)$ and $G(u, v)$ are closely related to conformal blocks, as discussed in e.g. [11].

2.1.5 Operator product expansion

In this section, we introduce the operator product expansion (OPE), which always plays a central role in the analysis of CFTs. For this purpose, let us consider two primary operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(0)$ inserted at positions x and 0 , respectively (see Fig. 2.1). When these operators are sufficiently

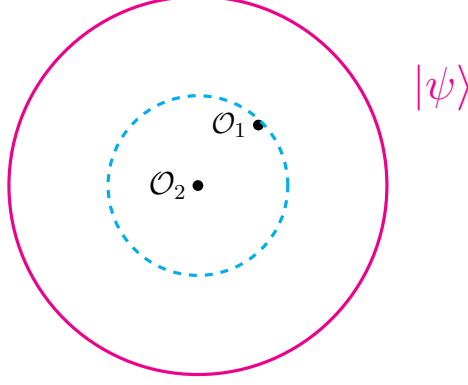


Figure 2.1: The state considered in the proof of the operator product expansion. The operator $\mathcal{O}_2$ is inserted at the origin in radial quantization.

close, it is expected that their product can be expanded in terms of local operators. Schematically, if $\mathcal{V}$ denotes the set of all local operators, we can write

$$\mathcal{O}_1(x) \times \mathcal{O}_2(0) = \sum_{\mathcal{O}_i \in \mathcal{V}} D_{12}^i(x) \mathcal{O}_i(0). \quad (2.91)$$

This expansion is known as the OPE [16], and it has been widely used in various contexts of quantum field theory, such as the analysis of hadrons.¹² Very interestingly, in CFTs, the OPE can be justified as follows. For simplicity, let us assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ are scalar primary operators.

Proof of OPE in CFT. Consider the setup depicted in Fig. 2.1, where the two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are inserted inside the pink $(d-1)$ -dimensional sphere. The state on this Hilbert space slice, denoted by $|\psi\rangle$, can be written as

$$|\psi\rangle = \mathcal{O}_1(x) \mathcal{O}_2(0) |0\rangle. \quad (2.92)$$

Using the state-operator correspondence (2.52), the operator $\mathcal{O}_2(0)$ inserted at the origin corresponds to the state $|\mathcal{O}_2\rangle$, so that the state $|\psi\rangle$ can equivalently be written as

$$|\psi\rangle = \mathcal{O}_1(x) |\mathcal{O}_2\rangle. \quad (2.93)$$

As mentioned in Sec. 2.1.2, in unitary CFTs, a complete basis of the Hilbert space can be formed by primary states and their descendants [11]. Consequently, the identity operator can be expressed as

$$1 = \sum_{\mathcal{O}_I \in \text{primary}} |\mathcal{O}_I\rangle \langle \mathcal{O}_I| + (\text{descendants}), \quad (2.94)$$

where I labels the primary operators by their conformal dimension and spin. Using this decomposition, the state $|\psi\rangle$ can be expanded as

$$|\psi\rangle = \sum_{\mathcal{O}_I \in \text{primary}} C_{12I}(x, \mathbf{P}^\mu) |\mathcal{O}_I\rangle, \quad (2.95)$$

¹²Note that the definition in (2.91) should be understood as an operator equation.

where descendant states can be generated by acting with momentum operators on primary states. Combining (2.92) and (2.95) and applying the state-operator correspondence in reverse, we obtain the operator equation

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_{\mathcal{O}_I \in \text{primary}} C_{12I}(x, \partial) \mathcal{O}_I(0). \quad (2.96)$$

Thus, we can arrive at the desired OPE. Note that, being an operator equation, the OPE can be used inside correlation functions.

Next, let us examine how conformal symmetry constrains the form of $C_{12I}(x, \partial)$ in (2.96). From scale invariance, $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_I$ must scale consistently. Combined with rotational invariance, we then expect

$$C_{12I}(x, \partial)\mathcal{O}_I(0) = \frac{c_{12I}}{|x|^{\Delta_1+\Delta_2-\Delta_I}}(\mathcal{O}_I(0) + \dots), \quad (2.97)$$

where $\dots$ denote descendants of $\mathcal{O}_I$. As a consistency check, one can act with the dilatation operator on both sides and verify that the scaling dimensions match. Furthermore, by imposing special conformal invariance, the structure of $C_{12I}(x, \partial)$, including the descendant terms, is highly constrained. For instance, for scalar primary contributions, one can write the full-order expression [17]:

$$|x|^{\Delta_1+\Delta_2-\Delta} \mathcal{C}_{12\Delta}(x, \partial) = \frac{1}{B(a_+, a_-)} \int_0^1 dt t^{a_+-1} (1-t)^{a_--1} \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{4}x^2 t(1-t)\partial^\mu \partial_\mu\right)^m}{m!(\Delta+1-\frac{d}{2})_m} e^{tx \cdot \partial},$$

$$a_{\pm} \equiv \frac{1}{2}(\Delta \pm (\Delta_1 - \Delta_2)), \quad B(a, b) \equiv \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (2.98)$$

Finally, the overall coefficients c_{12I} are not fixed by conformal symmetry alone; they are related to the normalization of three-point functions (2.87). To see this, let us consider the three-point function $\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\mathcal{O}_3(z) \rangle$ for $|x| < |z|$ and apply the OPE to $\mathcal{O}_1(x)\mathcal{O}_2(0)$. One can then find

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\mathcal{O}_3(z) \rangle = \sum_{\mathcal{O}_I} c_{12I} \mathcal{C}_{12I}(x, \partial_y) \langle \mathcal{O}_I(y)\mathcal{O}_3(z) \rangle \Big|_{y=0} \quad (2.99)$$

Since the two-point function between a spinning primary and a scalar primary vanishes, only scalar primaries need to be summed over. Focusing on the part with conformal dimension Δ_3 , we have¹³

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\mathcal{O}_3(z) \rangle = c_{123} \mathcal{C}_{123}(x, \partial_y) \langle \mathcal{O}_3(y)\mathcal{O}_3(z) \rangle \Big|_{y=0}. \quad (2.100)$$

Using the explicit expressions for two- and three-point functions:

$$\langle \mathcal{O}_3(0) \mathcal{O}_3(z) \rangle = \frac{c(\mathcal{O}_3, \mathcal{O}_3)}{|z|^{2\Delta_3}},$$

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(0) \mathcal{O}_3(z) \rangle = \frac{c(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)}{|x|^{\Delta_1+\Delta_2-\Delta_3} |z|^{\Delta_2+\Delta_3-\Delta_1} |x-z|^{\Delta_3+\Delta_1-\Delta_2}} \quad (2.101)$$

$$\underset{|x| \sim 0}{\sim} \frac{c(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)}{|x|^{\Delta_1+\Delta_2-\Delta_3} |z|^{2\Delta_3}},$$

¹³In general, the operator spectrum exhibits degeneracies.

we find the relation between the OPE coefficient and the overall normalization of two- and three-point functions:

$$c_{123} = \frac{c(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)}{c(\mathcal{O}_3, \mathcal{O}_3)} . \quad (2.102)$$

Hence, the OPE coefficients essentially play the same role as the three-point function coefficients in characterizing the theory.

2.1.6 CFT Data

The forms of correlation functions and the functions $\mathcal{C}_{12I}(x, \partial)$ are entirely fixed by conformal symmetry and thus independent of the specific details of a theory. Nevertheless, there exist physical quantities that cannot be fixed by symmetry alone; these quantities encode the dynamical content of the theory. For instance, the conformal dimension Δ and the $\text{SO}(d)$ spin ℓ clearly depend on the details of theory. In addition, the OPE coefficients c_{123} introduced in the previous subsection are also theory-dependent. No new independent quantities appear in n -point functions with $n \geq 4$, because repeated applications of the OPE reduce such correlators to products of lower-point functions, where only previously introduced OPE coefficients appear. Therefore, the quantities not fixed by conformal symmetry are precisely conformal dimensions, spin, OPE coefficients, which are of central importance in characterizing the theory. These three quantities are collectively referred to as the *CFT data*:¹⁴

CFT data: conformal dimension, $\text{SO}(d)$ spin, OPE coefficient .

Determining these CFT data provides a complete characterization of the theory and allows one to reconstruct all correlation functions through the OPE and conformal symmetry. In particular, the modern conformal bootstrap program exploits the consistency conditions imposed by crossing symmetry and unitarity to constrain or even solve for these data. Various numerical and analytical techniques have been developed to extract CFT data in both perturbative and non-perturbative regimes.

2.2 Defects in CFT

In this subsection, our goal is to review the systematics of defect CFT. In brief, a defect CFT is a QFT in d -dimensional spacetime $\mathbb{R}^{d15}$ in which a defect such as boundary or impurity is present. A defect CFT is expected to appear as a fixed point of the RG flow of a QFT with a defect. Moreover, even if a defect is present in $\mathbb{R}^d$, points in the bulk sufficiently far from the defect are expected to be unaffected by its presence¹⁶. This implies that the existence of a defect does not modify the

¹⁴Furthermore, in theories with a global symmetry, the representation of each operator under this symmetry constitutes an additional crucial piece of CFT data.

¹⁵We assume Euclidean spacetime in this subsection.

¹⁶For example, consider a point-like source j localized at x_0 in d -dimensional spacetime. A scalar field ϕ satisfies

$$\square\phi(x) = j(x), \quad j(x) \sim \delta^{(d)}(x - x_0).$$

Here, x_0 can be regarded as a defect, but clearly the bulk points $x \neq x_0$ follow a free theory and are unaffected by the defect.

criticality of the bulk, a property referred to as *bulk criticality*.¹⁷ Throughout this thesis, we assume this bulk criticality. Consequently, the bulk CFT data (e.g. conformal dimensions of bulk operators) are independent of the defect. While this assumption may seem to trivialize the dynamics of defect CFT, it does not: even when the bulk universality class is fixed, localized degrees of freedom on the defect broaden the range of possible universality classes! In the following, we will see various model-independent properties of defect CFT following [19].

2.2.1 Residual conformal symmetry

In general, when a p -dimensional conformal defect $\widehat{\mathcal{D}}^{(p)}$ is present, the d -dimensional conformal group $\text{SO}(1, d+1)$ is reduced to the subgroup that leaves the defect invariant, namely its shape and location. For simplicity, we focus on planar conformal defect, i.e. $\widehat{\mathcal{D}}^{(p)} = \mathbb{R}^p$. Correspondingly, it is convenient to decompose a spacetime point x^μ in $\mathbb{R}^d$ into components parallel and perpendicular to the defect:

$$x^\mu = (\hat{x}^a, x_\perp^i), \quad a = 1, \dots, p, \quad i = p+1, \dots, d, \quad (2.103)$$

where the defect is located at $x_\perp^i = 0$.¹⁸ Next, we consider the generators of the conformal algebra $\mathfrak{so}(1, d+1)$ that leave the defect invariant:

- Translations: $\mathbf{P}_\mu = (\mathbf{P}_a, \mathbf{P}_i)$.
Only translations $\mathbf{P}_a$ along the defect are preserved. However, $\mathbf{P}_i$ clearly moves the defect in perpendicular directions and is therefore broken.
- Rotation generators: $\mathbf{M}_{\mu\nu} = \{\mathbf{M}_{ab}, \mathbf{M}_{ai}, \mathbf{M}_{ij}\}$.
The generators $\mathbf{M}_{ai}$ generically rotate the defect in a way that changes its position and therefore are broken.¹⁹ On the other hand, $\mathbf{M}_{ab}$ and $\mathbf{M}_{ij}$ leave the defect invariant and hence remain unbroken.
- Dilatation generator: $\mathbf{D}$.
The dilatation generator $\mathbf{D}$ preserves both the position and the shape of the defect and therefore remains a symmetry.
- Special conformal generators: $\mathbf{K}_\mu = (\mathbf{K}_a, \mathbf{K}_i)$.
This might be somewhat non-trivial, so we give a detailed explanation. Recall that the defect $\widehat{\mathcal{D}}^{(p)}$ extends along the coordinates $x^\mu \equiv (\hat{x}^a, x_\perp^i = 0)$. Under the SCT in the transverse i -direction, the defect location is transformed as

$$x_\perp^i = \frac{-b^i \hat{x}^2}{1 - 2\hat{b} \cdot \hat{x} + b^2 \hat{x}^2}. \quad (2.104)$$

From this expression, we see that the position of the defect remains invariant only when $b^i = 0$. Consequently, the generators $\mathbf{K}_i$ are broken, while $\mathbf{K}_a$ remain unbroken.

¹⁷A detailed discussion on bulk criticality can be found in the review [18].

¹⁸In this thesis, quantities related to the defect are denoted with a hat.

¹⁹Strictly speaking, these symmetries may not be completely broken but can be reduced to discrete subgroups. For example, the π rotation of the defect generated by $\exp(i\pi \mathbf{M}_{ai})$ may remain as a symmetry. Such possibilities will not be considered in this thesis.

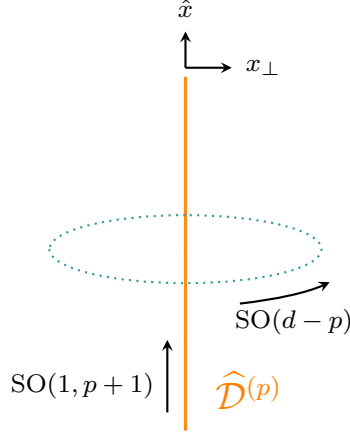


Figure 2.2: The illustration of residual conformal symmetry.

From the above discussion, the generators of the conformal algebra that leave the position of the defect invariant are

$$\{\mathbf{P}_a, \mathbf{M}_{ab}, \mathbf{D}, \mathbf{K}_a\}, \quad \{\mathbf{M}_{ij}\}. \quad (2.105)$$

Interestingly, the former set of generators forms the p -dimensional conformal group $\text{SO}(1, p+1)$ acting along the defect, while the latter generates the rotational symmetry $\text{SO}(d-p)$ around the defect. Therefore, upon inserting a p -dimensional defect, the full conformal group $\text{SO}(1, d+1)$ is broken as follows (see Fig. 2.2):

$$\text{SO}(1, d+1) \longrightarrow \text{SO}(1, p+1) \times \text{SO}(d-p). \quad (2.106)$$

The structure of this symmetry breaking becomes particularly transparent when compared with (5.14). In terms of the generators of $\mathfrak{so}(1, d+1)$, the residual symmetry algebra in defect CFT can be represented as

$$\mathbf{J}_{MN} = \begin{matrix} \begin{matrix} M \setminus N \rightarrow \\ \downarrow \end{matrix} & \begin{matrix} -1 & 0 & b & j \end{matrix} \\ \begin{matrix} -1 \\ 0 \\ a \\ i \end{matrix} & \begin{pmatrix} 0 & \mathbf{D} & \frac{1}{2}(\mathbf{P}_b - \mathbf{K}_b) & \frac{1}{2}(\mathbf{P}_j - \mathbf{K}_j) \\ -\mathbf{D} & 0 & \frac{1}{2}(\mathbf{P}_b + \mathbf{K}_b) & \frac{1}{2}(\mathbf{P}_j + \mathbf{K}_j) \\ -\frac{1}{2}(\mathbf{P}_a - \mathbf{K}_a) & -\frac{1}{2}(\mathbf{P}_a + \mathbf{K}_a) & \mathbf{M}_{ab} & \mathbf{M}_{aj} \\ -\frac{1}{2}(\mathbf{P}_i - \mathbf{K}_i) & -\frac{1}{2}(\mathbf{P}_i + \mathbf{K}_i) & \mathbf{M}_{ib} & \mathbf{M}_{ij} \end{pmatrix} \end{matrix}. \quad (2.107)$$

Here, the slashed entries indicate broken generators. This perspective is particularly useful when applying the embedding space formalism to defect CFTs as demonstrated in Chap. 5.

2.2.2 Local operators in defect CFT

In defect CFTs, we clearly have the two distinct types of local operators appear:²⁰

- bulk local operators $\mathcal{O}(x^\mu)$, inserted at points in the bulk,

²⁰As mentioned in Footnote 11, quantities associated with the defect will be denoted with a hat.

- defect local operators $\widehat{\mathcal{O}}(\hat{x}^a)$, inserted on the defect (i.e. at $x_\perp^i = 0$).

In QFTs, local fields are classified according to representations of the symmetry group. It is therefore necessary to understand under which symmetry groups bulk and defect local operators transform, and how they are organized into irreducible representations.

Let us begin with bulk local operators. Since they are inserted in the bulk, and taking into account the assumption of bulk criticality, it is appropriate to classify them according to representations of the d -dimensional conformal symmetry group $\text{SO}(1, d+1)$. As discussed in Sec. 2.1.2, bulk local operators are characterized by their conformal dimension Δ and their $\text{SO}(d)$ spin ℓ . We therefore denote them as

bulk local operators : $\mathcal{O}_{\Delta, \ell}$.

Next, we turn to defect local operators. Since defect operators are supported on the defect, they should be classified not by the full conformal group, but by representations of the preserved symmetry group $\text{SO}(1, p+1) \times \text{SO}(d-p)$. An important feature of defect local operators is that, in addition to their conformal dimension $\widehat{\Delta}$, they are characterized by two types of spin: an $\text{SO}(p)$ spin j and an $\text{SO}(d-p)$ spin s . In much of the literature, the former is referred to as the *parallel spin*, while the latter is called the *transverse spin*. Taking this into account, we denote defect local operators as

defect local operators : $\widehat{\mathcal{O}}_{\widehat{\Delta}, j, s}$.

For codimension-one (i.e. $d-p=1$) defects, such as boundaries or interfaces, the transverse spin becomes trivial, and defect local operators are classified solely by their conformal dimension and parallel spin. On the other hand, for line defects (i.e. $p=1$), the parallel spin is trivial, and defect local operators are instead characterized by their conformal dimension and transverse spin.

Since $\mathbf{K}_a$ acts as a lowering operator for the conformal dimension, as shown in (2.54), a defect *primary* state is defined by

$$\left| \widehat{\mathcal{O}}_{\widehat{\Delta}, j, s} \right\rangle \text{ is a defect primary } \stackrel{\text{def}}{\iff} \mathbf{K}_a \left| \widehat{\mathcal{O}}_{\widehat{\Delta}, j, s} \right\rangle = 0. \quad (2.108)$$

The corresponding descendant states are obtained by acting with the translation generators $\mathbf{P}_a$ along the defect, as in (2.57).

2.2.3 Correlation functions in the presence of defect

Next, consider the $(n+m)$ -point correlation function with n bulk local operators and m defect local operators inserted:

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \widehat{\mathcal{O}}_1(\hat{y}_1) \cdots \widehat{\mathcal{O}}_m(\hat{y}_m) \rangle. \quad (2.109)$$

This correlator is defined using the vacuum $|0\rangle$, which is invariant under the full conformal group $\text{SO}(1, d+1)$, as follows [20]:

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \widehat{\mathcal{O}}_1(\hat{y}_1) \cdots \widehat{\mathcal{O}}_m(\hat{y}_m) \rangle \equiv \frac{1}{\langle \widehat{\mathcal{D}}^{(p)} \rangle_0} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \widehat{\mathcal{O}}_1(\hat{y}_1) \cdots \widehat{\mathcal{O}}_m(\hat{y}_m) \widehat{\mathcal{D}}^{(p)} \rangle_0. \quad (2.110)$$

Here, $\langle \cdots \rangle_0 \equiv \langle 0 | \cdots | 0 \rangle$ denotes the vacuum expectation value in the absence of the defect, and $\widehat{\mathcal{D}}^{(p)}$ is an extended operator corresponding to the insertion of the defect. Note that the right-hand side can be written explicitly in terms of the usual path integral formulation. As in the ordinary CFTs, correlation functions in defect CFT satisfy the CWTIs associated with infinitesimal conformal transformations:

$$\left\langle \left[\mathbf{J}_{AB}, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \widehat{\mathcal{O}}_1(\hat{y}_1) \cdots \widehat{\mathcal{O}}_m(\hat{y}_m) \right] \right\rangle = 0, \quad (2.111)$$

$$\left\langle \left[\mathbf{J}_{IJ}, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \widehat{\mathcal{O}}_1(\hat{y}_1) \cdots \widehat{\mathcal{O}}_m(\hat{y}_m) \right] \right\rangle = 0, \quad (2.112)$$

where $A, B = -1, 0, 1, \cdots, p$ and $I, J = p+1, \cdots, d$.

To illustrate how these reduced symmetry constraints operate in practice, let us consider the simplest example: the one-point function of a scalar bulk primary operator $\mathcal{O}_\Delta(x)$ with conformal dimension Δ ,

$$\langle \mathcal{O}_\Delta(x) \rangle. \quad (2.113)$$

The CWTI associated to the translation along the defect is equivalent to the following differential equation:

$$\partial_a \langle \mathcal{O}_\Delta(x) \rangle = 0. \quad (2.114)$$

This constraint means that the bulk one-point function is independent of the parallel coordinates $\hat{x}^a$. Combining this with the CWTI associated to the $\text{SO}(d-p)$ rotational symmetry around the defect, the bulk one-point function can be written as $\langle \mathcal{O}_\Delta(x) \rangle = f(|x_\perp|)$. Finally, the scale invariance implies that the form of correlation function is fixed to be

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{a_{\mathcal{O}}}{|x_\perp|^\Delta}. \quad (2.115)$$

Recall that in an ordinary CFT without defects, conformal symmetry implies that the one-point function of any local operator vanishes unless the operator is the identity as in (2.79). Interestingly, nontrivial structures in correlation functions already appear at the level of the one-point function!

Similarly, by solving the above CWTIs, one can constrain the functional forms of correlation functions to a large extent. Although these constraints are weaker than those in ordinary CFTs due to the reduced symmetry, they are still powerful enough to fix the spacetime dependence of many correlators up to a finite set of constants. In what follows, we restrict ourselves to listing several selected correlation functions that will be relevant for later discussions.²¹

²¹It is also worth emphasizing that correlation functions involving only defect local operators are essentially identical in structure to those of an ordinary CFT without defects. This is because defect local operators are thought to be organized by the p -dimensional conformal symmetry $\text{SO}(1, p+1)$ acting along the defect with the global symmetry $\text{SO}(d-p)$, and their correlators are completely constrained by this symmetry in the same way as in a standalone p -dimensional CFT.

Bulk-defect two-point function

$$\langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta},j,s}^{j_1 \dots j_j; i_1 \dots i_s}(0) \rangle = b(\mathcal{O}, \widehat{\mathcal{O}}^s) \delta_{j,0} \frac{x_\perp^{(i_1 \dots i_s)}}{|x|^{2\widehat{\Delta}} |x_\perp|^{\Delta - \widehat{\Delta} + s}}, \quad (2.116)$$

where $b(\mathcal{O}, \widehat{\mathcal{O}}^s)$ is some coefficient and $(\dots)$ means to take the symmetric traceless projection. It is crucial to note that this two-point function vanishes whenever the defect local operator carries non-trivial parallel spin.

Bulk-bulk two-point function

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \frac{F(\xi, \phi)}{|x_{1\perp}|^{\Delta_1} |x_{2\perp}|^{\Delta_2}}, \quad (2.117)$$

where the residual conformal invariant cross-ratios ξ and ϕ are defined by

$$\xi \equiv \frac{|x_1 - x_2|^2}{|x_{1\perp}| |x_{2\perp}|}, \quad \cos \phi \equiv \frac{x_{1\perp} \cdot x_{2\perp}}{|x_{1\perp}| |x_{2\perp}|}. \quad (2.118)$$

Bulk-defect-defect three-point function

$$\langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}_1}(\hat{y}_1) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}(\hat{y}_2) \rangle = \frac{g(v)}{|x_\perp|^\Delta |x - \hat{y}_1|^{\widehat{\Delta}_1 - \widehat{\Delta}_2} |x - \hat{y}_2|^{\widehat{\Delta}_2 - \widehat{\Delta}_1} |\hat{y}_{12}|^{\widehat{\Delta}_1 + \widehat{\Delta}_2}}, \quad (2.119)$$

where v is the residual conformal invariant cross-ratio given by

$$v = \frac{|x_\perp|^2 |\hat{y}_{12}|^2}{|x - \hat{y}_1|^2 |x - \hat{y}_2|^2}. \quad (2.120)$$

Finally, although the functions $F(\xi, \phi)$ and $g(v)$ appear arbitrary at first sight, they are in fact closely related to conformal block expansions and their forms are also highly constrained. The former will be discussed in detail in Sec. 2.2.6, while the latter will be analyzed in App. ???. Before turning to these topics, however, we first explain an important structural operator relation known as the bulk–defect operator expansion.

2.2.4 Bulk-defect operator expansion

As mentioned at the beginning, in defect CFTs there exists, in addition to the bulk OPE, the bulk-defect operator expansion (OE). To define it, let us consider a situation in which a bulk local operator $\mathcal{O}(x)$ approaches the defect, namely $|x_\perp| \sim 0$. In this limit, a bulk local operator can be regarded, loosely speaking, as a superposition of defect local operators (see Fig. 2.3).

In what follows, we restrict our attention to bulk scalar primary operators. In this case, the bulk-defect OE takes the form

$$\mathcal{O}_\Delta(x) = \sum_{\widehat{\mathcal{O}}_{\widehat{\Delta},s}} b_{\mathcal{O}\widehat{\mathcal{O}}} \mathcal{B}(|x_\perp|, \partial_a)_{i_1 i_2 \dots i_s} \widehat{\mathcal{O}}_{\widehat{\Delta}}^{s, i_1 i_2 \dots i_s}(\hat{x}). \quad (2.121)$$

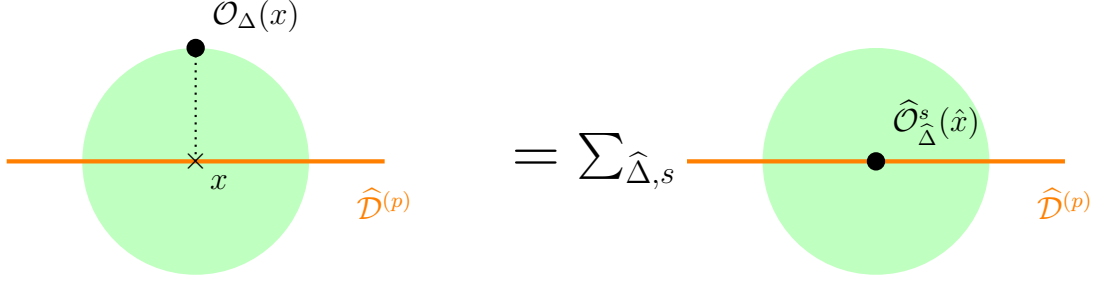


Figure 2.3: Schematic picture of the DOE. A bulk local operator $\mathcal{O}_\Delta(x)$ can be expanded by using the defect local ones $\hat{\mathcal{O}}_\Delta^s(\hat{x})$. This expansion is valid only when no local operators are inserted inside of the virtual ball (green region) whose center is $\hat{x}$ and radius is $|x_\perp|$.

Remarkably, only operators carrying transverse spin contribute to the expansion because the two-point function of bulk scalar primary operator and defect spinning operator with non-trivial parallel spin vanishes as (2.116). In the rest of this subsection, we will see that the function $\mathcal{B}(|x_\perp|, \partial_a)$ is completely fixed from the residual conformal symmetry while the coefficient $b_{\mathcal{O}\hat{\mathcal{O}}}$ cannot be determined purely from symmetry and carries dynamical information. To this end, we evaluate the bulk-defect two-point function

$$\langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_\Delta^{s, j_1 j_2 \dots j_s}(0) \rangle \quad (2.122)$$

in two different ways. The first one is to use the CWTIs for residual conformal symmetry and fix the form of correlation function. This has already been done in the last section, and its form is given by (2.116). The second one is to substitute the operator equation (2.121) into (2.122). We can compare these two different expressions, which leads to fixing the function $\mathcal{B}(|x_\perp|, \partial_a)$. This is our strategy which will be performed below.

Plugging (2.121) into (2.122), we obtain

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_\Delta^{s, j_1 \dots j_s}(0) \rangle &= \sum_{\hat{\mathcal{O}}^{s'}} b_{\mathcal{O}\hat{\mathcal{O}}^{s'}} \mathcal{B}(|x_\perp|, \partial_a)^{i_1 i_2 \dots i_{s'}} \langle \hat{\mathcal{O}}_{\Delta, i_1 i_2 \dots i_{s'}}^{s'}(\hat{x}) \hat{\mathcal{O}}_\Delta^{s, j_1 \dots j_s}(0) \rangle \\ &= b_{\mathcal{O}\hat{\mathcal{O}}^s} \mathcal{B}(|x_\perp|, \partial_a)^{i_1 i_2 \dots i_s} \frac{\delta_{i_1}^{(j_1} \dots \delta_{i_s}^{j_s)} c(\hat{\mathcal{O}}^s, \hat{\mathcal{O}}^s)}{|\hat{x}|^{2\hat{\Delta}}} \\ &= b_{\mathcal{O}\hat{\mathcal{O}}^s} \mathcal{B}(|x_\perp|, \partial_a)^{(j_1 \dots j_s)} \frac{c(\hat{\mathcal{O}}^s, \hat{\mathcal{O}}^s)}{|\hat{x}|^{2\hat{\Delta}}}. \end{aligned} \quad (2.123)$$

In the second line, we have used

$$\langle \hat{\mathcal{O}}_{\Delta, i_1 i_2 \dots i_s}^s(\hat{x}) \hat{\mathcal{O}}_\Delta^{s, j_1 \dots j_s}(0) \rangle = \frac{\delta_{i_1}^{(j_1} \dots \delta_{i_s}^{j_s)} c(\hat{\mathcal{O}}^s, \hat{\mathcal{O}}^s)}{|\hat{x}|^{2\hat{\Delta}}}, \quad (2.124)$$

where $c(\widehat{\mathcal{O}}^s, \widehat{\mathcal{O}}^s)$ is an overall normalization constant. Comparing (2.116) and (2.123), we arrive at

$$\begin{aligned} b_{\mathcal{O}\widehat{\mathcal{O}}^s} c(\widehat{\mathcal{O}}^s, \widehat{\mathcal{O}}^s) \left[\mathcal{B}(|x_\perp|, \partial_a)^{j_1 \dots j_s} \frac{1}{|\hat{x}|^{2\widehat{\Delta}}} \right] &= b(\mathcal{O}, \widehat{\mathcal{O}}^s) \frac{x_\perp^{j_1} \dots x_\perp^{j_s}}{|x_\perp|^{\Delta-\widehat{\Delta}+s} \{|\hat{x}|^s + |x_\perp|^2\}^{\widehat{\Delta}}} \\ &= b(\mathcal{O}, \widehat{\mathcal{O}}^s) \frac{x_\perp^{j_1} \dots x_\perp^{j_s}}{|x_\perp|^{\Delta-\widehat{\Delta}+s} |\hat{x}|^{2\widehat{\Delta}}} \sum_{n=0}^{\infty} \frac{(\widehat{\Delta})_n}{n!} \left(-\frac{|x_\perp|^2}{|\hat{x}|^2} \right)^n. \end{aligned} \quad (2.125)$$

In the second line, we have expanded for $|x_\perp| < |\hat{x}|$, which is appropriate when considering the bulk-defect OE. (See also Fig. 2.3.) Using (2.125) as a guide, we can determine the explicit form of $\mathcal{B}(|x_\perp|, \partial_a)$. For this purpose, we employ the identity

$$\widehat{\square}^n \frac{1}{|\hat{x}|^{2\widehat{\Delta}}} = \frac{4^n (\widehat{\Delta})_n \left(\widehat{\Delta} - \frac{p-2}{2} \right)_n}{|\hat{x}|^{2\widehat{\Delta}+2n}}, \quad \widehat{\square} \equiv \partial_a \partial^a. \quad (2.126)$$

Rewriting the series on the right-hand side of (2.125) using this identity, we find

$$\text{RHS of (2.125)} = b(\mathcal{O}, \widehat{\mathcal{O}}^s) \left[\frac{1}{|x_\perp|^{\Delta-\widehat{\Delta}}} \sum_{n=0}^{\infty} \frac{(-4)^{-n}}{n! (\widehat{\Delta} - \frac{p-2}{2})_n} \frac{x_\perp^{j_1} \dots x_\perp^{j_s}}{|x_\perp|^s} \left(|x_\perp|^2 \widehat{\square} \right)^n \frac{1}{|\hat{x}|^{2\widehat{\Delta}}} \right]. \quad (2.127)$$

Therefore, $\mathcal{B}(|x_\perp|, \partial_a)^{i_1 \dots i_s}$ is completely fixed by defect conformal symmetry as desired,

$$\mathcal{B}(|x_\perp|, \partial_a)^{i_1 \dots i_s} = \frac{1}{|x_\perp|^{\Delta-\widehat{\Delta}}} \sum_{n=0}^{\infty} \frac{(-4)^{-n}}{n! (\widehat{\Delta} - \frac{p-2}{2})_n} \frac{x_\perp^{i_1} \dots x_\perp^{i_s}}{|x_\perp|^s} \left(|x_\perp|^2 \widehat{\square} \right)^n. \quad (2.128)$$

Moreover, the relation between the bulk-defect OE coefficient $b_{\mathcal{O}\widehat{\mathcal{O}}^s}$ and the normalization constants of the two-point functions, $b(\mathcal{O}, \widehat{\mathcal{O}}^s)$ and $c(\widehat{\mathcal{O}}^s, \widehat{\mathcal{O}}^s)$, is found to be

$$b_{\mathcal{O}\widehat{\mathcal{O}}^s} = \frac{b(\mathcal{O}, \widehat{\mathcal{O}}^s)}{c(\widehat{\mathcal{O}}^s, \widehat{\mathcal{O}}^s)}. \quad (2.129)$$

For later reference, we summarize the results below:

Bulk-defect OE : When a bulk scalar primary operator $\mathcal{O}_\Delta(x)$ approaches the defect, it can be expanded in terms of defect local operators as

$$\mathcal{O}_\Delta(x) = \sum_{\widehat{\mathcal{O}}_{\Delta,s}} b_{\mathcal{O}\widehat{\mathcal{O}}} \mathcal{B}(|x_\perp|, \widehat{\square})_{i_1 i_2 \dots i_s} \widehat{\mathcal{O}}_{\Delta}^{s, i_1 i_2 \dots i_s}(\hat{x}). \quad (2.130)$$

The function $\mathcal{B}(|x_\perp|, \partial_a)_{i_1 i_2 \dots i_s}$ is completely fixed by defect conformal symmetry:

$$\mathcal{B}(|x_\perp|, \widehat{\square})_{i_1 \dots i_s} = \frac{1}{|x_\perp|^{\Delta-\widehat{\Delta}}} \sum_{n=0}^{\infty} \frac{(-4)^{-n}}{n! (\widehat{\Delta} - \frac{p-2}{2})_n} \frac{x_\perp^{i_1} \dots x_\perp^{i_s}}{|x_\perp|^s} \left(|x_\perp|^2 \widehat{\square} \right)^n. \quad (2.131)$$

In this way, the bulk-defect OE provides a precise relation between bulk local operators, classified by irreducible representations of $SO(1, d + 1)$, and defect local operators, classified by irreducible representations of $SO(1, p + 1) \times SO(d - p)$. The bulk-defect OE is one of the most fundamental ingredients in defect CFTs, and appears again and again throughout this thesis. Its importance will become even more apparent later.

2.2.5 Defect CFT data

In the previous subsection, we introduced the bulk–defect operator expansion and saw that the OE coefficients $b_{\mathcal{O}\widehat{\mathcal{O}}_s}$ are not fixed solely by defect conformal symmetry, but instead depend on the specific theory under consideration. Moreover, the conformal dimensions of defect local operators, as well as their transverse and parallel spins, are not determined by symmetry alone. In this sense, defect CFTs possess additional theory-dependent data beyond the bulk CFT data discussed in Sec. 2.1.6.

Thus, in defect CFTs, there exist CFT data characterizing the dynamics of the defect itself, such as the bulk–defect OE coefficients. In this thesis, we collectively refer to all theory-dependent parameters defined both in the bulk and on the defect as *defect CFT data*:

Defect CFT data: The parameters characterizing the dynamics of a defect CFT consist of the following sets associated with the bulk theory and the defect theory, respectively:

- bulk: {conformal dimensions, $SO(d)$ spins, OPE coefficients, ...},
- defect: {conformal dimensions, $SO(p)$ spins, $SO(d - p)$ spins, OE coefficients, ...}.

As long as the bulk theory remains critical, the bulk CFT data are not affected by the presence of the defect. Consequently, results from ordinary conformal field theory without defects can be directly applied to the analysis of the bulk sector. For this reason, studies of the dynamics of defect CFTs typically focus on the analysis of defect-related CFT data, which encode genuinely new physical information introduced by the defect and characterize the defect universality class!

2.2.6 Conformal blocks in defect CFTs: bulk and defect conformal blocks

In Sec. 2.1.5 and Sec. 2.2.4, we have seen that two distinct operator expansions exist in defect CFTs: the ordinary OPE and the bulk–defect OE. In this subsection, we show that, as a consequence, defect CFTs admit two corresponding types of conformal blocks, namely bulk conformal blocks and defect conformal blocks.²²

To define conformal blocks in defect CFTs, we consider the following two types of correlation

²²In ordinary CFTs on $\mathbb{R}^d$, conformal blocks are characterized as eigenfunctions of the quadratic Casimir operator of $\mathfrak{so}(1, d + 1)$. In the presence of a defect, an analogous structure persists: bulk conformal blocks are eigenfunctions of the $\mathfrak{so}(1, d + 1)$ Casimir, while defect conformal blocks are eigenfunctions of the $\mathfrak{so}(1, p + 1) \oplus \mathfrak{so}(q)$ Casimir. As in the defect-free case, one can in principle determine the explicit form of these conformal blocks by solving the associated Casimir differential equations [19]. In this section, we determine the conformal blocks using a more direct approach.

functions:²³

- bulk two-point functions,
- bulk–defect–defect three-point functions.

In what follows, we analyze the bulk two-point function and derive the bulk and defect conformal blocks. (We relegate the discussion on the conformal block of the bulk–defect–defect three-point function to App. ??.) As discussed in Sec. 2.2.3, the bulk two-point function in the presence of a defect takes the form

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \frac{F(\xi, \phi)}{|x_{1\perp}|^{\Delta_1} |x_{2\perp}|^{\Delta_2}} . \quad (2.132)$$

The conformal invariant cross-ratios ξ and ϕ are given in (2.118). To evaluate this correlator using operator expansions, it is important to note that there are two inequivalent choices:

1. apply the bulk–defect OE to both $\mathcal{O}_{\Delta_1}(x_1)$ and $\mathcal{O}_{\Delta_2}(x_2)$,
2. apply the bulk OPE to the pair $\mathcal{O}_{\Delta_1}(x_1)$ and $\mathcal{O}_{\Delta_2}(x_2)$.

These two channels give rise to defect conformal blocks and bulk conformal blocks, respectively.

Defect channel and defect conformal blocks. Applying the bulk–defect OE (2.130) to both operators, we obtain

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \frac{1}{|x_{1\perp}|^{\Delta_1} |x_{2\perp}|^{\Delta_2}} \sum_{\widehat{\mathcal{O}}_{\Delta}} \frac{b(\mathcal{O}_1, \widehat{\mathcal{O}}^s) b(\mathcal{O}_2, \widehat{\mathcal{O}}^s)}{c(\widehat{\mathcal{O}}^s, \widehat{\mathcal{O}}^s)} \hat{f}_{\widehat{\Delta}, 0, s}(\xi, \phi) , \quad (2.133)$$

where $\hat{f}_{\widehat{\Delta}, 0, s}(\xi, \phi)$ is called the defect conformal block, defined by

$$\hat{f}_{\widehat{\Delta}, 0, s}(\xi, \phi) \equiv |x_{1\perp}|^{\Delta_1} |x_{2\perp}|^{\Delta_2} \mathcal{B} \left(|x_{1\perp}|, \widehat{\square}_1 \right)^{(i_1 i_2 \dots i_s)} \mathcal{B} \left(|x_{2\perp}|, \widehat{\square}_2 \right)_{i_1 i_2 \dots i_s} \frac{1}{|\hat{x}_{12}|^{2\widehat{\Delta}}} . \quad (2.134)$$

²³This choice is motivated by the method of images [21], according to which these correlators are kinematically equivalent to four-point functions in an ordinary CFT.

This expression can be evaluated explicitly by substituting (2.131):

$$\begin{aligned}
\hat{f}_{\hat{\Delta},0,s}(\xi, \phi) &= |x_{1\perp}|^{\hat{\Delta}} |x_{2\perp}|^{\hat{\Delta}} \sum_{n,m=0}^{\infty} \frac{(-4)^{-n} (-4)^{-m}}{n! m! (\hat{\Delta} - \frac{p-2}{2})_n (\hat{\Delta} - \frac{p-2}{2})_m} \\
&\quad \times \frac{x_{1\perp}^{(i_1)} \cdots x_{1\perp}^{(i_s)} x_{2\perp i_1} \cdots x_{2\perp i_s}}{|x_{1\perp}|^s |x_{2\perp}|^s} |x_{1\perp}|^{2n} |x_{2\perp}|^{2m} \hat{\square}_1^n \hat{\square}_2^m \frac{1}{|\hat{x}_{12}|^{2\hat{\Delta}}} \\
&= |x_{1\perp}|^{\hat{\Delta}} |x_{2\perp}|^{\hat{\Delta}} \frac{x_{1\perp}^{(i_1)} \cdots x_{1\perp}^{(i_s)} x_{2\perp i_1} \cdots x_{2\perp i_s}}{|x_{1\perp}|^s |x_{2\perp}|^s} \\
&\quad \times \sum_{n,m=0}^{\infty} \frac{(\hat{\Delta})_m (\hat{\Delta} + m)_n \left(\hat{\Delta} - \frac{p-2}{2} + m \right)_n}{n! m! (\hat{\Delta} - \frac{p-2}{2})_n} \frac{(-|x_{1\perp}|^2)^n (-|x_{2\perp}|^2)^m}{|\hat{x}_{12}|^{2\hat{\Delta}+2m+2n}} \\
&= |x_{1\perp}|^{\hat{\Delta}} |x_{2\perp}|^{\hat{\Delta}} \frac{x_{1\perp}^{(i_1)} \cdots x_{1\perp}^{(i_s)} x_{2\perp i_1} \cdots x_{2\perp i_s}}{|x_{1\perp}|^s |x_{2\perp}|^s} \\
&\quad \times \sum_{n,m=0}^{\infty} \frac{(\hat{\Delta})_{m+n} \left(\hat{\Delta} - \frac{p-2}{2} \right)_{m+n}}{n! m! (\hat{\Delta} - \frac{p-2}{2})_n (\hat{\Delta} - \frac{p-2}{2})_m} \frac{(-|x_{1\perp}|^2)^n (-|x_{2\perp}|^2)^m}{|\hat{x}_{12}|^{2\hat{\Delta}+2m+2n}}.
\end{aligned} \tag{2.135}$$

In the second line, we used the identity (2.126), while in the final line we employed the following relation for the Pochhammer symbol:

$$(u + m)_n = \frac{(u)_{m+n}}{(u)_m}. \tag{2.136}$$

Next, in order to evaluate the double sum in (2.135), we apply the following formula [22]:

$$\sum_{n,m=0}^{\infty} \frac{1}{n! m!} \frac{(\lambda)_{m+n} (\kappa)_{m+n}}{(\kappa)_n (\kappa)_m} \frac{A^m B^n}{z^{2\lambda+2m+2n}} = \frac{1}{C^\lambda} \sum_{m=0}^{\infty} \frac{(\lambda)_{2m}}{m! (\kappa)_m} \left(\frac{AB}{C^2} \right)^m, \quad z^2 \equiv A + B + C. \tag{2.137}$$

In the present case, we note that

$$\frac{AB}{C^2} = \frac{1}{\chi^2}, \quad \chi \equiv \xi - 2 \cos \phi. \tag{2.138}$$

Using this relation, the defect conformal block can be written as

$$\begin{aligned}
\hat{f}_{\hat{\Delta},0,s}(\chi, \phi) &= \chi^{-\hat{\Delta}} \frac{x_{1\perp}^{(i_1)} \cdots x_{1\perp}^{(i_s)} x_{2\perp i_1} \cdots x_{2\perp i_s}}{|x_{1\perp}|^s |x_{2\perp}|^s} \sum_{m=0}^{\infty} \frac{(\hat{\Delta})_{2m}}{m! (\hat{\Delta} - \frac{p-2}{2})_m} \left(\frac{1}{\chi^2} \right)^m \\
&= \chi^{-\hat{\Delta}} \frac{x_{1\perp}^{(i_1)} \cdots x_{1\perp}^{(i_s)} x_{2\perp i_1} \cdots x_{2\perp i_s}}{|x_{1\perp}|^s |x_{2\perp}|^s} {}_2F_1 \left(\frac{\hat{\Delta}}{2}, \frac{\hat{\Delta}+1}{2}, \hat{\Delta} - \frac{p-2}{2}; \frac{4}{\chi^2} \right).
\end{aligned} \tag{2.139}$$

Finally, when the transverse spin s is integer, using the Gegenbauer polynomial $C_s^{(\frac{q}{2}-1)}(\cos \phi)$, we note that

$$\frac{x_{1\perp}^{(i_1)} \cdots x_{1\perp}^{(i_s)} x_{2\perp i_1} \cdots x_{2\perp i_s}}{|x_{1\perp}|^s |x_{2\perp}|^s} = \frac{\Gamma(\frac{q}{2}-1) \Gamma(s+1)}{2^s \Gamma(\frac{q}{2}+s-1)} C_s^{(\frac{q}{2}-1)}(\cos \phi), \quad q \equiv d - p. \tag{2.140}$$

With this identity, the defect conformal block is finally given by the closed form [19],

$$\begin{aligned} \hat{f}_{\hat{\Delta},0,s}(\chi, \phi) &= a_{s,q} C_s^{\left(\frac{q}{2}-1\right)} (\cos \phi) {}_2F_1 \left(\frac{\hat{\Delta}}{2}, \frac{\hat{\Delta}+1}{2}, \hat{\Delta} - \frac{p-2}{2}; \frac{4}{\chi^2} \right), \\ a_{s,q} &\equiv \frac{\Gamma\left(\frac{q}{2}-1\right) \Gamma(s+1)}{2^s \Gamma\left(\frac{q}{2}+s-1\right)}. \end{aligned} \quad (2.141)$$

Bulk channel and bulk conformal blocks. Let us now turn to the second channel, in which we apply the bulk OPE to $\mathcal{O}_{\Delta_1}(x_1)$ and $\mathcal{O}_{\Delta_2}(x_2)$. In this case, the two-point function can be written as

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \frac{1}{|x_{1\perp}|^{\Delta_1} |x_{2\perp}|^{\Delta_2}} \xi^{-\frac{\Delta_1+\Delta_2}{2}} \sum_{\mathcal{O}_{\Delta,J}} \frac{c(\mathcal{O}_{\Delta_1}, \mathcal{O}_{\Delta_2}, \mathcal{O}_{\Delta,J})}{c(\mathcal{O}_{\Delta,J}, \mathcal{O}_{\Delta,J})} a_{\mathcal{O}} f_{\Delta,J}(\xi, \phi). \quad (2.142)$$

where $a_{\mathcal{O}}$ is the coefficient of the bulk one-point function of $\mathcal{O}_{\Delta,\ell}^{\mu_1 \dots \mu_\ell}$ and $f_{\Delta,\ell}(\xi, \phi)$ is referred to as the bulk conformal block and defined by

$$\xi^{-\frac{\Delta_1+\Delta_2}{2}} f_{\Delta,\ell}(\xi, \phi) \equiv |x_{1\perp}|^{\Delta_1} |x_{2\perp}|^{\Delta_2} \mathcal{C}_{12\mathcal{O}}(x_1 - x_2, \partial_2)_{\mu_1 \dots \mu_J} \frac{x_{2\perp}^{(\mu_1} \dots x_{2\perp}^{\mu_s)}}{|x_{2\perp}|^{\Delta+s}}. \quad (2.143)$$

In the above expression, $\mathcal{C}_{12\mathcal{O}}$ is given by (2.98). Unfortunately, in contrast to defect conformal blocks, bulk conformal blocks in defect CFTs do not admit a simple closed-form expression in general. This can already be seen in the scalar channel $\ell = 0$, where the block takes the form of an infinite series involving integrals and hypergeometric functions and much involved. Nevertheless, closed expressions are available in special kinematic limits $\phi = 0$. In particular, for $q = 1$ defects such as boundaries or interfaces, one automatically has $\phi = 0$, and the bulk conformal block for scalar channel (i.e. $\ell = 0$) simplifies to

$$f_{\Delta,\ell=0}(\xi, \phi = 0) = \xi^{\Delta/2} {}_2F_1 \left(a_+, a_-, \Delta + 1 - \frac{d}{2}; -\frac{\xi}{4} \right), \quad (2.144)$$

where $a_{\pm}$ are defined in (2.98). This result is in agreement with earlier results [17, 23].

Crossing relation and defect bootstrap Since the bulk two-point function must be independent of the choice of operator expansion, the defect-channel and bulk-channel decompositions must agree. This leads to the crossing equation

$$\sum_{\hat{\mathcal{O}}_{\Delta}^s} b_{\mathcal{O}_1 \hat{\mathcal{O}}^s} b_{\mathcal{O}_2 \hat{\mathcal{O}}^s} c(\hat{\mathcal{O}}^s, \hat{\mathcal{O}}^s) \hat{f}_{\hat{\Delta},0,s}(\xi, \phi) = \xi^{-\frac{\Delta_1+\Delta_2}{2}} \sum_{\mathcal{O}_{\Delta,\ell}} c_{12\mathcal{O}} a_{\mathcal{O}} f_{\Delta,\ell}(\xi, \phi). \quad (2.145)$$

This equation constitutes the bootstrap equation of defect CFTs, imposing nontrivial constraints on the defect CFT data. There is much literature focusing on analyzing this crossing relation to extract the spectrum and OPE coefficients of defect local operators. A diagrammatic representation of the crossing equation is shown in Fig. 2.4.

In particular, for boundary CFTs, it is known that truncations of the bootstrap equation can be used to analytically determine the scaling dimensions of boundary operators [23]. More generally, defect bootstrap equations provide a powerful framework for probing the dynamics of conformal defects.

$$\sum_{\hat{\Delta}, s} \begin{array}{c} \mathcal{O}_{\Delta_1} \quad \mathcal{O}_{\Delta_2} \\ \bullet \quad \bullet \\ | \quad | \\ \hline \hat{\mathcal{O}}_{\hat{\Delta}, s} \end{array} = \sum_{\Delta, \ell} \begin{array}{c} \mathcal{O}_{\Delta_1} \quad \mathcal{O}_{\Delta_2} \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \\ | \\ \hline \mathcal{O}_{\Delta, \ell} \end{array}$$

Figure 2.4: Diagrammatic representation of the crossing equation. The left diagram represents the defect channel, while the right diagram represents the bulk channel.

Chapter 3

Line Defect in $O(N)$ Wilson-Fisher CFT

This chapter mainly follows [24].

3.1 Overview

In the previous chapter, we presented the review on the general aspects of DCFT. In this chapter, we aim to extract the DCFT data, in particular scaling dimensions of defect local operators, by utilizing the systematics explained there. To be more concrete, we deal with the $O(N)$ vector model with a line defect in $d = (4 - \epsilon)$ dimensions [25, 26].²⁴

$$I = \int d^d x \left[\frac{1}{2(d-2)\Omega_{d-1}} |\partial\Phi_1|^2 + \frac{\lambda\mu^\epsilon}{4!} |\Phi_1|^4 \right] - h\mu^{\frac{\epsilon}{2}} \int d\hat{x}^1 \Phi_1^1, \quad (3.1)$$

where Φ_1^α ($\alpha = 1, \dots, N$) is an $O(N)$ vector field and $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a $(d-1)$ -sphere. The line defect extends parallel to the first axis $\hat{x}^1$ and breaks the $O(N)$ symmetry down to $O(N-1)$. Hence, we will make a distinction between the first $O(N)$ index $\alpha = 1$ and the rest $\alpha = \hat{\alpha} = 2, \dots, N$. The coordinates of $\mathbb{R}^d$ are also decomposed into the parallel and transverse directions to the defect:

$$x^\mu = (\hat{x}^1, x_\perp^i), \quad i = 2, \dots, d. \quad (3.2)$$

This model shows critical behaviors when we tune the bulk coupling λ and defect coupling h to the values:

$$\lambda_* = \frac{3}{\pi^2(N+8)}\epsilon + O(\epsilon^2), \quad h_*^2 = \frac{N+8}{4\pi^2} + O(\epsilon). \quad (3.3)$$

We refer to the fixed point with $(\lambda, h) = (\lambda_*, h_*)$ as the Wilson-Fisher fixed point. At criticality, the symmetry group of this model enhances to

$$SL(2, \mathbb{R}) \times SO(d-1) \times O(N-1), \quad (3.4)$$

²⁴We normalize the kinematic part of the action so that the two-point function of Φ_1^α is unit-normalized when $h = 0$. This normalization is different from the one used in [25, 26].

where $\text{SL}(2, \mathbb{R})$ is the conformal symmetry parallel to the line defect and $\text{SO}(d-1)$ is the rotational symmetry around the defect. Defect local operators are classified according to the symmetry group (3.4) and the anomalous dimensions of several defect operators have been calculated by [26] in perturbation theory.

A textbook approach to critical phenomena is based on renormalization group flow with diagrammatic calculations. On the other hand, the recent developments of conformal bootstrap have shown that conformal symmetry is strong enough to constrain the dynamics of the theory at fixed points [27, 28]. The bootstrap approach has applications to the $\text{O}(N)$ vector model in the presence of a boundary [23, 29–32] or defects [33–35]. More recently, Rychkov and Tan developed an axiomatic method to examine critical behaviors [36] with the aim to bridge between diagrammatic and bootstrap approaches. By postulating a set of axioms, they succeeded in reproducing the leading anomalous dimensions of local operators in the $(4-\epsilon)$ -dimensional $\text{O}(N)$ model at the Wilson-Fisher fixed point without resorting to perturbation theory. Their axiomatic method has been extended and applied to various models [30, 37–46].

In this chapter, we leverage the Rychkov-Tan method to investigate the $\text{O}(N)$ vector model with a line defect (3.1) at criticality. We assume the bulk coupling is tuned to the value λ_* at the Wilson-Fisher fixed point and the whole system is at a conformal fixed point with the defect coupling h undetermined. One of our main results is to show that the critical defect coupling is uniquely fixed by defect conformal symmetry to the value h_* at the Wilson-Fisher fixed point (3.3). We note that the axioms we employ incorporate the equation of motion for bulk operators but do not introduce any direct relation between defect operators. Nevertheless, by combining the axioms with defect operator expansions (DOEs) we obtain nontrivial identities for defect operators, one of which turns out to yield the critical value h_* . We believe this machinery is not limited to the $\text{O}(N)$ vector model but is applicable to more general classes of DCFTs.

We also derive the leading anomalous dimensions of various defect local operators including composite ones by the axiomatic approach. While it is straightforward to derive the dimensions of the lowest-lying operators, the same strategy does not apply to higher-order (composite) operators unless considering the analyticity of correlation functions. Reassuringly, the resulting dimensions precisely agree with those obtained by diagrammatic calculations in [26]. The properties of the defect local operators considered in this chapter are summarized in table 3.1.

This chapter is organized as follows. In Sec. 3.2, we review the axiomatic approach to critical phenomena by Rychkov and Tan, followed by the extension to the case with defects. In Sec. 3.3, we proceed to examine the correlation functions and the structures of DOE in the free theory in four dimensions. In Sec. 3.4.1, we reproduce the critical defect coupling and the leading anomalous dimensions of the lowest-lying defect local operator: $\widehat{W}_1^\alpha$. In Sec. 3.4.2, we calculate the leading anomalous dimensions of the defect local operators with transverse spin associated with the rotation group $\text{SO}(d-1)$ around the line defect: $\widehat{U}_s^\alpha$. In Sec. 3.4.3, we compute the leading anomalous dimensions of defect composite operators: $\widehat{S}_\pm$, $\widehat{V}^{\hat{\alpha}}$, $\widehat{T}^{\hat{\alpha}\hat{\beta}}$, which are in scalar, vector and tensor representations under the global symmetry $\text{O}(N-1)$, respectively. Appendix ?? is devoted to the review of the conformal block expansion of the bulk-defect-defect three-point functions in DCFT.

Operators	Dimension	SO($d - 1$) rep.	O($N - 1$) rep.	Free limit
$\widehat{W}_1^1$	(3.66)	scalar	singlet	$\widehat{\Phi}_1^1$
$\widehat{W}_1^{\hat{\alpha}}$	(3.66)	scalar	vector	$\widehat{\Phi}_1^{\hat{\alpha}}$
$\widehat{U}_{i_1 \dots i_s}^1$	(3.71)	tensor	singlet	$\widehat{\Phi}_{s+1, i_1 \dots i_s}^1$
$\widehat{U}_{i_1 \dots i_s}^{\hat{\alpha}}$	(3.71)	tensor	vector	$\widehat{\Phi}_{s+1, i_1 \dots i_s}^{\hat{\alpha}}$
$\widehat{S}_{\pm}$	(3.88)	scalar	singlet	$\left\{ \widehat{\Phi}_1^1 ^2, \widehat{\Phi}_1^{\hat{\alpha}} ^2 \right\}$
$\widehat{V}^{\hat{\alpha}}$	(3.86)	scalar	vector	$\widehat{\Phi}_1^1 \widehat{\Phi}_1^{\hat{\alpha}}$
$\widehat{T}^{\hat{\alpha}\hat{\beta}}$	(3.87)	scalar	tensor	$\widehat{\Phi}_1^{(\hat{\alpha}} \widehat{\Phi}_1^{\hat{\beta})}$

Table 3.1: Summary of the conformal dimensions of defect local operators reproduced in this chapter. They are classified by the symmetry group on the line defect (3.4). The free limits of $\widehat{S}_{\pm}$ become linear combinations of $|\widehat{\Phi}_1^1|^2$ and $|\widehat{\Phi}_1^{\hat{\alpha}}|^2$.

3.2 Review of Rychkov-Tan method

This section describes Rychkov-Tan's axiomatic approach [36] to critical phenomena. After a brief review of their framework, we show that their axioms can be generalized to the case with a conformal defect of planar or spherical shape with some modifications. We note that all the statements below are valid for any dimensional conformal defect, including a boundary.

3.2.1 Axioms in CFT

The first axiom Rychkov-Tan postulates is about the conformal symmetry at the Wilson-Fisher fixed point:

Axiom I. The theory at the Wilson-Fisher fixed point has conformal symmetry.

It follows that the operator product expansions (OPEs) can be used at the Wilson-Fisher fixed point. For scalar operators, they are schematically written as

$$\mathcal{O}_{\Delta_1}(x) \times \mathcal{O}_{\Delta_2}(0) \supset \frac{c_{12k}}{|x|^{\Delta_1 + \Delta_2 - \Delta_k}} [1 + c_1 x^\mu \partial_\mu + \dots] \mathcal{O}_{\Delta_k}(0), \quad (3.5)$$

where c_1 is a coefficient fixed by conformal symmetry.

The second axiom is about the relation between operators at the free fixed point ($\epsilon = 0$) and the Wilson-Fisher fixed point ($\epsilon \neq 0$):

Axiom II. For every local operator $\mathcal{O}_{\text{free}}$ in the free theory ($\epsilon = 0$), there exists a local operator at the Wilson-Fisher fixed point ($\epsilon \neq 0$), $\mathcal{O}_{\text{WF}}$, which tends to $\mathcal{O}_{\text{free}}$ in the free limit $\epsilon \rightarrow 0$: $\lim_{\epsilon \rightarrow 0} \mathcal{O}_{\text{WF}} = \mathcal{O}_{\text{free}}$.

Axiom II implies that e.g. for free field operators Φ_1^α and $\Phi_3^\alpha \equiv \Phi_1^\alpha |\Phi_1|^2$ there exist corresponding operators W_1^α and W_3^α at the Wilson-Fisher fixed point.

Note that both the $(4 - \epsilon)$ -dimensional free theory and the theory at the Wilson-Fisher point satisfy Axioms I and II. Hence, to make a distinction between the two theories we must add the third axiom:

Axiom III. At the Wilson-Fisher fixed point, W_1^α and W_3^α are related by the following equation of motion:

$$\square_x W_1^\alpha(x) = \kappa W_3^\alpha(x), \quad (3.6)$$

where $\square_x$ is the Laplacian in $d = (4 - \epsilon)$ dimensions.

Two operators Φ_1^α and Φ_3^α are primary in the free theory and constitute conformal multiplets independently. Axiom III asserts that at the Wilson-Fisher fixed point W_3^α turns into a descendant of W_1^α and their conformal multiplets recombine. This multiplet recombination makes the interacting theory different from the free one.

With these axioms, Rychkov and Tan have determined κ in (3.6) and reproduced the leading anomalous dimensions as follows [36].

First, Axiom I allows us to consider the OPE of W_1^α :

$$W_1^\alpha(x) \times W_1^\beta(0) \supset \frac{\delta^{\alpha\beta} c_1}{|x|^{2\Delta_{W_1}}} \mathbf{1} + \dots, \quad (3.7)$$

with $\mathbf{1}$ being the identity operator. Then, Axiom II requires that the free limit of (3.7) should be

$$\Phi_1^\alpha(x) \times \Phi_1^\beta(0) \supset \frac{\delta^{\alpha\beta}}{|x|^2} \mathbf{1} + \dots. \quad (3.8)$$

We thus conclude $c_1 = 1 + O(\epsilon)$ and $\Delta_{W_1} = 1 + O(\epsilon)$.

Next, acting the Laplacian on the LHS of (3.7) and using Axiom III, we find

$$\begin{aligned} & W_3^\alpha(x) \times W_3^\beta(0) \\ & \supset \frac{4 \Delta_{W_1} (\Delta_{W_1} + 1) (2\Delta_{W_1} + 2 - d) (2\Delta_{W_1} + 4 - d)}{\kappa^2} \frac{\delta^{\alpha\beta} c_1}{|x|^{2\Delta_{W_1} + 4}} \mathbf{1} + \dots. \end{aligned} \quad (3.9)$$

On the other hand, when $\epsilon = 0$, the LHS of (3.9) should reduce to $\Phi_3^\alpha(x) \times \Phi_3^\beta(0)$ (Axiom II), which can be calculated by Wick's theorem as

$$\Phi_3^\alpha(x) \times \Phi_3^\beta(0) \supset \frac{2(N+2)\delta^{\alpha\beta}}{|x|^6} \mathbf{1} + \dots. \quad (3.10)$$

Comparing (3.9) and (3.10), we end up with

$$\Delta_{W_1} = \frac{d-2}{2} + \frac{N+2}{16} \kappa^2 + (\text{higher order terms in } \epsilon) . \quad (3.11)$$

Similar considerations for the OPEs of composite operators such as $\Phi_{2p} \times \Phi_{2p+1}^\alpha$ and $\Phi_{2p+1}^\alpha \times \Phi_{2p+2}$ with $\Phi_{2p} \equiv |\Phi_1|^{2p}$, $\Phi_{2p+1}^\alpha \equiv \Phi_1^\alpha |\Phi_1|^{2p}$ give rise to further constraints on the constants κ and Δ_{W_1} . After solving them one successfully reproduces the known diagrammatic results:

$$\kappa = \frac{2}{N+8} \epsilon + O(\epsilon^2) , \quad \Delta_{W_1} = \frac{d-2}{2} + \frac{N+2}{4(N+8)^2} \epsilon^2 + O(\epsilon^3) . \quad (3.12)$$

3.2.2 Axioms in DCFT

We move on to the modified axioms adapted to studying critical phenomena in the presence of a defect [42].

First, Axiom I is replaced by

Axiom I'. In the presence of a defect, the theory at the Wilson-Fisher fixed point has the **defect conformal symmetry**.

This axiom implies that the theory is described by DCFT, which allows two types of local operators; bulk and defect local operators. Axiom I' allows us to exploit DOE as an operator identity between bulk and defect local operators.

We postulate that, in taking $\epsilon \rightarrow 0$, the Wilson-Fisher DCFT should be reduced to the free theory with a defect. Axiom II is not modified in essence but should be restated to include bulk and defect local operators.

Axiom II'. For a bulk/defect local operator $\mathcal{O}_{\text{free}}/\widehat{\mathcal{O}}_{\text{free}}$ in the free theory with a defect, there exists a local operator $\mathcal{O}_{\text{WF}}/\widehat{\mathcal{O}}_{\text{WF}}$ at the Wilson-Fisher fixed point, which tends to $\mathcal{O}_{\text{free}}/\widehat{\mathcal{O}}_{\text{free}}$ in the limit $\epsilon \rightarrow 0$.

Axiom III holds as it stands without introducing any relation for defect operators:

Axiom III'. At the Wilson-Fisher fixed point, **two bulk operators** W_1^α and W_3^α are related by the following equation of motion:

$$\square_x W_1^\alpha(x) = \kappa W_3^\alpha(x) , \quad (3.13)$$

where $\square_x$ is the Laplacian in $d = (4 - \epsilon)$ dimensions.

Notice that we can use the OPEs for bulk operators (3.5) in DCFT, hence by repeating the same discussion as in the previous section we find

$$\kappa = \frac{2}{N+8} \epsilon + O(\epsilon^2) , \quad \Delta_{W_1} = \frac{d-2}{2} + O(\epsilon^2) . \quad (3.14)$$

3.2.3 Structures of DCFT

We list our notations and give a brief review on DCFT with a p -dimensional defect in d dimensions [19, 47].

On the d -dimensional flat spacetime $\mathbb{R}^d$, we place a p -dimensional planar defect at $x^i = 0$ with $i = p + 1, \dots, d$. We express a bulk point by x^μ and decompose its coordinate into parallel and transverse directions to the defect:

$$x^\mu = (\hat{x}^a, x_\perp^i), \quad a = 1, \dots, p, \quad i = p + 1, \dots, d, \quad (3.15)$$

while we use $\hat{y}$ to denote a point on the defect;

$$\hat{y}^\mu = (\hat{y}^a, 0). \quad (3.16)$$

We classify the bulk local operators according to the representations of the full conformal group $\text{SO}(1, d + 1)$ and, throughout this chapter, focus only on the scalar ones $\mathcal{O}_\Delta$ characterized by the conformal dimension Δ . On the other hand, the defect local operators are labeled by the defect conformal group that is a direct product of the conformal group parallel to the defect $\text{SO}(1, p + 1)$ and the rotation around the defect $\text{SO}(d - p)$. We are particularly interested in the scalar defect local operators labeled by conformal dimension $\hat{\Delta}$: $\hat{\mathcal{O}}_{\hat{\Delta}}$, and the ones carrying transverse spin s : $\hat{\mathcal{O}}_{\hat{\Delta}, i_1 \dots i_s}^s$.

The correlation functions of bulk and defect scalars are fixed as

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{y}) \rangle &= \frac{b(\mathcal{O}, \hat{\mathcal{O}})}{|x - \hat{y}|^{2\hat{\Delta}} |x_\perp|^{\Delta - \hat{\Delta}}}, \\ \langle \hat{\mathcal{O}}_{\hat{\Delta}_1}(\hat{y}_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(\hat{y}_2) \hat{\mathcal{O}}_{\hat{\Delta}_3}(\hat{y}_3) \rangle &= \frac{c(\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2, \hat{\mathcal{O}}_3)}{|\hat{y}_{12}|^{\hat{\Delta}_{12}^+ - \hat{\Delta}_3} |\hat{y}_{23}|^{\hat{\Delta}_{23}^+ - \hat{\Delta}_1} |\hat{y}_{13}|^{\hat{\Delta}_{13}^+ - \hat{\Delta}_2}}, \\ \langle \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{y}_1) \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{y}_2) \rangle &= \frac{c(\hat{\mathcal{O}}, \hat{\mathcal{O}})}{|\hat{y}_{12}|^{2\hat{\Delta}}}, \end{aligned} \quad (3.17)$$

where we used the following shorthand notations:

$$\hat{\Delta}_{ij}^\pm \equiv \hat{\Delta}_i \pm \hat{\Delta}_j, \quad \hat{y}_{12} \equiv \hat{y}_1 - \hat{y}_2. \quad (3.18)$$

The two-point functions involving defect local operators with transverse spin are

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}}^{s, i_1 \dots i_s}(\hat{y}) \rangle &= \frac{x_\perp^{(i_1} \dots x_\perp^{i_s)} b(\mathcal{O}, \hat{\mathcal{O}}^s)}{|x - \hat{y}|^{2\hat{\Delta}} |x_\perp|^{\Delta - \hat{\Delta} + s}}, \\ \langle \hat{\mathcal{O}}_{\hat{\Delta}}^{s, i_1 \dots i_s}(\hat{y}_1) \hat{\mathcal{O}}_{\hat{\Delta}, j_1 \dots j_s}^s(\hat{y}_2) \rangle &= \frac{\delta_{j_1}^{(i_1} \dots \delta_{j_s}^{i_s)} c(\hat{\mathcal{O}}^s, \hat{\mathcal{O}}^s)}{|\hat{y}_{12}|^{2\hat{\Delta}}}. \end{aligned} \quad (3.19)$$

Throughout this chapter, the parenthesis appearing in the indices stands for the symmetric and traceless structure:

$$x_\perp^{(i} x_\perp^{j)} = x_\perp^i x_\perp^j - \frac{\delta^{ij}}{d - p} |x_\perp|^2. \quad (3.20)$$

A bulk local operator can be expanded in terms of defect local ones:

$$\mathcal{O}_\Delta(x) \supset \sum_{\widehat{\mathcal{O}}^s} \frac{b(\mathcal{O}, \widehat{\mathcal{O}}^s)/c(\widehat{\mathcal{O}}^s, \widehat{\mathcal{O}}^s)}{|x_\perp|^{\Delta-\widehat{\Delta}+s}} x_\perp^{(i_1} \cdots x_\perp^{i_s)} \widehat{\mathcal{O}}_{\widehat{\Delta}, i_1 \dots i_s}^s(\hat{x}). \quad (3.21)$$

The symbol $\supset$ stands for the DOE, and we do not bother to write descendant terms. This expansion is fixed for all orders in $|x_\perp|$ to be compatible with (3.19) (see (3.119) for the full form for the scalar case).

We also give a quick review of bulk-defect-defect three-point functions $\langle \mathcal{O}_\Delta \widehat{\mathcal{O}}_{\widehat{\Delta}_1} \widehat{\mathcal{O}}_{\widehat{\Delta}_2} \rangle$ that are of great importance in Sec. 3.4.3, whose details are relegated to appendix ???. For simplicity we place $\widehat{\mathcal{O}}_{\widehat{\Delta}_1}$ and $\widehat{\mathcal{O}}_{\widehat{\Delta}_2}$ at the origin and infinity by using conformal symmetry.²⁵ The bulk-defect-defect three-point function depends on the cross-ratio $v = |x_\perp|^2/|x|^2$ and has the following conformal block expansion:

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}_1}(0) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}(\infty) \rangle &= \frac{1}{|x_\perp|^\Delta |x|^{\widehat{\Delta}_{12}^-}} \sum_{\widehat{\mathcal{O}}} \frac{b(\mathcal{O}, \widehat{\mathcal{O}}) c(\widehat{\mathcal{O}}, \widehat{\mathcal{O}}_1, \widehat{\mathcal{O}}_2)}{c(\widehat{\mathcal{O}}, \widehat{\mathcal{O}})} G_{\widehat{\Delta}}^{\widehat{\Delta}_{12}^-} \left(\frac{|x_\perp|^2}{|x|^2} \right), \\ G_{\widehat{\Delta}}^{\widehat{\Delta}_{12}^-}(v) &= v^{\widehat{\Delta}/2} {}_2F_1 \left(\frac{\widehat{\Delta} + \widehat{\Delta}_{12}^-}{2}, \frac{\widehat{\Delta} - \widehat{\Delta}_{12}^-}{2}; \widehat{\Delta} + 1 - \frac{p}{2}; v \right). \end{aligned} \quad (3.22)$$

Note that only scalar operators contribute to the conformal block expansion in (3.22) as the defect three-point functions of two scalars and one spinning operator vanish.

3.3 The free $O(N)$ model in four dimensions with a line defect

In this section, we investigate the free $O(N)$ model ($\lambda = 0$) in four dimensions with the line defect defined by (3.1). We focus on the correlation functions with the line defect and DOEs associated with the bulk operators Φ_1^α and Φ_3^α , which are necessary for the later section.

3.3.1 Correlation functions

Let $\langle \cdots \rangle_0$ be a correlation function in the free theory with no defect coupling.²⁶

$$\langle \cdots \rangle_0 \equiv \int \mathcal{D}\Phi_1(\cdots) \exp \left(-\frac{1}{8\pi^2} \int d^4x |\partial\Phi_1|^2 \right). \quad (3.23)$$

Then, we define correlation functions in the presence of the defect by [19, 20]

$$\langle \cdots \rangle \equiv \frac{\langle \cdots e^{h \int d\hat{y}^1 \Phi_1^1} \rangle_0}{\langle e^{h \int d\hat{y}^1 \Phi_1^1} \rangle_0}. \quad (3.24)$$

²⁵The defect local operator at the infinity is defined by $\widehat{\mathcal{O}}_{\widehat{\Delta}}(\infty) = \lim_{|\hat{y}| \rightarrow \infty} |\hat{y}|^{2\widehat{\Delta}} \widehat{\mathcal{O}}_{\widehat{\Delta}}(\hat{y})$.

²⁶The path integral measure is normalized such that the expectation value of the identity operator is one: $\langle \mathbf{1} \rangle_0 = 1$.

We begin by considering the two-point function without a defect, which satisfies the following differential equation:

$$\square_{x_1} \langle \Phi_1^\alpha(x_1) \Phi_1^\beta(x_2) \rangle_0 = 4\pi^2 \delta^{\alpha\beta} \delta^d(x_1 - x_2) . \quad (3.25)$$

The solution to this equation is given by

$$\langle \Phi_1^\alpha(x_1) \Phi_1^\beta(x_2) \rangle_0 = \frac{\delta^{\alpha\beta}}{|x_1 - x_2|^2} . \quad (3.26)$$

Once the defect coupling is turned on, the one-point function of Φ_1^α no longer vanishes [19, section 5.4]:

$$\langle \Phi_1^\alpha(x) \rangle = \frac{\delta^{\alpha 1} \hat{h}}{|x_\perp|} , \quad (3.27)$$

where we define $\hat{h}$ by

$$\hat{h} \equiv \pi h . \quad (3.28)$$

In what follows, we use $\hat{h}$ instead of the defect coupling h for convenience. The bulk two-point function can be calculated similarly:

$$\langle \Phi_1^\alpha(x_1) \Phi_1^\beta(x_2) \rangle = \frac{\delta^{\alpha 1} \delta^{\beta 1} \hat{h}^2}{|x_{1,\perp}| |x_{2,\perp}|} + \frac{\delta^{\alpha\beta}}{|x_1 - x_2|^2} . \quad (3.29)$$

We now define a defect local operator $\hat{\Phi}_1^\alpha$ by

$$\hat{\Phi}_1^\alpha(\hat{x}) \equiv \lim_{|x_\perp| \rightarrow 0} \Phi_1^\alpha(x) , \quad (3.30)$$

where the limit in the right hand side is defined inside the correlation functions such that $\langle \hat{\Phi}_1^\alpha(\hat{x}) \cdots \rangle = \lim_{|x_\perp| \rightarrow 0} \langle \Phi_1^\alpha(x) \cdots \rangle$ respect defect conformal invariance. For instance, applying (3.30) to the bulk two-point function (3.29), we obtain

$$\begin{aligned} \langle \Phi_1^\alpha(x_1) \hat{\Phi}_1^\beta(\hat{x}_2) \rangle &= \lim_{|x_{2,\perp}| \rightarrow 0} \left[\frac{\delta^{\alpha 1} \delta^{\beta 1} \hat{h}^2}{|x_{1,\perp}| |x_{2,\perp}|} + \frac{\delta^{\alpha\beta}}{|x_1 - x_2|^2} \right] \\ &= \frac{\delta^{\alpha\beta}}{|x_1 - \hat{x}_2|^2} , \end{aligned} \quad (3.31)$$

where we discarded the singular term which violates the defect conformal invariance. We can similarly calculate the two-point function of $\hat{\Phi}_1^\alpha$ from (3.29). Thus, we find

$$\langle \Phi_1^\alpha(x) \hat{\Phi}_1^\beta(\hat{y}) \rangle = \frac{\delta^{\alpha\beta}}{|x - \hat{y}|^2} , \quad \langle \hat{\Phi}_1^\alpha(\hat{y}_1) \hat{\Phi}_1^\beta(\hat{y}_2) \rangle = \frac{\delta^{\alpha\beta}}{|\hat{y}_{12}|^2} . \quad (3.32)$$

We are also interested in the defect local operators with transverse spin:

$$\hat{\Phi}_{s+1}^{\alpha, i_1 \cdots i_s}(\hat{x}) \equiv \lim_{|x_\perp| \rightarrow 0} \partial^{(i_1} \cdots \partial^{i_s)} \Phi_1^\alpha(x) . \quad (3.33)$$

Some of their correlation functions are given by

$$\langle \widehat{\Phi}_{s+1}^{\alpha, i_1 \dots i_s}(\hat{y}) \Phi_1^\beta(x) \rangle = 2^s s! \delta^{\alpha\beta} \frac{x_\perp^{(i_1} \dots x_\perp^{i_s)}}{|x - \hat{y}|^{2(s+1)}}, \quad (3.34)$$

$$\langle \widehat{\Phi}_{s+1}^{\alpha, i_1 \dots i_s}(\hat{y}_1) \widehat{\Phi}_{s+1}^{\beta, j_1 \dots j_s}(\hat{y}_2) \rangle = 2^s (s!)^2 \delta^{\alpha\beta} \frac{\delta_{j_1}^{(i_1} \dots \delta_{j_s}^{i_s)}}{|\hat{y}_{12}|^{2(s+1)}}. \quad (3.35)$$

Using the result obtained so far, any correlation functions can be computed via Wick's theorem. In the following, we enumerate the results relevant to this chapter.

One- and two-point functions. We define $\widehat{\Phi}_3^\alpha(\hat{x})$ by

$$\widehat{\Phi}_3^\alpha(\hat{x}) \equiv \lim_{|x_\perp| \rightarrow 0} \Phi_3^\alpha(x). \quad (3.36)$$

Then, we obtain the correlation functions below:

$$\begin{aligned} \langle \Phi_3^\alpha(x) \rangle &= \frac{\delta^{\alpha 1} \hat{h}^3}{|x_\perp|^3}, & \langle \Phi_3^\alpha(x) \widehat{\Phi}_1^\beta(\hat{y}) \rangle &= \frac{\hat{h}^2 (1 + 2 \delta^{\alpha 1}) \delta^{\alpha\beta}}{|x - \hat{y}|^2 |x_\perp|^2}, \\ \langle \Phi_3^\alpha(x) \widehat{\Phi}_3^\beta(\hat{y}) \rangle &= \frac{2(N+2) \delta^{\alpha\beta}}{|x - \hat{y}|^6}, & \langle \widehat{\Phi}_3^\alpha(\hat{y}_1) \widehat{\Phi}_3^\beta(\hat{y}_2) \rangle &= \frac{2(N+2) \delta^{\alpha\beta}}{|\hat{y}_{12}|^6}, \end{aligned} \quad (3.37)$$

$$\langle \widehat{\Phi}_{s+1}^{\alpha, i_1 \dots i_s}(\hat{y}) \Phi_3^\beta(x) \rangle = \hat{h}^2 (1 + 2 \delta^{\alpha 1}) 2^s s! \delta^{\alpha\beta} \frac{x_\perp^{(i_1} \dots x_\perp^{i_s)}}{|x - \hat{y}|^{2(s+1)}}. \quad (3.38)$$

Three-point functions. We consider the following three-point functions:

$$\langle \Phi_1^\alpha(x) \widehat{\Phi}_1^\beta(\hat{y}_1) \widehat{\Phi}_2(\hat{y}_2) \rangle, \quad \langle \Phi_3^\alpha(x) \widehat{\Phi}_1^\beta(\hat{y}_1) \widehat{\Phi}_2(\hat{y}_2) \rangle, \quad (3.39)$$

with $\widehat{\Phi}_2$ being the following composite defect local operators:

$$\widehat{\Phi}_2 \in \left\{ |\widehat{\Phi}_1^1|^2, |\widehat{\Phi}_1^{\hat{\gamma}}|^2, \widehat{\Phi}_1^1 \widehat{\Phi}_1^{\hat{\gamma}}, \widehat{\Phi}_1^{(\hat{\gamma} \widehat{\Phi}_1^{\hat{\sigma}})} \right\}. \quad (3.40)$$

The first two operators are $O(N-1)$ scalars, whereas the third one is an $O(N-1)$ vector. The last operator $\widehat{\Phi}_1^{(\hat{\gamma} \widehat{\Phi}_1^{\hat{\sigma}})}$ is a rank two $O(N-1)$ symmetric traceless tensor:

$$\begin{aligned} |\widehat{\Phi}_1^1|^2(\hat{x}) &\equiv \lim_{|x_\perp| \rightarrow 0} \Phi_1^1 \Phi_1^1(x), & |\widehat{\Phi}_1^{\hat{\gamma}}|^2(\hat{x}) &\equiv \lim_{|x_\perp| \rightarrow 0} \sum_{\hat{\gamma}=2}^N \Phi_1^{\hat{\gamma}} \Phi_1^{\hat{\gamma}}(x), \\ \widehat{\Phi}_1^1 \widehat{\Phi}_1^{\hat{\gamma}}(\hat{x}) &\equiv \lim_{|x_\perp| \rightarrow 0} \Phi_1^1 \Phi_1^{\hat{\gamma}}(x), & \widehat{\Phi}_1^{(\hat{\gamma} \widehat{\Phi}_1^{\hat{\sigma}})}(\hat{x}) &\equiv \lim_{|x_\perp| \rightarrow 0} \Phi_1^{(\hat{\gamma} \widehat{\Phi}_1^{\hat{\sigma}})}(x). \end{aligned} \quad (3.41)$$

For any operators listed in (3.40), the three-point functions take similar forms:

$$\langle \Phi_1^\alpha(x) \widehat{\Phi}_1^\beta(\hat{y}_1) \widehat{\Phi}_2(\hat{y}_2) \rangle = \frac{c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)}{|x - \hat{y}_2|^2 |\hat{y}_{12}|^2}, \quad (3.42)$$

$$\langle \Phi_3^\alpha(x) \widehat{\Phi}_1^\beta(\hat{y}_1) \widehat{\Phi}_2(\hat{y}_2) \rangle = \frac{(1 + 2 \delta^{\alpha 1}) \hat{h}^2 c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)}{|x - \hat{y}_2|^2 |\hat{y}_{12}|^2 |x_\perp|^2} + \frac{c(\Phi_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)}{|x - \hat{y}_1|^2 |x - \hat{y}_2|^4}. \quad (3.43)$$

$\widehat{\Phi}_2$	$c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)$	$c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)$
$ \widehat{\Phi}^1 ^2$	$2 \delta^{\alpha 1} \delta^{\beta 1}$	$2 \delta^{\alpha \beta} + 4 \delta^{\alpha 1} \delta^{\beta 1}$
$ \widehat{\Phi}_1^{\hat{\alpha}} ^2$	$2 \delta^{\alpha \beta} - 2 \delta^{\alpha 1} \delta^{\beta 1}$	$2 (N+1) \delta^{\alpha \beta} - 4 \delta^{\alpha 1} \delta^{\beta 1}$
$\widehat{\Phi}^1 \widehat{\Phi}^{\hat{\gamma}}$	$\delta^{\alpha 1} \delta^{\beta \hat{\gamma}} + \delta^{\beta 1} \delta^{\alpha \hat{\gamma}}$	$\delta^{\alpha 1} \delta^{\beta \hat{\gamma}} + 2 \delta^{\beta 1} \delta^{\alpha \hat{\gamma}}$
$\widehat{\Phi}^{(\hat{\gamma}} \widehat{\Phi}^{\hat{\sigma})}$	$2 \delta^{\alpha(\hat{\gamma}} \delta^{\hat{\sigma})\beta}$	$4 \delta^{\alpha(\hat{\gamma}} \delta^{\hat{\sigma})\beta}$

Table 3.2: List of defect three-point coefficients in (3.44).

Here, $c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)$ and $c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)$ are the defect three-point coefficients listed in table 3.2:

$$\begin{aligned}
\langle \widehat{\Phi}_1^\alpha(\hat{x}) \widehat{\Phi}_1^\beta(\hat{y}_1) \widehat{\Phi}_2(\hat{y}_2) \rangle &= \frac{c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)}{|\hat{x} - \hat{y}_2| |\hat{y}_{12}|^2}, \\
\langle \widehat{\Phi}_3^\alpha(\hat{x}) \widehat{\Phi}_1^\beta(\hat{y}_1) \widehat{\Phi}_2(\hat{y}_2) \rangle &= \frac{c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)}{|\hat{x} - \hat{y}_1|^2 |\hat{x} - \hat{y}_2|^4}.
\end{aligned} \tag{3.44}$$

Three-point functions for $N = 1$. When $N = 1$ (Sec. 3.4.4), the defect three-point functions can be calculated as follows:

$$\langle \widehat{\Phi}_1(\hat{x}) \widehat{\Phi}_p(\hat{y}_1) \widehat{\Phi}_{p+1}(\hat{y}_2) \rangle = \frac{c(\widehat{\Phi}_1, \widehat{\Phi}_p, \widehat{\Phi}_{p+1})}{|\hat{x} - \hat{y}_2|^2 |\hat{y}_{12}|^{2p}}, \tag{3.45}$$

$$\langle \widehat{\Phi}_3(\hat{x}) \widehat{\Phi}_p(\hat{y}_1) \widehat{\Phi}_{p+1}(\hat{y}_2) \rangle = \frac{c(\widehat{\Phi}_1, \widehat{\Phi}_p, \widehat{\Phi}_{p+1})}{|\hat{x} - \hat{y}_2|^2 |\hat{x} - \hat{y}_2|^4 |\hat{y}_{12}|^{2p-2}}, \tag{3.46}$$

where

$$\widehat{\Phi}_p(\hat{x}) \equiv \lim_{|x_\perp| \rightarrow 0} |\Phi_1|^p(x), \tag{3.47}$$

and

$$c(\widehat{\Phi}_1, \widehat{\Phi}_p, \widehat{\Phi}_{p+1}) = (p+1)!, \quad c(\widehat{\Phi}_3, \widehat{\Phi}_p, \widehat{\Phi}_{p+1}) = 3p(p+1)!. \tag{3.48}$$

The bulk-defect-defect three-point functions are

$$\langle \Phi_1(x) \widehat{\Phi}_p(\hat{y}_1) \widehat{\Phi}_{p+1}(\hat{y}_2) \rangle = \frac{c(\widehat{\Phi}_1, \widehat{\Phi}_p, \widehat{\Phi}_{p+1})}{|x - y_2|^2 |\hat{y}_{12}|^{2p}}, \tag{3.49}$$

$$\langle \Phi_3(x) \widehat{\Phi}_p(\hat{y}_1) \widehat{\Phi}_{p+1}(\hat{y}_2) \rangle = \frac{3 \hat{h} c(\widehat{\Phi}_1, \widehat{\Phi}_p, \widehat{\Phi}_{p+1})}{|x - y_2|^2 |\hat{y}_{12}|^{2p} |x_\perp|^2} + \frac{c(\widehat{\Phi}_1, \widehat{\Phi}_p, \widehat{\Phi}_{p+1})}{|x - y_2|^2 |x - y_2|^4 |\hat{y}_{12}|^{2p-2}}. \tag{3.50}$$

3.3.2 Defect operator expansions

We proceed to spell out the DOEs of two bulk local operators Φ_1^α and Φ_3^α . They are determined by comparing the general form of DOEs (3.21) with the correlation functions derived in the last subsection.

Defect operator expansion of Φ_1^α

The DOE of Φ_1^α turns out to be

$$\Phi_1^\alpha(x) = \frac{\delta^{\alpha 1} \hat{h}}{|x_\perp|} \mathbf{1} + \hat{\Phi}_1^\alpha(\hat{x}) + \sum_{s=1}^{\infty} \frac{1}{s!} x_\perp^{(i_1} \cdots x_\perp^{i_s)} \hat{\Phi}_{s+1, i_1 \dots i_s}^\alpha(\hat{x}) + (\text{descendants}) . \quad (3.51)$$

One can show that the other defect local operators are absent in this DOE [19, appendix B.1.1]. To see this, let $\hat{\mathcal{O}}_{\hat{\Delta}_s, i_1 \dots i_s}^\alpha(\hat{y})$ be a defect local operator with conformal dimension $\hat{\Delta}_s$ and s symmetric traceless indices of $\text{SO}(3)$. From the defect conformal symmetry, we have

$$\Phi_1^\alpha(x) \supset A \frac{x_\perp^{(i_1} \cdots x_\perp^{i_s)}}{|x_\perp|^{s+1-\hat{\Delta}_s}} \hat{\mathcal{O}}_{\hat{\Delta}_s, i_1 \dots i_s}^\alpha(\hat{x}) , \quad (3.52)$$

with A being some nonzero constant. When we act the Laplacian $\square_x$ on the LHS of (3.52), it should vanish due to the Klein-Gordon equation:

$$\square_x \Phi_1^\alpha(x) = 0 . \quad (3.53)$$

On the other hand, the RHS becomes

$$\square_x A \frac{x_\perp^{(i_1} \cdots x_\perp^{i_s)}}{|x_\perp|^{s+1-\hat{\Delta}_s}} \hat{\mathcal{O}}_{\hat{\Delta}_s, i_1 \dots i_s}^\alpha(\hat{x}) = A (\hat{\Delta}_s - s - 1)(\hat{\Delta}_s + s) \frac{x_\perp^{(i_1} \cdots x_\perp^{i_s)}}{|x_\perp|^{s+3-\hat{\Delta}_s}} \hat{\mathcal{O}}_{\hat{\Delta}_s, i_1 \dots i_s}^\alpha(\hat{x}) , \quad (3.54)$$

where we used the following identity for the d -dimensional Laplacian $\square_x$ in the presence of a p -dimensional planar defect:

$$\square_x \frac{x_\perp^{(i_1} \cdots x_\perp^{i_s)}}{|x_\perp|^\delta} = \delta(\delta + 2 - d + p - 2s) \frac{x_\perp^{(i_1} \cdots x_\perp^{i_s)}}{|x_\perp|^{\delta+2}} . \quad (3.55)$$

For (3.54) to be zero for a finite A , we should have $\hat{\Delta}_s = s + 1$ or $\hat{\Delta}_s = -s$. The former corresponds to $\hat{\Phi}_1^\alpha(\hat{x})$ for $s = 0$ and $\hat{\Phi}_{s+1, i_1 \dots i_s}^\alpha(\hat{x})$ for $s \geq 1$ as anticipated. On the other hand, the latter case $\hat{\Delta}_s = -s$ is below the unitarity bound of a one-dimensional CFT on the defect except for the identity operator $\mathbf{1}$ with $s = 0$.

Defect operator expansion of Φ_3^α

We find that an infinite number of operators contribute to the DOE of Φ_3^α :

$$\begin{aligned} \Phi_3^\alpha(x) \supset & \frac{\delta^{\alpha 1} \hat{h}^3}{|x_\perp|^3} \mathbf{1} + \frac{(1 + 2\delta^{\alpha 1}) \hat{h}^2}{|x_\perp|^2} \hat{\Phi}_1^\alpha(\hat{x}) \\ & + \frac{(1 + 2\delta^{\alpha 1}) \hat{h}^2}{|x_\perp|^2} \sum_{s=1}^{\infty} \frac{1}{s!} x_\perp^{(i_1} \cdots x_\perp^{i_s)} \hat{\Phi}_{s+1, i_1 \dots i_s}^\alpha(\hat{x}) \\ & + \sum_{n=0}^{\infty} \frac{b(\Phi_3^\alpha, \hat{\mathcal{O}}_{2n+3}^\alpha)}{c(\hat{\mathcal{O}}_{2n+3}^\alpha, \hat{\mathcal{O}}_{2n+3}^\alpha)} |x_\perp|^{2n} \hat{\mathcal{O}}_{2n+3}^\alpha(\hat{x}) . \end{aligned} \quad (3.56)$$

Here, $\widehat{\mathcal{O}}_3^\alpha$ can be identified with $\widehat{\Phi}_3^\alpha$ and the coefficients in the last line are subject to the relations:

$$\frac{b(\Phi_3^\alpha, \widehat{\mathcal{O}}_{2n+3}^\alpha) c(\widehat{\mathcal{O}}_{2n+3}^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)}{c(\widehat{\mathcal{O}}_{2n+3}^\alpha, \widehat{\mathcal{O}}_{2n+3}^\alpha)} = c(\Phi_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2) \frac{(-1)^n (2)_n}{(n + 5/2)_n}. \quad (3.57)$$

To see how this equality is obtained, let us perform the conformal block expansion of the bulk-defect-defect three-point function (3.43) and denote the intermediate operators with odd conformal dimensions $2n + 3$ ($n = 0, 1, \dots$) by $\widehat{\mathcal{O}}_{2n+3}^\alpha$.²⁷

$$\begin{aligned} \langle \Phi_3^\alpha(x) \widehat{\Phi}_1^\beta(0) \widehat{\Phi}_2(\infty) \rangle &= \frac{|x|}{|x_\perp|^3} \left[(1 + 2\delta^{\alpha 1}) \hat{h}^2 c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2) G_1^{-1}(v) \right. \\ &\quad \left. + c(\Phi_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2) \sum_{n=0}^{\infty} \frac{(-1)^n (2)_n}{(n + 5/2)_n} G_{2n+3}^{-1}(v) \right]. \end{aligned} \quad (3.58)$$

Comparing (3.58) with (3.22), we find that the coefficients associated with $\widehat{\mathcal{O}}_{2n+3}^\alpha$ must satisfy the equalities (3.57).

3.4 DCFT data on the line defect

We are now in a position to apply to the critical line defect the axiomatic method described in Sec. 3.2.2 and read off the conformal dimensions of several defect composite operators as the DCFT data.

3.4.1 Lowest-lying defect local operator and critical defect coupling

We consider the lowest-lying defect local operator $\widehat{W}_1^\alpha$, which reduces to $\widehat{\Phi}_1^\alpha$ in the $\epsilon \rightarrow 0$ limit. We assume that the critical defect coupling $\hat{h}$ is an $O(\epsilon^0)$ parameter to be fixed by defect conformal symmetry. At the zero-th order in ϵ , the conformal dimension of $\widehat{W}_1^\alpha$ should be

$$\widehat{\Delta}_{\widehat{W}_1^\alpha} = 1 + O(\epsilon). \quad (3.59)$$

We expect that the symmetry breaking $O(N) \rightarrow O(N - 1)$ on the defect makes $\widehat{\Delta}_{\widehat{W}_1}$ and $\widehat{\Delta}_{\widehat{W}^\alpha}$ different at the first order in ϵ .

We first apply Axiom I' to fix the DOE of W_1^α as

$$W_1^\alpha(x) \supset C_0^\alpha \frac{1}{|x_\perp|^{\Delta_{W_1}}} \mathbf{1} + C_1^\alpha \frac{1}{|x_\perp|^{\Delta_{W_1} - \widehat{\Delta}_{\widehat{W}_1^\alpha}}} \widehat{W}_1^\alpha(\hat{x}). \quad (3.60)$$

For (3.60) to be identified with (3.51) in the $\epsilon \rightarrow 0$ limit (Axiom II'), we have

$$C_1^\alpha = 1 + O(\epsilon), \quad C_0^\alpha = \delta^{\alpha 1} \hat{h} + O(\epsilon). \quad (3.61)$$

²⁷We have used $G_1^{-1}(v) = v^{1/2}$, $G_{2n+3}^{-1}(v) = v^{n+3/2} {}_2F_1(1+n, 2+n; 7/2+2n; v)$ and the following hypergeometric identity [48, equation (9.1.32)]; $1 = \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n (\beta)_n}{(n+\lambda)_n n!} z^n {}_2F_1(\alpha+n, \beta+n; \lambda+1+2n; z)$.

Combining all these, we now derive the leading anomalous dimension of $\widehat{W}_1^\alpha$. The equation of motion (3.13) with the DOE (3.60) (and the identity (3.55)) yields

$$W_3^\alpha(x) \supset \frac{C_0^\alpha}{\kappa} \frac{\Delta_{W_1} (\Delta_{W_1} - 1 + \epsilon)}{|x_\perp|^{\Delta_{W_1}+2}} \mathbf{1} + \frac{C_1^\alpha}{\kappa} \frac{(\Delta_{W_1} - \widehat{\Delta}_{\widehat{W}_1^\alpha})(\Delta_{W_1} - \widehat{\Delta}_{\widehat{W}_1^\alpha} - 1 + \epsilon)}{|x_\perp|^{\Delta_{W_1} - \widehat{\Delta}_{\widehat{W}_1^\alpha} + 2}} \widehat{W}_1^\alpha(\hat{x}) . \quad (3.62)$$

Axiom **II'** requires that this DOE should match (3.56) in the $\epsilon \rightarrow 0$ limit:

$$\hat{h}^3 \kappa = \hat{h} \Delta_{W_1} (\Delta_{W_1} - 1 + \epsilon) + O(\epsilon^2) , \quad (3.63)$$

$$(1 + 2\delta^{\alpha 1}) \hat{h}^2 \kappa = (\Delta_{W_1} - \widehat{\Delta}_{\widehat{W}_1^\alpha})(\Delta_{W_1} - \widehat{\Delta}_{\widehat{W}_1^\alpha} - 1 + \epsilon) + O(\epsilon^2) . \quad (3.64)$$

With the bulk parameters (3.14) substituted, the first equation (3.63) gives the critical value of the defect coupling:

$$\hat{h}^2 = \frac{N+8}{4} + O(\epsilon) . \quad (3.65)$$

Plugging this result into the second equation (3.64), we obtain the conformal dimension:

$$\begin{aligned} \widehat{\Delta}_{\widehat{W}_1^\alpha} &= \Delta_{W_1} + \frac{1+2\delta^{\alpha 1}}{2} \epsilon + O(\epsilon^2) \\ &= 1 + \epsilon \delta^{\alpha 1} + O(\epsilon^2) , \end{aligned} \quad (3.66)$$

which agrees with the diagrammatic results at order $O(\epsilon)$ [26, equation (3.19) and (3.21)].

3.4.2 Defect local operators with transverse spin

Next, we consider defect local operators with transverse spin, $\widehat{U}_{i_1 \dots i_s}^1$ and $\widehat{U}_{i_1 \dots i_s}^{\hat{\alpha}}$, of conformal dimension $\widehat{\Delta}_{\widehat{U}_s^1}$ and $\widehat{\Delta}_{\widehat{U}_s^{\hat{\alpha}}}$, respectively. In the free theory limit, they reduce to the free theory operators as follows:

$$\lim_{\epsilon \rightarrow 0} \widehat{U}_{i_1 \dots i_s}^1 = \widehat{\Phi}_{s+1, i_1 \dots i_s}^1 , \quad \lim_{\epsilon \rightarrow 0} \widehat{U}_{i_1 \dots i_s}^{\hat{\alpha}} = \widehat{\Phi}_{s+1, i_1 \dots i_s}^{\hat{\alpha}} . \quad (3.67)$$

The defect conformal symmetry restricts the form of the DOE of W_1^α to the defect operators with spin as

$$W_1^\alpha(x) \supset C_s^\alpha \frac{x_\perp^{(i_1} \dots x_\perp^{i_s)}}{|x_\perp|^{\Delta_{W_1} - \widehat{\Delta}_{\widehat{U}_s^\alpha} + s}} \widehat{U}_{i_1 \dots i_s}^\alpha(\hat{x}) . \quad (3.68)$$

Compared with the DOE (3.51) in the free limit, the coefficient C_s^α must be

$$C_s^\alpha = \frac{1}{s!} + O(\epsilon) . \quad (3.69)$$

Using the equation of motion (3.13) together with (3.55), one has

$$W_3^\alpha(x) \supset \frac{C_s^\alpha}{\kappa} (\Delta_{W_1} - \widehat{\Delta}_{\widehat{U}^\alpha} + s)(\Delta_{W_1} - \widehat{\Delta}_{\widehat{U}^\alpha} - s - 1 + \epsilon) \frac{x_\perp^{(i_1 \dots i_s)}}{|x_\perp|^{\Delta_{W_1} - \widehat{\Delta}_{\widehat{U}^\alpha} + s + 2}} \widehat{U}_{i_1 \dots i_s}^\alpha(\hat{x}) . \quad (3.70)$$

This DOE should reduce to (3.56) in the free limit. Repeating a similar analysis to the last section, we arrive at

$$\begin{aligned} \widehat{\Delta}_{\widehat{U}^\alpha} &= \Delta_{W_1} + s + \frac{1 + 2\delta^{\alpha 1}}{2(s+2)} \epsilon + O(\epsilon^2) \\ &= s + 1 + \frac{2\delta^{\alpha 1} - s - 1}{2(s+2)} \epsilon + O(\epsilon^2) . \end{aligned} \quad (3.71)$$

Reassuringly, this is consistent with the results obtained by other methods in [33, 35]. Notice that (3.71) is in agreement with the universal behavior of defect local operators with large transverse spins [49]:

$$\widehat{\Delta}_{\widehat{U}^\alpha} \simeq \Delta_{W_1} + s , \quad s \rightarrow \infty . \quad (3.72)$$

3.4.3 Defect composite operators

Let us move on to the composite operators which tend to $\widehat{\Phi}_2$ listed in (3.40) in the free limit:

$$\widehat{W}_2 \in \left\{ \widehat{V}^{\hat{\alpha}} , \widehat{T}^{\hat{\alpha}\hat{\beta}} , \widehat{S}_+ , \widehat{S}_- \right\} . \quad (3.73)$$

We denote the conformal dimension of $\widehat{W}_2$ by $\widehat{\Delta}_{\widehat{W}_2}$ and focus on its leading correction $\Gamma_{\widehat{W}_2}$:

$$\widehat{\Delta}_{\widehat{W}_2} = 2 + \Gamma_{\widehat{W}_2} \epsilon + O(\epsilon^2) . \quad (3.74)$$

The first two operators $\widehat{V}^{\hat{\alpha}}$ and $\widehat{T}^{\hat{\alpha}\hat{\beta}}$ are an $O(N-1)$ vector and a symmetric traceless tensor, respectively. In the free theory limit,

$$\lim_{\epsilon \rightarrow 0} \widehat{V}^{\hat{\alpha}} = \widehat{\Phi}_1^1 \widehat{\Phi}_1^{\hat{\alpha}} , \quad \lim_{\epsilon \rightarrow 0} \widehat{T}^{\hat{\alpha}\hat{\beta}} = \widehat{\Phi}_1^{(\hat{\alpha}} \widehat{\Phi}_1^{\hat{\beta})} . \quad (3.75)$$

The last two operators $\widehat{S}_+$ and $\widehat{S}_-$ are $O(N-1)$ scalars and can be identified as linear combinations of $|\widehat{\Phi}_1^1|^2$ and $|\widehat{\Phi}_1^{\hat{\gamma}}|^2$ in the free limit.²⁸ It is convenient to use the following parametrization:

$$\lim_{\epsilon \rightarrow 0} \begin{pmatrix} \widehat{S}_+ \\ \widehat{S}_- \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} |\widehat{\Phi}_1^1|^2 \\ \frac{1}{\sqrt{2(N-1)}} |\widehat{\Phi}_1^{\hat{\gamma}}|^2 \end{pmatrix} , \quad (3.76)$$

²⁸There is no operator mixing between $\widehat{S}_\pm$ and the other operators ($\widehat{V}^{\hat{\alpha}}$ and $\widehat{T}^{\hat{\alpha}\hat{\beta}}$) as they are in different irreducible representations of the symmetry group (3.4).

so that two-point functions of $\widehat{S}_\pm$ are unit-normalized and are orthogonal to each other. We take the same strategy as [46, section 5.2] to calculate the conformal dimensions of these composite operators.

Firstly, we employ the equation of motion (3.13) to determine the DOE of W_1^α at the first order in ϵ . It turns out that a series of operators $\widehat{O}'_{2n+3}{}^\alpha$ with conformal $\widehat{\Delta}'_{2n+3} = 2n+3 + O(\epsilon)$ that reduce to $\widehat{O}_{2n+3}$ in the free limit appear in the DOE of W_1^α .²⁹

$$W_1^\alpha(x) \supset \frac{C_1^\alpha}{|x_\perp|^{\Delta_1 - \widehat{\Delta}_{\widehat{W}_1^\alpha}}} \widehat{W}_1^\alpha(\hat{x}) + \sum_{n=0}^{\infty} \frac{b(W_1^\alpha, \widehat{O}'_{2n+3}{}^\alpha) / c(\widehat{O}'_{2n+3}{}^\alpha, \widehat{O}'_{2n+3}{}^\alpha)}{|x_\perp|^{\Delta_1 - \widehat{\Delta}'_{2n+3}}} \widehat{O}'_{2n+3}{}^\alpha(\hat{x}). \quad (3.77)$$

$C_1^\alpha = 1 + O(\epsilon)$ is the same constant as in Sec. 3.4.1. By applying the equation of motion (3.13), we have

$$W_3^\alpha(x) \supset \sum_{n=0}^{\infty} \frac{b(W_1^\alpha, \widehat{O}'_{2n+3}{}^\alpha)}{\kappa c(\widehat{O}'_{2n+3}{}^\alpha, \widehat{O}'_{2n+3}{}^\alpha)} \frac{(\Delta_1 - \widehat{\Delta}'_{2n+3})(\Delta_1 - \widehat{\Delta}'_{2n+3} - 1 + \epsilon)}{|x_\perp|^{\Delta_1 - \widehat{\Delta}'_{2n+3} + 2}} \widehat{O}'_{2n+3}{}^\alpha(\hat{x}). \quad (3.78)$$

Taking $\epsilon \rightarrow 0$ limit and comparing with (3.56), we find

$$\frac{b(W_1^\alpha, \widehat{O}'_{2n+3}{}^\alpha)}{c(\widehat{O}'_{2n+3}{}^\alpha, \widehat{O}'_{2n+3}{}^\alpha)} = \frac{\kappa}{2(n+1)(2n+3)} \frac{b(\Phi_3^\alpha, \widehat{O}_{2n+3}^\alpha)}{c(\widehat{O}_{2n+3}^\alpha, \widehat{O}_{2n+3}^\alpha)} + O(\epsilon^2). \quad (3.79)$$

With the DOE (3.77) of W_1^α , the three-point function $\langle W_1^\alpha \widehat{W}_1^\beta \widehat{W}_2 \rangle$ can be calculated as

$$\begin{aligned} \langle W_1^\alpha(x) \widehat{W}_1^\beta(0) \widehat{W}_2(\infty) \rangle &= \frac{1}{|x_\perp|^{\Delta_1} |x|^{|\widehat{\Delta}_{\widehat{W}_1^\beta} - \widehat{\Delta}_{\widehat{W}_2}|}} \\ &\cdot \left[C_1^\alpha \cdot c(\widehat{W}_1^\alpha, \widehat{W}_1^\beta, \widehat{W}_2) G_{\widehat{\Delta}_{\widehat{W}_1^\alpha}}^{\widehat{\Delta}_{\widehat{W}_1^\beta} - \widehat{\Delta}_{\widehat{W}_2}} \left(\frac{|x_\perp|^2}{|x|^2} \right) \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{b(W_1^\alpha, \widehat{O}'_{2n+3}{}^\alpha) c(\widehat{O}'_{2n+3}{}^\alpha, \widehat{W}_1^\beta, \widehat{W}_2)}{c(\widehat{O}'_{2n+3}{}^\alpha, \widehat{O}'_{2n+3}{}^\alpha)} G_{\widehat{\Delta}'_{2n+3}}^{\widehat{\Delta}_{\widehat{W}_1^\beta} - \widehat{\Delta}_{\widehat{W}_2}} \left(\frac{|x_\perp|^2}{|x|^2} \right) \right]. \end{aligned} \quad (3.80)$$

With (3.66), we expand the first term in the parenthesis in powers of ϵ as:

$$\begin{aligned} G_{\widehat{\Delta}_{\widehat{W}_1^\alpha}}^{\widehat{\Delta}_{\widehat{W}_1^\beta} - \widehat{\Delta}_{\widehat{W}_2}}(v) &= v^{\widehat{\Delta}_{\widehat{W}_1^\alpha}/2} {}_2F_1 \left(\frac{\delta^{\alpha 1} + \delta^{\beta 1} - \Gamma_{\widehat{W}_2}}{2} \epsilon, 1; 3/2; v \right) + O(\epsilon^2) \\ &= v^{\widehat{\Delta}_{\widehat{W}_1^\alpha}} + \frac{\delta^{\alpha 1} + \delta^{\beta 1} - \Gamma_{\widehat{W}_2}}{3} \epsilon v^{3/2} {}_2F_1(1, 1; 5/2; v) + O(\epsilon^2). \end{aligned} \quad (3.81)$$

²⁹Due to the symmetry breaking on the defect $O(N) \rightarrow O(N-1)$, the conformal multiplet of $\widehat{O}_{2n+3}^\alpha$ in general splits into two parts $\widehat{O}'_{2n+3}{}^1$ and $\widehat{O}'_{2n+3}{}^{\hat{\alpha}}$, and their conformal dimensions are different at order $O(\epsilon)$. Nevertheless, such differences in conformal dimensions do not affect our arguments and can be ignored in subsequent discussions.

On the other hand, the second term turns out to be³⁰

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{b(W_1^\alpha, \widehat{O}_{2n+3}'^\alpha) c(\widehat{O}_{2n+3}'^\alpha, \widehat{W}_1^\beta, \widehat{W}_2)}{c(\widehat{O}_{2n+3}'^\alpha, \widehat{O}_{2n+3}'^\alpha)} G_{\widehat{\Delta}_{2n+3}}^{\widehat{\Delta}_{\widehat{W}_1}^\beta - \widehat{\Delta}_{\widehat{W}_2}}(v) \\ = \frac{\kappa c(\Phi_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)}{6} v^{3/2} {}_2F_1(1, 1; 5/2; v) + O(\epsilon^2). \end{aligned} \quad (3.82)$$

By plugging (3.81) and (3.82) into (3.80), we end up with³¹

$$\begin{aligned} \langle W_1^\alpha(x) \widehat{W}_1^\beta(0) \widehat{W}_2(\infty) \rangle \\ = c(\widehat{W}_1^\alpha, \widehat{W}_1^\beta, \widehat{W}_2) \frac{C_1^\alpha}{|x_\perp|^{\Delta_{W_1} - \widehat{\Delta}_{\widehat{W}_1}^\beta} |x|^{\widehat{\Delta}_{\widehat{W}_1}^\alpha + \widehat{\Delta}_{\widehat{W}_1}^\beta - \widehat{\Delta}_{\widehat{W}_2}}} \\ + \frac{\epsilon}{3(N+8)} \left[(N+8)(\delta^{\alpha 1} + \delta^{\beta 1} - \Gamma_{\widehat{W}_2}) c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2) + c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2) \right] \\ \cdot \frac{|x_\perp|^2}{|x|^2} {}_2F_1\left(1, 1; \frac{5}{2}; \frac{|x_\perp|^2}{|x|^2}\right) + O(\epsilon^2). \end{aligned} \quad (3.83)$$

Let us take $\hat{x} \rightarrow 0$ in (3.83) where the bulk operator is still distant from defects. We observe (3.83) is not analytic in the limit due to the odd integer powers of $|\hat{x}|$. This non-analytic behavior originates from the asymptotic form of the Gauss's hypergeometric function:³²

$${}_2F_1\left(1, 1; \frac{5}{2}; \frac{|x_\perp|^2}{|x|^2}\right) \xrightarrow{\hat{x} \sim 0} \frac{3\pi}{2} \cdot \frac{|\hat{x}|}{|x|} + \dots \quad (3.84)$$

which contradicts the holomorphy of Euclidean correlators away from the coincidence of points. The only way to resolve this tension is to set the coefficient in front of the singular term in (3.83) to zero:

$$(N+8)(\delta^{\alpha 1} + \delta^{\beta 1} - \Gamma_{\widehat{W}_2}) c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2) + c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2) = 0. \quad (3.85)$$

³⁰We used (3.79) and (3.57), and the sum rule for the hypergeometric function:

$${}_2F_1(1, 1; 5/2; z) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(2n/3+1)(n+5/2)_n} z^n {}_2F_1(n+1, n+2; 2n+7/2; z),$$

which can be proved by expanding ${}_2F_1$ in the RHS and rearranging in powers of z .

³¹We use the following relation:

$$C_1^\alpha \cdot c(\widehat{W}_1^\alpha, \widehat{W}_1^\beta, \widehat{W}_2) = c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2) + O(\epsilon).$$

³²Use Kummer's connection formula for hypergeometric functions:

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} {}_2F_1(\alpha, \beta; \alpha+\beta-\gamma+1; 1-z) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1-z)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-z). \end{aligned}$$

By plugging the defect three-point coefficients $c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)$ and $c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, \widehat{\Phi}_2)$ listed in table 3.2 into the above equation, we obtain the conformal dimension of the operators at the first order in ϵ together with the mixing angle θ of the scalar operators (3.76):³³

$$\widehat{\Delta}_{\widehat{V}} = 2 + \frac{N+10}{N+8} \epsilon + O(\epsilon^2), \quad (3.86)$$

$$\widehat{\Delta}_{\widehat{T}} = 2 + \frac{2}{N+8} \epsilon + O(\epsilon^2), \quad (3.87)$$

$$\widehat{\Delta}_{\widehat{S}_\pm} = 2 + \frac{3N+20 \pm \sqrt{N^2+40N+320}}{2(N+8)} \epsilon + O(\epsilon^2), \quad (3.88)$$

$$\tan \theta = \frac{N+18 + \sqrt{N^2+40N+320}}{2\sqrt{N-1}}. \quad (3.89)$$

These are in agreement with the perturbative results [26, section 3.5].

3.4.4 Defect operator spectrum for $N = 1$ (Ising DCFT)

Finally, consider the case with $N = 1$ (Ising CFT with a localized magnetic field). Note that there are no flavor symmetries, hence no symmetry breaking on the defect. We can compute the anomalous dimensions of the defect operators $\widehat{W}_p(\hat{x})$ that tend to $\widehat{\Phi}_p(\hat{x})$ in $\epsilon \rightarrow 0$ limit. (Recall (3.47).)

Performing a similar analysis to the last subsection for $\langle W_1 \widehat{W}_p \widehat{W}_{p+1} \rangle$ leads

$$9(\Gamma_{\widehat{W}_1} + \Gamma_{\widehat{W}_p} - \Gamma_{\widehat{W}_{p+1}})c(\widehat{\Phi}_1, \widehat{\Phi}_p, \widehat{\Phi}_{p+1}) + c(\widehat{\Phi}_3, \widehat{\Phi}_p, \widehat{\Phi}_{p+1}) = 0, \quad (3.90)$$

with

$$c(\widehat{\Phi}_1, \widehat{\Phi}_p, \widehat{\Phi}_{p+1}) = (p+1)!, \quad c(\widehat{\Phi}_3, \widehat{\Phi}_p, \widehat{\Phi}_{p+1}) = 3p(p+1)!, \quad (3.91)$$

and

$$\widehat{\Delta}_{\widehat{W}_p} = p + \Gamma_{\widehat{W}_p} \epsilon + O(\epsilon^2). \quad (3.92)$$

(3.66) gives $\Gamma_{\widehat{W}_1} = 1$. By solving the recursion relation $\Gamma_{\widehat{W}_{p+1}} = \Gamma_{\widehat{W}_p} + 1 + p/3$ that follows from (3.90), we obtain

$$\widehat{\Delta}_{\widehat{W}_p} = p + \frac{p(p+5)}{6} \epsilon + O(\epsilon^2). \quad (3.93)$$

³³The conformal dimensions of $\widehat{\Delta}_{\widehat{S}_\pm}$ and the mixing angle θ are derived by solving the following simultaneous equations:

$$\begin{aligned} (N+8)(\Gamma_{\widehat{S}_+} - \delta^{\alpha 1} - \delta^{\beta 1}) & \left[\sqrt{N-1} c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, |\widehat{\Phi}_1^1|^2) - \tan \theta c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, |\widehat{\Phi}_1^\gamma|^2) \right] \\ & = \sqrt{N-1} c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, |\widehat{\Phi}_1^1|^2) - \tan \theta c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, |\widehat{\Phi}_1^\gamma|^2), \\ (N+8)(\Gamma_{\widehat{S}_-} - \delta^{\alpha 1} - \delta^{\beta 1}) & \left[\sqrt{N-1} \tan \theta c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, |\widehat{\Phi}_1^1|^2) + c(\widehat{\Phi}_1^\alpha, \widehat{\Phi}_1^\beta, |\widehat{\Phi}_1^\gamma|^2) \right] \\ & = \sqrt{N-1} \tan \theta c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, |\widehat{\Phi}_1^1|^2) + c(\widehat{\Phi}_3^\alpha, \widehat{\Phi}_1^\beta, |\widehat{\Phi}_1^\gamma|^2). \end{aligned}$$

There are two sets of solutions and we have chosen the ones satisfying $\Gamma_{\widehat{S}_+} \geq \Gamma_{\widehat{S}_-}$.

Note that this is consistent with (3.66) and the $O(\epsilon)$ correction to the conformal dimension of $\widehat{W}_2$ is identical to that of $\widehat{S}_+$ with $N = 1$.

3.A Bulk-defect-defect three-point function

Following [50, appendix C], we expand on the scalar bulk-defect-defect three-point functions in DCFT.³⁴ In the same manner as Sec. 3.2.3, we let the spacetime dimensions d and defect dimensions p be general for future reference.

We first introduce the embedding space formalism for DCFT (appendix 3.4.4) and derive the conformal block expansion for scalar bulk-defect-defect three-point function by solving conformal Casimir equation (appendix 3.4.4). Then, we apply the DOEs inside the three-point function in appendix 3.4.4. After summing over DOEs for all orders, we confirm that it reproduces the conformal block expansion derived in appendix 3.4.4.

3.A.1 Embedding space formalism

We embed the physical coordinate system $\mathbb{R}^d$ onto the projective null cone on $X^M \in \mathbb{R}^{1,d+1}$ with $M, N = -, +, 1, \dots, d$ [53]:

$$\begin{aligned} ds_{\mathbb{R}^{1,d+1}}^2 &= \eta_{MN} dX^M dX^N = -dX^+ dX^- + \delta_{\mu\nu} dX^\mu dX^\nu, \\ \delta_{\mu\nu} &= \text{diag}(1, \dots, 1), \quad \eta_{MN} X^M X^N = 0, \quad X^M \sim \lambda X^M, \quad \lambda \in \mathbb{R}_+. \end{aligned} \quad (3.94)$$

We denote the embedding space coordinates on and around the defect by $Q^M = (Q^A, Q^I = 0)$ and $P^M = (P^A, P^I)$ with $A = +, -, 1, \dots, p$ and $I = p + 1, \dots, d$. In going back to the physical coordinates, one should make the following replacements:

$$P^M = (P^+, P^-, P^a, P^i) \mapsto (1, x^2, \hat{x}^a, x_\perp^i), \quad Q^A = (Q^+, Q^-, Q^a) \mapsto (1, \hat{y}^2, \hat{y}^a). \quad (3.95)$$

The bulk and defect local scalars in the embedding space $\mathcal{O}_\Delta(P)$ and $\widehat{\mathcal{O}}_{\widehat{\Delta}}(Q)$ are specified by the homogeneity associated with their conformal dimensions Δ and $\widehat{\Delta}$:

$$P^M \frac{\partial}{\partial P^M} \mathcal{O}_\Delta(P) = -\Delta \mathcal{O}_\Delta(P), \quad Q^A \frac{\partial}{\partial Q^A} \widehat{\mathcal{O}}_{\widehat{\Delta}}(Q) = -\widehat{\Delta} \widehat{\mathcal{O}}_{\widehat{\Delta}}(Q). \quad (3.96)$$

³⁴The readers interested in more on group theoretical perspectives are also referred to [51]. A similar analysis to [50] has been carried out in [52] for boundary CFT, which fails to choose the proper solution to the conformal Casimir equation.

The generators of defect conformal group $\mathbf{J}_{AB}, \mathbf{J}_{IJ}$ act as rotation differential operators on the embedding space operators:

$$\begin{aligned}
[\mathbf{J}_{AB}, \mathcal{O}_\Delta(P)] &= -\mathcal{J}_{AB}(P) \mathcal{O}_\Delta(P) = -\left(P_A \frac{\partial}{\partial P^B} - P_B \frac{\partial}{\partial P^A}\right) \mathcal{O}_\Delta(P), \\
[\mathbf{J}_{IJ}, \mathcal{O}_\Delta(P)] &= -\mathcal{J}_{IJ}(P) \mathcal{O}_\Delta(P) = -\left(P_I \frac{\partial}{\partial P^J} - P_J \frac{\partial}{\partial P^I}\right) \mathcal{O}_\Delta(P), \\
[\mathbf{J}_{AB}, \hat{\mathcal{O}}_{\hat{\Delta}}(Q)] &= -\mathcal{J}_{AB}(Q) \hat{\mathcal{O}}_{\hat{\Delta}}(Q) = -\left(Q_A \frac{\partial}{\partial Q^B} - Q_B \frac{\partial}{\partial Q^A}\right) \hat{\mathcal{O}}_{\hat{\Delta}}(Q), \\
[\mathbf{J}_{IJ}, \hat{\mathcal{O}}_{\hat{\Delta}}(Q)] &= -\mathcal{J}_{IJ}(Q) \hat{\mathcal{O}}_{\hat{\Delta}}(Q) = -\left(Q_I \frac{\partial}{\partial Q^J} - Q_J \frac{\partial}{\partial Q^I}\right) \hat{\mathcal{O}}_{\hat{\Delta}}(Q),
\end{aligned} \tag{3.97}$$

We also use the following shorthanded notations to express $\text{SO}(1, p+1) \times \text{SO}(d-p)$ invariant inner products:

$$X \cdot X' = X^M X'_M, \quad X \bullet X' = X^A X'_A, \quad X \circ X' = X^I X'_I. \tag{3.98}$$

One should pay attention to the following relations:

$$P \cdot P = P \bullet P + P \circ P = 0, \quad Q \cdot Q = Q \bullet Q = 0. \tag{3.99}$$

3.A.2 Conformal block expansion

The three-point function of one bulk scalar $\mathcal{O}_{\Delta_1}$ and two defect scalars $\hat{\mathcal{O}}_{\hat{\Delta}_1}, \hat{\mathcal{O}}_{\hat{\Delta}_2}$ is fixed by defect conformal symmetry;

$$\begin{aligned}
&\langle \mathcal{O}_\Delta(P) \hat{\mathcal{O}}_{\hat{\Delta}_1}(Q_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(Q_2) \rangle \\
&= \frac{g(v)}{(P \circ P)^{\frac{\Delta}{2}} (-2P \bullet Q_1)^{\frac{\hat{\Delta}_{12}^-}{2}} (-2P \bullet Q_2)^{\frac{\hat{\Delta}_{21}^-}{2}} (-2Q_1 \bullet Q_2)^{\frac{\hat{\Delta}_{12}^+}{2}}}.
\end{aligned} \tag{3.100}$$

In the physical space, we have

$$\langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}_1}(\hat{y}_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(\hat{y}_2) \rangle = \frac{g(v)}{|x_\perp|^\Delta |x - \hat{y}_1|^{\hat{\Delta}_{12}^-} |x - \hat{y}_2|^{\hat{\Delta}_{21}^-} |\hat{y}_{12}|^{\hat{\Delta}_{12}^+}}. \tag{3.101}$$

Here $\hat{\Delta}_{ij}^\pm = \hat{\Delta}_i \pm \hat{\Delta}_j$ and $g(v)$ is some function of the $\text{SO}(1, p+1) \times \text{SO}(d-p)$ invariant v defined by

$$v = \frac{(P \circ P) (-2Q_1 \bullet Q_2)}{(-2P \bullet Q_1) (-2P \bullet Q_2)} \xrightarrow[\text{space}]{\text{physical}} \frac{|x_\perp|^2 |\hat{y}_{12}|^2}{|x - \hat{y}_1|^2 |x - \hat{y}_2|^2}. \tag{3.102}$$

Note that in the Euclidean regime, we have

$$0 \leq v \leq 1. \tag{3.103}$$

To see why this condition holds, consider the area of a triangle S with the vertices at $x, \hat{y}_1$ and $\hat{y}_2$ (see Fig. 3.1). On one hand, we have $S = \frac{1}{2} |x_\perp| |\hat{y}_{12}|$, while using the angle φ between

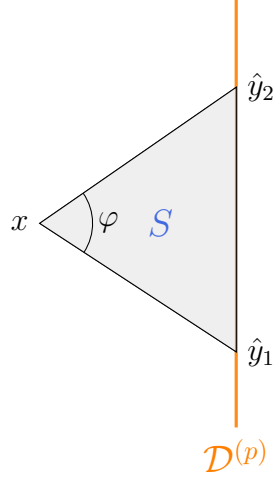


Figure 3.1: The triangle spanned by one point x on the bulk and two points $\hat{y}_1, \hat{y}_2$ on the defect $\mathcal{D}^{(p)}$.

$(x - y_1)$ and $(x - y_2)$, the area can be written by $S = \frac{1}{2} |\sin \varphi| |x - \hat{y}_1| |x - \hat{y}_2|$. Therefore, $2S = |x_\perp| |\hat{y}_{12}| = |\sin \varphi| |x - \hat{y}_1| |x - \hat{y}_2|$ and $0 \leq v = |\sin \varphi|^2 \leq 1$.

We now take the origin of the radial quantization at x , draw a circle tangent to the defect, and insert a completeness relation that diagonalizes the $\text{SO}(1, p+1) \times \text{SO}(d-p)$ Casimir (see Fig. 3.2). Because the three-point function (3.100) and cross-ratio (3.102) are $\text{SO}(d-p)$ singlet, we have

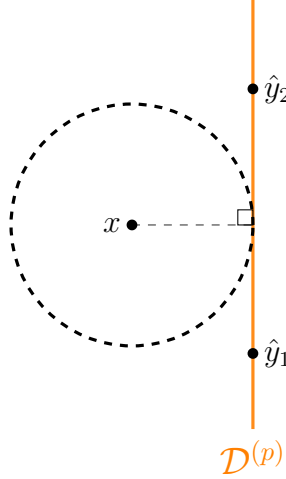


Figure 3.2: The completeness relation is inserted on the dashed circle centered at x .

only to consider the $\text{SO}(1, p+1)$ part of the Casimir. In what follows, we consider the $\text{SO}(1, p+1)$ eigenstates that are spanned by the defect local scalars, namely

$$\langle \mathcal{O}_\Delta(P) \hat{\mathcal{O}}_{\hat{\Delta}_1}(Q_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(Q_2) \rangle = \sum_{\hat{\Delta}} \langle \hat{\mathcal{D}} | \mathcal{R} \{ \hat{\mathcal{O}}_{\hat{\Delta}_1}(Q_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(Q_2) \} | \hat{\mathcal{O}}_{\hat{\Delta}} | \mathcal{O}_\Delta(P) | \Omega \rangle, \quad (3.104)$$

where $\mathcal{R}$ means the appropriate radial ordering and $\langle \hat{\mathcal{D}} |$ is the defect vacuum. Then, from (3.96)

and (3.97) the Casimir eigenvalue for $|\hat{\mathcal{O}}_{\hat{\Delta}}|$ is given by³⁵

$$\frac{1}{2} \mathbf{J}^{AB} \mathbf{J}_{AB} |\hat{\mathcal{O}}_{\hat{\Delta}}| = -\hat{\Delta} (\hat{\Delta} - p) |\hat{\mathcal{O}}_{\hat{\Delta}}|. \quad (3.105)$$

Let us define the eigenfunction $G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v)$ through

$$\begin{aligned} & \langle \hat{\mathcal{D}} | \mathcal{R}\{\hat{\mathcal{O}}_{\hat{\Delta}_1}(Q_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(Q_2)\} | \hat{\mathcal{O}}_{\hat{\Delta}} | \mathcal{O}_{\Delta}(P) | \Omega \rangle \\ &= \frac{b(\mathcal{O}, \hat{\mathcal{O}}) c(\hat{\mathcal{O}}, \hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2) / c(\hat{\mathcal{O}}, \hat{\mathcal{O}})}{(P \circ P)^{\frac{\Delta}{2}} (-2P \bullet Q_1)^{\frac{\hat{\Delta}_{12}^-}{2}} (-2P \bullet Q_2)^{\frac{\hat{\Delta}_{21}^-}{2}} (-2Q_1 \bullet Q_2)^{\frac{\hat{\Delta}_{12}^+}{2}}} G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v), \end{aligned} \quad (3.106)$$

where we denoted the bulk-defect two-point coefficient, defect three-point coefficient, and defect two-point coefficient by $b(\mathcal{O}, \hat{\mathcal{O}})$, $c(\hat{\mathcal{O}}, \hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2)$ and $c(\hat{\mathcal{O}}, \hat{\mathcal{O}})$ (see (3.17)). Inserting the $\text{SO}(1, p+1)$ Casimir in between $\hat{\mathcal{O}}_{\hat{\Delta}_2}(Q_2)$ and $|\hat{\mathcal{O}}_{\hat{\Delta}}|$ above and using (3.105) and (3.97), we have

$$\begin{aligned} & \langle \hat{\mathcal{D}} | \mathcal{R}\{\hat{\mathcal{O}}_{\hat{\Delta}_1}(Q_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(Q_2)\} \left(\frac{1}{2} \mathbf{J}^{AB} \mathbf{J}_{AB} \right) | \hat{\mathcal{O}}_{\hat{\Delta}} | \mathcal{O}_{\Delta}(P) | \Omega \rangle \\ &= -\hat{\Delta} (\hat{\Delta} - p) \langle \hat{\mathcal{D}} | \mathcal{R}\{\hat{\mathcal{O}}_{\hat{\Delta}_1}(Q_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(Q_2)\} | \hat{\mathcal{O}}_{\hat{\Delta}} | \mathcal{O}_{\Delta}(P) | \Omega \rangle \\ &= \frac{1}{2} \mathcal{J}_{AB}(P) \mathcal{J}^{AB}(P) \langle \hat{\mathcal{D}} | \mathcal{R}\{\hat{\mathcal{O}}_{\hat{\Delta}_1}(Q_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(Q_2)\} | \hat{\mathcal{O}}_{\hat{\Delta}} | \mathcal{O}_{\Delta}(P) | \Omega \rangle. \end{aligned} \quad (3.107)$$

Comparing these two expressions, we find

$$\begin{aligned} & \left[\frac{1}{2} \mathcal{J}_{AB}(P) \mathcal{J}^{AB}(P) + \hat{\Delta} (\hat{\Delta} - p) \right] \\ & \cdot \frac{G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v)}{(P \circ P)^{\frac{\Delta}{2}} (-2P \bullet Q_1)^{\frac{\hat{\Delta}_{12}^-}{2}} (-2P \bullet Q_2)^{\frac{\hat{\Delta}_{21}^-}{2}} (-2Q_1 \bullet Q_2)^{\frac{\hat{\Delta}_{12}^+}{2}}} = 0. \end{aligned} \quad (3.108)$$

The differential operator commutes with $P \circ P$ and $(-2Q_1 \bullet Q_2)$ and (3.108) can be reduced to

$$\left[\frac{1}{2} \mathcal{J}_{AB}(P) \mathcal{J}^{AB}(P) + \hat{\Delta} (\hat{\Delta} - p) \right] \frac{G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v)}{(-2P \bullet Q_1)^{\hat{\Delta}_{12}^-/2} (-2P \bullet Q_2)^{\hat{\Delta}_{21}^-/2}} = 0. \quad (3.109)$$

A short calculation shows

$$\left\{ 4v^2(1-v)\partial_v^2 + [4(1-v) - 2p] v \partial_v + (\hat{\Delta}_{12}^-)^2 v - \hat{\Delta} (\hat{\Delta} - p) \right\} G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v) = 0. \quad (3.110)$$

After setting $G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v) = v^{\hat{\Delta}/2} f(v)$ and some manipulations, it turns out that $f(v)$ satisfies the following hypergeometric differential equation:

$$\left\{ v(1-v) \partial_v^2 + \left[\hat{\Delta} + 1 - \frac{p}{2} - (\hat{\Delta} + 1) v \partial_v - \frac{\hat{\Delta} + \hat{\Delta}_{12}^-}{2} \frac{\hat{\Delta} - \hat{\Delta}_{12}^-}{2} \right] \right\} f(v) = 0. \quad (3.111)$$

³⁵This comes from $\frac{1}{2} \mathbf{J}^{AB} \mathbf{J}_{AB} \hat{\mathcal{O}}_{\hat{\Delta}}(Q) = -Q \bullet \frac{\partial}{\partial Q} \left(p + Q \bullet \frac{\partial}{\partial Q} \right) \hat{\mathcal{O}}_{\hat{\Delta}}(Q) = -\hat{\Delta} (\hat{\Delta} - p) \hat{\mathcal{O}}_{\hat{\Delta}}(Q)$.

To see the proper boundary condition to this differential equation, consider the DOE of $\mathcal{O}_\Delta$:

$$\mathcal{O}_\Delta(x) \supset \frac{b(\mathcal{O}, \hat{\mathcal{O}})/c(\hat{\mathcal{O}}, \hat{\mathcal{O}})}{|x_\perp|^{\Delta-\hat{\Delta}}} \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x}), \quad (3.112)$$

Taking v to zero is equivalent to defect OPE limit $|x_\perp| \rightarrow 0$, where

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}_1}(\hat{y}_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(\hat{y}_2) \rangle &\xrightarrow{|x_\perp| \rightarrow 0} \sum_{\hat{\Delta}} \frac{b(\mathcal{O}, \hat{\mathcal{O}})/c(\hat{\mathcal{O}}, \hat{\mathcal{O}})}{|x_\perp|^{\Delta-\hat{\Delta}}} \langle \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x}) \hat{\mathcal{O}}_{\hat{\Delta}_1}(\hat{y}_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(\hat{y}_2) \rangle \\ &= \sum_{\hat{\Delta}} \frac{b(\mathcal{O}, \hat{\mathcal{O}}) c(\hat{\mathcal{O}}, \hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2)/c(\hat{\mathcal{O}}, \hat{\mathcal{O}})}{|x_\perp|^{\Delta-\hat{\Delta}}} \frac{1}{|\hat{x} - \hat{y}_1|^{\hat{\Delta}+\hat{\Delta}_{12}^-} |\hat{x} - \hat{y}_2|^{\hat{\Delta}+\hat{\Delta}_{21}^-} |\hat{y}_{12}|^{\hat{\Delta}_{12}^+-\hat{\Delta}}}. \end{aligned} \quad (3.113)$$

Combining this with (3.104) and (3.106) leads

$$G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v) \xrightarrow{v \rightarrow 0} v^{\hat{\Delta}/2}. \quad (3.114)$$

This boundary condition (3.114) singles out the proper solution to this differential equation (3.111) by $f(v) = {}_2F_1\left(\frac{\hat{\Delta}+\hat{\Delta}_{12}^-}{2}, \frac{\hat{\Delta}-\hat{\Delta}_{12}^-}{2}; \hat{\Delta}+1-\frac{p}{2}; v\right)$.

To sum up, the conformal block expansion for the bulk-defect-defect three-point function is

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}_1}(\hat{y}_1) \hat{\mathcal{O}}_{\hat{\Delta}_2}(\hat{y}_2) \rangle \\ = T_{\hat{\Delta}}^{\hat{\Delta}_1, \hat{\Delta}_2}(x, \hat{y}_1, \hat{y}_2) \times \sum_{\hat{\Delta}} \frac{b(\mathcal{O}, \hat{\mathcal{O}}) c(\hat{\mathcal{O}}, \hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2)}{c(\hat{\mathcal{O}}, \hat{\mathcal{O}})} G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v), \end{aligned} \quad (3.115)$$

with $T_{\hat{\Delta}}^{\hat{\Delta}_1, \hat{\Delta}_2}(x, \hat{y}_1, \hat{y}_2)$ being some function that transforms covariantly with three-point functions under the residual conformal group $\text{SO}(1, p+1) \times \text{SO}(d-p)$:

$$T_{\hat{\Delta}}^{\hat{\Delta}_1, \hat{\Delta}_2}(x, \hat{y}_1, \hat{y}_2) = \frac{1}{|x_\perp|^\Delta |x - \hat{y}_1|^{\hat{\Delta}_{12}^-} |x - \hat{y}_2|^{\hat{\Delta}_{21}^-} |\hat{y}_{12}|^{\hat{\Delta}_{12}^+}}. \quad (3.116)$$

The conformal block $G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v)$ is given by

$$G_{\hat{\Delta}}^{\hat{\Delta}_{12}^-}(v) = v^{\hat{\Delta}/2} {}_2F_1\left(\frac{\hat{\Delta} + \hat{\Delta}_{12}^-}{2}, \frac{\hat{\Delta} - \hat{\Delta}_{12}^-}{2}; \hat{\Delta} + 1 - \frac{p}{2}; v\right), \quad (3.117)$$

where the cross-ratio v is

$$v = \frac{|x_\perp|^2 |\hat{y}_{12}|^2}{|x - \hat{y}_1|^2 |x - \hat{y}_2|^2}. \quad (3.118)$$

3.A.3 Reconstruction of conformal block from defect operator expansions

We now derive the conformal block expansion (3.115) more directly, by summing DOEs for all orders. The calculation is almost parallel to the ones presented in [54] and [55] about conformal four-point functions.

Defect operator expansion. The DOE's are completely fixed to reproduce bulk-defect two-point functions with the normalization of the defect two-point functions (3.17). In particular, for the defect scalar channel, the expression reads (see appendix B.1 of [19])

$$\mathcal{O}_\Delta(x) \supset \sum_{\hat{\mathcal{O}}} \frac{b(\mathcal{O}, \hat{\mathcal{O}})/c(\hat{\mathcal{O}}, \hat{\mathcal{O}})}{|x_\perp|^{\Delta-\hat{\Delta}}} \sum_{n=0}^{\infty} \frac{(-1)^n |x_\perp|^{2n}}{2^{2n} (\hat{\Delta} + 1 - p/2)_n n!} (\hat{\partial}_x^2)^n \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x}), \quad (3.119)$$

with $\hat{\partial}_x^2 = \partial^2/\partial\hat{x}^a\partial\hat{x}_a$. One can check the validity of this expansion by applying it inside the bulk-to-defect two-point functions:³⁶

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{y}) \rangle &= \frac{b(\mathcal{O}, \hat{\mathcal{O}})/c(\hat{\mathcal{O}}, \hat{\mathcal{O}})}{|x_\perp|^{\Delta-\hat{\Delta}}} \sum_{n=0}^{\infty} \frac{(-1)^n |x_\perp|^{2n}}{2^{2n} (\hat{\Delta} + 1 - p/2)_n n!} (\hat{\partial}_x^2)^n \langle \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x}) \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{y}) \rangle \\ &= \frac{b(\mathcal{O}, \hat{\mathcal{O}})}{|x_\perp|^{\Delta-\hat{\Delta}}} \sum_{n=0}^{\infty} \frac{(-1)^n |x_\perp|^{2n}}{2^{2n} (\hat{\Delta} + 1 - p/2)_n n!} (\hat{\partial}_x^2)^n \frac{1}{|\hat{x} - \hat{y}|^{2\hat{\Delta}}} \\ &= \frac{b(\mathcal{O}, \hat{\mathcal{O}})}{|x_\perp|^{\Delta-\hat{\Delta}} |\hat{x} - \hat{y}|^{2\hat{\Delta}}} \sum_{n=0}^{\infty} \frac{(\hat{\Delta})_n}{n!} \left(-\frac{|x_\perp|^2}{|\hat{x} - \hat{y}|^2} \right)^n \\ &= \frac{b(\mathcal{O}, \hat{\mathcal{O}})}{(|\hat{x} - \hat{y}|^2 + |x_\perp|^2)^{\hat{\Delta}} |x_\perp|^{\Delta-\hat{\Delta}}}. \end{aligned} \quad (3.120)$$

By considering the Fourier transformation of $\hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x})$ defined through the following relation:

$$\hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x}) = \int \frac{d^p \hat{q}}{(2\pi)^p} e^{i\hat{q}\cdot\hat{x}} \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{q}), \quad (3.121)$$

we obtain an alternative expression of the DOE:

$$\begin{aligned} \mathcal{O}_\Delta(x) \supset \sum_{\hat{\mathcal{O}}} \frac{b(\mathcal{O}, \hat{\mathcal{O}})/c(\hat{\mathcal{O}}, \hat{\mathcal{O}})}{|x_\perp|^{\Delta-\hat{\Delta}}} \Gamma(\hat{\Delta} + 1 - p/2) \left(\frac{|x_\perp|}{2} \right)^{p/2-\hat{\Delta}} \\ \times \int \frac{d^p \hat{q}}{(2\pi)^p} e^{i\hat{q}\cdot\hat{x}} |\hat{q}|^{p/2-\hat{\Delta}} I_{\hat{\Delta}-p/2}(|x_\perp| |\hat{q}|) \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{q}), \end{aligned} \quad (3.122)$$

Here, $I_\nu(z)$ is the modified Bessel function $I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\nu+n+1)n!} \left(\frac{z}{2}\right)^{\nu+2n}$.

³⁶The following two formulas are used;

$$(\hat{\partial}_x^2)^n \frac{1}{|\hat{x} - \hat{y}|^{2\hat{\Delta}}} = \frac{2^{2n} (\hat{\Delta})_n (\hat{\Delta} + 1 - p/2)_n}{|\hat{x} - \hat{y}|^{2\hat{\Delta}+2n}}, \quad \sum_{n=0}^{\infty} \frac{(\hat{\Delta})_n}{n!} \left(-\frac{|x_\perp|^2}{|\hat{x} - \hat{y}|^2} \right)^n = \frac{1}{(1 + |x_\perp|^2/|\hat{x} - \hat{y}|^2)^{\hat{\Delta}}}.$$

Summing over defect operator expansions for all orders. Let us apply the DOE inside bulk-defect-defect three-point function using (3.122):

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}_1}(\hat{y}_1) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}(\hat{y}_2) \rangle &= \sum_{\widehat{\mathcal{O}}} \frac{b(\mathcal{O}, \widehat{\mathcal{O}})/c(\widehat{\mathcal{O}}, \widehat{\mathcal{O}})}{2^{p/2-\widehat{\Delta}} |x_\perp|^{\Delta-p/2}} \cdot \Gamma(\widehat{\Delta} + 1 - p/2) \\ &\cdot \int \frac{d^p \hat{q}}{(2\pi)^p} e^{i\hat{q} \cdot \hat{x}} |\hat{q}|^{p/2-\widehat{\Delta}} I_{\widehat{\Delta}-p/2}(|x_\perp| |\hat{q}|) \langle \widehat{\mathcal{O}}_{\widehat{\Delta}}(\hat{q}) \widehat{\mathcal{O}}_{\widehat{\Delta}_1}(\hat{y}_1) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}(\hat{y}_2) \rangle. \end{aligned} \quad (3.123)$$

The partial Fourier transformation of the three-point function is given by (see e.g. [56, equation (3.41)])

$$\begin{aligned} \langle \widehat{\mathcal{O}}_{\widehat{\Delta}}(\hat{q}) \widehat{\mathcal{O}}_{\widehat{\Delta}_1}(\hat{y}_1) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}(\hat{y}_2) \rangle &= \int d^p \hat{x} e^{-i\hat{q} \cdot \hat{x}} \langle \widehat{\mathcal{O}}_{\widehat{\Delta}}(\hat{x}) \widehat{\mathcal{O}}_{\widehat{\Delta}_1}(\hat{y}_1) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}(\hat{y}_2) \rangle \\ &= \frac{c(\widehat{\mathcal{O}}, \widehat{\mathcal{O}}_1, \widehat{\mathcal{O}}_2)}{|\hat{y}_{12}|^{\Delta_{12}^+-\widehat{\Delta}}} \frac{2\pi^{p/2}}{\Gamma\left(\frac{\widehat{\Delta}+\widehat{\Delta}_{12}^-}{2}\right) \Gamma\left(\frac{\widehat{\Delta}+\widehat{\Delta}_{21}^-}{2}\right)} \left(\frac{|\hat{q}|}{2|\hat{y}_{12}|}\right)^{\widehat{\Delta}-p/2} \\ &\cdot \int_0^1 d\xi \xi^{\frac{\widehat{\Delta}_{12}^-+p/2}{2}-1} (1-\xi)^{\frac{\widehat{\Delta}_{21}^-+p/2}{2}-1} e^{-i\hat{q} \cdot [\xi \hat{y}_1 + (1-\xi) \hat{y}_2]} K_{\widehat{\Delta}-p/2}\left(\sqrt{\xi(1-\xi)} |\hat{q}| |\hat{y}_{12}|\right). \end{aligned} \quad (3.124)$$

Hence,

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}_1}(\hat{y}_1) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}(\hat{y}_2) \rangle &= \sum_{\widehat{\mathcal{O}}} \frac{b(\mathcal{O}, \widehat{\mathcal{O}}) c(\widehat{\mathcal{O}}, \widehat{\mathcal{O}}_1, \widehat{\mathcal{O}}_2)}{c(\widehat{\mathcal{O}}, \widehat{\mathcal{O}})} \\ &\cdot \frac{2\pi^{p/2} \Gamma(\widehat{\Delta} + 1 - p/2)}{|\hat{y}_{12}|^{\widehat{\Delta}_{12}^+-p/2} |x_\perp|^{\Delta-p/2} \Gamma\left(\frac{\widehat{\Delta}+\widehat{\Delta}_{12}^-}{2}\right) \Gamma\left(\frac{\widehat{\Delta}+\widehat{\Delta}_{21}^-}{2}\right)} \cdot \int_0^1 d\xi \xi^{\frac{\widehat{\Delta}_{12}^-+p/2}{2}-1} (1-\xi)^{\frac{\widehat{\Delta}_{21}^-+p/2}{2}-1} \\ &\cdot \int \frac{d^p \hat{q}}{(2\pi)^p} e^{i\hat{q} \cdot [\xi(\hat{x}-\hat{y}_1) + (1-\xi)(\hat{x}-\hat{y}_2)]} I_{\widehat{\Delta}-p/2}(|x_\perp| |\hat{q}|) K_{\widehat{\Delta}-p/2}\left(\sqrt{\xi(1-\xi)} |\hat{q}| |\hat{y}_{12}|\right). \end{aligned} \quad (3.125)$$

We first perform the angular part of the $\hat{q}$ -integral as follows³⁷

$$\begin{aligned} \int \frac{d^p \hat{q}}{(2\pi)^p} e^{-i\hat{q} \cdot \hat{x}} \dots &= \text{Vol}(\mathbb{S}^{p-2}) \int_0^\infty d|\hat{q}| |\hat{q}|^{p-1} \int_0^\pi d\theta (\sin \theta)^{p-2} e^{-i|\hat{q}| |\hat{x}| \cos \theta} \dots \\ &= \frac{1}{|\hat{x}|^{p/2-1}} \cdot \int_0^\infty \frac{d|\hat{q}|}{(2\pi)^{p/2}} |\hat{q}|^{p/2} J_{p/2-1}(|\hat{x}| |\hat{q}|) \dots \end{aligned} \quad (3.126)$$

³⁷The relevant formula is $\int_0^\pi d\theta (\sin \theta)^a e^{-i b \cos \theta} = \int_0^\pi d\theta (\sin \theta)^a \cos(b \cos \theta) = \left(\frac{b}{2}\right)^{-a/2} \sqrt{\pi} \Gamma(a/2 + 1/2) J_{a/2}(b)$. Also, $\text{Vol}(\mathbb{S}^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ is the volume of unit n -sphere.

Then, the remaining $\hat{q}$ -integral in the last line of (3.125) can be integrated to give a hypergeometric function:³⁸

(The last line of (3.125))

$$= \frac{\Gamma(\hat{\Delta}) [\xi(1-\xi)]^{-\frac{p/2}{2}}}{2^{\hat{\Delta}+1} \pi^{p/2} \Gamma(\hat{\Delta} - p/2 + 1) |x_{\perp}|^{p/2} |\hat{y}_{12}|^{p/2}} \frac{1}{\tilde{u}^{\hat{\Delta}}} {}_2F_1 \left(\frac{\hat{\Delta}}{2}, \frac{\hat{\Delta}+1}{2}; \hat{\Delta} - p/2 + 1; \frac{1}{\tilde{u}^2} \right). \quad (3.127)$$

Here we defined $\tilde{u}$ by

$$\tilde{u} = \frac{|x_{\perp}|^2 + |\hat{x} - \hat{y}_1|^2 \xi + |\hat{x} - \hat{y}_2|^2 (1-\xi)}{2 |x_{\perp}| |\hat{y}_{12}| \sqrt{\xi(1-\xi)}}. \quad (3.128)$$

Plugging this expression into the last two lines of (3.125) and using a Mellin–Barnes integral representation of the hypergeometric function

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s, \quad (3.129)$$

we have

(The last two lines of (3.125))

$$= T_{\hat{\Delta}}^{\hat{\Delta}_1, \hat{\Delta}_2}(x, \hat{y}_1, \hat{y}_2) v^{\hat{\Delta}} \cdot \frac{\Gamma(\hat{\Delta} - p/2 + 1)}{\Gamma\left(\frac{\hat{\Delta} + \hat{\Delta}_{12}^-}{2}\right) \Gamma\left(\frac{\hat{\Delta} + \hat{\Delta}_{21}^-}{2}\right)} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma\left(\frac{\hat{\Delta} + \hat{\Delta}_{12}^-}{2} + s\right) \Gamma\left(\frac{\hat{\Delta} + \hat{\Delta}_{21}^-}{2} + s\right) \Gamma(-s)}{\Gamma(\hat{\Delta} - p/2 + s + 1)} (-v)^s, \quad (3.130)$$

where we have implemented the following relation to perform the ξ -integral

$$\int_0^1 dx \frac{x^{\mu-1} (1-x)^{\nu-1}}{[ax + b(1-x) + c]^{\mu+\nu}} = \frac{\Gamma(\mu) \Gamma(\nu)}{(a+c)^{\mu} (b+c)^{\nu} \Gamma(\mu+\nu)}. \quad (3.131)$$

The last line of (3.130) can be identified as the Mellin–Barnes integral representation of hypergeometric function (3.129) and we end up with the conformal block expansion (3.115).

³⁸The following integration formula is used (see e.g. [57, equation (6.578.11)])

$$\int_0^\infty dx x^{\nu+1} K_\mu(ax) I_\mu(bx) J_\nu(cx) = \frac{\Gamma(\mu+\nu)}{2^{\mu+1} \Gamma(\mu+1)} \cdot \frac{c^\nu}{(ab)^{\nu+1} u^{\mu+\nu+1}} {}_2F_1 \left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \mu+1; \frac{1}{u^2} \right),$$

with $2abu = a^2 + b^2 + c^2$.

Chapter 4

Subdimensional Disorder and Logarithmic Defect

This chapter is mostly based on [58].

4.1 Overview

Random systems have long played a central role in condensed matter and statistical physics, as the introduction of disorder often gives rise to rich and fascinating phenomena. One of the earliest and most influential studies in this context is Anderson’s seminal work on localization [59], which demonstrated that quenched disorder can fundamentally alter the transport properties of waves including electromagnetic waves and spin waves on lattice, leading to the absence of diffusion in low-dimensional systems. Since then, a wide variety of disordered models have been explored, revealing phenomena, e.g. Griffiths phases [60] and spin glass phases [61, 62] (see e.g. [63–65] for the review of recent developments of these topics).

In recent years, the study of disorder has gained renewed interest in the context of holography and strongly correlated quantum systems. In particular, the Sachdev-Ye-Kitaev (SYK) model [66–68] and its variants, namely one-dimensional systems of fermions with randomly distributed coupling constants, have been extensively investigated. These models exhibit maximally chaotic behavior [66, 69] and nontrivial conformal symmetry in the infrared [66, 70, 71], providing a solvable playground to explore the interplay between randomness and quantum dynamics. Moreover, the SYK model is believed to be holographically dual to the nearly AdS_2 gravity [66, 67], providing one of the simplest models for quantum gravity. This duality highlights the essential role of ensemble averaging in establishing a consistent gravitational description, where Euclidean wormholes naturally compute averaged boundary observables (see e.g. [72–75]).

In quantum computing and quantum information, quenched disorder also provides a unifying framework for understanding crucial aspects such as quantum error correction [76–78] and measurement-induced phase transitions [79–81]. In particular, there is often a mapping such that the effects of quantum noise and dephasing can be captured by classical quenched disorder in

an effective statistical system.³⁹ For example, the study of error recovery using quantum error-correcting codes can be mapped onto classical disordered spin models, such as the random-bond Ising model (see e.g. [82–85]) with binary distribution, where the accuracy threshold (which measures the efficacy of the code) coincides with the critical point on the Nishimori line [82, 83],⁴⁰ defined by matching error probabilities and effective thermal fluctuations [76, 77]. Likewise, in monitored quantum circuits, averaging over measurement outcomes generates effective spacetime disorder in the corresponding statistical mechanics representations of entanglement dynamics (see e.g. [90–92]). Intuitively, via this quantum-to-classical mapping, the interplay between unitary entanglement growth and measurement-induced disentangling mirrors the competition between ordering and disorder in classical systems.

While these developments underscore the significant role of disorder across various branches of physics, it is worth noting that most previous studies have focused on cases where the random couplings are distributed throughout the entire space. In such cases, the disorder averaging effectively acts as a global modification of the theory, and the resulting averaged theory generally differs drastically from the original one. In this work, we propose to study a different but closely related scenario: instead of distributing disorder across the whole space, we consider the quenched disorder *localized along a subspacetime*.⁴¹ In other words, the coupling constants fluctuate randomly only on a lower-dimensional subsystem, while the rest of the system remains unaffected. This problem was first investigated in the 2d Ising model with a random boundary magnetic field [95–98] (see also the review [99]). More recent developments on boundary disorder can be found in [100–102]. In the present work, we extend the discussion to general codimensions. Concretely, we consider a d -dimensional Euclidean spacetime $\mathbb{R}^d$, where the UV fixed point is described by a conformal field theory (CFT) with action S_{CFT} . We deform this UV theory by adding the following localized term on a p -dimensional subspacetime $\mathbb{R}^p$ ($p < d$):

$$S = S_{\text{CFT}} + \int_{\mathbb{R}^p} d^p \hat{x} g(\hat{x}) O(\hat{x}), \quad (4.1)$$

where $\hat{x}$ denotes the coordinates on $\mathbb{R}^p$, and O is a scalar primary operator in the UV CFT. Throughout this chapter, we assume that the coupling g is a random variable distributed according to some probability functional $P[g]$. A natural and intriguing question then arises: what is the fate of this localized deformation in the infrared, particularly after performing the ensemble average over the random couplings? More specifically, under what conditions does such a spatially localized

³⁹An important difference is that, unlike ordinary quenched disorder which is uncorrelated, the disorder distribution obtained from the quantum-to-classical mapping follows Born rule and thus is highly correlated [79–81]. In the replica approach to quenched disorder with n replica copies, the former amounts to taking the limit $n \rightarrow 0$ whereas the latter corresponds to $n \rightarrow 1$, as is expected to produce von Neumann entropies.

⁴⁰Unlike the well-known Ising fixed points (in 2d and 3d) described by unitary CFTs [13, 86], the Nishimori critical point [82, 83] is believed to be logarithmic [87] and has not been solved. See recent work [88, 89] and references therein for recent progress.

⁴¹Throughout this chapter, we focus on CFT in Euclidean spacetime $\mathbb{R}^d$. Without loss of generality, we assume the disorder coupling is turned on in the subspacetime including the Euclidean time and a subspace in the Euclidean space. Here we study classical disorder (see e.g. [93]), where the random couplings vary on this subspacetime. In principle, one could also study quantum disorder, where the couplings fluctuate only in space. The latter case leads to novel phenomena, for example the appearance of Lifshitz scaling (see e.g. [94]). We leave the exploration of quantum disorder in subdimensional spacetime for future work.

disorder flow to a *conformal defect* (Figure 4.1)? Furthermore, what are the special properties of such a conformal defect?

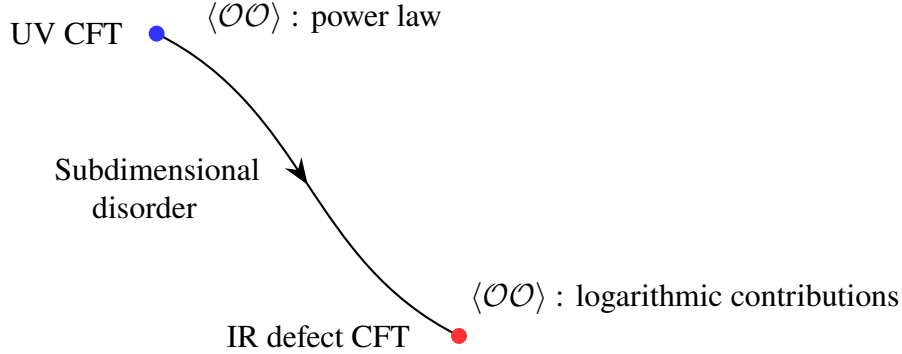


Figure 4.1: Illustration of the defect setup studied in this chapter. In the UV CFT, the correlation functions show usual power law, whereas in the IR defect CFT they develop logarithmic behavior due to subdimensional disorder.

To begin, the relevance of conventional disorder in the infrared is governed by the Harris criterion [103] (see also [93] for a review). As we explain in Section 4.2.2, this criterion admits a direct analog for subdimensional disorder, which depends on the conformal dimension Δ_O of the disorder operator in (4.1),⁴²

$$\text{Relevant : } \Delta_O < \frac{p}{2}, \quad \text{Irrelevant : } \Delta_O > \frac{p}{2}, \quad \text{Marginal : } \Delta_O = \frac{p}{2}. \quad (4.2)$$

This implies, for unitary CFTs, new defect fixed points from subdimensional disorder is only possible for $p \geq d - 2$.

To better understand the defect fixed points arising from ensemble averaging, we elucidate the systematics of conformal defects with logarithmic behavior on their worldvolume in Section 4.3, under the assumption that localized disorder flows to such a defect in the IR. The meaning of these logarithmic features will be reviewed below.

Recall that in a d -dimensional Euclidean spacetime, the conformal symmetry is described by the group $\text{SO}(1, d + 1)$. The introduction of a p -dimensional conformal defect breaks this symmetry as

$$\text{SO}(1, d + 1) \longrightarrow \text{SO}(1, p + 1) \times \text{SO}(d - p), \quad (4.3)$$

where $\text{SO}(1, p + 1)$ corresponds to the residual conformal symmetry along the defect worldvolume, while $\text{SO}(d - p)$ represents the rotational symmetry in the transverse directions [17, 19, 20, 23, 105]. Consequently, there exist two types of local operators: bulk local operators $\mathcal{O}(x)$ in the ambient spacetime and defect local operators $\hat{\mathcal{O}}(\hat{x})$ confined to the subspacetime. In conventional unitary defect CFT, bulk operators are classified by primary operators under $\text{SO}(1, d + 1)$, while defect operators are classified by primaries under $\text{SO}(1, p + 1) \times \text{SO}(d - p)$. In particular, the dilatation

⁴²We emphasize that the Harris criterion (4.2) applies only to Gaussian disorder, which we focus on in this chapter. For other quenched disorders such as long-range correlated disorder, the Harris criterion is modified [104].

operator D acts *diagonally* on these primaries. For instance, when two primaries $\mathcal{O}_1$ and $\mathcal{O}_2$ have the same conformal dimension Δ , the dilatation operator acts as

$$\text{ordinary CFT : } D \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \end{pmatrix}. \quad (4.4)$$

This diagonal structure makes correlation functions exhibit standard power-law behavior.

However, conformal defects generated by ensemble averaging over localized disorder are expected to fall outside this standard framework. Indeed, quenched disorder is often observed to induce logarithmic behavior in correlation functions [93, 106–109]. This is one of the hallmarks of logarithmic CFTs (logCFTs), in which the dilatation operator D acts *non-diagonally* on operators (see e.g. [110–115] for reviews on logCFTs). For example, when two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ share the same conformal dimension Δ , the dilatation operator may act as

$$\text{logarithmic CFT : } D \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \end{pmatrix} = \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \end{pmatrix}, \quad (4.5)$$

up to a change of basis. Clearly, this matrix is a Jordan cell and cannot be diagonalized. As reviewed in the main text, this non-diagonal structure gives rise to logarithmic terms in correlators. We refer to the lowest conformal weight operators $(\mathcal{O}_1, \mathcal{O}_2)$ satisfying the above property as the *logarithmic primaries* of a *logarithmic multiplet* with rank two and conformal dimension Δ . For the above reason, in the present work we expect that the conformal defects arising from localized ensemble averaging exhibit the following structure of operator spectrum: bulk local operators behave as ordinary $\text{SO}(1, d+1)$ primaries, while defect local operators form logarithmic representations of $\text{SO}(1, p+1) \times \text{SO}(d-p)$. Henceforth we will refer to such defects as *logarithmic conformal defects*. One of our main goals is to establish model-independent systematics for such logarithmic conformal defects. In particular, we obtain general results for the form of correlation functions, the conformal block decomposition, and the expansion of bulk operators in terms of defect operators, namely the logarithmic defect-operator-expansion (logDOE), using only symmetry considerations and representation theory.

As a concrete realization of the outlined systematic analysis, we consider a d -dimensional free scalar field theory as the UV CFT, deformed by a localized disorder as below,⁴³

$$S = \frac{1}{2} \int_{\mathbb{R}^d} d^d x (\partial \sigma)^2 + \int_{\mathbb{R}^p} d^p \hat{x} \hat{h}(\hat{x}) \sigma(\hat{x}). \quad (4.6)$$

For specificity, we assume that the coupling $\hat{h}(\hat{x})$ is sampled from a Gaussian distribution with variance c which measures the strength of the disorder, which is marginal according to the Harris criterion (4.2).

It is instructive to contrast our setup with the localized magnetic (or pinning) defects studied in [25, 26], where the defect coupling is uniform rather than randomly fluctuating. In that case, for $d < 4$ and $p = 1$, the coupling grows indefinitely under the RG flow, leading to runaway

⁴³Note that the defect coupling $\hat{h}$ used here is different from the one used in (3.28). We believe that there is no confusion about this notation.

behavior without an IR-stable defect CFT description [26]. There, the instability can be remedied by introducing bulk interactions. By contrast, in our setup the instability is avoided through disorder averaging. In fact, as we will show in the main text, using the replica method, the disorder-averaged theory can exhibit an IR-stable fixed line parametrized by the disorder strength c even though the bulk theory remains free. This is also to be contrasted with the earlier studies of boundary disorder in the 2d Ising model [95–97] where the disorder is marginally irrelevant and does not produce new boundary CFT.

The correlation functions for the disordered system in (4.6) can be computed explicitly for general p , allowing a direct comparison with the model-independent systematics derived in Section 4.3. This comparison provides an explicit diagnostic for whether the disorder produces a nontrivial conformal defect or not, in this free theory. Interestingly, we find that infrared behavior matches that of a logarithmic conformal defect only when the defect codimension is two, i.e. $p = d - 2$. Moreover, combining the bulk equations of motion, we find that various defect operators organize into logarithmic multiplets. Notably, the displacement operator, which encodes the response to local deformations of the defect’s position, also forms a part of the logarithmic multiplet.

We further extend this analysis to the natural generalization of a single scalar field: an $O(N)$ free scalar theory, deformed by

$$S = \frac{1}{2} \int_{\mathbb{R}^d} d^d x (\partial \vec{\sigma})^2 + \int_{\mathbb{R}^p} d^p \hat{x} \, \hat{h}(\hat{x}) \vec{n} \cdot \vec{\sigma}(\hat{x}). \quad (4.7)$$

Here, $\vec{n}$ is the fixed vector, and the bulk $O(N)$ symmetry is broken to $O(N - 1)$ around the vector $\vec{n}$. This setup also perfectly matches the expected properties of a logarithmic conformal defect only when $p = d - 2$. In particular, the tilt operator, which arises as a universal operator due to the breaking of the bulk $O(N)$ symmetry, is found to form a logarithmic multiplet as well.

The rest of the chapter is structured as follows. In Section 4.2, we introduce the notion of subdimensional quenched disorder and specify the conditions under which it gives rise to conformal defects. As a concrete example, we study random-field defects in free scalar field theories, which are conformal if and only if $p = d - 2$. We then analyze the associated defect RG flow, establishing the existence of a line of disordered codimension-two fixed points in the free theory and highlighting the emergence of logarithmic behavior. In Section 4.3, we develop the model-independent systematics of logarithmic defect CFTs: beginning with a brief review of logarithmic conformal representation theory, we derive the logarithmic defect operator-product-expansion and discuss general constraints from bulk two-point functions and bootstrap considerations. Section 4.4 returns to the free-theory setting, where we provide explicit realizations of logarithmic conformal defects. There we analyze the structure of logarithmic multiplets, extract detailed CFT data, and confirm consistency with the general framework developed earlier. Technical details are collected in the App. ??.

4.2 Subdimensional quenched disorder

In this section, we introduce the notion of subdimensional quenched disorder, and analyze the concrete models described in Overview. In Section 4.2.1, we set up our notation and formulate the subdimensional disorder in its most general form. In Section 4.2.2, we formulate the Harris-type criterion for conformality in the presence of subdimensional disorder. To illustrate this

criterion, we consider the disordered theory (4.6) with bulk dynamics governed by a free scalar field. We show that subdimensional disorder produces a conformal defect only when $p = d - 2$, as evidenced by the behavior of certain bulk correlators. In Section 4.2.3, we analyze the fixed-point structure using the replica method. In particular, we derive the beta function for the disorder strength c and find that it is exactly marginal, implying that the disordered theory defines a defect conformal manifold parametrized by c . Moreover, through explicit computations of disorder-averaged correlation functions, we uncover logarithmic behavior, which serves as a characteristic signature of disordered fixed points.

4.2.1 Defect from subdimensional quenched disorder

We now set up the general framework of subdimensional quenched disorder in d -dimensional Euclidean spacetime $\mathbb{R}^d$. For later convenience, we split the coordinate for $\mathbb{R}^d$ as

$$x^\mu = (\hat{x}^a, x_\perp^i), \quad a = 1, 2, \dots, p, \quad i = p + 1, \dots, d, \quad (4.8)$$

where $\hat{x}^a$ parameterize the p -dimensional subspace $\mathbb{R}^p$ on which the disorder is taken, while $x_\perp^i$ denote the remaining transverse directions. As mentioned in [Overview](#), we consider the disordered system defined by the following action,

$$S[X, g] \equiv S_{\text{CFT}} + \int_{\mathbb{R}^p} d^p \hat{x} g(\hat{x}) O(\hat{x}), \quad (4.9)$$

where S_{CFT} is the UV CFT action, $O(\hat{x})$ is some primary operator in the UV CFT and X represents the bulk fields collectively. Whenever the transverse coordinates are suppressed, they are taken to be at $x_\perp^i = 0$, which is the location of the disorder.

We assume that the coupling $g(\hat{x})$ is randomly fluctuating over $\mathbb{R}^p$, and the probability distribution functional $P[g]$ satisfies the following normalization condition:

$$\int \mathcal{D}g P[g] = 1. \quad (4.10)$$

For a fixed configuration of the subdimensional coupling g , one can define the correlation function as

$$\langle \dots \rangle_g \equiv \frac{1}{Z_g} \int \mathcal{D}X (\dots) e^{-S[X, g]}, \quad Z_g \equiv \int \mathcal{D}X e^{-S[X, g]}, \quad (4.11)$$

where $\dots$ means some inserted operators, and the subscript g is attached just to emphasize that g is fixed. In terms of this, we introduce the subdimensional disorder-averaged correlator by

$$\overline{\langle \dots \rangle} \equiv \int \mathcal{D}g P[g] \langle \dots \rangle_g. \quad (4.12)$$

In principle, one may consider various probability distributions. In this chapter, for simplicity, we exclusively consider the Gaussian distribution defined by

$$P[g] \propto \exp \left\{ \left[-\frac{1}{2c^2} \int_{\mathbb{R}^p} d^p \hat{x} (g(\hat{x}))^2 \right] \right\}, \quad (4.13)$$

where c is the width of the Gaussian function which characterizes the strength of the disorder. The omitted overall factor above can be fixed from the normalization condition (4.10). In this case, the one and two-point functions of subdimensional coupling g is written as

$$\overline{\langle g(\hat{x}) \rangle} = 0, \quad \overline{\langle g(\hat{x})g(\hat{y}) \rangle} = c^2 \delta^p(\hat{x} - \hat{y}). \quad (4.14)$$

One can deduce all other correlation functions from this two-point function via the Wick's theorem. For instance, the four-point function reads

$$\begin{aligned} & \overline{\langle g(\hat{x}_1)g(\hat{x}_2)g(\hat{x}_3)g(\hat{x}_4) \rangle} \\ &= c^4 [\delta^p(\hat{x}_1 - \hat{x}_2)\delta^p(\hat{x}_3 - \hat{x}_4) + \delta^p(\hat{x}_1 - \hat{x}_3)\delta^p(\hat{x}_2 - \hat{x}_4) + \delta^p(\hat{x}_1 - \hat{x}_4)\delta^p(\hat{x}_2 - \hat{x}_3)]. \end{aligned} \quad (4.15)$$

4.2.2 Conformality from disorder and free theory example

A natural way to assess the relevance of quenched disorder is to adapt the Harris criterion [103] to the defect setting (see also [93] for a review). The strength of disorder is characterized by a dimensionless variance parameter c , which requires the random coupling $g(\hat{x})$ to have dimension $p/2$, as dictated by (4.13). It then follows that the subdimensional disorder in (4.9) is relevant if $\Delta_O < p/2$, marginal if $\Delta_O = p/2$, and irrelevant if $\Delta_O > p/2$, as summarized in (4.2). We therefore expect disorder to generate nontrivial conformal defects only when $\Delta_O \leq p/2$.⁴⁴ For a unitary CFT, however, this Harris bound is in tension with the unitarity bound for scalar operators, which imposes $p \geq d - 2$. The borderline case is realized by the free scalar theory, which we now turn to.

As a concrete illustration of conformal defects arising from subdimensional disorder, let us consider the simplest bulk theory: a free massless scalar field in d dimensions. We introduce quenched disorder by coupling the scalar to a localized random magnetic field $\hat{h}(\hat{x})$ (corresponding to $g(\hat{x})$ in (4.9)) along a p -dimensional subspace:

$$S = \frac{1}{2} \int_{\mathbb{R}^d} d^d x (\partial \sigma)^2 + \int_{\mathbb{R}^p} d^p \hat{x} \hat{h}(\hat{x}) \sigma(\hat{x}), \quad (4.16)$$

where $\hat{h}(\hat{x})$ is drawn from a Gaussian distribution. This model can be regarded as a disordered generalization of a pinning defect, where the scalar couples to a random local magnetic field. We will show that scale invariance is preserved only when the defect has codimension two, i.e. $p = d - 2$. From the Harris criterion discussed above, this corresponds precisely to the marginal case, since the free scalar has scaling dimension $\Delta_\sigma = \frac{d-2}{2}$. This provides the first indication that the codimension-two case is distinguished and potentially admits a conformal fixed point.

To make this relation between scale invariance and the dimensional constraint $p = d - 2$ more concrete, let us examine the bulk two-point function of σ explicitly. For a given coupling $\hat{h}$, we obtain

$$\langle \sigma(x)\sigma(y) \rangle_{\hat{h}} = \frac{C_d}{|x - y|^{d-2}} + C_d^2 \int d^p \hat{z} d^p \hat{w} \frac{\hat{h}(\hat{z})\hat{h}(\hat{w})}{|x - \hat{z}|^{d-2}|y - \hat{w}|^{d-2}}, \quad (4.17)$$

⁴⁴See [93] for a discussion of “dangerously irrelevant” disorder, which can still affect infrared correlation functions through contact terms.

where

$$C_d \equiv \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}}. \quad (4.18)$$

Then, the averaging over the coupling h gives rise to

$$\overline{\langle \sigma(x)\sigma(y) \rangle} = \frac{C_d}{|x-y|^{d-2}} + c^2 C_d^2 \int d^p \hat{z} \frac{1}{|x-\hat{z}|^{d-2}|y-\hat{z}|^{d-2}}. \quad (4.19)$$

Suppose that the scale invariance does hold even with the disorder, the two-point function must be subject to the following Ward-Takahashi identity:

$$\overline{\langle \sigma(\lambda x)\sigma(\lambda y) \rangle} = \lambda^{-(d-2)} \overline{\langle \sigma(x)\sigma(y) \rangle}. \quad (4.20)$$

However the second term in the RHS of (4.19) shows the different scaling response in general

$$\overline{\langle \sigma(\lambda x)\sigma(\lambda y) \rangle}_{\hat{h}} = \lambda^{-(d-2)} \frac{C_d}{|x-y|^{d-2}} + \lambda^{p-2d+4} \hat{c}^2 C_d^2 \int d^p \hat{z} \frac{1}{|x-\hat{z}|^{d-2}|y-\hat{z}|^{d-2}}. \quad (4.21)$$

This implies that the scale invariance for subdimensional disorder in the free theory requires that the bulk and defect dimensions must be related as $p = d - 2$.

4.2.3 Half-line of logarithmic fixed points in free theory

In the previous section, we saw that conformality of subdimensional disorder in the free scalar theory can only occur at codimension two (i.e. $p = d - 2$). In this section, we turn to a fixed-point analysis for this case. Under this dimensional constraint, not only the localized magnetic field but also a localized mass term becomes marginal. It is therefore necessary to analyze the fixed points of the theory including both contributions. To this end, let us consider the following action:

$$S = S_{\text{bulk}}[\sigma] + S_{\text{D}}[\sigma, \hat{h}], \quad (4.22)$$

where S_{bulk} and S_{D} are defined by

$$S_{\text{bulk}} \equiv \frac{1}{2} \int_{\mathbb{R}^d} d^d x (\partial \sigma)^2, \quad S_{\text{D}} \equiv \int_{\mathbb{R}^p} d^p \hat{x} \hat{h}(\hat{x}) \sigma(\hat{x}) + \frac{\gamma}{2} \int_{\mathbb{R}^p} d^p \hat{x} (\sigma(\hat{x}))^2. \quad (4.23)$$

Here, γ is the defect coupling constant (to the mass term) and together with the disorder strength c , they are the deformation parameters of the theory. In what follows, we compute the corresponding beta functions, and discuss the stability of the fixed points. We proceed by employing the replica method as in [116–121], to reformulate the disorder-averaged theory in terms of a conventional field theory, which allows us to define beta functions in the usual sense and study the RG flow of c and γ . To be more concrete, let us consider the disorder-averaged correlation function of a local operator $\mathcal{O}(\sigma)$ made up of σ , and deform this as follows:

$$\begin{aligned} \overline{\langle \mathcal{O}(\sigma) \rangle} &= \overline{\langle \mathcal{O}(\sigma) \rangle \langle \mathbf{1} \rangle^{n-1}} \\ &= \int \mathcal{D} \hat{h} \frac{P[\hat{h}]}{Z_h^n} \int \mathcal{D} \vec{\sigma} \mathcal{O}(\sigma_1) e^{-\sum_{A=1}^n (S_{\text{bulk}}[\sigma_A] - S_{\text{D}}[\sigma_A, \hat{h}])}, \end{aligned} \quad (4.24)$$

where $\vec{\sigma} \equiv (\sigma_1, \sigma_2, \dots, \sigma_n)$. In the first line, we have simply inserted one, and in the second line, we have used the definition of the disorder-averaged correlator (4.12). Applying the replica trick for the quenched disorder amounts to taking the limit $n \rightarrow 0$, in which $Z_{\hat{h}}^n \rightarrow 1$. Since the distribution $P[\hat{h}]$ is Gaussian, we can straightforwardly integrate out the disorder field $\hat{h}$, and obtain the effective replica action S_{rep} describing interactions among different replicas:

$$S_{\text{rep}}[\vec{\sigma}] \equiv \sum_{A=1}^n S_{\text{bulk}}[\sigma_A] + \sum_{A,B} \int d^p \hat{x} m_{AB} \sigma_A(\hat{x}) \sigma_B(\hat{x}), \quad (4.25)$$

where m_{AB} is defined by

$$m_{AB} \equiv -\frac{c^2}{2} + \gamma \delta_{AB}. \quad (4.26)$$

With the replica action (4.25) at hand, the beta functions for defect couplings c^2 and γ can now be defined in the standard QFT sense as follows:

$$\beta_{c^2,n} \equiv \frac{dc^2}{d \log \mu}, \quad \beta_{\gamma,n} \equiv \frac{d\gamma}{d \log \mu}, \quad (4.27)$$

where the subscript n signals that they are defined in the n -times replicated theory. These beta functions are determined by solving the Callan–Symanzik equation for the bulk one-point function $\langle \sigma_A \sigma_B(x) \rangle_{\text{rep}}$. The explicit evaluation of $\beta_{c^2,n}$ and $\beta_{\gamma,n}$ is presented in Appendix 4.4.4, and the results are summarized here,

$$\beta_{c^2,n} = C_d S_{p-1} (2\gamma c^2 - n c^4) + \dots, \quad \beta_{\gamma,n} = C_d S_{p-1} \gamma^2 + \dots, \quad (4.28)$$

where $S_{p-1} \equiv 2\pi^{p/2}/\Gamma(\frac{p}{2})$ is the area of a unit p -sphere. In order to extract the beta functions β_{c^2} and β_{γ} for the original disordered system, we take the replica limit $n \rightarrow 0$,

$$\beta_{c^2,0} = 2C_d S_{p-1} \gamma c^2 + \dots, \quad \beta_{\gamma,0} = C_d S_{p-1} \gamma^2 + \dots. \quad (4.29)$$

From these expressions, one immediately finds that there exists a half-line (due to the identification $c \rightarrow -c$) of fixed points given by

$$\gamma^* = 0, \quad c : \text{arbitrary}. \quad (4.30)$$

This half-line of fixed points corresponds precisely to the exactly marginal disorder strength c at $p = d - 2$.⁴⁵

4.2.4 Free energy of quenched disorder defect and RG monotone

In the last subsection, we found the half-line of conformal fixed points (4.30) by replica method. Since this conformal manifold includes the trivial point without any defect, one may expect that the averaged defect free energy, defined below, should vanish along the entire manifold, as what must

⁴⁵Without disorder, namely at $c = 0$, it is known that the σ^2 deformation in codimension-two is marginally irrelevant (for $\gamma > 0$) [50].

happen when monotonicity theorems for defect RG are obeyed [122–131]. In fact, this is expected from the replica approach (see below) to quenched disorder as we take the number of replica copies $n \rightarrow 0$ in the end.

Indeed, for ordinary quenched disorder in 2d CFTs, this leads to the familiar vanishing central charge $c = 0$ upon disordering (see e.g. [87, 107]). To have a more interesting RG monotone in the disordered context, it was suggested in [132] (see also [133]) to use the effective central charge $c_{\text{eff}} \equiv \left. \frac{dc(n)}{dn} \right|_{n=0}$ defined by a derivative of the central charge for the n -th replicated theory. Here we will also propose a direct analog for subdimensional disorder in general dimensions.

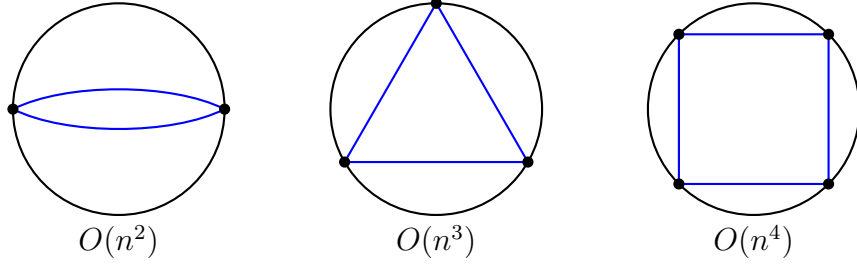


Figure 4.2: Sample diagrams contributing to the averaged defect free energy. There are no contributions at order $O(n)$.

We first confirm the vanishing of the averaged defect free energy F_D defined below,

$$F_D \equiv -\log \overline{\frac{Z_{\hat{h}}}{Z_{\hat{h}=0}}}, \quad (4.31)$$

where $Z_{\hat{h}}$ is the partition function in the presence of the spherical defect on S^p . Since the logarithm obstructs a direct evaluation of the above quantity, we employ the replica method as before. Namely, we introduce n copies of the system and write

$$F_D = -\lim_{n \rightarrow 0} F_D(n), \quad F_D(n) \equiv \frac{1}{n} \left[\overline{\left(\frac{Z_{\hat{h}}}{Z_{\hat{h}=0}} \right)^n} - 1 \right]. \quad (4.32)$$

The averaged quantity defined above can be evaluated by first integrating over the disorder field $\hat{h}$, which yields

$$\overline{\left(\frac{Z_{\hat{h}}}{Z_{\hat{h}=0}} \right)^n} = \left\langle e^{\frac{c^2}{2} \sum_{A,B=1}^n \int d^p \hat{x} \sigma_A(\hat{x}) \sigma_B(\hat{x})} \right\rangle_{\text{rep}, c=0} \quad (4.33)$$

where $\langle \dots \rangle_{\text{rep}, c=0}$ denotes the partition function of the disorder-less replicated theory,

$$\langle \dots \rangle_{\text{rep}, c=0} \equiv \frac{\int \mathcal{D}\vec{\sigma}(\dots) e^{-\sum_{A=1}^n S_{\text{bulk}}[\sigma_A]}}{\int \mathcal{D}\vec{\sigma} e^{-\sum_{A=1}^n S_{\text{bulk}}[\sigma_A]}}, \quad (4.34)$$

with S_{bulk} given in (4.23). This expression admits a straightforward perturbative evaluation. Note that by (4.32), only the terms proportional to n from (4.33) contribute nontrivially to the defect free

energy in the replica limit $n \rightarrow 0$. However, the perturbative expansion admits no linear term in n , and the contributions starts from order $O(n^2)$ (see Figure 5.13). Therefore, the universal part of the defect free energy always vanishes for our disordered free scalar, as is expected by the exact marginality of the disorder strength.

On the other hand, to have a more interesting RG monotone, we define the effective free energy for the disordered defect by the derivative in n at $n = 0$,

$$F_D^{\text{eff}} \equiv \left. \frac{dF_D(n)}{dn} \right|_{n=0}, \quad (4.35)$$

generalizing the work of [132, 133] for bulk disorder. It would be interesting to investigate whether (4.35) is indeed monotonic under disordered defect RG flows.

4.2.5 Symmetry restoration by subdimensional disorder

It is often observed that a symmetry that is explicitly broken in the absence of disorder becomes restored once disorder is introduced.⁴⁶ In this short section, we confirm this phenomenon in our pinning defect model (4.16). Before introducing the subdimensional disorder, the bulk spin-flip $\mathbb{Z}_2$ symmetry is explicitly broken on the defect. Indeed, if we compute the bulk one-point function of σ in a fixed background configuration of the defect coupling $\hat{h}$, it turns out to be nonzero:

$$\langle \sigma(x) \rangle_{\hat{h}} = -C_d \int d^p \hat{y} \frac{\hat{h}(\hat{y})}{|x - \hat{y}|^p}. \quad (4.36)$$

However, once we average over $\hat{h}$ according to the Gaussian distribution, this one-point function vanishes,

$$\overline{\langle \sigma(x) \rangle} = 0. \quad (4.37)$$

This clearly indicates that the $\mathbb{Z}_2$ symmetry is restored by the inclusion of the subdimensional disorder. Physically, this is reasonable because the Gaussian distribution is symmetric under the transformation $\hat{h} \rightarrow -\hat{h}$.

4.2.6 First pass: logarithmic behavior of correlation functions

We now turn to the study of correlation functions on the fixed line described in (4.30). Since the fixed line occurs at $\gamma = 0$, it is sufficient to analyze the original action (4.16). In this subsection, we focus exclusively on defect two-point functions, while correlation functions involving bulk local operators will be discussed in the next section. Our main goal here is to demonstrate explicitly that the disorder-averaged CFT correlators exhibit logarithmic dependence in contrast to the power-law behavior in usual CFTs.

⁴⁶See [134] for discussions on this phenomenon from the view point of topological defects.

Two-point function: $\overline{\langle \sigma(\hat{x})\sigma(0) \rangle}$

We firstly investigate the disorder-averaged two-point function $\overline{\langle \sigma(\hat{x})\sigma(0) \rangle}$ on the defect, which can be deduced by setting $x_\perp \rightarrow 0$ and $y \rightarrow 0$ in (4.19):

$$\overline{\langle \sigma(\hat{x})\sigma(0) \rangle} = \frac{C_d}{|\hat{x}|^p} + c^2 C_d^2 \int d^p \hat{z} \frac{1}{|\hat{x} - \hat{z}|^p |\hat{z}|^p}. \quad (4.38)$$

The remaining integral clearly develops logarithmic singularities near the operator insertions at $\hat{z} = 0$ and $\hat{z} = \hat{x}$. Indeed, upon evaluating the integral, one finds that it behaves as

$$\overline{\langle \sigma(\hat{x})\sigma(0) \rangle} = \frac{C_d}{|\hat{x}|^p} + \frac{C_d c^2 \log \mu^2 |\hat{x}|^2}{2\pi |\hat{x}|^p}, \quad (4.39)$$

where μ is an energy scale from regulating the integral. This logarithmic behavior is a distinctive feature of disorder averaging, and it clearly indicates that $\sigma(\hat{x})$ does not behave as an ordinary CFT primary operator.

Two-point function: $\overline{\langle \tilde{D}_i(\hat{x})\tilde{D}_j(0) \rangle}$

As a next step, we turn to a slightly more non-trivial example by considering a composite defect local operator $\tilde{D}_i(\hat{x}) \equiv \sigma \partial_i \sigma(\hat{x})$ that carries transverse spin one. Its two-point function can be computed from

$$\overline{\langle \tilde{D}_i(\hat{x})\tilde{D}_j(\hat{y}) \rangle} \equiv \frac{1}{4} \frac{\partial}{\partial x_\perp^i} \frac{\partial}{\partial y_\perp^j} \overline{\langle \sigma^2(x)\sigma^2(y) \rangle} \Big|_{x_\perp=y_\perp=0}. \quad (4.40)$$

After a straightforward calculation, one finds that the above two-point function can be expressed in terms of an integral of the form:

$$\overline{\langle \tilde{D}_i(\hat{x})\tilde{D}_j(0) \rangle} = \frac{p C_d^2 \delta_{ij}}{|\hat{x}|^{2p+2}} + \frac{p c^2 C_d^3 \delta_{ij}}{|\hat{x}|^{p+2}} \int d^p \hat{z} \frac{1}{|\hat{x} - \hat{z}|^p |\hat{z}|^p}. \quad (4.41)$$

This integral is the same as the one that we already encountered in the scalar two-point function (4.38). In particular, it exhibits a logarithmic divergence, confirming once again that the correlation functions do not follow a simple power-law behavior but instead display logarithmic corrections:

$$\overline{\langle \tilde{D}_i(\hat{x})\tilde{D}_j(0) \rangle} = p C_d^2 \delta_{ij} \left(\frac{1}{|\hat{x}|^{2p+2}} + \frac{c^2 \log \mu^2 |\hat{x}|^2}{2\pi |\hat{x}|^{2p+2}} \right). \quad (4.42)$$

In Section 4.4, we will see that these two-point functions are consistent with the structure of a logarithmic conformal defect, where defect local operators form nontrivial logarithmic multiplets. In particular, $\tilde{D}_i$ will be an operator in the logarithmic displacement multiplet.

4.3 Systematics of logarithmic defect CFT

From the defect two-point functions discussed in the previous sections and general discussion in [Overview](#), we expect defect local operators generated by subdimensional disorder to exhibit

logarithmic behavior in their correlation functions. In particular, the appearance of logarithmic corrections indicates that these operators are naturally organized into logarithmic multiplets in logarithmic CFT (logCFT), rather than ordinary primaries and their descendants. Motivated by this perspective, we will refer to conformal defects whose local operator spectrum is built out of logarithmic multiplets as *logarithmic defects*. The main goal of this section is to extract the model-independent features of such logarithmic conformal defects. In Section 4.3.1, we review the representation theory of logCFT by focusing on primaries and the structures of correlation functions. In Section 4.3.3, we discuss how ordinary bulk CFT primary operators can be expanded in terms of logarithmic defect operators. Particularly, we determine the possible form of the defect-operator-expansion just from the representation theory. In Section 4.3.4, we also discuss the bootstrap equation that constrains the CFT data associated with logarithmic defects.

4.3.1 Review of logarithmic conformal representation theory

In this subsection, we present a brief review on the representation theory of the logCFTs, following [115]. Readers who wish to explore this subject in more detail are referred to e.g. [109, 115]. The conformal algebra $\mathfrak{so}(1, p+1)$ is generated by the dilatation D , translation P^a , special conformal transformation K^a , and rotation $M^{ab} = -M^{ba}$ in p dimensions. In logCFTs, primary operators with conformal dimension $\hat{\Delta}$ form a logarithmic multiplet of rank $N_{\hat{\Delta}} \geq 1$, and such states are denoted as $|\hat{\mathcal{O}}_I\rangle$ ($I = 1, 2, \dots, N_{\hat{\Delta}}$).⁴⁷ They satisfy the following highest-weight condition,

$$K^a |\hat{\mathcal{O}}_I\rangle = 0. \quad (4.43)$$

The descendant states are obtained by acting the translation operators P^a on the primary states, just like in the standard CFT. The action of the dilatation operator D on this logarithmic multiplet takes the following form,

$$D |\hat{\mathcal{O}}_I\rangle = -i \hat{\Delta}_I^J |\hat{\mathcal{O}}_J\rangle, \quad (4.44)$$

where $\hat{\Delta}$ is a matrix of size $N_{\hat{\Delta}}$. The crucial point is that this matrix cannot be diagonalized by any choice of basis. Instead, by fixing a basis, it can be brought into a Jordan block form

$$\hat{\Delta} = \begin{pmatrix} \hat{\Delta} & 1 & 0 & \dots & 0 \\ 0 & \hat{\Delta} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \hat{\Delta} & 1 \\ 0 & 0 & 0 & 0 & \hat{\Delta} \end{pmatrix}. \quad (4.45)$$

It should be emphasized that, as will become evident later, the logarithmic behavior in logCFTs originates from the off-diagonal structure of the dilatation operator. By using the state-operator

⁴⁷As our primary aim in this subsection is to arrange *defect* primaries into logarithmic multiplets, we adopt the convention of attaching a hat to most logarithmic variables.

correspondence and translations, we can define the primary operator $\widehat{\mathcal{O}}_I(\hat{x})$, which transforms as

$$\begin{aligned}
i[P_a, \widehat{\mathcal{O}}_I(\hat{x})] &= \frac{\partial}{\partial \hat{x}^a} \widehat{\mathcal{O}}_I(\hat{x}), \\
i[M_{ab}, \widehat{\mathcal{O}}_I(\hat{x})] &= \left(\Sigma_{ab} + \hat{x}_a \frac{\partial}{\partial \hat{x}^b} - \hat{x}_b \frac{\partial}{\partial \hat{x}^a} \right) \widehat{\mathcal{O}}_I(\hat{x}), \\
i[D, \widehat{\mathcal{O}}_I(\hat{x})] &= \left(\widehat{\Delta}_I^J + \delta_I^J \hat{x} \cdot \frac{\partial}{\partial \hat{x}} \right) \widehat{\mathcal{O}}_J(\hat{x}), \\
i[K_a, \widehat{\mathcal{O}}_I(\hat{x})] &= -2\hat{x}_a \left(\widehat{\Delta}_I^J + \delta_I^J \hat{x} \cdot \frac{\partial}{\partial \hat{x}} \right) \widehat{\mathcal{O}}_J(\hat{x}) + |\hat{x}|^2 \frac{\partial}{\partial \hat{x}^a} \widehat{\mathcal{O}}_I(\hat{x}) + 2i\hat{x}^b \Sigma_{ba} \widehat{\mathcal{O}}_I(\hat{x}),
\end{aligned} \tag{4.46}$$

where Σ_{ab} is the matrix representation of the p -dimensional rotation group associated to the primary operator $\widehat{\mathcal{O}}_I(\hat{x})$. Notably, the action of D and K_a deviates from that in ordinary CFT, and the non-diagonal structure of Δ implies mixing among the operators within a logarithmic multiplet. This mixing structure underlies the distinctive feature of logarithmic CFTs and will play a central role in the construction of correlation functions in the following sections.

Finally, we present the finite form of the conformal transformation. From the above infinitesimal transformation rules (4.46), one can deduce the finite transformation of logarithmic primaries. For a rank $N_{\widehat{\Delta}}$ logarithmic scalar primary operator $\widehat{\mathcal{O}}_I(\hat{x})$, the finite transformation under $\hat{x} \rightarrow \hat{x}'$ reads

$$\widehat{\mathcal{O}}'_I(\hat{x}') = \Omega(\hat{x})^{-\widehat{\Delta}/p} \Lambda_I^J \widehat{\mathcal{O}}_J(\hat{x}), \tag{4.47}$$

where $\Omega(\hat{x})$ is the Jacobian of the coordinate transformation,

$$\Omega(\hat{x}) \equiv \left| \frac{\partial(\hat{x}')}{\partial(\hat{x})} \right|, \tag{4.48}$$

and Λ_I^J is an upper triangular matrix of the form

$$\Lambda_I^J = \frac{(-\ln \Omega)^{J-I}}{(J-I)!}, \quad J \geq I. \tag{4.49}$$

For instance, in the case of $N_{\widehat{\Delta}} = 2$, the above transformation rule (4.47) reads,

$$\begin{aligned}
\widehat{\mathcal{O}}'_1(\hat{x}') &= \Omega(\hat{x})^{-\widehat{\Delta}/p} \left(\widehat{\mathcal{O}}_1(\hat{x}) - \ln \Omega \widehat{\mathcal{O}}_2(\hat{x}) \right), \\
\widehat{\mathcal{O}}'_2(\hat{x}') &= \Omega(\hat{x})^{-\widehat{\Delta}/p} \widehat{\mathcal{O}}_2(\hat{x}).
\end{aligned} \tag{4.50}$$

4.3.2 Selected correlation functions

In this subsection, we investigate how certain correlation functions are constrained by the logarithmic conformal representation theory developed in the previous section. As discussed in [Overview](#), we consider a setup in which ordinary CFT primaries $\mathcal{O}_{\Delta}(x)$ reside in the bulk, while logarithmic primary operators $\widehat{\mathcal{O}}_{\widehat{\Delta}}^I(\hat{x})$ are localized on a logarithmic defect. In general, correlation functions in such a defect CFT must satisfy the conformal Ward-Takahashi (WT) identities,⁴⁸

$$\langle [G, \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \cdots \widehat{\mathcal{O}}_{\widehat{\Delta}_1}^{I_1}(\hat{x}_1) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}^{I_2}(\hat{x}_2) \cdots] \rangle = 0, \tag{4.51}$$

⁴⁸We assume the identity operator on the defect is in an ordinary representation (invariant under the residual conformal symmetry).

where G is a generator of the residual conformal symmetry $\text{SO}(1, p+1) \times \text{SO}(d-p)$. To illustrate these constraints explicitly, we consider two representative cases in order. First, we study two-point functions of logarithmic primaries living on the defect, showing how their structure is fixed by the conformal Ward-Takahashi identities (4.51). Second, we examine bulk-defect two-point functions and demonstrate how their form is similarly determined by these identities. This is without loss of generality since more general correlators are all determined by these basic correlators by operator-product-expansion (OPE). Through these examples, we also highlight the characteristic features of logarithmic correlations and the interplay between bulk CFT primaries and defect logarithmic primaries.

Defect two-point Function

To begin with, we determine the defect scalar two-point function⁴⁹

$$\langle \widehat{\mathcal{O}}_{\widehat{\Delta}_1}^{I_1}(\hat{x}) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}^{I_2}(0) \rangle, \quad I_i = 1, 2, \dots, N_i, \quad (4.52)$$

where $N_i \equiv N_{\widehat{\Delta}_i}$ is the rank of a logarithmic multiplet of conformal dimension $\widehat{\Delta}_i$. Firstly, the translation and rotational symmetries imply that the defect two-point function depends only on $s \equiv |\hat{x}|$:

$$\langle \widehat{\mathcal{O}}_{\widehat{\Delta}_1}^{I_1}(\hat{x}) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}^{I_2}(0) \rangle = \frac{B_{I_1 I_2}(s)}{s^{\widehat{\Delta}_1 + \widehat{\Delta}_2}}. \quad (4.53)$$

Here, we pull out the factor $s^{\widehat{\Delta}_1 + \widehat{\Delta}_2}$ for convenience, and $B_{I_1 I_2}$ is an $N_1 \times N_2$ matrix satisfying

$$B_{I_1 I_2} = 0, \quad I_1 > N_1 \text{ or } I_2 > N_2. \quad (4.54)$$

Moreover, the conformal WT identities associated to the scale and special conformal transformations produce the following constraints,

$$s \frac{d}{ds} B_{I_1 I_2}(s) = -B_{(I_1+1)I_2}(s) - B_{I_1(I_2+1)}(s), \quad (4.55)$$

$$\left(s \frac{d}{ds} + \widehat{\Delta}_1 - \widehat{\Delta}_2 \right) B_{I_1 I_2}(s) = -2B_{(I_1+1)I_2}(s), \quad (4.56)$$

where we have fixed the basis as in (4.45). From these constraints, one can readily obtain

$$(-\widehat{\Delta}_1 + \widehat{\Delta}_2) B_{I_1 I_2}(s) = B_{(I_1+1)I_2}(s) - B_{I_1(I_2+1)}(s). \quad (4.57)$$

This implies that, in order for $B_{I_1 I_2}(s)$ to be non-trivial, the conformal dimensions for the two operators must be the same $\widehat{\Delta} \equiv \widehat{\Delta}_1 = \widehat{\Delta}_2$, thereby the ranks are also the same $N \equiv N_1 = N_2$. We thus obtain the result, familiar in the case of ordinary CFT, that two-point functions are only nontrivial for operators in the same conformal multiplet, which holds also in the logarithmic setting. The constraint (4.57) further implies that $B_{I_1 I_2}(s)$ only depends on $I_1 + I_2$:

$$B_{I_1 I_2}(s) \equiv \beta_{I_1 + I_2 - 1}(s). \quad (4.58)$$

⁴⁹The determination of the defect two-point function follows essentially the same logic as in the case of logarithmic CFT without a defect, see [115, section 2.2]. Therefore, readers who are already familiar with the analysis in ordinary homogeneous logCFT may safely skip to Section 4.3.2.

Note that the condition (4.54) demands that $\beta_n = 0$ for $n > N$. In terms of $\beta_n(s)$, the conformal WT identity (4.56) takes the simpler form,

$$s \frac{d}{ds} \beta_n(s) = -2\beta_{n+1}(s). \quad (4.59)$$

By using $\beta_{N+1} = 0$, we can iteratively solve this differential equation as

$$\beta_n(s) = \sum_{m=0}^{N-n} \frac{k_{n+m}}{m!} (-\log s^2)^m. \quad (4.60)$$

In summary, the defect scalar two-point function in logCFT can be fixed as

$$\langle \widehat{\mathcal{O}}_{\widehat{\Delta}_1}^{I_1}(\hat{x}) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}^{I_2}(0) \rangle = \frac{\delta_{N_1, N_2} \delta_{\widehat{\Delta}_1, \widehat{\Delta}_2}}{|\hat{x}|^{2\widehat{\Delta}_1}} \times \begin{cases} \beta_{I_1+I_2-1}(|\hat{x}|), & 2 \leq I_1 + I_2 \leq N_1 + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.61)$$

We now extend the above discussion to defect two-point functions of logarithmic primaries $\widehat{\mathcal{O}}_{\widehat{\Delta}_1, i_1 i_2, \dots, i_s}^{I_1}$ with transverse spin s associated to the rotation group $\text{SO}(d-p)$. For simplicity, we focus on operators transforming as symmetric traceless tensors under $\text{SO}(d-p)$. This case is sufficient for the examples treated in the present chapter. For defect operators, the rotational group $\text{SO}(d-p)$ behaves as a global symmetry, and the only invariant tensor is

$$\mathcal{I}_{i_1 i_2, \dots, i_s}^{j_1 j_2, \dots, j_s} \equiv \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \dots \delta_{i_s}^{j_s}, \quad (4.62)$$

where the parenthesis is understood to impose the symmetric and traceless combination of indices. In terms of this invariant tensor, the spinning defect two-point function (4.132) can be determined as

$$\langle \widehat{\mathcal{O}}_{\widehat{\Delta}_1, i_1 i_2, \dots, i_s}^{I_1}(\hat{x}) \widehat{\mathcal{O}}_{\widehat{\Delta}_2}^{I_2, j_1 j_2, \dots, j_s}(0) \rangle = \frac{\Delta_{N_1, N_2} \delta_{\widehat{\Delta}_1, \widehat{\Delta}_2}}{|\hat{x}|^{2\widehat{\Delta}_1}} \mathcal{I}_{i_1 i_2, \dots, i_s}^{j_1 j_2, \dots, j_s} \times \begin{cases} \beta_{I_1+I_2-1}(|\hat{x}|), & 2 \leq I_1 + I_2 \leq N_1 + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.63)$$

Bulk-defect two-point function

We next determine the two-point function of a bulk scalar primary $\mathcal{O}_\Delta$ and a defect scalar logarithmic primary $\widehat{\mathcal{O}}_{\widehat{\Delta}}^I$. Here, we set the position of defect local operator to the origin by using the translation symmetry along the defect. Firstly, recalling the form of the bulk-defect two-point function in normal defect CFT [19], it is convenient to write,

$$\langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}}^I(0) \rangle = \frac{C_{\Delta, \widehat{\Delta}}^I(x)}{|x_\perp|^{\Delta - \widehat{\Delta}} |x|^{2\widehat{\Delta}}}. \quad (4.64)$$

Our remaining task is to fix the function $C_{\Delta, \widehat{\Delta}}^I(x)$ from the conformal WT identities (4.51). Scale invariance forces this function to be subject to the following differential equation,

$$x \cdot \frac{\partial}{\partial x} C_{\Delta, \widehat{\Delta}}^I(x) = -C_{\Delta, \widehat{\Delta}}^{I+1}(x). \quad (4.65)$$

On the other hand, special conformal transformation invariance requires,

$$-2\hat{x}^a \left(\Delta + x \cdot \frac{\partial}{\partial x} \right) F^a(x) + x^2 \frac{\partial}{\partial \hat{x}^a} F^a(x) = 0, \quad (4.66)$$

which is equivalent to

$$-2\hat{x}^a x \cdot \frac{\partial}{\partial x} C_{\Delta, \hat{\Delta}}^I(x) + |x|^2 \frac{\partial}{\partial \hat{x}^a} C_{\Delta, \hat{\Delta}}^I(x) = 0. \quad (4.67)$$

The residual Poincaré invariance constrains the function $C_{\Delta, \hat{\Delta}}^I(x)$ to depend only on the two variables⁵⁰

$$\tau \equiv \log \frac{|x_\perp|}{|x|^2}, \quad \tilde{\tau} \equiv \log |x_\perp|^2. \quad (4.68)$$

Although at first sight the correlator seems to depend on both τ and $\tilde{\tau}$, it does depend only on τ . Indeed, by using the following Leibniz rule,

$$\frac{\partial}{\partial x^\mu} = \left(+ \frac{x_\perp^i \delta_\mu^i}{|x_\perp|^2} - \frac{2x_\mu}{|x|^2} \right) \frac{\partial}{\partial \tau} + 2 \frac{x_\perp^i \delta_\mu^i}{|x_\perp|^2} \frac{\partial}{\partial \tilde{\tau}}, \quad (4.69)$$

the differential equation (4.67) reduces to

$$\frac{\partial}{\partial \tilde{\tau}} C_{\Delta, \hat{\Delta}}^I(\tau, \tilde{\tau}) = 0, \quad (4.70)$$

which clearly implies that the function $C_{\Delta, \hat{\Delta}}^I$ depends only on τ and (4.65) becomes,

$$\frac{\partial}{\partial \tau} C_{\Delta, \hat{\Delta}}^I(\tau) = C_{\Delta, \hat{\Delta}}^{I+1}(\tau). \quad (4.71)$$

This can be readily solved by the iterative method as before:

$$C_{\Delta, \hat{\Delta}}^I(\tau) = \sum_{k=0}^{N_{\hat{\Delta}}-I} \frac{\lambda_{I+k} \tau^k}{k!}, \quad (4.72)$$

where the coefficients λ_{I+k} remain undetermined by residual conformal symmetry alone. We comment that this result is consistent with the method of images [21], where the bulk-defect two-point function is *kinematically* equivalent to the three-point function in the absence of a defect. More precisely,

$$\langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}}^I(0) \rangle_{\text{DCFT}} \cong \langle \mathcal{O}_{\Delta/2}(x) \mathcal{O}_{\Delta/2}(\bar{x}) \mathcal{O}_{\hat{\Delta}}^I(0) \rangle_{\text{CFT}}, \quad (4.73)$$

where $\bar{x} \equiv (\hat{x}^a, -x_\perp^i)$. The correlation function on the right hand side has been determined in [115, equation (2.34)], and completely matches with our result (4.64) and (4.72) including the τ -dependence.

⁵⁰Remark that $\log |\hat{x}|$ can be written in terms of τ and $\tilde{\tau}$.

We can also extend the above analysis to the case where the defect operator carries spin. For simplicity, we keep the bulk primary to be a scalar. In general, a defect local primary can carry either parallel and transverse spins associated to the rotational groups $SO(p)$ and $SO(d-p)$, respectively. However, one can show that the bulk–defect two-point function involving a bulk scalar and a defect operator with parallel spin vanishes identically. Therefore, it suffices to consider defect primaries with transverse spin. Let s be the transverse spin, the result takes the following form

$$\langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}, i_1 i_2 \dots i_s}^I(0) \rangle = \frac{x_\perp^{(i_1} x_\perp^{i_2} \dots x_\perp^{i_s)}}{|x_\perp|^{\Delta - \hat{\Delta} + s} |x|^{2\hat{\Delta}}} C_{\Delta, \hat{\Delta}}^I(x), \quad (4.74)$$

where $C_{\Delta, \hat{\Delta}}^I$ has the same structure in (4.72), and the parenthesis is to impose the symmetric and traceless combination of indices.

4.3.3 Logarithmic defect operator expansion

In homogeneous CFTs, it is well known that when two local operators approach each other, their product can be systematically expanded into a sum of local operators through the operator-product-expansion (OPE). In the presence of a defect, this framework is enriched by the so-called defect-operator-expansion (DOE), in which a bulk local operator can be expanded in terms of local operators residing on the defect. The purpose of this subsection is to explore the logarithmic defect-operator-expansion (logDOE), which generalizes the standard DOE to the case where the defect supports logCFT local operators. The significance of this expansion lies in the fact that it bridges the representations of the bulk CFT with those of the defect, thereby providing a powerful tool to analyze the interplay between bulk and defect degrees of freedom. This feature will play an important role in later sections.

For simplicity, in this work we focus on the logDOE of a bulk scalar primary $\mathcal{O}_\Delta$. From the analysis in the Section 4.3.2, it is clear that both scalar defect operators and spinning defect operators with transverse spin can contribute to the expansion. Accordingly, we have the following general ansatz for the logDOE:

$$\mathcal{O}_\Delta(x) = \sum_{\hat{\Delta}, s} \sum_{I=1}^{N_{\hat{\Delta}, s}} \frac{x_\perp^{i_1} x_\perp^{i_2} \dots x_\perp^{i_s}}{|x_\perp|^{\Delta - \hat{\Delta} + s}} B_{\Delta, \hat{\Delta}}^I(|x_\perp|, \hat{\partial}) \hat{\mathcal{O}}_{\hat{\Delta}, i_1 i_2 \dots i_s}^I(\hat{x}), \quad (4.75)$$

where the sum runs over defect logarithmic primaries (both scalar and spinning), and $N_{\hat{\Delta}, s}$ represents the rank of a defect logarithmic primary with dimension $\hat{\Delta}$ and transverse spin s . Also, the function $B_{\Delta, \hat{\Delta}}^I(|x_\perp|, \hat{\partial})$ encodes the dependence on the transverse distance as well as derivatives along the defect. For convenience, in the above expression we have explicitly extracted the overall power of $|x_\perp|$ together with the tensor structure associated with the transverse spin s .⁵¹

Our next task is to determine the coefficient functions $B_{\Delta, \hat{\Delta}}^I(|x_\perp|, \hat{\partial})$ in (4.75). Since we have already factored out the tensor structures associated with the transverse spin, it suffices to focus only

⁵¹This separation makes the covariance under the transverse rotation group $SO(d-p)$ manifest.

on the scalar contributions:

$$\mathcal{O}_\Delta(x) \stackrel{\text{scalar}}{=} \sum_{\hat{\Delta}} \sum_{I=1}^{N_{\hat{\Delta}}} \frac{1}{|x_\perp|^{\Delta-\hat{\Delta}}} B_{\Delta,\hat{\Delta}}^I(|x_\perp|, \hat{\partial}) \hat{\mathcal{O}}_{\hat{\Delta}}^I(\hat{x}). \quad (4.76)$$

The essential requirement is that both sides of (4.76) must transform covariantly under the dilatation $x \rightarrow x' = \lambda x$,

$$\mathcal{O}'_\Delta(x') \stackrel{\text{scalar}}{=} \sum_{\hat{\Delta}} \sum_{I=1}^{N_{\hat{\Delta}}} \frac{1}{|x'_\perp|^{\Delta-\hat{\Delta}}} B_{\Delta,\hat{\Delta}}^I(|x'_\perp|, \hat{\partial}') \hat{\mathcal{O}}_{\hat{\Delta}}^I(\hat{x}'). \quad (4.77)$$

By recalling the transformation law (4.47), the above relation is reduced to

$$\mathcal{O}_\Delta(x) \stackrel{\text{scalar}}{=} \sum_{\hat{\Delta}} \sum_{I,J=1}^{N_{\hat{\Delta}}} \frac{1}{|x_\perp|^{\Delta-\hat{\Delta}}} B_{\Delta,\hat{\Delta}}^I(\lambda|x_\perp|, \lambda^{-1}\hat{\partial}) \Lambda_J^I \hat{\mathcal{O}}_{\hat{\Delta}}^J(\hat{x}), \quad (4.78)$$

where the matrix element Λ_J^I is given in (4.49). By comparing this with (4.76), we have

$$\sum_{J=1}^{N_{\hat{\Delta}}} B_{\Delta,\hat{\Delta}}^J(\lambda|x_\perp|, \lambda^{-1}\hat{\partial}) \Lambda_J^I = B_{\Delta,\hat{\Delta}}^I(|x_\perp|, \hat{\partial}). \quad (4.79)$$

The most general solution is given by

$$B_{\Delta,\hat{\Delta}}^I(|x_\perp|, \hat{\partial}) = \sum_{n=0}^{I-1} \frac{(\log|x_\perp|)^n}{n!} g_{I-n}(|x_\perp|^2 \square_{\hat{x}}), \quad (4.80)$$

where $g_{I-n}(|x_\perp|^2 \square_{\hat{x}})$ is non-singular under the limit $x_\perp \rightarrow 0$, and expanded as

$$g_{I-n}(|x_\perp|^2 \square_{\hat{x}}) = \sum_{k=0}^{\infty} A_{I-n,k}^{N_{\hat{\Delta}}} |x_\perp|^{2k} \square_{\hat{x}}^k. \quad (4.81)$$

Here the index k encodes the level of the descendant operators that enter in the DOE.

What remains is to be fixed are the coefficients $A_{I-n,k}^{N_{\hat{\Delta}}}$ in (4.81). In principle, imposing covariance of the logDOE under the full set of residual special conformal transformations parallel to the defect determines all of these coefficients except for $A_{I-n,0}^{N_{\hat{\Delta}}}$ for the primary. Rather than following this route here, we adopt an alternative and more direct method. In practice, this is carried out as follows. First, we evaluate the bulk-defect two-point function $\langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}}^I(0) \rangle$ by employing the logDOE together with the defect two-point function (4.61) derived in the previous subsection. Second, we compare the resulting expression with the general form of the bulk-defect two-point function (4.64) obtained earlier. This comparison uniquely fixes the coefficients $A_{I-n,k}^{N_{\hat{\Delta}}}$ (up to $A_{I-n,0}^{N_{\hat{\Delta}}}$).

By plugging the expression of logDOE (4.76) into the bulk-defect two-point function, we obtain

$$\langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}}^I(0) \rangle = \sum_{\widehat{\Delta}'}^{N_{\widehat{\Delta}'}} \sum_{J=1}^{J-1} \sum_{n=0}^{J-1} \frac{(\log |x_\perp|)^n}{n!} \frac{g_{J-n}(|x_\perp|^2 \square_{\widehat{x}})}{|x_\perp|^{\Delta - \widehat{\Delta}'}} \langle \widehat{\mathcal{O}}_{\widehat{\Delta}'}^J(\widehat{x}) \widehat{\mathcal{O}}_{\widehat{\Delta}}^I(0) \rangle. \quad (4.82)$$

Moreover, by using the explicit form of the defect two-point function (4.61), the above equation is rewritten as

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}}^I(0) \rangle &= \sum_{J=1}^{N_{\widehat{\Delta}} - I + 1} \sum_{n=0}^{J-1} \sum_{m=0}^{N_{\widehat{\Delta}} - I - J + 1} \sum_{\ell=0}^{\infty} \frac{(-1)^m (\log |x_\perp|)^n}{n! m!} \frac{k_{I+J-1+m} A_{J-n,\ell}^{N_{\widehat{\Delta}}}}{|x_\perp|^{\Delta - \widehat{\Delta} - 2\ell}} \square_{\widehat{x}}^\ell \frac{(\log |\widehat{x}|^2)^m}{|\widehat{x}|^{2\widehat{\Delta}}} \\ &= \sum_{\ell=0}^{\infty} \sum_{k=0}^{N_{\widehat{\Delta}} - I} \sum_{J=1}^{N_{\widehat{\Delta}} - I - k + 1} \sum_{n=0}^k \frac{(\log |x_\perp|)^n}{n! (k-n)!} \frac{k_{I+J+k-1} A_{J,\ell}^{N_{\widehat{\Delta}}}}{|x_\perp|^{\Delta - \widehat{\Delta} - 2\ell}} \square_{\widehat{x}}^\ell \frac{(-\log |\widehat{x}|^2)^{k-n}}{|\widehat{x}|^{2\widehat{\Delta}}}, \end{aligned} \quad (4.83)$$

where in the second line, we have rearranged the sum by,

$$\sum_{J=1}^{N_{\widehat{\Delta}} - I + 1} \sum_{n=0}^{J-1} \sum_{m=0}^{N_{\widehat{\Delta}} - I - J + 1} f(J, n, m) = \sum_{k=0}^{N_{\widehat{\Delta}} - I} \sum_{J=1}^{N_{\widehat{\Delta}} - I - k + 1} \sum_{n=0}^k f(J + n, n, k - n). \quad (4.84)$$

In what follows, we first focus on the coefficients $A_{J,0}^{N_{\widehat{\Delta}}}$, which are associated with the contributions of the defect primaries, and reorganize the above expression using the binomial expansion formula,

$$\langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}}^I(0) \rangle = \frac{1}{|x_\perp|^{\Delta - \widehat{\Delta}} |\widehat{x}|^{2\widehat{\Delta}}} \sum_{k=0}^{N_{\widehat{\Delta}} - I} \sum_{J=1}^{N_{\widehat{\Delta}} - I - k + 1} \frac{k_{I+J+k-1} A_{J,0}^{N_{\widehat{\Delta}}}}{k!} \left(\log \frac{|x_\perp|}{|\widehat{x}|^2} \right)^k + O \left(\frac{|x_\perp|^2}{|\widehat{x}|^2} \right). \quad (4.85)$$

On the other hand, by expanding (4.64) and (4.72) around $|x_\perp|/|\widehat{x}| = 0$, we obtain

$$\langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}}^I(0) \rangle = \frac{1}{|x_\perp|^{\Delta - \widehat{\Delta}} |\widehat{x}|^{2\widehat{\Delta}}} \sum_{k=0}^{N_{\widehat{\Delta}} - I} \frac{\lambda_{I+k}}{k!} \left(\log \frac{|x_\perp|}{|\widehat{x}|^2} \right)^k + O \left(\frac{|x_\perp|^2}{|\widehat{x}|^2} \right). \quad (4.86)$$

By comparing this expression with (4.85), we find the relation

$$\lambda_I = \sum_{J=1}^{N_{\widehat{\Delta}} - I + 1} k_{I+J-1} A_{J,0}^{N_{\widehat{\Delta}}}, \quad 1 \leq I \leq N_{\widehat{\Delta}}. \quad (4.87)$$

In the same manner, one can determine the coefficients $A_{J,\ell}^{N_{\widehat{\Delta}}}$ ($\ell \geq 1$) associated to the descendant operators. Since the derivation is rather involved and highly technical, we relegate the detailed computation to the Appendix 4.4.4 and only present the final results here,

$$A_{J,\ell}^{N_{\widehat{\Delta}}} = \frac{(-4)^{-\ell}}{\ell! \left(\widehat{\Delta} + z - \frac{p-2}{2} \right)_\ell} \sum_{m=1}^J (-1)^{J+m} A_{m,0}^{N_{\widehat{\Delta}}} \sum_{0 \leq k_1 < \dots < k_{J-m} \leq \ell-1} \prod_{r=1}^{J-m} \left(\widehat{\Delta} - \frac{p-2}{2} + k_r \right)^{-1}. \quad (4.88)$$

Note that, as is the case for the two-point functions of logarithmic primaries, the logDOE coefficients depend on the choice of basis among the multiple primaries in a logarithmic multiplet. In Section 4.3.4, we introduce a specific “gauge choice” following [115] that simplifies various expressions and highlights the gauge-independent physical data encoded in the DOE. In Section 4.4.2, we will give also explicit expressions for these logDOE coefficients for the logarithmic defect that emerges from subdimensional disorder in free theory.

4.3.4 Bulk two-point function and bootstrap constraints

In this subsection we consider the bulk scalar two-point function $\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle$ in the presence of a logarithmic defect. The residual conformal invariance allows us to write the bulk two-point function as

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \frac{F(\xi, \phi)}{|x_{1\perp}|^{\Delta_1} |x_{2\perp}|^{\Delta_2}}, \quad (4.89)$$

where ξ and ϕ are cross-ratios defined by

$$\xi \equiv \frac{|x_{12}|^2}{|x_{1\perp}| |x_{2\perp}|}, \quad \cos \phi \equiv \frac{x_{1\perp} \cdot x_{2\perp}}{|x_{1\perp}| |x_{2\perp}|}. \quad (4.90)$$

Studying this correlator provides the starting point for deriving the defect bootstrap equations. The bootstrap equations encode the consistency of OPE: the result obtained by first fusing the two bulk operators must coincide with the one obtained by first expanding a bulk operator into defect operators. Since the bulk theory is an ordinary CFT, the corresponding bulk conformal blocks are identical to the well-known expressions in the literature [19]. On the other hand, the defect channel exhibits a much richer structure, and the associated defect conformal blocks develop new features characteristic of the logarithmic defect (the discussions on defect conformal blocks of ordinary defect CFT can be found in [19]). The main goal of this subsection is to derive the logarithmic defect conformal block explicitly.

Specific Gauge Choice

The two-point functions of logarithmic multiplets contain redundancies that correspond to a change of basis among the primaries that preserve the form of the dilatation operator in (4.44) (see [115, Appendix. A] for a detailed discussion). To simplify the presentation below, we further make use of this freedom to fix $k_1 = k_2 = \dots = k_{N_{\hat{\Delta}}-1} = 0$ and $k_{\hat{\mathcal{O}}} \equiv k_{N_{\hat{\Delta}}} \neq 0$ in (4.60) which enter in the two-point functions (4.61) and (4.63) so that,

$$\langle \hat{\mathcal{O}}_{\hat{\Delta}_1, i_1 i_2, \dots, i_s}^{I_1}(\hat{x}) \hat{\mathcal{O}}_{\hat{\Delta}_2}^{I_2, j_1 j_2, \dots, j_s}(0) \rangle = \frac{k_{\hat{\mathcal{O}}} \cdot \delta_{\hat{\Delta}_1, \hat{\Delta}_2} \mathcal{I}_{i_1 i_2, \dots, i_s}^{j_1 j_2, \dots, j_s}}{(N_{\hat{\Delta}, s} - I_1 - I_2 + 1)!} \frac{(-\log |\hat{x}|^2)^{N_{\hat{\Delta}, s} - I_1 - I_2 + 1}}{|\hat{x}|^{2\hat{\Delta}}}, \quad (4.91)$$

and vanishes when $N_{\hat{\Delta}} + 1 \leq I_1 + I_2$. In the same basis, the relation (4.87) is simplified to

$$\lambda_I = k_{\hat{\mathcal{O}}} A_{N_{\hat{\Delta}, s} - I + 1, 0}^{N_{\hat{\Delta}, s}}. \quad (4.92)$$

Gram Matrix

As preparation for the defect conformal block expansion, we introduce the Gram matrix

$$G_{I\vec{a};J\vec{b}} \equiv \left\langle \widehat{\mathcal{O}}_{\widehat{\Delta},s}^I; \vec{a} \middle| \widehat{\mathcal{O}}_{\widehat{\Delta},s}^J; \vec{b} \middle| \widehat{\mathcal{O}}_{\widehat{\Delta},s}^I; \vec{a} \middle| \widehat{\mathcal{O}}_{\widehat{\Delta},s}^J; \vec{b} \right\rangle, \quad (4.93)$$

where $\vec{b} = \{b_1, b_2, \dots\}$ denotes collectively the vector indices tangential to the defect, and the descendant state $\left| \widehat{\mathcal{O}}_{\widehat{\Delta},s}^J; \vec{b} \right\rangle$ is written in terms of parallel momenta operators,

$$\left| \widehat{\mathcal{O}}_{\widehat{\Delta},s}^J; \vec{b} \right\rangle = P_{b_1} P_{b_2} \dots \left| \widehat{\mathcal{O}}_{\widehat{\Delta},s}^J \right\rangle. \quad (4.94)$$

Also, its BPZ conjugate state can be expressed as

$$\left\langle \widehat{\mathcal{O}}_{\widehat{\Delta},s}^I; \vec{a} \right| = \left\langle \widehat{\mathcal{O}}_{\widehat{\Delta},s}^I \middle| K_{a_1} K_{a_2} \dots \right. \quad (4.95)$$

By using the commutation relation P_b and K_a and the conformal dimension matrix (4.45), one can express this Gram matrix as follows:⁵²

$$G_{I\vec{a};J\vec{b}} = k_{\widehat{\mathcal{O}}} \delta_{\vec{a},\vec{b}} V_{IJ}(\partial_{\widehat{\Delta}}) g_{\vec{a}}(\widehat{\Delta}, s), \quad (4.96)$$

where $g_{\vec{a}}(\widehat{\Delta}, s)$ depends only on $\widehat{\Delta}$ and s , and can be fixed by conformal algebra. Also, the matrix V_{IJ} is defined by

$$V_{IJ}(\partial_{\widehat{\Delta}}) = \frac{1}{(N_{\widehat{\Delta},s} - I_1 - I_2 + 1)!} \times \begin{cases} \partial_{\widehat{\Delta}}^{N_{\widehat{\Delta},s} - I_1 - I_2 + 1}, & 2 \leq I_1 + I_2 \leq N_{\widehat{\Delta},s} + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.97)$$

Moreover, the inverse Gram matrix is given by

$$\widetilde{G}_{I\vec{a};J\vec{b}} = k_{\widehat{\mathcal{O}}}^{-1} \delta_{\vec{a},\vec{b}} \widetilde{V}_{IJ}(\partial_{\widehat{\Delta}}) g_{\vec{a}}^{-1}(\widehat{\Delta}, s), \quad (4.98)$$

where $\widetilde{V}_{IJ}$ is defined by

$$\widetilde{V}_{IJ}(\partial_{\widehat{\Delta}}) = \frac{1}{(I_1 + I_2 - N_{\widehat{\Delta},s} - 1)!} \times \begin{cases} \partial_{\widehat{\Delta}}^{I_1 + I_2 - N_{\widehat{\Delta},s} - 1}, & N_{\widehat{\Delta},s} + 1 \leq I_1 + I_2 \leq 2N_{\widehat{\Delta},s}, \\ 0, & \text{otherwise.} \end{cases} \quad (4.99)$$

One can readily check that the above matrix (4.98) satisfies the following desired relation,

$$\sum_{K,\vec{c}} G_{I\vec{a};K\vec{c}} \widetilde{G}_{K\vec{c};J\vec{b}} = \delta_{I,J} \delta_{\vec{a},\vec{b}}. \quad (4.100)$$

⁵²Note that the commutation algebra between P_b and K_a concern not only D but also M_{ab} which is the generator of $\text{SO}(p)$ group. However, since we are interested in the defect local operator with trivial $\text{SO}(p)$ spin, the contributions from M_{ab} do not appear.

Logarithmic Defect Conformal Block Expansion

With the preparation complete, we are now ready to determine the defect conformal block expansion of the bulk two-point function. The first step is to insert a complete set of defect primary operators:⁵³

$$1 = \sum_{\hat{\Delta}, s} \sum_{I, J=1}^{N_{\hat{\Delta}, s}} \sum_{\vec{a}, \vec{b}} \tilde{G}_{I\vec{a}; J\vec{b}} \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^I; \vec{a} \right\rangle \left\langle \hat{\mathcal{O}}_{\hat{\Delta}, s}^J; \vec{b} \right| \quad (4.101)$$

into (4.89). This allows us to decompose the bulk two-point function as follows:

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \sum_{\hat{\Delta}, s} W_{\hat{\mathcal{O}}}^{N_{\hat{\Delta}, s}}(x_1, x_2; \Delta_1, \Delta_2), \quad (4.102)$$

where $W_{\hat{\mathcal{O}}}^{N_{\hat{\Delta}, s}}$ is the conformal partial wave defined by

$$W_{\hat{\mathcal{O}}}^{N_{\hat{\Delta}, s}} \equiv \sum_{I, J=1}^{N_{\hat{\Delta}, s}} \sum_{\vec{a}, \vec{b}} \tilde{G}_{I\vec{a}; J\vec{b}} \langle \mathcal{D} | \mathcal{O}_{\Delta_1}(x_1) \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^I; \vec{a} \right\rangle \left\langle \hat{\mathcal{O}}_{\hat{\Delta}, s}^J; \vec{b} \right| \mathcal{O}_{\Delta_2}(x_2) | \mathcal{D} \rangle. \quad (4.103)$$

Here, $|\mathcal{D}\rangle$ represents the defect vacuum, and the radial quantization is performed around a point on the defect. We next show that this conformal partial wave satisfies differential equations. We firstly recall that the residual conformal Lie algebra $\mathfrak{so}(1, p+1) \oplus \mathfrak{so}(d-p)$ has two quadratic Casimir operators L^2 and S^2 :

$$L^2 \equiv iD(iD - p) + \frac{1}{2} M_{ab} M^{ab} + P^a K_a, \quad S^2 \equiv \frac{1}{2} M_{ij} M^{ij}. \quad (4.104)$$

These Casimir operators act on the logarithmic multiplet as follows,

$$\begin{aligned} L^2 \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^I; \vec{a} \right\rangle &= \hat{C}_{\hat{\Delta}, 0} \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^I; \vec{a} \right\rangle + \partial_{\hat{\Delta}} \hat{C}_{\hat{\Delta}, 0} \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^{I+1}; \vec{a} \right\rangle + \frac{1}{2} \partial_{\hat{\Delta}}^2 \hat{C}_{\hat{\Delta}, 0} \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^{I+2}; \vec{a} \right\rangle, \\ S^2 \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^I; \vec{a} \right\rangle &= \hat{C}_{0, s} \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^I; \vec{a} \right\rangle, \end{aligned} \quad (4.105)$$

where $\hat{C}_{\hat{\Delta}, s} = \hat{\Delta}(\hat{\Delta} - p) + s(s + d - p - 2)$. In contrast to the case of an ordinary CFT, the Casimir operator for $\mathfrak{so}(1, d+1)$ does not act diagonally on the space of states in an irreducible logarithmic multiplet. Instead, they induce a mixing among states with different Jordan labels I due to the non-diagonal property of conformal dimension matrix $\hat{\Delta}$. Noting that, for a state of arbitrary rank $N_{\hat{\Delta}, s}$, the operator $(L^2 - \hat{C}_{\hat{\Delta}, 0})^{N_{\hat{\Delta}, s}}$ annihilates it, the above Casimir equations (4.105) are written as

$$\begin{aligned} (L^2 - \hat{C}_{\hat{\Delta}, 0})^{N_{\hat{\Delta}, s}} \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^I; \vec{a} \right\rangle &= 0, \quad \forall I, \\ (S^2 - \hat{C}_{0, s}) \left| \hat{\mathcal{O}}_{\hat{\Delta}, s}^I; \vec{a} \right\rangle &= 0. \end{aligned} \quad (4.106)$$

⁵³Here, we again neglect the contribution from the defect local primaries with $\text{SO}(p)$ spin. This is because the correlation function of bulk scalar operator and defect operator with $\text{SO}(p)$ spin vanishes.

By combining these equations with (4.103), it turns out that the conformal partial wave $W_{\hat{\mathcal{O}}}^{N_{\hat{\Delta},s}}$ satisfies the following differential equations:

$$\begin{aligned} (\mathcal{L}_1^2 - \hat{C}_{\hat{\Delta},0})^{N_{\hat{\Delta},s}} W_{\hat{\mathcal{O}}}^{N_{\hat{\Delta},s}} &= 0, \\ (\mathcal{S}_1^2 - \hat{C}_{0,s}) W_{\hat{\mathcal{O}}}^{N_{\hat{\Delta},s}} &= 0, \end{aligned} \quad (4.107)$$

where $\mathcal{L}_1^2$ and $\mathcal{S}_1^2$ are differential representations of Casimir operators L^2 and S^2 acting on the coordinate x_1 , respectively. In line with (4.89), it is convenient to factor out the conformal factor as follows:

$$W_{\hat{\mathcal{O}}}^{N_{\hat{\Delta},s}} = \frac{G_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta},s}}(\chi) H_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta},s}}(\phi)}{|x_{1\perp}|^{\Delta_1} |x_{2\perp}|^{\Delta_2}}, \quad \chi \equiv \frac{|\hat{x}_{12}|^2 + |x_{1\perp}|^2 + |x_{2\perp}|^2}{|x_{1\perp}| |x_{2\perp}|}. \quad (4.108)$$

Then, the above differential equations (4.107) become

$$(\hat{\mathcal{D}}_{L^2})^{N_{\hat{\Delta},s}} G_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta},s}}(\chi) = 0, \quad (4.109)$$

$$\hat{\mathcal{D}}_{S^2} H_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta},s}}(\phi) = 0, \quad (4.110)$$

where $\hat{\mathcal{D}}_{L^2}$ and $\hat{\mathcal{D}}_{S^2}$ are differential operators:

$$\begin{aligned} \hat{\mathcal{D}}_{L^2} &\equiv (4 - \chi^2) \frac{\partial^2}{\partial \chi^2} - (p+1) \chi \frac{\partial}{\partial \chi} + \hat{C}_{\hat{\Delta},0}, \\ \hat{\mathcal{D}}_{S^2} &\equiv (1 - \cos^2 \phi) \frac{\partial^2}{\partial (\cos \phi)^2} + (1 - d + p) \cos \phi \frac{\partial}{\partial \cos \phi} + \hat{C}_{0,s}. \end{aligned} \quad (4.111)$$

We can immediately solve the second differential equation (4.110), and its physical conformal block is derived as⁵⁴

$$H_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta},s}}(\phi) = {}_2F_1 \left(\frac{d-p+s}{2} - 1, -\frac{s}{2}; \frac{d-p-1}{2}; \sin^2 \phi \right). \quad (4.112)$$

In particular, when the transverse spin is integer, the above block can be expressed in terms of the Gegenbauer polynomial,⁵⁵

$$H_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta},s}}(\phi) = \frac{\Gamma(s+1) \Gamma(d-p-2)}{\Gamma(d-p+s-2)} C_s^{(\frac{d-p}{2}-1)}(\cos \phi) \quad (4.113)$$

To solve (4.109), let us first consider the following differential equation, which applies in the ordinary DCFT case,

$$\hat{\mathcal{D}}_{L^2} G_{W_{\hat{\mathcal{O}}}}(\chi) = 0, \quad (4.114)$$

⁵⁴In general, not only physical conformal block but also its shadow block contribute to the conformal partial wave expansion. Throughout this chapter, we do not consider shadow block contributions.

⁵⁵When the defect codimension is two, it is legitimate to interpret this Gegenbauer expression by taking the limit $d-p = 2 + \epsilon$ with $\epsilon \rightarrow 0$. This rule is also applied to the expression of (4.119).

and its physical solution is readily computed as [19]

$$G_{W_{\hat{\mathcal{O}}}}(\chi) = \chi^{-\hat{\Delta}} {}_2F_1 \left(\frac{\hat{\Delta}+1}{2}, \frac{\hat{\Delta}}{2}; \hat{\Delta} - \frac{p-2}{2}; \frac{4}{\chi^2} \right). \quad (4.115)$$

In terms of this, the general solution of (4.109) can be expressed as follows:

$$G_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta},s}}(\chi) = \sum_{k=0}^{N_{\hat{\Delta},s}-1} \frac{C_k}{k!} \partial_{\hat{\Delta}}^k G_{W_{\hat{\mathcal{O}}}}(\chi), \quad (4.116)$$

where the logarithmic features are evident from the derivatives in $\hat{\Delta}$.

The constants C_k are to be fixed in terms of the bulk-defect two-point functions (equivalently the DOE coefficients). This can be achieved by bringing a bulk local operator close to the defect, i.e. taking $\chi \sim \infty$ while keeping $\cos \phi$ finite. In this limit, the bulk two-point function behaves as

$$W_{\hat{\Delta}}^{N_{\hat{\Delta},s}} \underset{\chi \sim \infty}{\sim} \frac{1}{|x_{1\perp}|^{\Delta_1-\hat{\Delta}} |x_{2\perp}|^{\Delta_2-\hat{\Delta}}} H_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta},s}}(\phi) \sum_{k=0}^{N_{\hat{\Delta},s}-1} \frac{C_k}{k!} \frac{(-\log \chi)^k}{|\hat{x}_{12}|^{2\hat{\Delta}}}. \quad (4.117)$$

On the other hand, we can evaluate the bulk two-point function using the logDOE obtained in the previous section:

$$W_{\hat{\Delta}}^{N_{\hat{\Delta},s}} \underset{\chi \sim \infty}{\sim} = \frac{a_s k_{\hat{\mathcal{O}}}^{-1}}{|x_{1\perp}|^{\Delta_1-\hat{\Delta}} |x_{2\perp}|^{\Delta_2-\hat{\Delta}}} H_{W_{\hat{\mathcal{O}}}}^N(\phi) \sum_{k=0}^{N-1} \sum_{q=1}^{N-k} \lambda_{N+1-q} \lambda_{k+q} \frac{1}{k!} \frac{(-\log \chi)^k}{|\hat{x}_{12}|^{2\hat{\Delta}}}, \quad (4.118)$$

where a_s is defined by⁵⁶

$$a_s \equiv \frac{\Gamma(d-p+s-2)\Gamma(\frac{d-p}{2}-1)}{2^s \Gamma(\frac{d-p}{2}+s-1)\Gamma(d-p-2)} \quad (4.119)$$

We relegate its derivation to Appendix 4.4.4. By comparing these two asymptotic formulas (4.117) and (4.118), one can relate C_k to the bulk-defect coupling:

$$C_k = a_s k_{\hat{\mathcal{O}}}^{-1} \sum_{q=1}^{N-k} \lambda_{N+1-q} \lambda_{k+q}. \quad (4.120)$$

Bulk Conformal Block Expansion

Finally, let us comment on the bulk conformal block expansion. As mentioned earlier, this discussion proceeds in essentially the same way as in an ordinary defect CFT [19] since the logarithmic features of interest here are confined to the defect worldvolume. For completeness and for the reader's convenience, we record the relevant previous results from [135, 136] below. Following [136], it is convenient to introduce new cross-ratios X and $\bar{X}$, which are related to ξ and ϕ via

$$\frac{X + \bar{X}}{2(X\bar{X})^{1/2}} = \cos \phi, \quad \frac{(1-X)(1-\bar{X})}{(X\bar{X})^{1/2}} = \xi. \quad (4.121)$$

⁵⁶See footnote 55.

The bulk-channel conformal block decomposition of the two-point function (4.89) reads,

$$F(X, \bar{X}) = \xi^{-\frac{\Delta_1 + \Delta_2}{2}} \sum_{\mathcal{O}_{\Delta, \ell} \in \mathcal{O}_{\Delta_1} \times \mathcal{O}_{\Delta_2}} c(\mathcal{O}_{\Delta_1}, \mathcal{O}_{\Delta_2}, \mathcal{O}_{\Delta, \ell}) a_{\mathcal{O}_{\Delta, \ell}} f_{\Delta, \ell}^{\text{bulk}}(X, \bar{X}), \quad (4.122)$$

where $c(\mathcal{O}_{\Delta_1}, \mathcal{O}_{\Delta_2}, \mathcal{O}_{\Delta, \ell})$ is the OPE coefficient from $\mathcal{O}_{\Delta_1} \times \mathcal{O}_{\Delta_2}$ to $\mathcal{O}_{\Delta, \ell}$, and $a_{\mathcal{O}_{\Delta, \ell}}$ is the coefficient in the bulk one-point function of $\mathcal{O}_{\Delta, \ell}$. The explicit form of each bulk-channel conformal block $f_{\Delta, \ell}^{\text{bulk}}(X, \bar{X})$ is [136, equation (2.7)]

$$f_{\Delta, \ell}^{\text{bulk}}(X, \bar{X}) = f_{\Delta, \ell}^{\text{HS}}(X, \bar{X}) + \frac{\Gamma(d + \ell - 2)\Gamma(-\ell - \frac{d-2}{2})}{\Gamma(\ell + \frac{d-2}{2})\Gamma(-\ell)} \frac{\Gamma(\frac{\ell+d-p-1}{2})\Gamma(-\frac{\ell-1}{2})}{\Gamma(\frac{\ell+d-1}{2})\Gamma(-\frac{\ell+p-1}{2})} f_{\Delta, 2-d-\ell}^{\text{HS}}(X, \bar{X}), \quad (4.123)$$

where $f_{\Delta, \ell}^{\text{HS}}(X, \bar{X})$ is the Harish-Chandra function defined by,

$$\begin{aligned} f_{\Delta, \ell}^{\text{HS}}(X, \bar{X}) &= (X\bar{X})^{\Delta_{21}/4} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_n(\Delta, \ell) h_m(1 - \ell, 1 - \Delta) \frac{\left(\frac{\Delta + \ell - \Delta_{21}}{2}\right)_{n-m}}{\left(\frac{\Delta + \ell - 1}{2}\right)_{n-m}} \frac{4^{n-m}}{n! m!} \\ &\times {}_4F_3\left(\begin{matrix} -n, -m, \frac{1-\Delta_{21}}{2}, \frac{\Delta - \Delta_{21} - \ell - d}{2} + 1 \\ -\frac{\Delta + \ell + \Delta_{21}}{2} + 1 - n, \frac{\Delta + \ell - \Delta_{21}}{2} - m, \frac{\Delta - \ell - d + 3}{2} \end{matrix}; 1\right) (1 - X\bar{X})^{\ell - 2m} [(1 - X)(1 - \bar{X})]^{\frac{\Delta - \ell}{2} + m + n} \\ &\times {}_2F_1\left(\frac{\Delta + \ell + \Delta_{21}}{2} - m + n, \frac{\Delta + \ell}{2} - m + n; \Delta + \ell - 2m + 2n; 1 - X\bar{X}\right), \end{aligned} \quad (4.124)$$

with $\Delta_{21} \equiv \Delta_2 - \Delta_1$ and $h_n(\Delta, \ell)$ is given by

$$h_n(\Delta, \ell) \equiv \frac{\left(\frac{\Delta - 1}{2}, \frac{\Delta - p}{2}, \frac{\Delta + \ell + \Delta_{21}}{2}\right)_n}{\left(\Delta - \frac{d-2}{2}, \frac{\Delta + \ell + 1}{2}\right)_n}, \quad (4.125)$$

where $(a_1, a_2, \dots, a_m)_n \equiv (a_1)_n (a_2)_n \dots (a_m)_n$ is a shorthand for the product of Pochhammer functions.

Bootstrap Constraints

To summarize, in the above, we have seen that the bulk two-point function (4.89) can be expanded in terms of defect conformal block (4.102) and (4.108). Its precise forms are given by (4.112) and (4.116) with (4.120).

The associativity of OPE in the presence of a logarithmic defect leads to the two-point function bootstrap equation which relates the bulk and the defect channel decompositions of the correlator [19, 137–139], as in Figure 4.3. More explicitly, the logarithmic defect bootstrap equation is written as

$$\sum_{\hat{\Delta}, s} G_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta}, s}}(\chi) H_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta}, s}}(\phi) = \xi^{-\frac{\Delta_1 + \Delta_2}{2}} \sum_{\mathcal{O}_{\Delta, \ell} \in \mathcal{O}_{\Delta_1} \times \mathcal{O}_{\Delta_2}} c(\mathcal{O}_{\Delta_1}, \mathcal{O}_{\Delta_2}, \mathcal{O}_{\Delta, \ell}) a_{\mathcal{O}_{\Delta, \ell}} f_{\Delta, \ell}^{\text{bulk}}(X, \bar{X}). \quad (4.126)$$

This equation ties together bulk OPE data and bulk-defect OPE data introduced above and provides a natural arena to explore constraints on and chart the space of logarithmic defects.

$$\sum_{\hat{\Delta}, s, I} \begin{array}{c} \mathcal{O}_{\Delta_1} \quad \mathcal{O}_{\Delta_2} \\ \vdots \quad \vdots \\ \hline \hat{\mathcal{O}}_{\hat{\Delta}, s}^I \end{array} = \sum_{\Delta, \ell} \begin{array}{c} \mathcal{O}_{\Delta_1} \quad \mathcal{O}_{\Delta_2} \\ \diagdown \quad \diagup \\ \hline \mathcal{O}_{\Delta, \ell} \end{array}$$

Figure 4.3: Schematic picture of logarithmic defect bootstrap equation.

Despite the absence of positivity, the bootstrap equation was analyzed in [32] using the determinant method developed in [140, 141], after truncating the spectrum of exchanged operators. Further applications of this approach can be found in [31, 142–150], where it has been employed in both bulk and defect bootstrap settings. Since this truncation method does not rely on unitarity, it is naturally well-suited for studying systems with logarithmic features. In the present work, we have set up the basic ingredients necessary to pursue further numerical investigations of logarithmic defects along these lines.

In two dimensions, an additional powerful constraint on defects (i.e. boundaries) is provided by the Cardy condition [138, 151], also known as open–closed duality for the annulus partition function. See e.g. [152–154] where this condition has been analyzed. Notably, the recent work [154] combines the Cardy condition with the two-point function bootstrap equation to yield a positive system in the unitary case, where semidefinite programming can be applied to derive stronger bounds on defect data. In the logarithmic setting, positivity is no longer guaranteed, but it would be interesting to explore possible generalizations of these methods to investigate logarithmic defects in 2d CFTs.

4.4 More details on logarithmic conformal defect in free theory

In the previous section, we have discussed the universal properties of logarithmic conformal defects, i.e. structural features that do not rely on a specific model. In this section, we circle back to our main focus on such defects obtained from subdimensional disorder and discuss how the framework developed above can be applied to the generalized pinning field model (4.16) with random fluctuations. As anticipated in Section 4.2.6, several defect local operators are expected to behave as logarithmic primaries. A natural question then arises: what is the rank of the corresponding logarithmic multiplets? To address this, in Section 4.4.1, we analyze in detail how various primary operators, including the displacement operator and the tilt operator, organize themselves into logarithmic multiplets, sometimes making use of the bulk equations of motion. In Section 4.4.2, we also clarify how the bulk-defect OPE data are determined in this setting. In Section 4.4.3 we present the bulk scalar two-point function together with its explicit conformal block decomposition, emphasizing both the similarities and the differences with respect to the disorder-free case. Finally in Section 4.4.4, we determine the defect one-point functions for bulk operators in the OPE of the scalar and particularly find that the stress energy one-point function vanishes for this non-topological defect!

4.4.1 Zoo of logarithmic multiplets in defect spectrum

In this subsection, we examine the spectrum of defect operators generated by subdimensional disorder in the free scalar theory. As we will see, the spectrum exhibits a variety of logarithmic structures, which can be directly compared with the general considerations developed in the previous section. This allows us to pinpoint how the universal features of logarithmic conformal defects are realized in the concrete setting of subdimensional disorder given by the generalized pinning field model with randomly fluctuating coupling.

Defect operators with $(\hat{\Delta}, s) = (p/2, 0)$

We firstly explore the structures of defect operators which are logDOE channels of the bulk scalar field σ . The logDOE of the bulk operator σ is given by⁵⁷

$$\sigma(x) \supset \sum_{I=1}^{N_{\hat{\Delta},s}} \sum_{n=0}^{I-1} \alpha_{I-n} \frac{x_{\perp}^{i_1} x_{\perp}^{i_2} \cdots x_{\perp}^{i_s} (\log |x_{\perp}|)^n}{|x_{\perp}|^{\Delta_{\sigma}-\hat{\Delta}+s}} \frac{1}{n!} \hat{\mathcal{O}}_{\hat{\Delta}, i_1 i_2 \dots i_s}^I(\hat{x}), \quad (4.127)$$

where $\Delta_{\sigma} = (d-2)/2 = p/2$ and $\alpha_{I-n} \equiv A_{I-n,0}^{N_{\hat{\Delta},s}}$. We can extract possible defect channels of σ by using the equation of motion [19, 23]. By acting the d -dimensional d'Alembertian on both hand sides, one can obtain

$$\square \sigma(x) \supset \sum_{\hat{\Delta},s} \sum_{I=1}^{N_{\hat{\Delta},s}} \sum_{n=0}^{I-1} C_{I-n} \frac{x_{\perp}^{i_1} x_{\perp}^{i_2} \cdots x_{\perp}^{i_s} (\log |x_{\perp}|)^n}{|x_{\perp}|^{\Delta_{\sigma}-\hat{\Delta}+s}} \frac{1}{n!} \hat{\mathcal{O}}_{\hat{\Delta}, i_1 i_2 \dots i_s}^I(\hat{x}), \quad (4.128)$$

where C_J ($1 \leq J \leq N_{\hat{\Delta},s}$) is given by

$$C_J \equiv \alpha_{J-2} + 2(\hat{\Delta} - \Delta_{\sigma})\alpha_{J-1} + (\hat{\Delta} - \Delta_{\sigma} + s)(\hat{\Delta} - \Delta_{\sigma} - s)\alpha_J. \quad (4.129)$$

The bulk equation of motion then implies that all coefficients C_J must vanish:

$$C_J = 0, \quad 1 \leq J \leq N_{\hat{\Delta},s}. \quad (4.130)$$

To make the subsequent statements more precise, let us denote by $\mathcal{V}_{\sigma}$ the module of defect local operators that appear in the DOE of σ . In general, this space can be expressed as

$$\mathcal{V}_{\sigma} = \bigoplus_{\hat{\Delta},s} \bigoplus_{N_{\hat{\Delta},s}} \mathcal{V}_{\sigma}^{N_{\hat{\Delta},s}}, \quad (4.131)$$

where $\mathcal{V}_{\sigma}^{N_{\hat{\Delta},s}}$ is the module associated to the defect primary operator with conformal dimension $\hat{\Delta}$, $\text{SO}(d-p)$ spin s and rank $N_{\hat{\Delta},s}$. In the following, we summarize the results that follow from the bulk equation of motion.

- There are no defect logarithmic multiplets in $\mathcal{V}_{\sigma}$ with non-trivial $\text{SO}(d-p)$ spin:⁵⁸

$$\mathcal{V}_{\sigma}^{N_{\hat{\Delta},s}} = \emptyset, \quad N_{\hat{\Delta},s} \geq 2, \quad s \geq 1. \quad (4.132)$$

⁵⁷We omit the descendant contributions, which do not matter in the discussions below.

⁵⁸The displacement multiplet is logarithmic and has rank 2 but does not appear in $\mathcal{V}_{\sigma}$. This is consistent with the fact that σ has vanishing one-point function with the defect due to the restored global symmetry after disorder averaging.

- Logarithmic scalar multiplets with conformal dimension $p/2$ and rank two appears in $\mathcal{V}_\sigma$. Moreover, there are no logarithmic scalar multiplets of higher rank $N_{\hat{\Delta},0} > 2$ in $\mathcal{V}_\sigma$:

$$\mathcal{V}_\sigma^{N_{p/2,0}=2} \neq \emptyset, \quad \mathcal{V}_\sigma^{N_{\hat{\Delta},0}>2} = \emptyset. \quad (4.133)$$

The proof of these statements is somewhat technical and is therefore deferred to the Appendix 4.4.4. Nonetheless, it is clear that the bulk equation of motion imposes non-trivial and strong constraints on the possible defect multiplets.

We now turn to the task of identifying the defect local operators that belong to the non-trivial logarithmic multiplet $\mathcal{V}_\sigma^{N_{p/2,0}=2}$. The most immediate candidate for a defect local operator with conformal dimension $p/2$ is simply the restriction of the bulk field σ to the defect

$$\hat{\sigma}(\hat{x}) \equiv \lim_{x_\perp \rightarrow 0} \sigma(x). \quad (4.134)$$

As we have seen in Section 4.2.6, the defect two-point function of σ exhibits a logarithmic behavior with exponent one,

$$\overline{\langle \hat{\sigma}(\hat{x}) \hat{\sigma}(0) \rangle} = \frac{C_d}{|\hat{x}|^p} + \frac{C_d c^2 \log \mu^2 |\hat{x}|^2}{2\pi |\hat{x}|^p}. \quad (4.135)$$

This indicates that $\hat{\sigma}$ does not form an ordinary conformal multiplet but instead belongs to a logarithmic multiplet of rank two. The natural question is then what operator serves as the logarithmic partner of $\hat{\sigma}$. The answer comes from the defect degrees of freedom themselves: the randomly fluctuating local magnetic field $\hat{h}$ precisely plays this role. Indeed, by computing the corresponding defect correlators (at separated points) one finds that they exhibit the structure characteristic of a logarithmic pair as

$$\overline{\langle \hat{\sigma}(\hat{x}) \hat{h}(0) \rangle} = -\frac{C_d c^2}{|\hat{x}|^p}, \quad \overline{\langle \hat{h}(\hat{x}) \hat{h}(0) \rangle} = 0, \quad (\hat{x} \neq 0). \quad (4.136)$$

thereby confirming that $\hat{\sigma}$ and $\hat{h}$ together generate the non-trivial multiplet $\mathcal{V}_\sigma^{N_{p/2,0}=2}$.

Finally, to move to the basis described in Section 4.3.4, we perform the following field redefinition:

$$\hat{\sigma}' = \hat{\sigma} + \frac{\hat{h}}{2c^2}, \quad \hat{h}' = \frac{\hat{h}}{2\pi}. \quad (4.137)$$

This brings the two-point functions given above to the form (4.91):

$$\overline{\langle \hat{\sigma}(\hat{x}) \hat{\sigma}(0) \rangle} = -k_{\{\hat{\sigma}, \hat{h}\}} \frac{\log \mu^2 |\hat{x}|^2}{|\hat{x}|^p}, \quad \overline{\langle \hat{\sigma}'(\hat{x}) \hat{h}'(0) \rangle} = \frac{k_{\{\hat{\sigma}, \hat{h}\}}}{|\hat{x}|^p}, \quad \overline{\langle \hat{h}'(\hat{x}) \hat{h}'(0) \rangle} = 0, \quad (4.138)$$

where the coefficient $k_{\{\hat{\sigma}, \hat{h}\}}$ is given by

$$k_{\{\hat{\sigma}, \hat{h}\}} = -\frac{C_d c^2}{2\pi}. \quad (4.139)$$

This change of basis will be convenient in the analysis of bulk two-point function in Section 4.4.3.

Defect operators with $(\hat{\Delta}, s) = (np/2, 0)$

As a mild generalization of the previous example, we consider defect multiplets with conformal dimension $\hat{\Delta} = np/2$ and transverse spin $s = 0$. These provide the simplest nontrivial examples of higher-rank logarithmic multiplets. Clearly, the composite field $\hat{\sigma}^n$ belongs to such a multiplet. Evaluating its defect two-point function, one finds the characteristic logarithmic structure, with the highest exponent of the logarithm equal to n :

$$\langle \hat{\sigma}^n(\hat{x}) \hat{\sigma}^n(0) \rangle = \# c^{2n} \frac{(\log |\hat{x}|)^n}{|\hat{x}|^{np}} + \dots \quad (4.140)$$

For the purposes of this discussion, we omit the explicit numerical coefficients, as they are not essential for the main conceptual point. The crucial observation is that the appearance of $(\log |\hat{x}|)^n$ indicates that $\hat{\sigma}^n$ forms a rank- $(n + 1)$ logarithmic multiplet. The logarithmic primary operators spanning this multiplet are clearly the composites $\hat{\sigma}^m \hat{h}^{n-m}$ with $0 \leq m \leq n$.

Displacement Operator Multiplets

A central and universal ingredient of any conformal defect (non-topological) is the displacement operator. This operator arises because the presence of the defect explicitly breaks the translational invariance of the bulk theory in the directions transverse to the defect. As a result, the Ward-Takahashi identity associated with transverse translations is not preserved in its usual form, but instead acquires an insertion localized on the defect. The operator that generates this insertion is precisely the displacement operator. To be more concrete, we recall that the (improved) energy-momentum tensor for the free theory is given by

$$T^{\mu\nu} = \partial^\mu \sigma \partial^\nu \sigma - \frac{1}{2} \delta^{\mu\nu} \partial^\rho \sigma \partial_\rho \sigma - \frac{d-2}{4(d-1)} (\partial^\mu \partial^\nu - \delta^{\mu\nu} \square) \sigma^2. \quad (4.141)$$

By using the equation of motion

$$\square \sigma(x) = \hat{h}(\hat{x}) \delta^{(d-p)}(x_\perp), \quad (4.142)$$

the conservation law of the energy-momentum tensor becomes

$$\partial_\mu T^{\mu i}(x) = \hat{h}(\hat{x}) \partial^i \sigma(x) \delta^{(d-p)}(x_\perp). \quad (4.143)$$

From this, we can read off the displacement operator D_i as

$$D_i(\hat{x}) \equiv \hat{h}(\hat{x}) \partial_i \hat{\sigma}. \quad (4.144)$$

The displacement operator transforms as a vector under the transverse rotation group $\text{SO}(d-p)$, and it always has fixed conformal dimension $p + 1$ from (4.143). Interestingly, in our disordered system, the displacement operator inherits direct contributions from the randomly fluctuating source $\hat{h}(\hat{x})$. This naturally raises the question of whether this universal defect local operator actually organizes into a logarithmic multiplet. To answer this, one should examine the corresponding defect two-point functions, which as we will see, have features that differ from those of conventional defects in curious ways.

A first observation is that the two-point function of the displacement operator vanishes identically at separated points,

$$\overline{\langle D_i(\hat{x}) D_j(0) \rangle} = 0, \quad \hat{x} \neq 0. \quad (4.145)$$

This signals that, due to the effect of disorder averaging, the displacement operator does not behave as an ordinary CFT primary operator. Instead, it is expected to participate in a logarithmic multiplet. A natural candidate for its logarithmic partner is $\tilde{D}_i(\hat{x}) \equiv \hat{\sigma}(\hat{x}) \partial_i \hat{\sigma}(\hat{x})$ which carries the same conformal dimension and $\text{SO}(d-p)$ spin as the displacement operator. (This defect local operator has already shown up in Section 4.2.6.) Computing the relevant correlation functions, one finds

$$\overline{\langle D_i(\hat{x}) \tilde{D}_j(0) \rangle} = -\frac{p C_d^2 c^2 \delta_{ij}}{|\hat{x}|^{2p+2}}, \quad (4.146)$$

$$\overline{\langle \tilde{D}_i(\hat{x}) \tilde{D}_j(0) \rangle} = p C_d^2 \delta_{ij} \left(\frac{1}{|\hat{x}|^{2p+2}} + \frac{C_d c^2 \log \mu^2 |\hat{x}|^2}{2\pi |\hat{x}|^{2p+2}} \right), \quad (4.147)$$

where the second correlator has already been computed in Section 4.2.6. These results, together with (4.145) and the general discussion in Section 4.3, demonstrate that the displacement operator D_i and its partner $\tilde{D}_i$ organize into a rank-2 logarithmic multiplet.

Although in this chapter we just focus on the free theory as an explicit example, the logarithmic nature of the displacement operator is expected to persist even in interacting theories, as long as the defect is generated by subdimensional disorder. For instance, it has been shown that in two-dimensional (interacting) CFTs deformed by disorder over the entire space, the energy-momentum tensor acquires the logarithmic structure [107, 109]. Importantly, this logarithmic structure emerges in the replica limit $n \rightarrow 0$. As discussed in Section 4.2.3, the replica trick can also be applied to subdimensional disorder, and it is therefore natural to expect that the displacement operator develops a similar logarithmic structure in the replica limit.

Tilt Operator Multiplets

Whenever a bulk continuous global symmetry is spontaneously broken on the defect, there exist universal defect local operators associated with the broken charges. To introduce the tilt operator explicitly in a example, we consider a natural generalization of the disordered single scalar theory to an $\text{O}(N)$ free scalar theory coupled to a defect via a generalized pinning field:

$$S = \frac{1}{2} \int_{\mathbb{R}^d} d^d x (\partial \vec{\sigma})^2 + \int_{\mathbb{R}^p} d^p \hat{x} \hat{h}(\hat{x}) \vec{n} \cdot \vec{\sigma}(\hat{x}), \quad (4.148)$$

where $\vec{n}$ is a fixed vector that selects a direction in the internal space, breaking the bulk $\text{O}(N)$ symmetry down to $\text{O}(N-1)$ around $\vec{n}$ on the defect. To achieve defect conformality, we again set the bulk dimension as $d = p + 2$. The bulk Noether current

$$J_{\alpha\beta}^\mu = \sigma_\alpha \partial^\mu \sigma_\beta - \partial^\mu \sigma_\alpha \sigma_\beta. \quad (4.149)$$

satisfies the modified conservation equation,

$$\partial_\mu J_{\alpha\beta}^\mu = \hat{h}(\hat{x}) (n_\beta \sigma_\alpha - n_\alpha \sigma_\beta) \delta^{(d-p)}(x_\perp), \quad (4.150)$$

and this defines the tilt operator $t_{\alpha\beta}$ as

$$t_{\alpha\beta}(\hat{x}) \equiv \hat{h}(\hat{x})(n_\beta \hat{\sigma}_\alpha(\hat{x}) - n_\alpha \hat{\sigma}_\beta(\hat{x})). \quad (4.151)$$

In the following, we choose the specific vector $\vec{n} = (0, 0, \dots, 0, 1)$ without loss of generality. Then, the above tilt operators can be written as

$$t_{\hat{\alpha}}(\hat{x}) = \hat{h}(\hat{x}) \hat{\sigma}_{\hat{\alpha}}(\hat{x}), \quad \hat{\alpha} = 1, \dots, N-1. \quad (4.152)$$

Note that from the conservation equation, the conformal dimension of tilt operators is fixed to be p . In a similar way to the displacement operator, one may then ask whether these tilt operators also participate in logarithmic multiplets. To address this question, we again study the defect two-point function, and the result is clearly

$$\overline{\langle t_{\hat{\alpha}}(\hat{x}) t_{\hat{\beta}}(0) \rangle} = 0, \quad \hat{x} \neq 0, \quad (4.153)$$

which implies again that the tilt operator is a nontrivial logarithmic primary. Its logarithmic partner must share the same quantum numbers, namely conformal dimension p and transformation as a vector under the residual $O(N-1)$ symmetry. The natural candidate is $\tilde{t}_\alpha = \hat{\sigma}_N \hat{\sigma}_\alpha$, and its relevant correlation functions can be easily computed

$$\begin{aligned} \overline{\langle t_{\hat{\alpha}}(\hat{x}) \tilde{t}_{\hat{\beta}}(0) \rangle} &= -\frac{C_d^2 c^2 \delta_{\hat{\alpha}\hat{\beta}}}{|\hat{x}|^{2p}}, \\ \overline{\langle \tilde{t}_{\hat{\alpha}}(\hat{x}) \tilde{t}_{\hat{\beta}}(0) \rangle} &= C_d^2 \delta_{\hat{\alpha}\hat{\beta}} \left(\frac{1}{|\hat{x}|^{2p}} + \frac{C_d c^2 \log \mu^2 |\hat{x}|^2}{2\pi |\hat{x}|^{2p}} \right). \end{aligned} \quad (4.154)$$

These results indicate that, after disorder averaging, the tilt operator $t_{\hat{\alpha}}$ belongs to a rank-2 logarithmic multiplet together with its partner $\tilde{t}_{\hat{\alpha}}$.

Despite the simplicity of the explicit example presented above, we expect that these special defect multiplets with protected conformal dimensions to become logarithmic in general interacting CFTs with subdimensional disorder. It would be interesting to understand whether the rank-two property of such multiplets found here continue to hold in the interacting examples.

Are there ordinary CFT primaries?

One may wonder whether, after disorder averaging, there still exist ordinary CFT primaries on the defect, namely rank-1 operators. Indeed, such multiplets do exist. They are given by defect local operators with nonzero transverse spin s , which take the following form:⁵⁹

$$\Psi_{i_1, i_2, \dots, i_s}(\hat{x}) \equiv \partial_{i_1} \dots \partial_{i_s} \hat{\sigma}(\hat{x}). \quad (4.155)$$

Its two-point function takes the following form:

$$\overline{\langle \Psi_{i_1, i_2, \dots, i_s}(\hat{x}) \Psi_{j_1, j_2, \dots, j_s}(0) \rangle} = \frac{2^s s! \left(\frac{p}{2}\right)_s C_d}{|\hat{x}|^{p+2s}} \mathcal{I}_{i_1 \dots i_s}^{j_1 \dots j_s}. \quad (4.156)$$

⁵⁹It turns out that only the primaries with transverse spin $s \geq 1$ show up in the defect operator spectrum. See Section 4.4.3 and remark at the end.

Interestingly, these two-point functions show no dependence on the disorder strength c (i.e. same as without the defect). This indicates that the disorder averaging procedure leaves untouched sectors with nonzero transverse spin quantum numbers. This observation is consistent with our earlier conclusion (4.132) from the DOE: channels with nonzero transverse spin can only accommodate ordinary CFT primaries, rather than logarithmic ones. This is suggestive of a general feature for subdimensional disorder with scalar operators. We leave the investigation of this feature and generalization to disorder terms involving spinning operators to future studies.

4.4.2 Bulk-defect OPE data from subdimensional disorder

In the previous section, we have seen how various defect local operators in our free theory model organize themselves into logarithmic multiplets. In a defect CFT, besides the usual conformal dimensions and spins, the DOE coefficients $A_{J,0}^{N_{\hat{\Delta}},s}$ constitute another set of essential CFT data that characterize the defect. In this section, we focus on explicitly computing the bulk-defect OPE coefficients in our model, with the goal of gaining a deeper understanding of the structure of logarithmic defects.

To derive the DOE coefficients, we firstly need to calculate the bulk-defect two-point function, and obtain the structure constants λ_I in (4.72). Then, we can translate this into DOE coefficients $A_{J,0}^{N_{\hat{\Delta}},s}$ via the relation (4.87) with the coefficients k_I of defect two-point functions. In what follows, we illustrate how this program works by focusing on the bulk operator σ . From the discussion in the previous section, we know that the defect local primary operators appearing in the DOE of the bulk field σ are precisely the following set:⁶⁰

$$\hat{\sigma}(\hat{x}), \quad \hat{h}(\hat{x}), \quad \Psi_{i_1 i_2 \dots i_s}(\hat{x}). \quad (4.157)$$

Therefore, to determine the DOE coefficients $A_{J,0}^{N_{\hat{\Delta}},s}$ associated with σ , it suffices to compute the bulk-defect two-point functions between σ and these defect local operators. In the following, we carry out this analysis in order.

We firstly analyze the two-point function of $\sigma(x)$ and $\hat{\sigma}(\hat{y})$:

$$\overline{\langle \sigma(x) \hat{\sigma}(0) \rangle} = \frac{C_d}{|x|^p} + c^2 C_d^2 \int d^p \hat{z} \frac{1}{|x - \hat{z}|^p |\hat{z}|^p}. \quad (4.158)$$

Using the Fourier transformation (4.221) and (4.222), we have

$$\int d^p \hat{z} \frac{1}{|x - \hat{z}|^p |\hat{z}|^p} = \frac{\pi^{p/2} \left(\psi\left(\frac{p}{2}\right) - \gamma_E + \log 4 \right)}{\Gamma\left(\frac{p}{2}\right) |x|^p} - \frac{4\pi^{p+1}}{\Gamma\left(\frac{p}{2}\right)^2} \mathcal{I}_1, \quad (4.159)$$

where γ_E is the Euler–Mascheroni constant and $\mathcal{I}_1$ is the following

$$\mathcal{I}_1 = \int \frac{d^p \hat{p}}{(2\pi)^{p+1}} K_0(|\hat{p}| |x_\perp|) \log |\hat{p}|^2 e^{i\hat{p} \cdot \hat{x}} = \frac{\Gamma\left(\frac{p}{2}\right)}{2\pi^{\frac{p}{2}+1} |x|^p} \left(\tau + \log 2 + \psi\left(\frac{p}{2}\right) \right). \quad (4.160)$$

⁶⁰See also footnote 59.

Here, K_0 is the modified Bessel function of the second kind, and τ is defined in (4.68). We give the detailed derivation of this integral formula in Appendix 4.4.4. Then, the two-point function is simplified to the following desired form:

$$\overline{\langle \sigma(x) \hat{\sigma}(0) \rangle} = \frac{\tilde{\lambda}_1 + \tilde{\lambda}_2 \tau}{|x|^p}, \quad (4.161)$$

where $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are given by

$$\tilde{\lambda}_1 = C_d \left(1 - \frac{c^2}{4\pi} H_{\frac{p}{2}-1} \right), \quad \tilde{\lambda}_2 = -C_d \frac{c^2}{2\pi}, \quad (4.162)$$

where $H_z \equiv \psi(z+1) + \gamma_E$ is the harmonic number continued to the complex plane and C_d is defined in (4.18). On the other hand, the two-point function of σ and $\hat{h}$ can be easily evaluated:

$$\overline{\langle \sigma(x) \hat{h}(0) \rangle} = -\frac{c^2 C_d}{|x|^p}. \quad (4.163)$$

At this stage, it is convenient to switch to a specific basis (4.138) in which the two-point functions take a simpler form. In this basis, the coefficients λ_1 and λ_2 appearing in bulk-defect two-point functions are given by

$$\lambda_1 = \frac{C_d}{2} \left(1 - \frac{c^2}{2\pi} H_{\frac{p}{2}-1} \right), \quad \lambda_2 = -\frac{C_d c^2}{2\pi} = k_{\{\hat{\sigma}, \hat{h}\}}. \quad (4.164)$$

Then, the relation (4.87) and the defect two-point functions (4.138) can translate λ_I into the OE coefficients $A_{J,0}^{N_{p/2,0}}$ (see Section 4.3.3 and (4.92)):

$$A_{1,0}^{N_{p/2,0}} = 1, \quad A_{2,0}^{N_{p/2,0}} = -\frac{\pi}{c^2} + \frac{1}{2} H_{\frac{p}{2}-1}. \quad (4.165)$$

Recall that the first subscript J of $A_{J,\ell}^{N_{\hat{\Delta}}}$ labels the primary operators in the logarithmic multiplet that enter in the DOE of rank $N_{\hat{\Delta}}$ and ℓ indexes the descendants (with $\ell = 0$ corresponding to the primaries).

In contrast, for the defect spinning operator $\Psi_{i_1 \dots i_s}$ discussed in the previous section, the situation is simpler: since it behaves as an ordinary CFT primary, the non-vanishing DOE coefficients are

$$A_{1,0}^{N_{s+p/2,s}} = \frac{1}{s!}. \quad (4.166)$$

We have thus completely fixed the logDOE coefficients of the bulk operator σ in the presence of subdimensional disorder.

4.4.3 Bulk two-point function and deformation from disorder

Finally, we analyze the bulk two-point function σ to illustrate the general discussion in Section 4.3.4.

$$\begin{aligned} \overline{\langle \sigma(x) \sigma(y) \rangle} &= \frac{C_d}{|x-y|^p} + c^2 C_d^2 \int d^p \hat{z} \frac{1}{|x-\hat{z}|^p |y-\hat{z}|^p} \\ &= \frac{C_d}{|x-y|^p} + \frac{16\pi^{p+2}}{\Gamma(p/2)^2} c^2 C_d^2 \mathcal{I}_2, \end{aligned} \quad (4.167)$$

where $\mathcal{I}_2$ is the momentum-space integral defined by

$$\mathcal{I}_2 = \int \frac{d^p \hat{p}}{(2\pi)^{p+2}} K_0(|\hat{p}||x_\perp|) K_0(|\hat{p}||y_\perp|) e^{i\hat{p} \cdot (\hat{x} - \hat{y})}, \quad (4.168)$$

which follows from the Fourier transformation (4.222). In the following, we will unpack the conformal block decomposition of (4.167) which includes logarithmic contributions.

The first term on the RHS of (4.167) is obviously the two-point function with a trivial defect and admits the following conformal block decomposition (as in ordinary defect CFT),

$$\frac{1}{|x - y|^p} = \frac{1}{|x_\perp|^{p/2} |y_\perp|^{p/2}} \left[\chi^{-\frac{p}{2}} {}_2F_1 \left(\frac{p}{4}, \frac{p+2}{4}; 1; \frac{4}{\chi^2} \right) \right. \quad (4.169)$$

$$\left. + \sum_{s=1}^{\infty} \frac{2(\frac{p}{2})_s}{s!} \cos(s\phi) {}_2F_1 \left(\frac{p+2s}{4}, \frac{p+2s+2}{4}; s+1; \frac{4}{\chi^2} \right) \right], \quad (4.170)$$

where χ is the cross-ratio in (4.108). Compared with the general discussion in Section 4.3.4 for logarithmic defects, the first term in the square bracket corresponds to (a part of) the defect conformal block with $(\hat{\Delta}, s) = (p/2, 0)$, while the second term corresponds to the spinning defect conformal block with $(\hat{\Delta}, s) = (p/2 + s, s)$ with $s \geq 1$. Using the notation of (4.112) and (4.115), this tree-level part of (4.167) can be rewritten as

$$\frac{C_d}{|x - y|^p} = \frac{1}{|x_\perp|^{p/2} |y_\perp|^{p/2}} \left(C_d \mathbf{G}_{\{\hat{\sigma}, \hat{h}\}}(\chi) + \sum_{s=1}^{\infty} H_{\Psi_s}^{N_{p/2+s, s}=1}(\phi) G_{\Psi_s}^{N_{p/2+s, s}=1}(\chi) \right). \quad (4.171)$$

Next, we turn to the analysis of the second term on the RHS of (4.167), which encodes the effects of the subdimensional disorder. To proceed, one needs to evaluate the integral in (4.168). We defer the details to the Appendix 4.4.4 and only present the final result here,

$$\mathcal{I}_2 = -\frac{\Gamma(p/2)}{8\pi^{\frac{p}{2}+2}} \frac{1}{|x_\perp|^{p/2} |y_\perp|^{p/2}} \left[H_{\frac{p}{2}-1} \mathbf{G}_{\hat{\Delta}}(\chi) + \partial_{\hat{\Delta}} \mathbf{G}_{\hat{\Delta}}(\chi) \right] \Big|_{\hat{\Delta}=p/2}, \quad (4.172)$$

where $\mathbf{G}_{\hat{\Delta}}(\chi)$ is given by (4.115). Combined this with the tree-level part (4.171), the full bulk two-point function can be written as,

$$\langle \sigma(x) \sigma(y) \rangle = \frac{1}{|x_\perp|^{p/2} |y_\perp|^{p/2}} \left(G_{\{\hat{\sigma}, \hat{h}\}}(\chi) + \sum_{s=1}^{\infty} H_{\Psi_s}^{N_{p/2+s, s}=1}(\phi) G_{\Psi_s}^{N_{p/2+s, s}=1}(\chi) \right), \quad (4.173)$$

where we used (4.116), (4.120) and (4.164). As desired, the resulting expression matches the structure of the defect conformal block expansion developed in Section 4.3.4. This expression also makes it clear that the disorder dependence (one-parameter family parametrized by the disorder strength c) is concentrated in the scalar defect channel by comparing to (4.4.3).

Here we make a brief comment regarding the spectrum of defect spinning operators found from the above analysis. In Section 4.4.1, when we impose constraints on the defect operator spectrum associated with bulk operator σ using the equation of motion, the spinning operators with $\hat{\Delta} = p/2 - |s|$ are a priori not excluded. This is because unitarity is lost in general on the disordered defect. However, the present analysis of the bulk two-point function makes it clear that such operators do not actually appear in the spectrum. This observation is consistent with the remark made at the end of Section 4.4.1, namely that the subdimensional disorder (by a local scalar deformation) does not affect the operators with nonzero transverse spin quantum numbers.

4.4.4 Bulk one-point functions and crossing

In the previous section, we have explicitly evaluated the bulk two-point function and verified that it takes the form consistent with the defect conformal block decomposition. In this section, we revisit the same two-point function from the viewpoint of the bulk conformal block expansion (see around (4.126)).

The bulk OPE of two identical free scalar operators σ takes the schematic form

$$\sigma \times \sigma \rightarrow \mathbf{1} + \sigma^2 + \sum_{\ell=1}^{\infty} \Phi_{\mu_1 \dots \mu_{2\ell}}, \quad (4.174)$$

where $\Phi_{\mu_1 \dots \mu_{2\ell}}(y)$ is bulk local primary operator with even spin 2ℓ ($\ell \in \mathbb{Z}_+$),

$$\Phi_{\mu_1 \dots \mu_{2\ell}} \equiv \sum_{k=0}^{2\ell} c_{2\ell,k} \partial_{\mu_1} \dots \partial_{\mu_k} \sigma \cdot \partial_{\mu_{k+1}} \dots \partial_{\mu_{2\ell}} \sigma. \quad (4.175)$$

with coefficients given in [155],

$$c_{2\ell,k} = \frac{4\sqrt{\pi}\Gamma(\frac{p}{2} + 2\ell)\Gamma(2\ell + p - 1)}{\Gamma(\frac{p-1}{2})} \frac{(-1)^k}{k!(2\ell - k)!\Gamma(k + \frac{p}{2})\Gamma(2\ell - k + \frac{p}{2})}. \quad (4.176)$$

They correspond to the higher even-spin conserved currents in the free theory.

In the following, we will show that only the identity operator and the scalar operator σ^2 in (4.174) contribute in the bulk-channel, by directly evaluating the relevant one-point functions in the presence of the subdimensional disorder by Wick contractions and taking into account normal ordering. We will find the result to be consistent with the other form (4.167) of the two-point function derived before.

For the scalar σ^2 , one finds,

$$\overline{\langle \sigma^2(x) \rangle} = \frac{\pi^{\frac{p}{2}} \Gamma(\frac{p}{2})}{\Gamma(p)} C_d^2 c^2 \frac{1}{|x_{\perp}|^p}. \quad (4.177)$$

For the higher spin currents $\Phi_{\mu_1 \dots \mu_{2\ell}}$, it is convenient to introduce a null polarization vector ζ (i.e. $\zeta \cdot \zeta = 0$) and define,

$$\Phi_{2\ell}(x, \zeta) \equiv \zeta^{\mu_1} \dots \zeta^{\mu_{2\ell}} \Phi_{\mu_1 \dots \mu_{2\ell}}(x). \quad (4.178)$$

Then the bulk one-point function of $\Phi_{2\ell}(x, \zeta)$ can be evaluated as follows,

$$\begin{aligned} \overline{\langle \Phi_{2\ell}(x, \zeta) \rangle} &= C_d c^2 \lim_{y \rightarrow x} \sum_{k=0}^{2\ell} c_{2\ell,k} (\zeta \cdot \partial_x)^{2\ell-k} (\zeta \cdot \partial_y)^k \int d^p \hat{z} \frac{1}{|x - \hat{z}|^p |y - \hat{z}|^p} \\ &= 2^{2\ell} C_d c^2 \sum_{k=0}^{2\ell} c_{2\ell,k} \left(\frac{p}{2}\right)_k \left(\frac{p}{2}\right)_{2\ell-k} \int d^p \hat{z} \frac{(\zeta \cdot (\hat{z} - x))^{2\ell}}{|x - \hat{z}|^{2p+2\ell}} \\ &= 0, \end{aligned} \quad (4.179)$$

where in the final line, we have used the following identity satisfied by (4.176),

$$\sum_{k=0}^{2\ell} c_{2\ell,k} \left(\frac{p}{2}\right)_k \left(\frac{p}{2}\right)_{2\ell-k} = 0. \quad (4.180)$$

The resulting bulk-channel conformal block decomposition of the scalar two-point function follows from the general discussion around (4.122) (see notations thereof),

$$F(X, \bar{X}) = C_d \xi^{-p/2} + \frac{\pi^{\frac{p}{2}} \Gamma(\frac{p}{2})}{\Gamma(p)} C_d^2 c^2 (X \bar{X})^{p/4} {}_2F_1\left(\frac{p}{2}, \frac{p}{2}; p; 1 - X \bar{X}\right). \quad (4.181)$$

As a consistency check, we also see the second term on the RHS reproduces the integral $\mathcal{I}_2$ in (4.168) (see Appendix 4.4.4 for details).

The above result should be contrasted with what happens in ordinary DCFT, where typically an infinite tower of spinning operators (e.g. double-twist operators) contribute in the bulk-channel conformal block decompositions. In particular, the one-point function of the energy-momentum tensor $T^{\mu\nu}$ generally acquires a nontrivial value [19, 20]. However, as demonstrated above, in our free scalar theory with subdimensional disorder, averaged bulk one-point functions of the spinning operators Φ_{2J} all vanish, in particular

$$\overline{\langle T^{\mu\nu}(x) \rangle} = 0. \quad (4.182)$$

In other words, even though the defect is non-topological, the one-point function of the energy-momentum tensor is zero! This interesting and peculiar property is clearly due to taking the subdimensional disorder. Another indication of the quasi-topological nature of the defect arises from the correlators of the displacement operators D_i , which vanish except for contact terms (see (4.145)). A similar phenomenon occurs for higher-spin symmetries. The corresponding Ward-Takahashi identities take the form

$$\partial_\mu \Phi^{\mu i_1 \dots i_{2J-1}}(x) = c_{2J} \hat{h}(\hat{x}) \partial^{i_1} \dots \partial^{i_{2J-1}} \sigma(x) \delta^{(d-p)}(x_\perp), \quad (4.183)$$

where c_{2J} is a constant. This generalizes (4.143) to the spin-2 case ($J = 1$), and the RHS defines the analog of the displacement operator D_i in (4.144) for higher-spin currents broken by the defect. Like their spin-2 counterparts, these higher-spin displacement operators also belong to rank-2 logarithmic multiplets and exhibit vanishing correlators among themselves.

4.A Computations of beta functions

In this Appendix, we perform the diagrammatic analysis to derive the beta function (4.28), using the Callan-Symanzik equation for the bulk one-point function of $\sigma_A \sigma_B(x)$,

$$\frac{d}{d \log \mu} \langle \sigma_A \sigma_B(x) \rangle_{\text{rep}} = 0, \quad (4.184)$$

where μ is a reference energy scale. Using the Wick theorem, we can perturbatively expand this bulk one-point function as a power series of the coupling m_{AB} in the replicated theory,

$$\langle \sigma_A \sigma_B(x) \rangle_{\text{rep}} = -C_d^2 m_{AB} I_1(x) + 4C_d^3 \sum_D m_{AD} m_{BD} I_2(x) + O(m^3), \quad (4.185)$$

where I_1 and I_2 are the integrals given by

$$\begin{aligned} I_1(x) &\equiv \int d^p \hat{y} \frac{1}{|x - \hat{y}|^{2p}}, \\ I_2(x) &\equiv \int d^p \hat{y} d^p \hat{z} \frac{1}{|x - \hat{y}|^p |\hat{y} - \hat{z}|^p |x - \hat{z}|^p}. \end{aligned} \quad (4.186)$$

The Feynman diagram corresponding to each integral is drawn in Figure 4.4. The first integral is

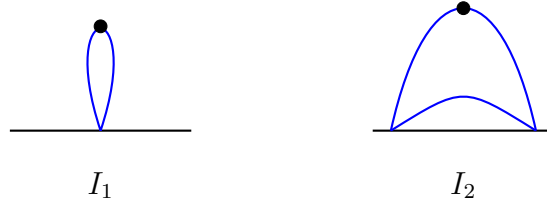


Figure 4.4: Feynman diagrams contributing to the bulk one-point function (4.185)

the contribution from tree diagram, and can be evaluated using the Schwinger's reparameterization,

$$I_1(x) = \frac{\pi^{p/2} \Gamma(p/2)}{\Gamma(p)} \frac{1}{|x_\perp|^p}. \quad (4.187)$$

On the other hand, the second integral I_2 comes from the loop contribution, and contains logarithmic divergence around $\hat{y} = \hat{z}$. With a proper regularization of this UV divergence, the scheme independent part of I_2 is written as

$$I_2 = \frac{\pi^p}{\Gamma(p)} \frac{\log \mu^2 |x_\perp|^2}{|x_\perp|^p}, \quad (4.188)$$

where we introduce the energy scale μ as a regulator. By plugging these results into (4.185), we obtain

$$\frac{\Gamma(p)}{\pi^{p/2} \Gamma(\frac{p}{2})} C_d^{-2} \langle \sigma_A \sigma_B(x) \rangle_{\text{rep}} = \left(c^2 - 2\lambda \delta_{AB} + \frac{8C_d \pi^{p/2}}{\Gamma(\frac{p}{2})} \left(\frac{n c^2}{4} - \lambda c^2 + \lambda^2 \delta_{AB} \right) \log \mu^2 |x_\perp|^2 \right) \frac{1}{|x_\perp|^p}. \quad (4.189)$$

The Callan-Symanzik equation (4.184) then immediately determines the beta functions as is given in (4.28).

Let us end by making a brief comment. Clearly, from the viewpoint of conformal symmetry, bulk one-point functions should not exhibit any logarithmic dependence. Indeed, upon setting $n \rightarrow 0$ and substituting the fixed point values (4.30) into the above results, one finds that the logarithmic term disappears, confirming the consistency of the analysis.

4.B Fixing full logDOE structures

In Section 4.3.3, we presented the derivation of the coefficients $A_{J,0}^{N_{\hat{\Delta}}}$ in (4.81). Here we provide detailed computation of general coefficients $A_{J,\ell}^{N_{\hat{\Delta}}}$ including the descendant contributions.

Evaluation of (4.83)

We note that the derivative part on the RHS in (4.83) can be written as

$$\square_{\hat{x}}^{\ell} \frac{(\log |\hat{x}|^2)^m}{|\hat{x}|^{2\hat{\Delta}}} = \frac{1}{|\hat{x}|^{2\hat{\Delta}+2\ell}} \sum_{q=0}^m a_{\ell,q,m} (\log |\hat{x}|^2)^q. \quad (4.190)$$

The coefficient $a_{\ell,q,m}$ can be determined as follows.

First we note that, in the special case with $m = 0$ in (4.190),

$$a_{\ell,0,0} = 4^{\ell} (\hat{\Delta})_{\ell} \left(\hat{\Delta} - \frac{p-2}{2} \right)_{\ell}. \quad (4.191)$$

By taking derivative $(-\partial_{\hat{\Delta}})^m$, we find

$$a_{\ell,0,m} = (-\partial_{\hat{\Delta}})^m a_{\ell,0,0}. \quad (4.192)$$

Next, we have the following useful recursion relation

$$a_{\ell,n,m} = \frac{m}{n} a_{\ell,n-1,m-1}, \quad (4.193)$$

and thus for $0 \leq q \leq m$,

$$a_{\ell,q,m} = \frac{m!}{q!(m-q)!} (-\partial_{\hat{\Delta}})^{m-q} a_{\ell,0,0}. \quad (4.194)$$

Using the expression (4.190), one can deform (4.83) as

$$\begin{aligned} \langle \mathcal{O}_{\Delta}(x) \hat{\mathcal{O}}_{\hat{\Delta}}^I(0) \rangle &= \sum_{\ell=0}^{\infty} \sum_{k=0}^{N_{\hat{\Delta}}-I} \sum_{J=1}^{N_{\hat{\Delta}}-I-k+1} \sum_{n=0}^k \frac{(\log |x_{\perp}|)^n}{n! (k-n)!} \frac{k_{I+J+k-1} A_{J,\ell}^{N_{\hat{\Delta}}}}{|x_{\perp}|^{\Delta-\hat{\Delta}-2\ell}} \square_{\hat{x}}^{\ell} \frac{(-\log |\hat{x}|^2)^{k-n}}{|\hat{x}|^{2\hat{\Delta}}} \\ &= \frac{1}{|x_{\perp}|^{\Delta-\hat{\Delta}} |\hat{x}|^{2\hat{\Delta}}} \sum_{\ell=0}^{\infty} t^{\ell} \sum_{k=0}^{N_{\hat{\Delta}}-I} \sum_{J=1}^{N_{\hat{\Delta}}-I-k+1} k_{I+J+k-1} A_{J,\ell}^{N_{\hat{\Delta}}} \\ &\quad \cdot \sum_{n=0}^k \sum_{q=0}^{k-n} \frac{(-1)^{k-n} a_{\ell,q,k-n}}{n! (k-n)!} (\log |x_{\perp}|)^n (\log |\hat{x}|^2)^q \\ &= \frac{1}{|x_{\perp}|^{\Delta-\hat{\Delta}} |\hat{x}|^{2\hat{\Delta}}} \sum_{\ell=0}^{\infty} t^{\ell} \sum_{k=0}^{N_{\hat{\Delta}}-I} \sum_{J=1}^{N_{\hat{\Delta}}-I-k+1} k_{I+J+k-1} A_{J,\ell}^{N_{\hat{\Delta}}} \\ &\quad \cdot \sum_{n=0}^k \sum_{q=0}^{k-n} \frac{(-1)^{k-n} a_{\ell,k-n-q,k-n}}{n! (k-n)!} (\log |x_{\perp}|)^n (\log |\hat{x}|^2)^{k-n-q}, \end{aligned} \quad (4.195)$$

where we introduce a new symbol $t \equiv |x_\perp|^2/|\hat{x}|^2$, and in the final line we changed the sum variables as

$$\sum_{q=0}^{k-n} f(q) = \sum_{q=0}^{k-n} f(k-n-q). \quad (4.196)$$

Finally, for later convenience, we swap the summations over n and q as

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}}^I(0) \rangle &= \frac{1}{|x_\perp|^{\Delta-\hat{\Delta}} |\hat{x}|^{2\hat{\Delta}}} \sum_{\ell=0}^{\infty} t^\ell \sum_{k=0}^{N_{\hat{\Delta}}-I} \sum_{J=1}^{N_{\hat{\Delta}}-I-k+1} k_{I+J+k-1} A_{J,\ell}^{N_{\hat{\Delta}}} \\ &\quad \cdot \sum_{q=0}^k \sum_{n=0}^{k-q} \frac{(-1)^{k-n} a_{\ell,k-n-q,k-n} (\log |x_\perp|)^n (\log |\hat{x}|^2)^{k-n-q}}{n! (k-n)!}. \end{aligned} \quad (4.197)$$

The last summand can be implemented explicitly using (4.194),

$$\sum_{q=0}^k \sum_{n=0}^{k-q} \frac{(-1)^{k-n-q} k!}{n! (k-n)!} a_{\ell,k-n-q,k-n} (\log |x_\perp|)^n (\log |\hat{x}|^2)^{k-n-q} = \sum_{q=0}^k a_{\ell,k-q,k} \left(\log \frac{|x_\perp|}{|\hat{x}|^2} \right)^{k-q}, \quad (4.198)$$

which takes the expected form from conformal symmetry. In summary, (4.197) is reduced to

$$\langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}}^I(0) \rangle = \frac{1}{|x_\perp|^{\Delta-\hat{\Delta}} |\hat{x}|^{2\hat{\Delta}}} \sum_{\ell=0}^{\infty} t^\ell \sum_{s=0}^{N_{\hat{\Delta}}-I} \left(\log \frac{|x_\perp|}{|\hat{x}|^2} \right)^s \sum_{J=1}^{N_{\hat{\Delta}}-I-s+1} \sum_{k=s}^{N_{\hat{\Delta}}-I-J+1} \frac{(-1)^{k+s} k_{I+J+k-1} a_{\ell,s,k}}{k!} A_{J,\ell}^{N_{\hat{\Delta}}}. \quad (4.199)$$

Evaluation of (4.64)

As mentioned in the main text, we fix $A_{J,\ell}^{N_{\hat{\Delta}}}$ by comparing (4.197) with the general structure of the bulk-defect two-point function which was given in (4.64). Therefore, what we should do next is to express (4.64) as a power series of $t \equiv |x_\perp|^2/|\hat{x}|^2$. For this purpose, we first note the following series expansion formulas:

$$\begin{aligned} \frac{1}{|x|^{2\hat{\Delta}}} &= \frac{1}{|\hat{x}|^{2\hat{\Delta}}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (\hat{\Delta})_m t^m, \\ \tau^k &= \sum_{n=0}^{\infty} \sum_{q=0}^k \frac{(-1)^n S_1(n,q) k!}{n! (k-q)!} \left(\log \frac{|x_\perp|}{|\hat{x}|^2} \right)^{k-q} t^n, \end{aligned} \quad (4.200)$$

where $S_1(n,q)$ is the unsigned Stirling number of first kind, and its generating function is given by

$$(\log(1+t))^q = q! \sum_{n=0}^{\infty} \frac{(-1)^{n-q} S_1(n,q)}{n!} t^n. \quad (4.201)$$

Making use of these Taylor expansions, one can deform (4.64) as

$$\begin{aligned}
\langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}}^I(0) \rangle &= \frac{1}{|x_\perp|^{\Delta-\hat{\Delta}}|x|^{2\hat{\Delta}}} \sum_{k=0}^{N_{\hat{\Delta}}-I} \frac{\lambda_{I+k} \tau^k}{k!} \\
&= \frac{1}{|x_\perp|^{\Delta-\hat{\Delta}}|\hat{x}|^{2\hat{\Delta}}} \sum_{k=0}^{N_{\hat{\Delta}}-I} \lambda_{I+k} \sum_{q=0}^k \sum_{m,n=0}^{\infty} \frac{(-1)^{n+m} (\hat{\Delta})_m S_1(n, q)}{m! n! (k-q)!} \left(\log \frac{|x_\perp|}{|\hat{x}|^2} \right)^{k-q} t^{n+m} \\
&= \frac{1}{|x_\perp|^{\Delta-\hat{\Delta}}|\hat{x}|^{2\hat{\Delta}}} \sum_{\ell=0}^{\infty} (-1)^\ell t^\ell \sum_{k=0}^{N_{\hat{\Delta}}-I} \lambda_{I+k} \sum_{q=0}^k \frac{1}{(k-q)!} \left(\log \frac{|x_\perp|}{|\hat{x}|^2} \right)^{k-q} F_{\ell,q} \\
&= \frac{1}{|x_\perp|^{\Delta-\hat{\Delta}}|\hat{x}|^{2\hat{\Delta}}} \sum_{\ell=0}^{\infty} (-1)^\ell t^\ell \sum_{s=0}^{N_{\hat{\Delta}}-I} \left(\log \frac{|x_\perp|}{|\hat{x}|^2} \right)^s \sum_{k=s}^{N_{\hat{\Delta}}-I} \frac{\lambda_{I+k}}{s!} F_{\ell,k-s},
\end{aligned} \tag{4.202}$$

where $F_{\ell,q}$ is defined by

$$F_{\ell,q} \equiv \sum_{m=0}^{\ell-q} \frac{(\hat{\Delta})_m S_1(\ell-m, q)}{m! (\ell-m)!}. \tag{4.203}$$

Noting that the $F_{\ell,q}$ is the coefficient of y^q of the following polynomial:

$$\frac{1}{\ell!} \sum_{m=0}^{\ell-q} \binom{\ell}{m} (\hat{\Delta})_m (y)_{\ell-m} = \frac{1}{\ell!} (\hat{\Delta} + y)_\ell, \tag{4.204}$$

and one can express $F_{\ell,q}$ as follows

$$F_{\ell,q} = \frac{1}{\ell! q!} \partial_{\hat{\Delta}}^q (\hat{\Delta})_\ell. \tag{4.205}$$

Finally, by comparing (4.199) with (4.202), one can obtain the constraints on $A_{J,\ell}^{N_{\hat{\Delta}}}$ as

$$\sum_{J=1}^{N_{\hat{\Delta}}-I_s+1} \sum_{k=0}^{N_{\hat{\Delta}}-I_s-J+1} \frac{k_{I_s+J+k-1} \partial_{\hat{\Delta}}^k a_{\ell,0,0}}{k!} A_{J,\ell}^{N_{\hat{\Delta}}} = (-1)^\ell \sum_{k=0}^{N_{\hat{\Delta}}-I_s} \lambda_{I_s+k} F_{\ell,k}, \tag{4.206}$$

where $I_s \equiv I + s$, and $I \leq I_s \leq N_{\hat{\Delta}}$. Notably, the dependence on I and s appears only through the combination I_s , which implies that there are $N_{\hat{\Delta}}$ independent constraints on $A_{J,\ell}^{N_{\hat{\Delta}}}$ at each level ℓ . As a consistency check, by setting $\ell = 0$, the above constraint becomes (4.87). After some algebras, by using the expression (4.87), the above constraint reduces to,

$$\sum_{k=0}^{J-1} G_k A_{J-k,\ell}^{N_{\hat{\Delta}}} = \frac{(-1)^\ell}{\ell!} \sum_{k=0}^{J-1} H_k A_{J-k,0}^{N_{\hat{\Delta}}}, \tag{4.207}$$

where G_k and H_k are defined

$$G_k \equiv \frac{\partial_{\hat{\Delta}}^k a_{\ell,0,0}}{k!}, \quad H_k \equiv \frac{\partial_{\hat{\Delta}}^k (\hat{\Delta})_\ell}{k!}. \tag{4.208}$$

To further simplify these constraints, we introduce the following formal generating functions:

$$\begin{aligned}
A_\ell(z) &\equiv \sum_{k=0}^{\infty} A_{k,\ell}^{N_{\hat{\Delta}}} z^k, \\
G(z) &\equiv \sum_{k=0}^{\infty} G_k z^k = 4^\ell (\hat{\Delta} + z)_\ell \left(\hat{\Delta} + z - \frac{p-2}{2} \right)_\ell, \\
H(z) &\equiv \sum_{k=0}^{\infty} H_k z^k = (\hat{\Delta} + z)_\ell.
\end{aligned} \tag{4.209}$$

Then, we should notice that, in (4.207), the left and right hand sides correspond to the z^J coefficients of the polynomials $G(z)A_\ell(z)$ and $H(z)A_0(z)$, respectively. This identification allows us to rewrite the above constraint in a drastically simpler form:

$$G(z)A_\ell(z) = \frac{(-1)^\ell}{\ell!} H(z)A_0(z). \tag{4.210}$$

Therefore, by using the explicit forms of $G(z)$ and $H(z)$ in (4.209), one can obtain the generating function of all logDOE coefficients:

$$A_\ell(z) = \frac{(-4)^{-\ell}}{\ell! \left(\hat{\Delta} + z - \frac{p-2}{2} \right)_\ell} A_0(z). \tag{4.211}$$

From this expression, we arrive at the explicit form of $A_{J,\ell}^{N_{\hat{\Delta}}}$:

$$A_{J,\ell}^{N_{\hat{\Delta}}} = \frac{(-4)^{-\ell}}{\ell! \left(\hat{\Delta} + z - \frac{p-2}{2} \right)_\ell} \sum_{m=1}^J (-1)^{J+m} A_{m,0}^{N_{\hat{\Delta}}} \sum_{0 \leq k_1 < \dots < k_{J-m} \leq \ell-1} \prod_{r=1}^{J-m} \left(\hat{\Delta} - \frac{p-2}{2} + k_r \right)^{-1}. \tag{4.212}$$

4.C Derivation of (4.118)

In this Appendix, we derive the asymptotic formula (4.118). For simplicity, we denote $N_{\hat{\Delta},s}$ simply by N below. Firstly, by using the expression of logDOE (4.75), we obtain

$$\begin{aligned}
W_{\hat{\Delta}}^N &\stackrel{\chi \sim \infty}{\sim} \frac{k_{\hat{\mathcal{O}}}}{|x_{1\perp}|^{\Delta_1 - \hat{\Delta}} |x_{2\perp}|^{\Delta_2 - \hat{\Delta}}} \frac{x_{1\perp}^{(i_1)} \dots x_{1\perp}^{(i_s)} x_{2\perp i_1} \dots x_{2\perp i_s}}{|x_{1\perp}|^s |x_{2\perp}|^s} \frac{1}{|\hat{x}_{12}|^{2\hat{\Delta}}} \\
&\cdot \sum_{I_1=1}^N \sum_{I_2=1}^{N-I_1+1} \sum_{n=0}^{I_1-1} \sum_{m=0}^{I_2-1} A_{I_1-n,0} A_{I_2-m,0} \frac{(\log |x_{1\perp}|)^n (\log |x_{2\perp}|)^m (-\log |\hat{x}_{12}|^2)^{N-I_1-I_2+1}}{n! m! (N - I_1 - I_2 + 1)!} \\
&= \frac{a_s k_{\hat{\mathcal{O}}}}{|x_{1\perp}|^{\Delta_1 - \hat{\Delta}} |x_{2\perp}|^{\Delta_2 - \hat{\Delta}}} H_{W_{\hat{\mathcal{O}}}}^N(\phi) \frac{1}{|\hat{x}_{12}|^{2\hat{\Delta}}} \\
&\cdot \sum_{q=1}^N \sum_{r=1}^{N+1-q} A_{q,0}^N A_{r,0}^N \sum_{I_1=1}^{N+1-r} \sum_{I_2=r}^{N-I_1+1} \frac{(\log |x_{1\perp}|)^{I_1-q} (\log |x_{2\perp}|)^{I_2-r} (-\log |\hat{x}_{12}|^2)^{N-I_1-I_2+1}}{(I_1 - q)! (I_2 - r)! (N - I_1 - I_2 + 1)!}.
\end{aligned} \tag{4.213}$$

In the second line, we change the order of summations and used the following identity:

$$\begin{aligned} \frac{x_{1\perp}^{(i_1)} \cdots x_{1\perp}^{(i_s)} x_{2\perp i_1} \cdots x_{2\perp i_s}}{|x_{1\perp}|^s |x_{2\perp}|^s} &= \frac{\Gamma(s+1)\Gamma(\frac{d-p}{2}-1)}{2^s \Gamma(\frac{d-p}{2}+s-1)} C_s^{(\frac{d-p}{2}-1)}(\cos \phi) \\ &= a_s H_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta}},s}(\phi), \end{aligned} \quad (4.214)$$

where $H_{W_{\hat{\mathcal{O}}}}^{N_{\hat{\Delta}},s}(\phi)$ is given by (4.113), and a_s is defined in (4.119). Moreover, by using the multinomial theorem:

$$\sum_{a+b+c=L} \frac{s^a t^b u^c}{a!b!c!} = \frac{(s+t+u)^L}{L!}, \quad (4.215)$$

the expression (4.213) is simplified to

$$\begin{aligned} W_{\hat{\Delta}}^N &\stackrel{\chi \sim \infty}{\sim} \frac{a_s k_{\hat{\mathcal{O}}}}{|x_{1\perp}|^{\Delta_1-\hat{\Delta}} |x_{2\perp}|^{\Delta_2-\hat{\Delta}}} H_{W_{\hat{\mathcal{O}}}}^N(\phi) \sum_{q=1}^N \sum_{r=1}^{N+1-q} A_{q,0}^N A_{r,0}^N \frac{1}{(N+1-q-r)!} \frac{(-\log \chi)^{N+1-q-r}}{|\hat{x}_{12}|^{2\hat{\Delta}}} \\ &= \frac{a_s k_{\hat{\mathcal{O}}}}{|x_{1\perp}|^{\Delta_1-\hat{\Delta}} |x_{2\perp}|^{\Delta_2-\hat{\Delta}}} H_{W_{\hat{\mathcal{O}}}}^N(\phi) \sum_{k=0}^{N-1} \sum_{q=1}^{N-k} A_{q,0}^N A_{N+1-k-q,0}^N \frac{1}{k!} \frac{(-\log \chi)^k}{|\hat{x}_{12}|^{2\hat{\Delta}}} \\ &= \frac{a_s k_{\hat{\mathcal{O}}}^{-1}}{|x_{1\perp}|^{\Delta_1-\hat{\Delta}} |x_{2\perp}|^{\Delta_2-\hat{\Delta}}} H_{W_{\hat{\mathcal{O}}}}^N(\phi) \sum_{k=0}^{N-1} \sum_{q=1}^{N-k} \lambda_{N+1-q} \lambda_{k+q} \frac{1}{k!} \frac{(-\log \chi)^k}{|\hat{x}_{12}|^{2\hat{\Delta}}}, \end{aligned} \quad (4.216)$$

where in the final line, we used the relation (4.92). By comparing this expression with (4.117), one can fix the coefficient C_k as (4.120).

4.D Constraints from bulk equation of motion

In this Appendix, we prove the statements described (4.132) and (4.133) in order.

Proof of (4.132)

We start with giving the proof of the statement (4.132). This can be shown by contradiction. Suppose that the associated module is non-trivial, i.e. $\mathcal{V}_{\sigma}^{N_{\hat{\Delta}},s} \neq \emptyset$, $N_{\hat{\Delta},s} \geq 2$ and $s \geq 1$. The constraint (4.130) at $J = 1$ reads

$$(\hat{\Delta} - \Delta_{\sigma} + s)(\hat{\Delta} - \Delta_{\sigma} - s)\alpha_1 = 0. \quad (4.217)$$

Our assumption implies that there is non-zero solution for α_1 , hence the above equation restricts the defect conformal dimensions as $\hat{\Delta} = \Delta_{\sigma} \pm s$. However, with these dimensions, one can show that there is no non-zero solution compatible with other constraints with $J \neq 1$. Indeed, the constraint (4.130) with $J = 2$ reads $s\alpha_1 = 0$, which forces $\alpha_1 = 0$ for $s \geq 1$. This completes the proof for (4.132). Also, the above proof is enough to show the statement (4.133) by setting $s = 0$.

Proof of (4.133)

We next move on to the proof for (4.133). For simplicity, we give the proof of $\mathcal{V}_\sigma^{N_{\hat{\Delta},0}=3} = \emptyset$, although the generalization to generic rank $N_{\hat{\Delta},0}$ is quite straightforward. In the case of rank three, the constraints associated to equation of motion read

$$(\hat{\Delta} - \Delta_\sigma)^2 \alpha_1 = 0, \quad (4.218)$$

$$2(\hat{\Delta} - \Delta_\sigma) \alpha_1 + (\hat{\Delta} - \Delta_\sigma)^2 \alpha_2 = 0, \quad (4.219)$$

$$\alpha_1 + 2(\hat{\Delta} - \Delta_\sigma) \alpha_2 + (\hat{\Delta} - \Delta_\sigma)^2 \alpha_3 = 0. \quad (4.220)$$

We show that α_1 must be trivial again by the contradiction. Suppose that $\alpha_1 \neq 0$, the first constraint (4.218) implies again that the defect conformal dimensions are restricted as $\hat{\Delta} = \Delta_\sigma$. Then, the final constraint (4.220) forces α_1 to be zero, which contradicts our assumption. This completes the proof.

4.E Derivation of selected integrals

We first list two integrals that are useful in performing the Fourier transforms:

$$\int \frac{d^n k}{(2\pi)^n} \frac{e^{ik \cdot x}}{|k|^{2\alpha}} = \frac{\Gamma(\frac{n}{2} - \alpha)}{\pi^{n/2} 4^\alpha \Gamma(\alpha)} \frac{1}{|x|^{n-2\alpha}}, \quad (4.221)$$

and the following generalization that involves mixed bulk-defect coordinates,

$$\frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{\alpha}{2})} \int \frac{d^p \hat{k}}{(2\pi)^p} e^{i\hat{k} \cdot \hat{x}} |k|^{\frac{\alpha-p}{2}} K_{\frac{p-\alpha}{2}}(|\hat{k}| |x_\perp|) = \frac{|2x_\perp|^{\frac{\alpha-p}{2}}}{(\hat{x}^2 + x_\perp^2)^{\frac{\alpha}{2}}}, \quad (4.222)$$

where $K_\nu(x)$ is the Bessel K -function.

4.E.1 Integral formula (4.160)

Here we present the derivation of (4.160) by solving the following integral:

$$\mathcal{I}_1 = \int \frac{d^p \hat{p}}{(2\pi)^{p+1}} K_0(|\hat{p}| |x_\perp|) \log |\hat{p}|^2 e^{i\hat{p} \cdot \hat{x}}. \quad (4.223)$$

In the following, we will make use of the Mellin representation of the Bessel K -function

$$K_0(2y) = \frac{1}{8\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} ds y^{-s} \Gamma\left(\frac{s}{2}\right)^2, \quad (4.224)$$

with $\epsilon > 0$ and the following two integral identities involving the Bessel J -function:

$$\int d\Omega_p e^{i\hat{p} \cdot \hat{x}} = \frac{(2\pi)^{\frac{p}{2}}}{(\rho|\hat{x}|)^{\frac{p-2}{2}}} J_{\frac{p-2}{2}}(\rho|\hat{x}|) \quad (4.225)$$

where the integral is over the unit sphere S^{p-1} and $\rho \equiv |\hat{p}|$, and from [156]

$$\int_0^\infty d\rho \rho^{\mu+\frac{1}{2}} \log \rho J_\nu(\rho r) = \frac{2^{\mu-\frac{1}{2}} \Gamma\left(\frac{\nu+\mu}{2} + \frac{3}{4}\right)}{r^{\mu+\frac{3}{2}} \Gamma\left(\frac{\nu-\mu}{2} + \frac{1}{4}\right)} \left(\psi\left(\frac{\nu+\mu}{2} + \frac{3}{4}\right) + \psi\left(\frac{\nu-\mu}{2} + \frac{1}{4}\right) - \log \frac{r^2}{4} \right), \quad (4.226)$$

where $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ is the digamma function.

In particular, with $\mu = \frac{p-1}{2} - s$ and $\nu = \frac{p-2}{2}$ in the above identities, we have

$$\mathcal{I}_1 = \frac{1}{|\hat{x}|^p} \frac{1}{\pi^{p/2+1}} \frac{1}{16\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} ds \left(\frac{|\hat{x}|}{|x_\perp|} \right)^s \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{p-s}{2}\right) \left(\psi\left(\frac{s}{2}\right) + \psi\left(\frac{p-s}{2}\right) - \log \frac{|\hat{x}|^2}{4} \right). \quad (4.227)$$

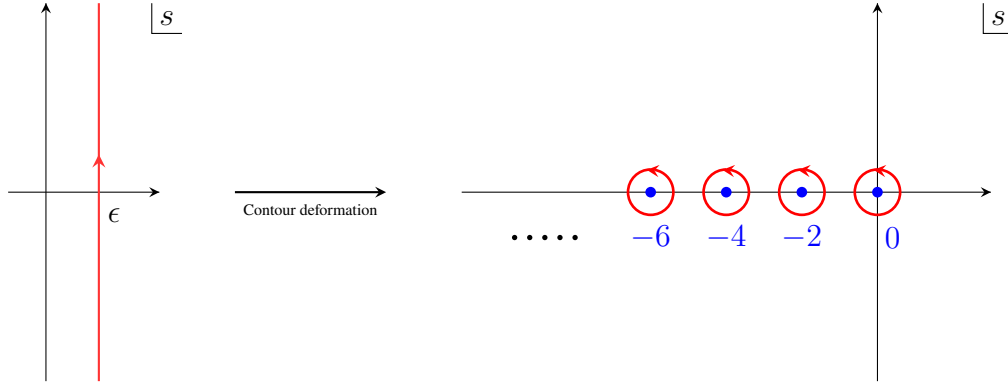


Figure 4.5: Contour deformation of the integral (4.227).

To evaluate the integral, we deform the contour to the left (assuming $|\hat{x}| > |x_\perp|$) and pick up the residues at the poles $s = 0, -2, -4, \dots$ (see also Figure 4.5):

$$\mathcal{I}_1 = \frac{1}{|\hat{x}|^p} \frac{1}{\pi^{p/2+1}} \frac{1}{8} \sum_{n=0}^{\infty} \frac{2(-1)^n}{n!} \left(\frac{|x_\perp|}{|\hat{x}|} \right)^{2n} \Gamma\left(n + \frac{p}{2}\right) \left(\log \frac{4x_\perp^2}{|\hat{x}|^4} + \psi\left(n + \frac{p}{2}\right) \right). \quad (4.228)$$

Performing the summation, this gives what we want to show,

$$\mathcal{I}_1 = \frac{\Gamma\left(\frac{p}{2}\right)}{2\pi^{\frac{p}{2}+1}|x|^p} \left(\log \frac{|x_\perp|}{|x|^2} + \log 2 + \psi\left(\frac{p}{2}\right) \right). \quad (4.229)$$

4.E.2 Integral formula (4.172)

Finally, we evaluate the following integral,

$$\mathcal{I}_2 = \int \frac{d^p \hat{p}}{(2\pi)^{p+2}} K_0(|\hat{p}||x_\perp|) K_0(|\hat{p}||y_\perp|) e^{i\hat{p} \cdot (\hat{x} - \hat{y})}. \quad (4.230)$$

We again use the Mellin representation of the Bessel- K function (4.224),

$$\mathcal{I}_2 = \frac{1}{(8\pi i)^2} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} ds \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dt \frac{2^{s+t}}{|x_\perp|^s |y_\perp|^t} \Gamma^2\left(\frac{s}{2}\right) \Gamma^2\left(\frac{t}{2}\right) \int \frac{d^p \hat{p}}{(2\pi)^{p+2}} \frac{e^{i\hat{p} \cdot (\hat{x} - \hat{y})}}{|\hat{p}|^{s+t}}. \quad (4.231)$$

Firstly, we perform the integral over $\hat{p}$ using (4.225) and the following formula,

$$\int_0^\infty d\rho \rho^\mu J_\nu(a\rho) = \frac{2^\mu}{a^{\mu+1}} \frac{\Gamma(\frac{1+\mu+\nu}{2})}{\Gamma(\frac{1+\nu-\mu}{2})}, \quad (4.232)$$

and the result is

$$\mathcal{I}_2 = -\frac{1}{256\pi^{p/2+4}} \frac{1}{|\hat{x} - \hat{y}|^p} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} ds \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dt \xi_1^{-s} \xi_2^{-t} \frac{\Gamma^2\left(\frac{s}{2}\right) \Gamma^2\left(\frac{t}{2}\right) \Gamma(\frac{p-s-t}{2})}{\Gamma(\frac{s+t}{2})}. \quad (4.233)$$

Here, ξ_1 and ξ_2 are the cross-ratios defined by

$$\xi_1 \equiv \frac{|x_\perp|}{|\hat{x} - \hat{y}|}, \quad \xi_2 \equiv \frac{|y_\perp|}{|\hat{x} - \hat{y}|}. \quad (4.234)$$

We are interested in the defect block expansion, namely with $0 < \xi_i < 1$ ($i = 1, 2$).

We next evaluate the following integral,

$$\tilde{\mathcal{I}}_2 \equiv \int_{-i\infty+\epsilon}^{i\infty+\epsilon} ds \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dt \xi_1^{-s} \xi_2^{-t} \frac{\Gamma^2\left(\frac{s}{2}\right) \Gamma^2\left(\frac{t}{2}\right) \Gamma(\frac{p-s-t}{2})}{\Gamma(\frac{s+t}{2})}, \quad (4.235)$$

by deforming the contour to the left and using the residue theorem. As in Fig. 4.5, this picks up the poles at $t = 0, -2, -4, \dots$ and gives,

$$\begin{aligned} & \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dt \xi_2^{-t} \frac{\Gamma^2\left(\frac{t}{2}\right) \Gamma(\frac{p-s-t}{2})}{\Gamma(\frac{s+t}{2})} \\ &= 4\pi i \sum_{n=0}^{\infty} \xi_2^{2n} \frac{\Gamma(n + \frac{p-s}{2})}{(n!)^2 \Gamma(-n + \frac{s}{2})} \left(2\psi(n+1) - \psi\left(n + \frac{p-s}{2}\right) - \psi\left(-n + \frac{s}{2}\right) - 2\log \xi_2 \right). \end{aligned} \quad (4.236)$$

We next deform the contour associated to s -integral to the left, and pick up the poles at $s = 0, -2, -4, \dots$. This enables us to express the integral $\tilde{\mathcal{I}}_2$ as a double sum,

$$\begin{aligned} \tilde{\mathcal{I}}_2 &= -32\pi^2 \sum_{n,m=0}^{\infty} \frac{(-\xi_1^2)^m (-\xi_2^2)^n}{(m+n)!} \binom{m+n}{m}^2 \Gamma\left(m+n + \frac{p}{2}\right) \\ &\quad \times \left(\psi(n+1) + \psi(m+1) - \psi(m+n+1) - \psi\left(m+n + \frac{p}{2}\right) - \log \xi_1 \xi_2 \right). \end{aligned} \quad (4.237)$$

In summary so far, we obtain the following expression of $\mathcal{I}_2$

$$\mathcal{I}_2 = \frac{1}{8\pi^{\frac{p}{2}+2}} \frac{1}{|x_\perp|^{p/2} |y_\perp|^{p/2}} [\mathcal{K}_1 - \mathcal{K}_2 - \log(1 + \xi_1^2 + \xi_2^2) \mathcal{K}_3 + \log \chi \mathcal{K}_3], \quad (4.238)$$

where $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are given by

$$\begin{aligned}
\mathcal{K}_1 &\equiv \chi^{-\frac{p}{2}} (1 + \xi_1^2 + \xi_2^2)^{\frac{p}{2}} \sum_{n,m=0}^{\infty} \frac{(-\xi_1^2)^m (-\xi_2^2)^n}{(m+n)!} \binom{m+n}{m}^2 \Gamma\left(m+n+\frac{p}{2}\right) \\
&\quad \cdot (\psi(n+1) + \psi(m+1) - \psi(m+n+1)) , \\
\mathcal{K}_2 &\equiv \chi^{-\frac{p}{2}} (1 + \xi_1^2 + \xi_2^2)^{\frac{p}{2}} \sum_{n,m=0}^{\infty} \frac{(-\xi_1^2)^m (-\xi_2^2)^n}{(m+n)!} \binom{m+n}{m}^2 \Gamma\left(m+n+\frac{p}{2}\right) \psi\left(m+n+\frac{p}{2}\right) , \\
\mathcal{K}_3 &\equiv \chi^{-\frac{p}{2}} (1 + \xi_1^2 + \xi_2^2)^{\frac{p}{2}} \sum_{n,m=0}^{\infty} \frac{(-\xi_1^2)^m (-\xi_2^2)^n}{(m+n)!} \binom{m+n}{m}^2 \Gamma\left(m+n+\frac{p}{2}\right)
\end{aligned} \tag{4.239}$$

Note that χ is written in terms of ξ_1 and ξ_2 as

$$\chi = \frac{1 + \xi_1^2 + \xi_2^2}{\xi_1 \xi_2} . \tag{4.240}$$

Introducing a new quantity $\mathcal{K}(\Delta)$ as

$$\mathcal{K}(\Delta) \equiv \chi^{-\frac{p}{2}} (1 + \xi_1^2 + \xi_2^2)^{\frac{p}{2}} \sum_{n,m=0}^{\infty} \frac{(-\xi_1^2)^m (-\xi_2^2)^n}{(m+n)!} \binom{m+n}{m}^2 \Gamma(m+n+\Delta) , \tag{4.241}$$

and the double sums $\mathcal{K}_2$ and $\mathcal{K}_3$ can be written in terms of $\mathcal{K}$ as

$$\mathcal{K}_2 = \partial_{\Delta} \mathcal{K} \left(\frac{p}{2} \right) , \quad \mathcal{K}_3 = \mathcal{K} \left(\frac{p}{2} \right) . \tag{4.242}$$

This motivate us to firstly evaluate $\mathcal{K}(\Delta)$,

$$\begin{aligned}
\mathcal{K}(\hat{\Delta}) &= \chi^{-\frac{p}{2}} (1 + \xi_1^2 + \xi_2^2)^{\frac{p}{2}} \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell + \hat{\Delta}) (\xi_2^2 - \xi_1^2)^{\ell}}{\ell!} P_{\ell} \left(\frac{\xi_1^2 + \xi_2^2}{\xi_1^2 - \xi_2^2} \right) \\
&= \chi^{-\frac{p}{2}} (1 + \xi_1^2 + \xi_2^2)^{\frac{p}{2}} \int_0^{\infty} dt t^{\hat{\Delta}-1} e^{-(1+\xi_1^2+\xi_2^2)t} I_0(2\xi_1\xi_2 t) \\
&= \Gamma(\hat{\Delta}) (1 + \xi_1^2 + \xi_2^2)^{\frac{p}{2}-\hat{\Delta}} \chi^{-\frac{p}{2}} {}_2F_1 \left(\frac{\hat{\Delta}}{2}, \frac{\hat{\Delta}+1}{2}; 1; \frac{4}{\chi^2} \right) .
\end{aligned} \tag{4.243}$$

From this, one can derive $\mathcal{K}_2$ and $\mathcal{K}_3$ as follows:

$$\begin{aligned}
\mathcal{K}_2 &= \Gamma\left(\frac{p}{2}\right) \left[\left(\psi\left(\frac{p}{2}\right) - \log(1 + \xi_1^2 + \xi_2^2) \right) \mathbf{G}_{\{\hat{\sigma}, \hat{h}\}}(\chi) + \chi^{-\frac{p}{2}} \partial_{\hat{\Delta}} {}_2F_1 \left(\frac{\hat{\Delta}}{2}, \frac{\hat{\Delta}+1}{2}; 1; \frac{4}{\chi^2} \right) \right]_{\hat{\Delta}=p/2} \\
\mathcal{K}_3 &= \Gamma\left(\frac{p}{2}\right) \chi^{-\frac{p}{2}} {}_2F_1 \left(\frac{p}{4}, \frac{p+2}{4}; 1; \frac{4}{\chi^2} \right) = \Gamma\left(\frac{p}{2}\right) \mathbf{G}_{\{\hat{\sigma}, \hat{h}\}}(\chi) .
\end{aligned} \tag{4.244}$$

By plugging these results into (4.239), one can obtain

$$\mathcal{I}_2 = -\frac{\Gamma(p/2)}{8\pi^{\frac{p}{2}+2}} \frac{1}{|x_\perp|^{p/2}|y_\perp|^{p/2}} \left[H_{\frac{p}{2}-1} \mathbf{G}_{\hat{\Delta}}(\chi) + \partial_{\hat{\Delta}} \mathbf{G}_{\hat{\Delta}}(\chi) \Big|_{\hat{\Delta}=p/2} - \mathcal{R} \right], \quad (4.245)$$

where $\mathcal{R}$ is given by

$$\mathcal{R} = \frac{1}{\Gamma(p/2)} \mathcal{K}_1 + \gamma_E \mathbf{G}_{\{\hat{\sigma}, \hat{h}\}}(\chi) + \chi^{-\frac{p}{2}} \frac{\partial}{\partial a} {}_2F_1 \left(\frac{p}{4}, \frac{p+2}{4}; a; \frac{4}{\chi^2} \right) \Big|_{a=1}. \quad (4.246)$$

By expanding $\mathcal{R}$ in powers of ξ_1 and ξ_2 and examining the coefficients, one finds that $\mathcal{R} = 0$. The desired integral formula (4.172) then follows.

4.F Bulk conformal block expansion in free scalar theory

In this Appendix, we derive the bulk-channel conformal block expansion (4.181) by directly evaluating the integral $\mathcal{I}_2$ that appears in the bulk two-point function (4.167). In fact we will solve the following more general integral,

$$I_{\alpha,\beta} \equiv \int d^p \hat{z} \frac{1}{|x - \hat{z}|^{2\alpha} |y - \hat{z}|^{2\beta}}, \quad (4.247)$$

which may be useful for more general DCFT setups. Later we will specialize to the case $\alpha = \beta = \frac{p}{2}$, which is relevant for (4.167) that describes the free scalar theory with subdimensional disorder.

By using the Schwinger reparameterization, the integral (4.247) becomes

$$\begin{aligned} I_{\alpha,\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int d^p \hat{z} \int_0^\infty du u^{\alpha-1} e^{-u|x-\hat{z}|^2} \cdot \int_0^\infty dv v^{\beta-1} e^{-v|y-\hat{z}|^2} \\ &= \frac{\pi^{p/2}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty dudv u^{\alpha-1} v^{\beta-1} (u+v)^{-p/2} e^{-\frac{uv}{u+v}|\hat{x}-\hat{y}|^2} e^{-u|x_\perp|^2-v|y_\perp|^2}, \end{aligned} \quad (4.248)$$

where, in the final equality, we have performed the integral over $\hat{z}$. Next we expand the last exponential in terms of the power of u and v ,

$$I_{\alpha,\beta} = \frac{\pi^{p/2}}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n,m=0}^\infty \frac{(-|x_\perp|^2)^n (-|y_\perp|^2)^m}{n! m!} \int_0^\infty dudv u^{n+\alpha-1} v^{m+\beta-1} (u+v)^{-p/2} e^{-\frac{uv}{u+v}|\hat{x}-\hat{y}|^2}. \quad (4.249)$$

By changing the integral variables as

$$u = yt, \quad v = y(1-t), \quad (4.250)$$

under which the integration measure turns into

$$\int_0^\infty dudv \mapsto \int_0^\infty y dy \int_0^1 dt, \quad (4.251)$$

the integral becomes

$$I_{\alpha,\beta} = \frac{\pi^{p/2}}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n,m=0}^{\infty} \frac{(-|x_{\perp}|^2)^n (-|y_{\perp}|^2)^m}{n! m!} \int_0^1 dt t^{n+\alpha-1} (1-t)^{m+\beta-1} \times \int_0^{\infty} dy y^{n+m+\alpha+\beta-\frac{p}{2}-1} e^{-yt(1-t)|\hat{x}-\hat{y}|^2}. \quad (4.252)$$

After another change of variable by,

$$y = \frac{\check{y}}{t(1-t)|\hat{x}-\hat{y}|^2}, \quad (4.253)$$

the above integral can be written as,

$$I_{\alpha,\beta} = \frac{\pi^{p/2} F_{\alpha,\beta}}{\Gamma(\alpha)\Gamma(\beta)|\hat{x}-\hat{y}|^{2\alpha+2\beta-p}}, \quad (4.254)$$

where $F_{\alpha,\beta}$ is defined by

$$F_{\alpha,\beta} \equiv \sum_{n,m=0}^{\infty} \frac{\Gamma(n+m+\alpha+\beta-\frac{p}{2})}{n! m!} A^n B^m \int_0^1 dt t^{\frac{p}{2}-\beta-m-1} (1-t)^{\frac{p}{2}-\alpha-n-1}, \quad (4.255)$$

$$A \equiv -\frac{|x_{\perp}|^2}{|\hat{x}-\hat{y}|^2}, \quad B \equiv -\frac{|y_{\perp}|^2}{|\hat{x}-\hat{y}|^2}.$$

We can evaluate $F_{\alpha,\beta}$ as

$$\begin{aligned} F_{\alpha,\beta} &\equiv \sum_{k=0}^{\infty} \sum_{n=0}^k \frac{\Gamma(k+\alpha+\beta-\frac{p}{2})}{n! (k-n)!} A^n B^{k-n} \int_0^1 dt t^{\frac{p}{2}-\beta-k+n-1} (1-t)^{\frac{p}{2}-\alpha-n-1} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha+\beta-\frac{p}{2})}{k!} \int_0^1 dt t^{\frac{p}{2}-\beta-k-1} (1-t)^{\frac{p}{2}-\alpha-1} \left(B + \frac{At}{1-t} \right)^k \\ &= \Gamma\left(\alpha+\beta-\frac{p}{2}\right) \int_0^1 dt t^{\frac{p}{2}-\beta-1} (1-t)^{\frac{p}{2}-\alpha-1} \left(1 - \frac{A}{1-t} - \frac{B}{t} \right)^{\frac{p}{2}-\alpha-\beta} \\ &= \Gamma\left(\alpha+\beta-\frac{p}{2}\right) (-B)^{\frac{p}{2}-\alpha-\beta} \int_0^1 dt t^{\alpha-1} (1-t)^{\beta-1} (1-tx')^{\frac{p}{2}-\alpha-\beta} (1-ty')^{\frac{p}{2}-\alpha-\beta}, \end{aligned} \quad (4.256)$$

where x' and y' satisfy

$$x'y' = \frac{1}{B}, \quad x' + y' = -\frac{A-B-1}{B}. \quad (4.257)$$

Then, by using the following integral formula [19, eq.(B.22)],

$$\int_0^1 dt t^{w-1} (1-t)^{r-1} (1-tx')^{-\rho} (1-ty')^{-\sigma} = B(w, r) F_1(w, \rho, \sigma, w+r; x', y'), \quad (4.258)$$

where F_1 is the Appel function, $F_{\alpha,\beta}$ can be written as

$$F_{\alpha,\beta} = B(\alpha, \beta) \Gamma\left(\alpha + \beta - \frac{p}{2}\right) (-B)^{\frac{p}{2}-\alpha-\beta} F_1\left(\alpha, \alpha + \beta - \frac{p}{2}, \alpha + \beta - \frac{p}{2}, \alpha + \beta; x'; y'\right). \quad (4.259)$$

Particularly, when $w + r = \rho + \sigma$, the Appel function is reduced to the hypergeometric function via the following identity,

$$F_1(w, \rho, \sigma, \rho + \sigma; x'; y') = (1 - y')^w {}_2F_1\left(w, \rho, \rho + \sigma; \frac{x' + y'}{1 - y'}\right). \quad (4.260)$$

In our case, $\alpha = \beta = p/2$, so $F_{\alpha,\beta}$ can be simplified to

$$F_{p/2,p/2} = B\left(\frac{p}{2}, \frac{p}{2}\right) \Gamma\left(\frac{p}{2}\right) \left(\frac{1 - y'}{-B}\right)^{p/2} {}_2F_1\left(\frac{p}{2}, \frac{p}{2}, p; \frac{x' + y'}{1 - y'}\right). \quad (4.261)$$

To sum up, the bulk two-point function of σ is given by

$$\overline{\langle \sigma(x) \sigma(y) \rangle} = \frac{C_d}{|x - y|^p} + \frac{\pi^{\frac{p}{2}} \Gamma(\frac{p}{2})}{\Gamma(p)} C_d^2 c^2 \frac{1}{|\hat{x} - \hat{y}|^p} \left(\frac{1 - y'}{-B}\right)^{p/2} {}_2F_1\left(\frac{p}{2}, \frac{p}{2}, p; \frac{x' + y'}{1 - y'}\right). \quad (4.262)$$

Finally, by using the following identities,

$$\frac{|x_\perp| |y_\perp|}{|\hat{x} - \hat{y}|^2} \left(\frac{1 - y'}{-B}\right) = (X\bar{X})^{1/2}, \quad \frac{x' + y'}{1 - y'} = 1 - X\bar{X}, \quad (4.263)$$

one can show that the expression (4.262) perfectly matches the desired form (4.181) from the bulk-channel conformal block expansion.

Chapter 5

Composite Defects in Conformal Field Theories

This chapter mainly follows [157, 158].

5.1 Overview

Conformal field theories (CFTs) have emerged as cornerstones in describing critical phenomena in our universe. They provide profound insights into the physical behaviors of systems at critical points, and play a crucial role in elucidating universal features that transcend microscopic details. However, our spacetime typically hosts defects whose shapes are distorted. Introducing these defects into spacetime leads to a loss of symmetries such as translation symmetries. Consequently, we cannot predict the dynamics of these defects in a controllable way. Nevertheless, non-trivial predictions can be made for such defect systems when spacetime possesses conformal symmetry and the shape of the defect is restricted to be planar or spherical. This kind of defect is referred to as the conformal defect, and the CFT in the presence of the conformal defect as a defect CFT. Defect CFTs serve as indispensable tools for characterizing the dynamics of impurities, boundaries, or interfaces at critical points, ranging from condensed matter physics to high energy physics. In recent years, studies on various conformal properties for a single conformal defect in d -dimensional CFT⁶¹ have been deeply investigated in various directions e.g. kinematical structures [19, 21, 47, 159–162], dynamics of conformal defects [17, 20, 24, 26, 29, 34, 42, 44–46, 50, 127, 153, 163–194], renormalization group flows [124–126, 128–131, 195–205].

However, it is also possible to consider critical systems exhibiting hierarchical structures of defects where a p -dimensional conformal defect $\widehat{\mathcal{D}}^{(p)}$ itself hosts an additional r -dimensional one $\check{\mathcal{D}}^{(r)}$ with $r < p < d$. (In Fig. 5.1, we illustrate this setup both for two-dimensional quantum systems and three-dimensional continuum theory.) Throughout this chapter, we refer to the former and latter defects as *defect* and *sub-defect*, respectively. Since this kind of defect can be conceptualized as a “composite defect” consisting of the defect $\widehat{\mathcal{D}}^{(p)}$ and the sub-defect $\check{\mathcal{D}}^{(r)}$, we call the CFT in the presence of a composite defect as *composite defect CFT*. The conformal structure of this

⁶¹Here, d is greater than two.

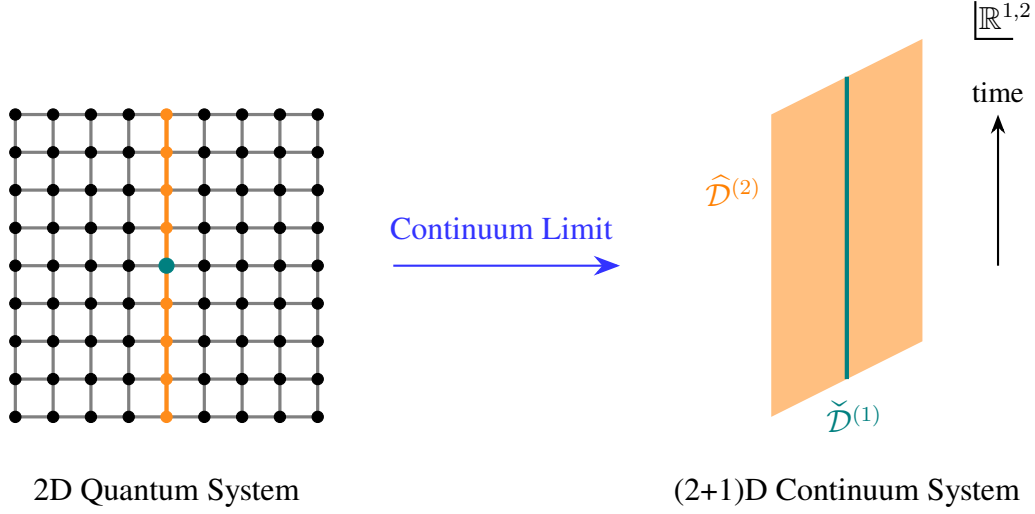


Figure 5.1: Our setup for composite defect CFTs. In the left panel, we illustrate a composite defect in a two-dimensional quantum system. In this system, a composite defect consists of an interface (orange line) that is contaminated by an impurity (teal point). In the right panel, we provide the schematic picture of the continuum theory which can be naively obtained by taking the limit of the left quantum system. In the continuum theory, an interface and an impurity become the worldsurface and worldline, respectively.

nested defect was firstly discussed in the context of wedge CFT [206] for specific defect dimensions $p = d - 1$ and $r = d - 2$, where a sub-defect is realized by folding a boundary at some angle. (See also [182, 207, 208].) We can naturally expect that the presence of a sub-defect gives rise to new universality classes that are unattainable within conventional defect CFTs and provides us with much richer phase structures of quantum field theories.

In this chapter, toward elucidating the conformal dynamics of composite defects, we firstly develop the theoretical framework in defect CFTs to encompass such composite defect CFTs, particularly paying our attention to symmetry structures, conformal correlators, operator expansions, and conformal block expansions. Our work is a generalization of the wedge CFT [206] (see also [182, 207]) to the CFTs in the presence of a nested defect with general co-dimensions.

Next, we also give the concrete models realizing composite defect CFTs. From the localized RG flow point of view, once relevant localized deformations are turned both on the higher-dimensional defect and the embedded one (the sub-defect), a conformal composite defect [157] arises when the two flows to non-trivial IR fixed points simultaneously. We aim to construct a simple model, where a non-trivial conformal composite defect appears at the Wilson-Fisher fixed point. This brought us to consider the $O(N)$ vector model. In $d = 4 - \varepsilon$ dimension, it is known that a line defect does not have a non-trivial IR fixed point in a free bulk theory [26]. Embedding such a line defect into a surface one in this dimension does not gain a non-trivial IR fixed point for the line defect either. Instead of turning on the bulk interaction, we consider a lower dimension with $d = 3 - \varepsilon$. To give

a conformal theory at the classical level, the action is required to be

$$I \equiv \frac{1}{2} \int_{\mathbb{R}^d} d^d x \partial_\mu \phi^I \partial^\mu \phi^I + \frac{h_0}{4!} \int_{\mathbb{R}^2} d^2 \hat{y} (\phi^I \phi^I)^2 + \frac{g_{0,IJ}}{2} \int_{\mathbb{R}} d\tilde{z} \phi^I \phi^J, \quad (5.1)$$

where x , $\hat{y}$ and $\tilde{z}$ denote the coordinates for the d -dimensional bulk $\mathbb{R}^d$, two-dimensional surface defect $\widehat{\mathcal{D}}^{(2)}$ and the embedded line defect $\check{\mathcal{D}}^{(1)}$ respectively.⁶² The configuration is drawn in Fig. 5.1. In the action (5.1), ϕ^I is the $O(N)$ vector field, with index I runs from 1 to N ; h_0 and $g_{0,IJ}$ are the bare couplings for the surface defect and line defect accordingly. We consider the case where the $O(N)$ symmetry is preserved on the surface defect, while on the line it may be broken depending on the form of the coupling $g_{0,IJ}$. This model allows for perturbative examination both for the surface and line couplings. As either one is turned off, the other one finds non-trivial IR fixed point, proportional to ε at one-loop order. When both couplings are turned on, the line coupling receives a contribution from the surface coupling and reaches the IR fixed points simultaneously with the surface one. In this sense, a conformal composite defect exists in the model we constructed, even when the bulk theory is free.⁶³ We further analyze the $O(N)$ structures in the model we constructed, with focus on the $O(N)$ breaking pattern $O(N) \rightarrow O(m) \times O(N-m)$ characterized by two coupling constants (g_0, g'_0) on the line defect. It turns out that conformal composite defects appear not only on the two $O(N)$ symmetric fixed points, but also on the two $O(N)$ symmetry breaking ones. Though in the latter case, N is required to be $N \geq 23$ to give a unitary theory. All the IR fixed points share the same value of the surface coupling h_* at one-loop level, but differs from each other by the set of values of the line couplings (g_*, g'_*) . To understand the localized RG flows among the all the fixed points quantitatively, we conjecture a $\mathcal{C}$ -theorem for the sub-defect, or for the submost defect of a system with more coexisting defects. This is a generalization of the defect $\mathcal{C}$ -theorem for a single defect system and serves as the other main result of this work.

This chapter is structured as follows. In Sec. 5.2, we elaborate on the various model-independent properties of composite defect CFTs. In Sec. 5.2.1, we discuss the residual conformal symmetry in the presence of a composite defect. In Sec. 5.2.2, we explain how we can determine the form of scalar correlation functions in composite defect CFTs only from the residual symmetry. In Sec. 5.2.3, we discuss the operator expansions appearing in composite defect CFTs. We particularly show that a bulk local field can be expanded in terms of sub-defect local ones. In Sec. 5.2.4, we derive the conformal block expansions by focusing on a bulk one-point function and a bulk-sub-defect two-point one. The followings are dedicated to the analysis of the concrete model (5.1). In Sec. 5.3, as a warm-up, we consider the RG fixed points and the flows between them in a single scalar field with a composite defect (i.e., we set $N = 1$ in the action (5.1)). In Sec. 5.4, we analyze to the $O(N)$ vector model we have constructed with composite defects, where the $O(N)$ symmetry can be broken on the sub-defect. In Sec. 5.5, we propose the sub-defect $\mathcal{C}$ -theorem and check the validity of our conjecture in the model we are studying.

⁶²While we fix the dimensionality of defects throughout this chapter, we can consider the case where the co-dimensions of defects are fixed as in e.g. [169, 209].

⁶³If we require that the theory is conformal classically, dimensional analysis allows only one type of combination in higher even dimension $d = 2n - \varepsilon$ ($n > 2$), which is to embed an $(n-1)$ -dimensional defect into a $(2n-2)$ -dimensional one. In these dimensions, a conformal composite defect cannot be constructed for a free bulk theory (shown in appendix 5.5.2). In higher odd dimensions $d = 2n - 1 - \varepsilon$ ($n > 2$), two localized classically conformal deformations cannot coexist. This makes $d = 3 - \varepsilon$ dimensions quite special.

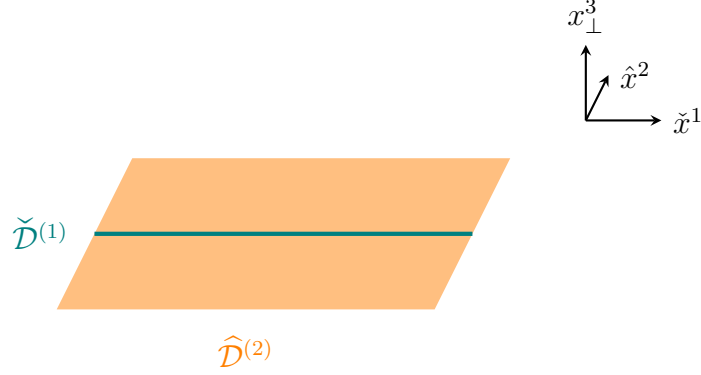


Figure 5.2: Illustrative example for a composite defect system. In this figure, we consider $d = 3$ bulk spacetime where a composite defect $\mathcal{D}^{(2,1)} \equiv \widehat{\mathcal{D}}^{(2)} \cup \check{\mathcal{D}}^{(1)}$ is inserted. Now, a three dimensional bulk coordinate is given by $x^\mu = (\check{x}^1, \hat{x}^2, x_\perp^3)$, where a two-dimensional defect $\widehat{\mathcal{D}}^{(2)}$ sits at $x_\perp^3 = 0$, and a one-dimensional defect $\check{\mathcal{D}}^{(1)}$ is stretched in the direction of the $\check{x}^1$ axis lying on $\widehat{\mathcal{D}}^{(2)}$.

5.2 Composite defect CFT

In this section, we discuss some conformal aspects of composite defect CFTs. We emphasize that the contents of this section can be applied to arbitrary composite defect CFTs. In Sec. 5.2.1, we explain how the full conformal symmetry in d -dimensional spacetime is broken into its subgroup by employing conformal maps. In Sec. 5.2.2, we determine the form of correlation functions by making use of the residual symmetry discussed in Sec. 5.2.1. In Sec. 5.2.3, we speculate on the operator expansions in composite defect CFTs. Finally, in Sec. 5.2.4, we derive differential equations that conformal blocks must satisfy, and give some analytical expressions of conformal blocks.

5.2.1 Symmetry in composite defect CFT

In this subsection, we investigate the residual conformal symmetry in composite defect CFTs. We consider a CFT in d -dimensional spacetime X^d whose coordinate is described by x^μ ($\mu = 1, 2, \dots, d$), and assume the existence of a composite defect $\mathcal{D}^{(p,r)}$:

$$\mathcal{D}^{(p,r)} \equiv \widehat{\mathcal{D}}^{(p)} \cup \check{\mathcal{D}}^{(r)} \quad , \quad r < p < d \quad , \quad (5.2)$$

in a bulk spacetime. Here, $\widehat{\mathcal{D}}^{(p)}$ is a p -dimensional conformal (i.e., planar or spherical) defect, and $\check{\mathcal{D}}^{(r)}$ is a r -dimensional conformal sub-defect which is embedded into $\widehat{\mathcal{D}}^{(p)}$. For simplicity, we confine ourselves to the planar composite defect, and set its location as follows:

$$\begin{aligned} \widehat{\mathcal{D}}^{(p)} : \quad & \check{x}^a \neq 0 \quad , \quad \hat{x}^\alpha \neq 0 \quad , \quad x_\perp^i = 0 \quad , \\ \check{\mathcal{D}}^{(r)} : \quad & \check{x}^a \neq 0 \quad , \quad \hat{x}^\alpha = 0 \quad , \quad x_\perp^i = 0 \quad , \end{aligned} \quad (5.3)$$

where the coordinate system is

$$x^\mu = (\check{x}^a, \hat{x}^\alpha, x_\perp^i) \quad , \quad a = 1 \sim r \quad , \quad \alpha = r + 1 \sim p \quad , \quad i = p + 1 \sim d \quad . \quad (5.4)$$

(In Fig. 5.2, we provide an illustrative example that helps the readers with imagining our setup.)

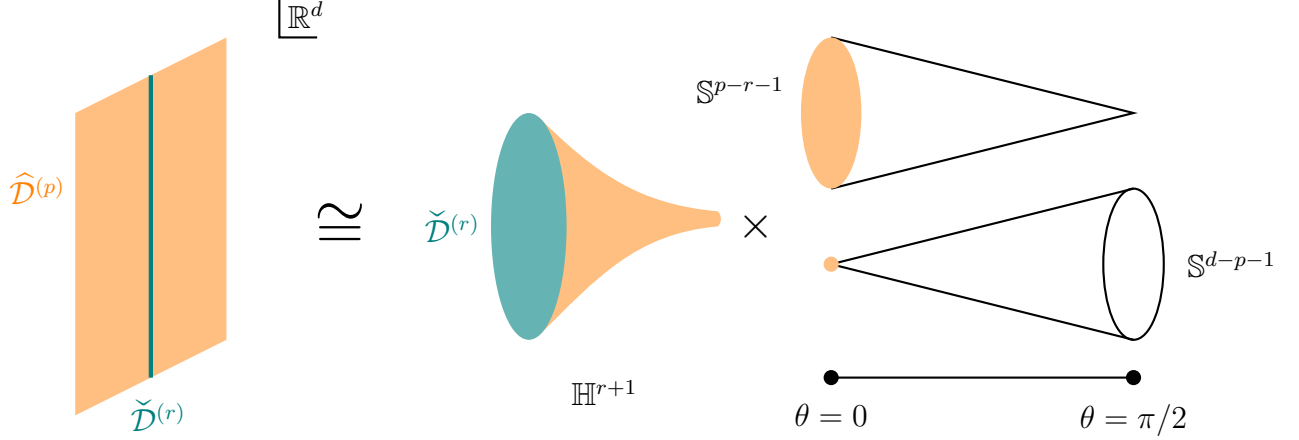


Figure 5.3: Flat space $\mathbb{R}^d$ with a composite defect $\hat{\mathcal{D}}^{(p,r)}$ is conformally equivalent ($\cong$) to the product of the $(r+1)$ -dimensional Euclidean AdS spacetime $\mathbb{H}^{r+1}$, and the spherical geometry which is characterized by (5.12). Particularly, the sub-defect $\check{\mathcal{D}}^{(r)}$ is mapped to the AdS boundary while the defect $\hat{\mathcal{D}}^{(p)}$ is to some $(r+1)$ -dimensional sub-manifold of $\mathbb{H}^{r+1}$ and $(p-r-1)$ -dimensional unit sphere $\mathbb{S}^{p-r-1}$ at $\theta = 0$.

A d -dimensional (Euclidean) CFT possesses the conformal symmetry $\text{SO}(1, d+1)$. Introducing some conformal defect typically breaks $\text{SO}(1, d+1)$ symmetry to its subgroup. If we introduce a composite defect $\mathcal{D}^{(p,r)}$ into a spacetime, the full conformal symmetry is broken as follows:

$$\text{insertion of } \mathcal{D}^{(p,r)} : \text{SO}(1, d+1) \longrightarrow \text{SO}(1, r+1) \times \text{SO}(p-r) \times \text{SO}(d-p), \quad (5.5)$$

where $\text{SO}(1, r+1)$ is the conformal symmetry along with the sub-defect $\check{\mathcal{D}}^{(r)}$, and $\text{SO}(d-p)$ and $\text{SO}(p-r)$ are rotational symmetries.

We here give an interpretation of this residual symmetry (5.5) from the viewpoint of the conformal map along with [210] (see also e.g. [20, 178, 211]). In our composite defect systems, we have the following flat metric:

$$ds^2 = \sum_{a=1}^r d\tilde{x}_a^2 + \sum_{\alpha=r+1}^p d\hat{x}_\alpha^2 + \sum_{i=p+1}^d dx_{\perp,i}^2, \quad (5.6)$$

where a composite defect $\mathcal{D}^{(p,r)}$ is located as (5.3). We first move on to the cylinder coordinates:

$$\sum_{\alpha=r+1}^p d\hat{x}_\alpha^2 = d\hat{z}^2 + \hat{z}^2 d\Omega_{p-r-1}^2, \quad \sum_{i=p+1}^d dx_{\perp,i}^2 = dz_\perp^2 + z_\perp^2 d\Omega_{d-p-1}^2, \quad (5.7)$$

where $\hat{z}, z_\perp \in [0, \infty)$ denote the radial coordinates, and $d\Omega_q^2$ is the metric for the q -dimensional unit sphere $\mathbb{S}^q$. After performing this coordinate transformation, the flat metric (5.6) becomes

$$ds^2 = \sum_{a=1}^r d\tilde{x}_a^2 + d\hat{z}^2 + dz_\perp^2 + \hat{z}^2 d\Omega_{p-r-1}^2 + z_\perp^2 d\Omega_{d-p-1}^2. \quad (5.8)$$

We moreover transfer from the $(\hat{z}, z_\perp)$ coordinates to the polar ones (z, θ) :

$$\hat{z} = z \cos \theta \quad , \quad z_\perp = z \sin \theta \quad , \quad (5.9)$$

where $z \geq 0$ and $0 \leq \theta \leq \pi/2$, and then the metric (5.8) becomes transformed into

$$ds^2 = z^2 (ds_{\mathbb{H}^{r+1}}^2 + ds_{\text{sph}}^2) \quad . \quad (5.10)$$

Here, $ds_{\mathbb{H}^{r+1}}^2$ is the metric for the $(r+1)$ -dimensional Euclidean AdS spacetime $\mathbb{H}^{r+1}$:

$$ds_{\mathbb{H}^{r+1}}^2 = \frac{\sum_{a=1}^r d\tilde{x}_a^2 + dz^2}{z^2} \quad , \quad (5.11)$$

and the sub-defect $\check{\mathcal{D}}^{(r)}$ is located at the AdS boundary (namely, $z = 0$). Also, ds_{sph}^2 is the spherical metric which is defined by:

$$ds_{\text{sph}}^2 = d\theta^2 + \cos^2 \theta d\Omega_{p-r-1}^2 + \sin^2 \theta d\Omega_{d-p-1}^2 \quad . \quad (5.12)$$

For a fixed value of θ , the above spherical geometry consists of the $(p-r-1)$ -dimensional sphere whose radius is $\cos \theta$ and the $(d-p-1)$ -dimensional one whose radius is $\sin \theta$. Therefore, by applying the Weyl rescaling to (5.10), the flat spacetime in the presence of a composite defect $\mathcal{D}^{(p,r)}$ is conformally equivalent to the product of the $(r+1)$ -dimensional Euclidean AdS spacetime (5.11) and the spherical geometry (5.12). We illustrate this picture in Fig. 5.3. We should notice that the metric (5.10) has the isometry $\text{SO}(1, r+1) \times \text{SO}(p-r) \times \text{SO}(d-p)$ which is exactly coincident with the residual conformal symmetry (5.5). This provides one way to understand the residual conformal symmetry for composite defect CFTs.

Finally, it is instructive to discuss which symmetry generators of $\text{SO}(1, d+1)$ do survive in the presence of a composite defect $\mathcal{D}^{(p,r)}$. The symmetry generators of the full conformal symmetry $\text{SO}(1, d+1)$ are described by $\mathbf{J}_{MN}$ ($M, N = -1, 0, 1, \dots, d$) satisfying the following commutation relations:

$$\begin{aligned} [\mathbf{J}_{MN}, \mathbf{J}_{RS}] &= -\eta_{MR} \mathbf{J}_{NS} + \eta_{NR} \mathbf{J}_{MS} + \eta_{MS} \mathbf{J}_{NR} - \eta_{NS} \mathbf{J}_{MR} \quad , \\ \eta_{MN} &\equiv \text{diag}(-1, 1, 1, \dots, 1) \quad . \end{aligned} \quad (5.13)$$

The relations between the generators $\mathbf{J}_{MN}$ and the dilatation, translation, rotation, and special conformal transformation generators $\{\mathbf{D}, \mathbf{P}_\mu, \mathbf{M}_{\mu\nu}, \mathbf{K}_\mu\}$ are given by

$$\mathbf{J}_{MN} = \begin{matrix} M \setminus N \rightarrow \\ \downarrow \end{matrix} \begin{matrix} & -1 & 0 & \nu \\ -1 & \begin{pmatrix} 0 & \mathbf{D} \\ 0 & -\mathbf{D} \end{pmatrix} & \begin{pmatrix} \frac{1}{2}(\mathbf{P}_\nu - \mathbf{K}_\nu) \\ \frac{1}{2}(\mathbf{P}_\nu + \mathbf{K}_\nu) \end{pmatrix} \\ \mu & \begin{pmatrix} -\frac{1}{2}(\mathbf{P}_\mu - \mathbf{K}_\mu) & -\frac{1}{2}(\mathbf{P}_\mu + \mathbf{K}_\mu) \end{pmatrix} & \mathbf{M}_{\mu\nu} \end{matrix} \quad . \quad (5.14)$$

Clearly from the symmetry breaking pattern (5.5), the symmetry generators for the residual sym-

metry $\text{SO}(1, r + 1) \times \text{SO}(p - r) \times \text{SO}(d - p)$ are block-diagonal parts of $\mathbf{J}_{MN}$ as follows:

$$\mathbf{J}_{MN} = \begin{matrix} M \setminus N \rightarrow \\ \downarrow \end{matrix} \begin{matrix} -1 & & 0 & & b & & \beta & & j \\ -1 & \begin{pmatrix} 0 & \mathbf{D} & \frac{1}{2}(\mathbf{P}_b - \mathbf{K}_b) & \frac{1}{2}(\mathbf{P}_\beta - \mathbf{K}_\beta) & \frac{1}{2}(\mathbf{P}_j - \mathbf{K}_j) \\ -\mathbf{D} & 0 & \frac{1}{2}(\mathbf{P}_b + \mathbf{K}_b) & \frac{1}{2}(\mathbf{P}_\beta + \mathbf{K}_\beta) & \frac{1}{2}(\mathbf{P}_j + \mathbf{K}_j) \end{pmatrix} \\ 0 & & & & & & & & \\ a & -\frac{1}{2}(\mathbf{P}_a - \mathbf{K}_a) & -\frac{1}{2}(\mathbf{P}_a + \mathbf{K}_a) & \mathbf{M}_{ab} & \mathbf{M}_{a\beta} & \mathbf{M}_{aj} \\ \alpha & -\frac{1}{2}(\mathbf{P}_\alpha - \mathbf{K}_\alpha) & -\frac{1}{2}(\mathbf{P}_\alpha + \mathbf{K}_\alpha) & \mathbf{M}_{\alpha b} & \mathbf{M}_{\alpha\beta} & \mathbf{M}_{\alpha j} \\ i & -\frac{1}{2}(\mathbf{P}_i - \mathbf{K}_i) & -\frac{1}{2}(\mathbf{P}_i + \mathbf{K}_i) & \mathbf{M}_{ib} & \mathbf{M}_{i\beta} & \mathbf{M}_{ij} \end{matrix} \Bigg) , \quad (5.15)$$

where the slashed generators correspond to the broken ones by the insertion of $\mathcal{D}^{(p,r)}$.

In composite defect CFTs, the number of symmetries is of course fewer compared to homogeneous CFTs or conventional defect CFTs. Therefore, it is non-trivial to determine how much restrictions can be imposed on composite defect CFTs only from the residual symmetry. In the rest of Sec. 5.2, we address this issue by focusing on conformal correlators (Sec. 5.2.2), operator expansions (Sec. 5.2.3), and conformal blocks (Sec. 5.2.4).

5.2.2 Scalar correlators in composite defect CFTs

In this section, we elaborate on how the correlation functions can be fixed only from the residual conformal symmetry $\text{SO}(1, r + 1) \times \text{SO}(p - r) \times \text{SO}(d - p)$. In composite defect CFTs, local operators are classified into the following three classes:

- Bulk local operator at $x^\mu \equiv (\tilde{x}^a, \hat{x}^\alpha, x_\perp^i)$:

$$\mathcal{O}_{\Delta, J}(x) , \quad (5.16)$$

where Δ is the conformal dimension and J is $\text{SO}(d)$ spin number. (In this expression, we omit tensorial indices to avoid some complicated notations. This remark is also applied to (5.17) and (5.18).)

- Defect local operator at $\hat{x}^\mu \equiv (\tilde{x}^a, \hat{x}^\alpha, x_\perp^i = 0)$:

$$\widehat{\mathcal{O}}_{\widehat{\Delta}, (\ell, s)}(\hat{x}) , \quad (5.17)$$

where $\widehat{\Delta}$ is the conformal dimension, and ℓ and s are the $\text{SO}(p)$ and $\text{SO}(d - p)$ spin numbers, respectively.

- Sub-defect local operator at $\check{x}^\mu \equiv (\tilde{x}^a, \hat{x}^\alpha = 0, x_\perp^i = 0)$:

$$\check{\mathcal{O}}_{\check{\Delta}, (\ell, s_1, s_2)}(\check{x}) , \quad (5.18)$$

where $\check{\Delta}$ is the conformal dimension, and ℓ , s_1 and s_2 are the $\text{SO}(r)$, $\text{SO}(p - r)$ and $\text{SO}(d - p)$ spin numbers, respectively.

In this subsection, we argue how the residual symmetry (5.5) puts restrictions on the conformal correlators composing scalar primaries.

Preliminaries: embedding space formalism

Our methodology to fix the conformal correlators relies on the *embedding space formalism*, which has been developed in [53, 212, 213]. In this subsection, we present a brief review of the embedding space formalism in the absence of any defects. (We recommend e.g. [12, 53, 214] for the readers who want to know more details.) If readers are familiar with the embedding space formalism in d -dimensional CFT, they may skip this subsection and move on to the next one.

In the embedding space formalism, we lift up the d -dimensional Euclidean spacetime $\mathbb{R}^d$ to the projective null cone $\mathbb{PNC}^{1,d+1}$ which is defined by

$$\mathbb{PNC}^{1,d+1} \equiv \frac{\{X^M \in \mathbb{R}^{1,d+1}; X \cdot X \equiv \eta_{MN} X^M X^N = 0\}}{\{X^M \sim \lambda X^M; \forall \lambda > 0\}}, \quad (5.19)$$

where η_{MN} is the Lorentzian metric given in (5.13), and $X^M \sim \lambda X^M$ means that we identify X^M up to positive scalar multiplication. On the projective null cone $\mathbb{PNC}^{1,d+1}$, the conformal transformation acts linearly on the embedding space coordinate X^M :

$$X^M \mapsto \Lambda_N^M X^N, \quad \Lambda_N^M \in \text{SO}(1, d+1), \quad (5.20)$$

and the scalar primary operator $\mathcal{O}_\Delta(X)$ satisfies

$$\mathcal{O}_\Delta(\lambda X) = \lambda^{-\Delta} \mathcal{O}_\Delta(X). \quad (5.21)$$

While the conformal transformations in the physical space non-linearly act on fields, those in the embedding space formalism does linearly. This fact makes it much easier to construct conformal correlators. One way to get back to the physical space $\mathbb{R}^d$ is to restrict the embedding space coordinate X^M as follows:

$$X^M|_{\text{phys}} = \left(\frac{x^2 + 1}{2}, \frac{x^2 - 1}{2}, x^\mu \right). \quad (5.22)$$

Under this restriction, a scalar primary operator $\mathcal{O}_\Delta(X)$ on the projective null cone $\mathbb{PNC}^{1,d+1}$ is reduced to the one $\mathcal{O}_\Delta(x)$ on the physical space:

$$\mathcal{O}_\Delta(x) = \mathcal{O}_\Delta(X|_{\text{phys}}). \quad (5.23)$$

Here, we give an example: the scalar two-point function $\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle_{\mathbb{R}^d}$. We first need to lift its correlator on the physical space $\mathbb{R}^d$ to the one on the projective null cone $\mathbb{PNC}^{1,d+1}$:

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle_{\mathbb{R}^d} \longrightarrow \langle \mathcal{O}_{\Delta_1}(X_1) \mathcal{O}_{\Delta_2}(X_2) \rangle_{\mathbb{PNC}^{1,d+1}}. \quad (5.24)$$

Now that we have the full conformal symmetry $\text{SO}(1, d+1)$, the conformal correlator should consist of the $\text{SO}(1, d+1)$ invariant quantities. In this case, $X_1 \cdot X_2$ is the clearly unique invariant up to scalar multiplications due to null properties: $X_1^2 = X_2^2 = 0$. Therefore, the two-point function should be some function of $X_1 \cdot X_2$:

$$\langle \mathcal{O}_{\Delta_1}(X_1) \mathcal{O}_{\Delta_2}(X_2) \rangle_{\mathbb{PNC}^{1,d+1}} = f(X_1 \cdot X_2). \quad (5.25)$$

From the homogeneity condition (5.21), the function $f(X_1 \cdot X_2)$ should satisfy

$$f(\lambda X_1 \cdot X_2) = \lambda^{-\Delta_1} f(X_1 \cdot X_2) = \lambda^{-\Delta_2} f(X_1 \cdot X_2) , \quad (5.26)$$

from which we can fix the form of the two-point function as follows:

$$\langle \mathcal{O}_{\Delta_1}(X_1) \mathcal{O}_{\Delta_2}(X_2) \rangle_{\mathbb{PNC}^{1,d+1}} = \frac{c \delta_{\Delta_1, \Delta_2}}{(-2X_1 \cdot X_2)^{\Delta_1}} , \quad (5.27)$$

where c is some constant, and -2 appearing in front of $X_1 \cdot X_2$ is just for later convenience. Finally, in order to come back to the physical space, we need to restrict the embedding space coordinate onto (5.22). By using (5.23) and the following relation:

$$-2X_1 \cdot X_2|_{\text{phys}} = |x_{12}|^2 , \quad x_{12} \equiv x_1 - x_2 , \quad (5.28)$$

we arrive at

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle_{\mathbb{R}^d} = \frac{c \delta_{\Delta_1, \Delta_2}}{|x_{12}|^{2\Delta_1}} , \quad (5.29)$$

which is a well-known result in d -dimensional CFT.

Scalar correlators in composite defect CFT

In the previous subsection, we briefly reviewed the embedding space formalism. In this subsection, we apply the embedding space formalism to composite defect CFTs, and explore conformal structures of correlators consisting of only scalar primaries.

As discussed in subSec. 5.2.1, we have the residual symmetry $\text{SO}(1, r+1) \times \text{SO}(p-r) \times \text{SO}(d-p)$. It is therefore convenient to introduce three kinds of inner products $\bullet$, $\diamond$ and $\circ$ which are defined as follows:

$$X \bullet Y \equiv \eta_{AB} X^A Y^B , \quad A, B = -1, 0, 1, \dots, r , \quad (5.30)$$

$$X \diamond Y \equiv \delta_{\alpha\beta} X^\alpha Y^\beta , \quad \alpha, \beta = r+1, r+2, \dots, p , \quad (5.31)$$

$$X \circ Y \equiv \delta_{ij} X^i Y^j , \quad i, j = p+1, r+2, \dots, d . \quad (5.32)$$

These three inner products are invariant under the residual symmetry transformations and play a crucial role in the construction of the conformal correlators in composite defect CFTs.

Bulk one-point function. We first discuss the bulk one-point function $\langle \mathcal{O}_\Delta(x) \rangle$. In the same way as the subSec. 5.2.2, we promote it to the correlator on the projective null cone $\mathbb{PNC}^{1,d+1}$, and we consider $\langle \mathcal{O}_\Delta(X) \rangle$ ⁶⁴. By noting the null condition:

$$X \cdot X = X \bullet X + X \diamond X + X \circ X = 0 , \quad (5.33)$$

we can construct two independent invariants:

$$X \diamond X , \quad X \circ X , \quad (5.34)$$

⁶⁴For simplicity, in the rest of this chapter, we leave the symbols $\mathbb{R}^d$ and $\mathbb{PNC}^{1,d+1}$ out of correlators.

from which we can also consider the cross-ratio ξ defined by

$$\xi \equiv \frac{X \circ X}{X \diamond X} . \quad (5.35)$$

Thanks to the homogeneity condition (5.21), the bulk one-point function in the embedding space can be fixed as

$$\langle \mathcal{O}_\Delta(X) \rangle = \frac{f(\xi)}{(X \circ X)^{\Delta/2}} , \quad (5.36)$$

where $f(\xi)$ is some function depending only on the cross-ratio ξ . In the physical space, the cross-ratio ξ and the inner product $X \circ X$ become

$$\xi \longrightarrow \frac{|x_\perp|^2}{|\hat{x}|^2} , \quad X \circ X \longrightarrow |x_\perp|^2 , \quad (5.37)$$

therefore the bulk one-point function in the physical space is given by⁶⁵

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{f(\xi)}{|x_\perp|^\Delta} , \quad \xi = \frac{|x_\perp|^2}{|\hat{x}|^2} . \quad (5.38)$$

Defect one-point function. We next consider the defect one-point function $\langle \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x}) \rangle$ where $\hat{x}^\mu = (\tilde{x}^a, \hat{x}^\alpha, x_\perp^i = 0)$. The corresponding correlator in the embedding space is given by

$$\langle \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{X}) \rangle , \quad \hat{X} \cdot \hat{X} = \hat{X} \bullet \hat{X} + \hat{X} \diamond \hat{X} = 0 . \quad (5.39)$$

In this case, there is only one invariant $\hat{X} \diamond \hat{X}$, and the homogeneity condition fixes the form of the defect one-point function:

$$\langle \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{X}) \rangle = \frac{c}{(\hat{X} \diamond \hat{X})^{\hat{\Delta}/2}} , \quad (5.40)$$

where c is some constant. By the reduction to the physical space, we have

$$\langle \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x}) \rangle = \frac{c}{|\hat{x}|^{\hat{\Delta}}} . \quad (5.41)$$

Sub-defect one-point function. Clearly, the sub-defect one-point function $\langle \check{\mathcal{O}}_{\check{\Delta}}(\check{x}) \rangle$ ($\check{x}^\mu = (\tilde{x}^a, \hat{x}^\alpha = 0, x_\perp^i = 0)$) vanishes except the case where the sub-defect local operator is the identity one **1**. We therefore have

$$\langle \check{\mathcal{O}}_{\check{\Delta}}(\check{x}) \rangle = c \delta_{\check{\mathcal{O}}, 1} , \quad (5.42)$$

where c is again some constant.

We should note that conformal correlators with only defect and sub-defect local operators insertions are subject to the constraints of a conventional defect CFT whose bulk and defect are p and r -dimensional flat spaces, respectively. Since those constraints have been well studied in [19, 47], we confine ourselves to the conformal correlators with at least one bulk local operator inserted below.

⁶⁵In the following, we employ the same symbols for both cross-ratios defined in physical and embedding space. We believe that readers will not be confused about this notation.

Bulk-sub-defect two-point function. A first non-trivial example of two-point functions is the bulk-sub-defect two-point function:

$$\langle \mathcal{O}_\Delta(X) \check{\mathcal{O}}_{\check{\Delta}}(\check{Y}) \rangle. \quad (5.43)$$

By noting $\check{Y} \bullet \check{Y} = 0$, there are three invariants:

$$X \circ X, \quad X \diamond X, \quad X \bullet \check{Y}, \quad (5.44)$$

and only one cross-ratio ξ defined in (5.35). We therefore have the conformal correlator in the embedding space:

$$\langle \mathcal{O}_\Delta(X) \check{\mathcal{O}}_{\check{\Delta}}(\check{Y}) \rangle = \frac{f(\xi)}{(X \circ X)^{(\Delta-\check{\Delta})/2} (-2X \bullet \check{Y})^{\check{\Delta}}}, \quad (5.45)$$

and in the physical space, it is reduced to

$$\langle \mathcal{O}_\Delta(x) \check{\mathcal{O}}_{\check{\Delta}}(\check{y}) \rangle = \frac{f(\xi)}{|x_\perp|^{\Delta-\check{\Delta}} |x - \check{y}|^{2\check{\Delta}}}. \quad (5.46)$$

Bulk-defect two-point function. We next consider the bulk-defect two-point function $\langle \mathcal{O}_\Delta(x) \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{y}) \rangle$ whose counterpart in the embedding space is

$$\langle \mathcal{O}_\Delta(X) \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{Y}) \rangle. \quad (5.47)$$

Since X and $\hat{Y}$ are of course subject to the null conditions, independent invariants constructed from them are given by

$$X \circ X, \quad X \diamond X, \quad \hat{Y} \diamond \hat{Y}, \quad X \cdot \hat{Y}, \quad X \diamond \hat{Y}. \quad (5.48)$$

Likewise in the discussion of the bulk one-point function, we can also construct three cross-ratios:

$$\xi_1 \equiv \frac{(X \circ X)^{1/2} (\hat{Y} \diamond \hat{Y})^{1/2}}{-2X \cdot \hat{Y}}, \quad \xi_2 \equiv \frac{(X \circ X)^{1/2} (\hat{Y} \diamond \hat{Y})^{1/2}}{X \diamond \hat{Y}}, \quad \xi_3 \equiv \xi, \quad (5.49)$$

where ξ is defined in (5.35). By imposing the homogeneity conditions, the bulk-defect two-point function is determined as

$$\langle \mathcal{O}_\Delta(X) \hat{\mathcal{O}}_{\hat{\Delta}}(\hat{Y}) \rangle = \frac{f(\xi_1, \xi_2, \xi_3)}{(X \circ X)^{(\Delta-\hat{\Delta})/2} (-2X \cdot \hat{Y})^{\hat{\Delta}}}. \quad (5.50)$$

In the physical space, three cross-ratios and inner products are reduced to

$$\xi_1 \longrightarrow \frac{|x_\perp| |\hat{y}|}{|x - \hat{y}|^2}, \quad \xi_2 \longrightarrow \frac{|x_\perp| |\hat{y}|}{\hat{x} \cdot \hat{y}}, \quad \xi_3 \longrightarrow \frac{|x_\perp|^2}{|\hat{x}|^2}, \quad (5.51)$$

$$X \circ X \longrightarrow |x_\perp|^2, \quad -2X \cdot \hat{Y} \longrightarrow |x - \hat{y}|^2, \quad (5.52)$$

hence the physical space correlator is fixed in the following way:

$$\langle \mathcal{O}_\Delta(x) \widehat{\mathcal{O}}_{\widehat{\Delta}}(\hat{\mathbf{y}}) \rangle = \frac{f(\xi_1, \xi_2, \xi_3)}{|x_\perp|^{\Delta-\widehat{\Delta}} |x - \hat{\mathbf{y}}|^{2\widehat{\Delta}}} . \quad (5.53)$$

We should notice that when $p - r = 1$, the above three cross-ratios are not independent since ξ_2 can be written in terms of ξ_3 . This can be intuitively understood as follows. For the case $p - r > 1$, we can define non-trivial angle by using ξ_2 and ξ_3 :

$$\cos \theta \equiv \frac{\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}}{|\hat{\mathbf{x}}| |\hat{\mathbf{y}}|} = \xi_2^{-1} \xi_3^{1/2} , \quad (5.54)$$

while for the case $p - r = 1$, the above angle becomes trivial due to the dimensional ground. Therefore, in that case, the number of cross-ratios is reduced to two rather than three. We can also determine the form of correlators by using the method of images [21].

5.2.3 Sub-defect operator expansions of bulk local scalar primary

Operator product expansions (OPEs) have always played a crucial role in the analysis of CFTs even in the presence of a conformal defect. For instance, the OPE ensures that when two bulk local scalar primaries $\mathcal{O}_{\Delta_1}(x_1)$ and $\mathcal{O}_{\Delta_2}(x_2)$ are enough to close to each other, we can expand their product in terms of primaries and their descendants:

$$\mathcal{O}_{\Delta_1}(x_1) \times \mathcal{O}_{\Delta_2}(x_2) = \sum_{\Delta_3} \frac{c_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}} \mathcal{O}_{\Delta_3}(x_2) + (\text{descendants}) . \quad (5.55)$$

The similar operator identities to the above OPE formula do hold for the product of two defect local operators $\widehat{\mathcal{O}}_{\widehat{\Delta}_1} \times \widehat{\mathcal{O}}_{\widehat{\Delta}_2}$ and two sub-defect ones $\check{\mathcal{O}}_{\check{\Delta}_1} \times \check{\mathcal{O}}_{\check{\Delta}_2}$. In addition, in defect CFTs, we can also expand a bulk local operator in terms of defect local ones. This expansion is often called *defect operator expansion* (DOE). In Sec. 5.2.3, we present a review of the notion of DOE in conventional defect CFTs. In Sec. 5.2.3, we apply the DOE to composite defect systems and show that a bulk local operator can also be expanded in terms of sub-defect local ones⁶⁶.

Defect operator expansions (DOEs)

To explain the concept of the DOE, let us consider a conventional defect CFT with a single p -dimensional conformal defect.⁶⁷ We can expand the bulk local scalar primary $\mathcal{O}_\Delta(x)$ in terms of defect one $\widehat{\mathcal{O}}_{\widehat{\Delta}}(\hat{\mathbf{x}})$ as follows:

$$\mathcal{O}_\Delta(x) = \sum_{\widehat{\Delta}} \frac{b_{\Delta, \widehat{\Delta}}}{|x_\perp|^{\Delta-\widehat{\Delta}}} \mathcal{B}_{\widehat{\Delta}}^{(d,p)}(|x_\perp|, \widehat{\square}_p) \widehat{\mathcal{O}}_{\widehat{\Delta}}(\hat{\mathbf{x}}) + (\text{spinning contributions}) , \quad (5.56)$$

⁶⁶In the context of wedge CFTs, the SDOE is discussed for concrete models [182, equation (2.13)], i.e., expanding a bulk local operator in terms of sub-defect ones in our language. We emphasize that our discussions here can be applied to any kind of composite defect CFTs.

⁶⁷See also Sec. 2.2.4 for the detailed derivation.

where $\hat{\square}_p$ is the p -dimensional Laplacian associated to the conformal defect, and $\mathcal{B}_{\hat{\Delta}}^{(d,p)}(|x_{\perp}|, \hat{\square}_p)$ is a differential operator which is given by [19, equation (B.3)]

$$\mathcal{B}_{\hat{\Delta}}^{(d,p)}(|x_{\perp}|, \hat{\square}_p) = \sum_{n=0}^{\infty} \frac{(-4)^{-n}}{n! \left(\hat{\Delta} - \frac{p-2}{2}\right)_n} |x_{\perp}|^{2n} \hat{\square}_p^n. \quad (5.57)$$

Also, coefficients $\{b_{\Delta, \hat{\Delta}}\}$ appearing in the DOE are often called DOE coefficients which cannot be fixed only from the symmetry and characterize the CFT data of the conformal defect. We should comment here that the DOE formula of $\mathcal{O}_{\Delta}(x)$ does hold only when no other local operators are inserted inside the virtual ball centered at $\hat{x}^{\mu}$ with radius $|x_{\perp}|$ (see Fig. 2.3.). This remark plays a crucial role in the next section.

Sub-defect operator expansions (SDOE)

In the previous section, we reviewed the DOE which is a specific operator identity to defect CFTs. By utilizing the DOE, we can also expand a bulk local operator in terms of sub-defect local operators in composite defect CFTs. We refer to this expansion as *sub-defect operator expansion* (SDOE) throughout this chapter. The main aim of this section is to derive the SDOE formula paying our attention to only sub-defect scalar channels.

Our strategy for deriving the SDOE formula for a bulk local primary $\mathcal{O}_{\Delta}(x)$ consists of the following two steps (see Fig. 5.4):

- **Step 1.** we expand a bulk local operator $\mathcal{O}_{\Delta}(x)$ located in $|x_{\perp}| < |\hat{x}|$ in terms of defect local ones $\{\hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x})\}$.
- **Step 2.** we further expand each defect local operator appearing in **Step 1** in terms of sub-defect ones $\{\check{\mathcal{O}}_{\check{\Delta}}(\check{x})\}$.

In **Step 1**, we carry out the DOE for a bulk local field $\mathcal{O}_{\Delta}(x)$ based on (5.56). Importantly, for this DOE to be well-defined, we need to put $\mathcal{O}_{\Delta}(x)$ at a bulk point such that $|x_{\perp}| < |\hat{x}|$. If we do so, indeed, we are sure to find a virtual ball in which the sub-defect does not exist as depicted in the left panel of Fig. 5.5. This implies that we can always expand a bulk local operator by the complete set of defect operators. On the other hand, however, if we place $\mathcal{O}_{\Delta}(x)$ such that $|x_{\perp}| < |\hat{x}|$, the sub-defect does always enter inside of the virtual ball as depicted in the right panel of Fig. 5.5. This means that we cannot expand a bulk local operator only in terms of defect ones, and we must take into account the effect of the sub-defect. For these reasons, whenever we consider the SDOE of a bulk local operator, we put a bulk local operator such that $|x_{\perp}| < |\hat{x}|$.

To implement **Step 2**, it is sufficient to make use of the DOE formula again:

$$\hat{\mathcal{O}}_{\hat{\Delta}}(\hat{x}) = \sum_{\check{\Delta}} \frac{b_{\hat{\Delta}, \check{\Delta}}}{|\hat{x}|^{\hat{\Delta}-\check{\Delta}}} \mathcal{B}_{\check{\Delta}}^{(p,r)}(|\hat{x}|, \check{\square}_r) \check{\mathcal{O}}_{\check{\Delta}}(\check{x}) + (\text{spinning contributions}), \quad (5.58)$$

where $\check{\square}_r$ is the r -dimensional Laplacian associated to the sub-defect $\check{\mathcal{D}}^{(r)}$. Just for simplicity, we particularly focus on sub-defect scalar channels in this chapter. We also leave the generalization to the sub-defect spinning ones for interesting future work. By combining the above two operator

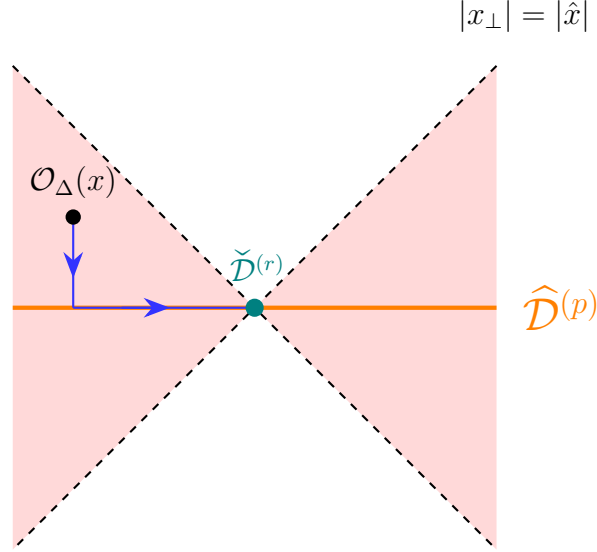


Figure 5.4: Our strategy for deriving the SDOE of a bulk local operator $\mathcal{O}_\Delta(x)$. We first expand $\mathcal{O}_\Delta(x)$ in terms of defect local operators (**Step 1**), and next expand defect local operators by using sub-defect operators (**Step 2**). To define the SDOE correctly, we need to place a bulk local operator $\mathcal{O}_\Delta(x)$ at a bulk point such that $|x_\perp| < |\hat{x}|$ (depicted as the red region). See also Fig. 5.5.

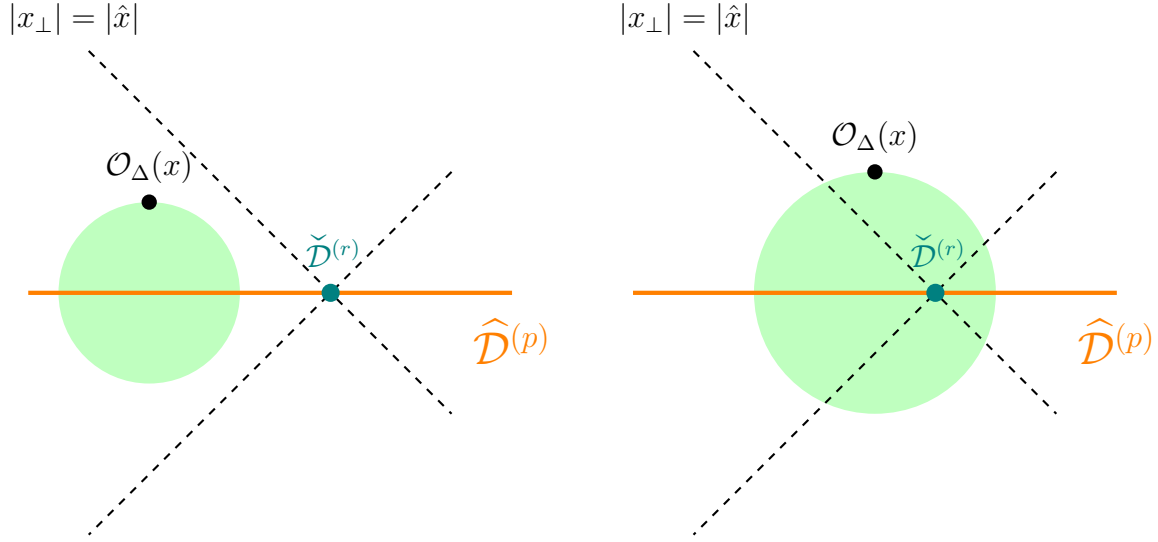


Figure 5.5: In order to implement our strategy for defining the SDOE, we need to place a bulk local operator $\mathcal{O}_\Delta(x)$ such that $|x_\perp| < |\hat{x}|$ as depicted in the left panel. If we do so, we can take the DOE of $\mathcal{O}_\Delta(x)$ without any issues since the sub-defect is out of the virtual ball (green region). On the one hand, if we put $\mathcal{O}_\Delta(x)$ such that $|x_\perp| > |\hat{x}|$ like in the right panel, we can no longer define the DOE because the sub-defect does exist inside of the virtual ball.

expansions (5.56) and (5.58), we can expand a bulk local operator in terms of sub-defect local

operators:

$$\mathcal{O}_\Delta(x) = \sum_{\check{\Delta}} \sum_{\hat{\Delta}} \frac{b_{\Delta,\hat{\Delta}} b_{\hat{\Delta},\check{\Delta}}}{|x_\perp|^{\Delta-\hat{\Delta}}} \mathcal{B}_{\hat{\Delta}}^{(d,p)}(|x_\perp|, \hat{\square}_p) \frac{1}{|\hat{x}|^{\hat{\Delta}-\check{\Delta}}} \mathcal{B}_{\check{\Delta}}^{(p,r)}(|\hat{x}|, \check{\square}_r) \check{\mathcal{O}}_{\check{\Delta}}(\check{x}). \quad (5.59)$$

By using the expression (5.57), the above SDOE formula is reduced to

$$\begin{aligned} \mathcal{O}_\Delta(x) &= \sum_{\check{\Delta}} \sum_{\hat{\Delta}} \frac{b_{\Delta,\hat{\Delta}} b_{\hat{\Delta},\check{\Delta}}}{|x_\perp|^{\Delta-\hat{\Delta}}} \sum_{L=0}^{\infty} \sum_{m=0}^L \frac{1}{(L-m)! m! \left(\hat{\Delta} - \frac{p-2}{2}\right)_{L-m} \left(\check{\Delta} - \frac{r-2}{2}\right)_m} \xi^{-m + \frac{\hat{\Delta}-\check{\Delta}}{2}} \\ &\quad \cdot {}_2F_1\left(-m + \frac{\hat{\Delta}-\check{\Delta}}{2}, 1 - m + \frac{-p+r+\hat{\Delta}-\check{\Delta}}{2}; 1 + L - m - \frac{p}{2} + \hat{\Delta}; -\xi\right) \left(-\frac{|\hat{x}|^2}{4} \check{\square}_r\right)^L \check{\mathcal{O}}_{\check{\Delta}}(\check{x}), \end{aligned} \quad (5.60)$$

where ξ is the cross-ratio which is defined in (5.35), and takes its value in the range of $0 < \xi < 1$. Also, ${}_2F_1$ is the usual Gauss Hypergeometric function. We should remark that while the index $L = 0$ corresponds to the sub-defect primary channel $\check{\mathcal{O}}_{\check{\Delta}}$, the index $L \geq 1$ represents the level of the sub-defect descendant $\check{\square}_r^L \check{\mathcal{O}}_{\check{\Delta}}$. Also, the above SDOE formula crucially depends on the cross-ratio ξ . To the best of our efforts, deriving the SDOE formula presented above into a closed form turns out to be quite challenging in general.

5.2.4 Conformal block expansions

In Sec. 5.2.2, we derived various conformal correlators in composite defect CFTs. In this section, we particularly focus on a bulk one-point function (5.36) and a bulk-sub-defect two-point one (5.45). These correlators are not completely fixed by the residual conformal symmetry, and they admit the conformal block expansions. We dedicate this section to derive the conformal block expansions for these correlators.

Bulk one-point function

We begin with exploring the conformal expansion of a bulk one-point function $\langle \mathcal{O}_\Delta(x) \rangle$. We now take the origin of the radial quantization at x , draw a $(d-1)$ -dimensional sphere tangent to the defect $\hat{\mathcal{D}}^{(p)}$, and insert a completeness relation:

$$\begin{aligned} 1 &= \sum_{\hat{\Delta}} |\hat{\mathcal{O}}_{\hat{\Delta}}| \\ &\equiv \sum_{\hat{\Delta}} \left(|\hat{\mathcal{O}}_{\hat{\Delta}}\rangle \langle \hat{\mathcal{O}}_{\hat{\Delta}}| + (\text{descendants}) \right). \end{aligned} \quad (5.61)$$

Here, the state $|\hat{\mathcal{O}}_{\hat{\Delta}}\rangle$ is the diagonal basis of the $\text{SO}(1, p+1) \times \text{SO}(d-p)$ Casimir operator $\widehat{Cas}$.⁶⁸

$$\widehat{Cas} |\hat{\mathcal{O}}_{\hat{\Delta}}\rangle = \hat{\Delta}(\hat{\Delta} - p) |\hat{\mathcal{O}}_{\hat{\Delta}}\rangle, \quad \widehat{Cas} \equiv -\frac{1}{2} \mathbf{J}_{AB} \mathbf{J}^{AB} - \frac{1}{2} \mathbf{J}_{ij} \mathbf{J}^{ij}, \quad (5.62)$$

⁶⁸Remark that the defect local primaries are classified by the representations of $\text{SO}(1, p+1) \times \text{SO}(d-p)$ rather than $\text{SO}(1, r+1) \times \text{SO}(d-p) \times \text{SO}(p-r)$

where $A, B = -1, 0, \dots, p$ and $i, j = p+1, p+2, \dots, d$. Since the Casimir operator $\widehat{Cas}$ particularly commutes with momentum operators along with $\widehat{\mathcal{D}}^{(p)}$, the following relation does hold:

$$\widehat{Cas} | \widehat{\mathcal{O}}_{\widehat{\Delta}} | = | \widehat{\mathcal{O}}_{\widehat{\Delta}} | \widehat{Cas} = \widehat{\Delta}(\widehat{\Delta} - p) | \widehat{\mathcal{O}}_{\widehat{\Delta}} |. \quad (5.63)$$

In the discussion below, we consider the embedding space correlator $\langle \mathcal{O}_{\Delta}(X) \rangle$ instead of the physical space one $\langle \mathcal{O}_{\Delta}(x) \rangle$ since the Casimir operator in the embedding space acts linearly on the operator. Under these preparations, we can expand a bulk one-point function $\langle \mathcal{O}_{\Delta}(X) \rangle$ in the following way:

$$\begin{aligned} \langle \mathcal{O}_{\Delta}(X) \rangle &= \sum_{\widehat{\Delta}} \langle \mathcal{D}^{(p,r)} | \widehat{\mathcal{O}}_{\widehat{\Delta}} | \mathcal{O}_{\Delta}(X) | 0 \rangle \\ &\equiv \sum_{\widehat{\Delta}} \frac{c_{\widehat{\Delta}}}{(X \circ X)^{\Delta/2}} G_{\widehat{\Delta}}(\xi), \end{aligned} \quad (5.64)$$

where $|\mathcal{D}^{(p,r)}\rangle$ and $|0\rangle$ are the conformal vacuum states which are invariant under the actions of $\text{SO}(1, r+1) \times \text{SO}(d-p) \times \text{SO}(p-r)$ and $\text{SO}(1, d+1)$, respectively. Also, $c_{\widehat{\Delta}}$ is some expansion coefficient, and $G_{\widehat{\Delta}}(\xi)$ is the what is called *conformal block*. In what follows, we derive the exact form of the conformal block $G_{\widehat{\Delta}}(\xi)$ and the relation between $c_{\widehat{\Delta}}$ and OE coefficients. By using the relation (5.63), we can derive a non-trivial differential equation as follows;

$$\begin{aligned} \widehat{\Delta}(\widehat{\Delta} - p) \langle \mathcal{D}^{(p,r)} | \widehat{\mathcal{O}}_{\widehat{\Delta}} | \mathcal{O}_{\Delta}(X) | 0 \rangle &= \langle \mathcal{D}^{(p,r)} | \widehat{\mathcal{O}}_{\widehat{\Delta}} | \widehat{Cas} \mathcal{O}_{\Delta}(X) | 0 \rangle \\ &= -\frac{1}{2} \mathcal{L}_{AB} \mathcal{L}^{AB}(X) \langle \mathcal{D}^{(p,r)} | \widehat{\mathcal{O}}_{\widehat{\Delta}} | \mathcal{O}_{\Delta}(X) | 0 \rangle, \end{aligned} \quad (5.65)$$

where $\mathcal{L}_{MN}(X)$ is the differential operator which is defined by

$$\mathcal{L}_{MN}(X) \equiv X_M \frac{\partial}{\partial X^N} - X_N \frac{\partial}{\partial X^M}. \quad (5.66)$$

Notice that both $X \circ X$ and ξ are $\text{SO}(d-p)$ invariant, hence we do not need to take into account the differential operator $\mathcal{L}_{ij} \mathcal{L}^{ij}(X)$. After somewhat tedious computations, the above equation (5.65) is reduced to

$$4\xi^2 (\xi + 1) \frac{d^2}{d\xi^2} G_{\widehat{\Delta}}(\xi) - 2\xi (p - 2 + (p - r - 4)\xi) \frac{d}{d\xi} G_{\widehat{\Delta}}(\xi) - \widehat{\Delta}(\widehat{\Delta} - p) G_{\widehat{\Delta}}(\xi) = 0. \quad (5.67)$$

To solve this differential equation, we need to specify the boundary condition for the conformal block $G_{\widehat{\Delta}}$ at $\xi = 0$. With the help from the DOE (5.56), the bulk one-point function should show the following asymptotic behavior:

$$\langle \mathcal{O}_{\Delta}(X) \rangle \xrightarrow{\xi \rightarrow 0} \sum_{\widehat{\Delta}} \frac{b_{\Delta, \widehat{\Delta}}}{(X \circ X)^{(\Delta - \widehat{\Delta})/2}} \langle \widehat{\mathcal{O}}_{\widehat{\Delta}}(\widehat{X}) \rangle = \sum_{\widehat{\Delta}} \frac{b_{\Delta, \widehat{\Delta}} b_{\widehat{\Delta}, 0}}{(X \circ X)^{\Delta/2}} \xi^{\widehat{\Delta}/2}, \quad (5.68)$$

from which we can read off the expansion coefficient $c_{\widehat{\Delta}}$ in (5.64), and the boundary condition for $G_{\widehat{\Delta}}$:

$$c_{\widehat{\Delta}} = b_{\Delta, \widehat{\Delta}} b_{\widehat{\Delta}, 0}, \quad G_{\widehat{\Delta}}(\xi) \xrightarrow{\xi \rightarrow 0} \xi^{\widehat{\Delta}/2}. \quad (5.69)$$

Under this boundary condition, the solution for the differential equation (5.67) can be computed to be⁶⁹

$$G_{\hat{\Delta}}(\xi) = \xi^{\hat{\Delta}/2} {}_2F_1\left(\frac{\hat{\Delta}}{2}, \frac{2-p+r+\hat{\Delta}}{2}; \frac{2-p}{2} + \hat{\Delta}; -\xi\right). \quad (5.70)$$

In summary, the conformal block expansion of a bulk one-point function in the physical space is given by

$$\langle \mathcal{O}_{\Delta}(x) \rangle = \sum_{\hat{\Delta}} \frac{b_{\Delta, \hat{\Delta}} b_{\hat{\Delta}, 0}}{|x_{\perp}|^{\Delta}} G_{\hat{\Delta}}(\xi), \quad (5.71)$$

where the conformal block $G_{\hat{\Delta}}(\xi)$ is given by (5.70). This expansion is completely consistent with the SDOE formula (5.60). Indeed, by taking the vacuum expectation value on both sides of (5.60), only vacuum sector contributes to the SDOE.

Bulk-sub-defect two-point function

We next consider the conformal block expansion of a bulk-sub-defect two-point function $\langle \mathcal{O}_{\Delta}(x) \check{\mathcal{O}}_{\check{\Delta}}(\check{y}) \rangle$. Since our methodology is the same as the case of a bulk one-point function, we just present the result. A bulk-sub-defect two point function admit the following conformal block expansion:

$$\langle \mathcal{O}_{\Delta}(X) \check{\mathcal{O}}_{\check{\Delta}}(\check{Y}) \rangle = \sum_{\hat{\Delta}} \frac{b_{\Delta, \hat{\Delta}} b_{\hat{\Delta}, \check{\Delta}}}{|x_{\perp}|^{\Delta-\check{\Delta}} |x - \check{y}|^{2\check{\Delta}}} \tilde{G}_{\hat{\Delta}, \check{\Delta}}(\xi), \quad (5.72)$$

where the conformal block $\tilde{G}_{\hat{\Delta}, \check{\Delta}}(\xi)$ is defined by

$$\tilde{G}_{\hat{\Delta}, \check{\Delta}}(\xi) = \xi^{(\hat{\Delta}-\check{\Delta})/2} {}_2F_1\left(\frac{\hat{\Delta}-\check{\Delta}}{2}, \frac{2-p+r+\hat{\Delta}-\check{\Delta}}{2}; \frac{2-p}{2} + \hat{\Delta}; -\xi\right), \quad (5.73)$$

As a consistency check, if we set the sub-defect primary to be identity operator, namely $\check{\Delta} = 0$, the above expansion is reduced to (5.70).

5.3 A single scalar with a composite defect

In this section, as a warm-up, we consider the model of a single scalar in $d = 3 - \varepsilon$ dimensions with both a surface defect and a line defect turned on,

$$I = \frac{1}{2} \int d^d x (\partial \phi)^2 + \frac{h_0}{4!} \int_{\mathbb{R}^2} d^2 \hat{y} \phi^4 + \frac{g_0}{2} \int_{\mathbb{R}} d\check{z} \phi^2, \quad (5.74)$$

where $x^{\mu} = (x_{\perp}, x_{\parallel}, x_L)$, with $x_{\perp}$ being the orthogonal direction to the surface defect, $x_{\parallel}$ being parallel to the surface but orthogonal to the line, and x_L being parallel to the line defect. The hatted

⁶⁹Since the differential equation (5.67) is invariant under $\hat{\Delta} \rightarrow p - \hat{\Delta}$, there is another independent solution which can be obtained by replacing $\hat{\Delta}$ with $p - \hat{\Delta}$ in (5.70). This solution corresponds to the conformal block of the defect shadow operator, and we exclude this solution by imposing the boundary condition (5.69).

coordinates cover the surface defect $\hat{y} = (x_\perp = 0, x_\parallel, x_L)$ and the tilde coordinates cover the line defect $\tilde{z} = (x_\perp = 0, x_\parallel = 0, x_L)$. In this way, the line defect is embedded inside the surface defect. We work in the minimal subtraction scheme; the dimensional regularization allows us to write the bare couplings in terms of the renormalized ones as

$$h_0 = M^{2\varepsilon} \left(h + \frac{\delta h}{\varepsilon} + \frac{\delta_2 h}{\varepsilon^2} + \dots \right), \quad (5.75)$$

$$g_0 = M^\varepsilon \left(g + \frac{\delta g}{\varepsilon} + \frac{\delta_2 g}{\varepsilon^2} + \dots \right). \quad (5.76)$$

To obtain the beta functions of the two couplings, it is enough to consider two types of bulk composite fields, $\phi^2(x)$ and the composite one $\phi^4(x)$. The bare fields and the renormalized ones are related as

$$\phi^2(x) = Z_{\phi^2} [\phi^2](x), \quad \phi^4(x) = Z_{\phi^4} [\phi^4](x), \quad (5.77)$$

with Z_{ϕ^2} and Z_{ϕ^4} being the wave-function renormalizations, however, for free bulk theory $Z_{\phi^2} = Z_{\phi^4} = 1$. The physical condition requires that the correlation of the renormalized fields to give a finite answer as $\varepsilon \rightarrow 0$, in the case of one-point functions, that is

$$\langle [\phi^2](x) \rangle = \text{finite}, \quad \langle [\phi^4](x) \rangle = \text{finite}. \quad (5.78)$$

Note that only the completely connected diagrams contribute.⁷⁰ As we shall see, the renormalized couplings at the non-trivial fixed points are of order $O(\varepsilon)$. This renders a perturbative way of calculating the one-point functions.

5.3.1 Diagrams for the ε -expansion

Up to the second order in both h_0 and g_0 , there are four diagrams contributing to the evaluation of one-point function of the quadratic composite operator $\phi^2(x)$ and four diagrams contributing to the quartic composite operator $\phi^4(x)$ as well, depicted in Fig. 5.6. The basic structure of each diagram, $D_{i,j}$ and $\bar{D}_{i,j}$ can be written as

$$D_{i,j} = (-h_0)^i (-g_0)^j M_{i,j} I_{i,j}(x), \quad (5.80)$$

$$\bar{D}_{i,j} = (-h_0)^i (-g_0)^j \bar{M}_{i,j} \bar{I}_{i,j}(x), \quad (5.81)$$

where the indices i and j denote the i -th order in h_0 and the j -th order in g_0 , $M_{i,j}$ ($\bar{M}_{i,j}$) denote the multiplication factor and $I_{i,j}$ ($\bar{I}_{i,j}$) denote the integrals of the propagators over the internal coordinates on the defect surface or line. The multiplication factor $M_{i,j}$ and $\bar{M}_{i,j}$ can be evaluated by quotienting out the symmetry factor, namely,

$$M_{i,j} = \frac{2!}{\text{symm. factor}}, \quad \bar{M}_{i,j} = \frac{4!}{\text{symm. factor}}, \quad (5.82)$$

⁷⁰More precisely, the correlation function in presence of the defects is defined as

$$\langle \phi^2(x) \rangle \equiv \frac{\langle \phi(x) D_L D_S \rangle}{\langle D_L D_S \rangle}, \quad \langle \phi^4(x) \rangle \equiv \frac{\langle \phi^2(x) D_L D_S \rangle}{\langle D_L D_S \rangle}, \quad (5.79)$$

where $D_{L,S} = e^{-I_{L,S}}$ with $I_{L,S}$ being the action of the line/surface deformation. In this way, it is clear that the partially connected diagrams are excluded.

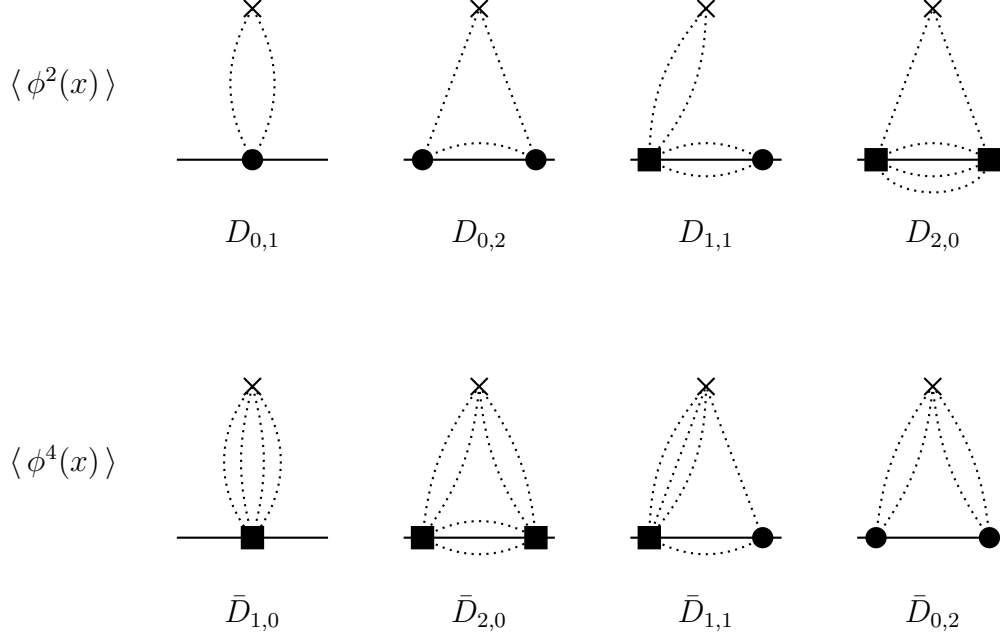


Figure 5.6: Diagrams up to the second order in both h_0 and g_0 contributing $\langle \phi^2(x) \rangle$ and $\langle \phi^4(x) \rangle$. The dashed line represents the propagator and the cross symbols represent the bulk operator insertions. Also, the circle and square represent the line and surface defect interactions, respectively.

where the factorials $2!$ and $4!$ in the numerator are results of the insertion of the quadratic and quartic operators respectively. For instance, the symmetry factor for $D_{0,1}$ is $2!$, due to the pair of the dotted lines, while for $D_{1,1}$, it is $2!2!$, as there are two pairs. The multiplication factors for all the diagrams under consideration are given as,

$$\begin{aligned} M_{0,1} = M_{0,2} = 1, \quad M_{1,1} = \frac{1}{2}, \quad M_{2,0} = \frac{1}{6}, \\ \bar{M}_{1,0} = 1, \quad \bar{M}_{2,0} = \frac{3}{2}, \quad \bar{M}_{1,1} = 4, \quad \bar{M}_{0,2} = 3. \end{aligned} \quad (5.83)$$

The integrals $I_{i,j}$ and $\bar{I}_{i,j}$ can be evaluated using the free bulk propagator,

$$G(x, y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y)}}{p^2} = \frac{C_{\phi\phi}}{|x-y|^{d-2}}, \quad C_{\phi\phi} = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}, \quad (5.84)$$

with $C_{\phi\phi}$ being the two-point function normalization of the free bulk field. Before giving the explicit formulas of those integrals, let us give an overview of their divergent structures. For the contributions to $\langle \phi^2(x) \rangle$, only $I_{0,2}(x)$ and $I_{1,1}(x)$ are divergent as $1/\varepsilon$. This means that to one-loop order, δg receives corrections from the renormalized surface coupling h , as $D_{1,1}$ is a mixed diagram. However, for the contributions to $\langle \phi^4(x) \rangle$, only $\bar{I}_{2,0}(x)$ is divergent as $O(1/\varepsilon)$, such that to one-loop order, δh is uninfluenced by the renormalized line coupling g . This suggests that the surface coupling has a higher hierarchy over the line coupling.

The relevant integrals for the corrections to the line defect coupling are $I_{0,1}(x)$, $I_{0,2}(x)$, $I_{1,1}(x)$, while to the surface coupling, the relevant ones are $\bar{I}_{1,0}(x)$ and $\bar{I}_{2,0}(x)$. The results for those integrals

are given as following which can be obtained using formulas listed in (5.175), (5.176) and (5.177) of appendix 5.5.2

$$I_{0,1}(x) = \frac{\pi^{\frac{1}{2}-d} \Gamma\left(\frac{d}{2} - 1\right)^2 \Gamma\left(d - \frac{5}{2}\right)}{16 \Gamma(d-2) \left(x_{\parallel}^2 + x_{\perp}^2\right)^{d-\frac{5}{2}}}, \quad (5.85)$$

$$I_{0,2}(x) = \frac{(4\pi)^{\frac{3}{2}(1-d)} \Gamma\left(\frac{3}{2} - \frac{d}{2}\right) \Gamma\left(d - \frac{5}{2}\right) \Gamma\left(\frac{3d}{2} - 4\right)}{\Gamma\left(\frac{d-1}{2}\right) \left(x_{\parallel}^2 + x_{\perp}^2\right)^{\frac{3d}{2}-4}}, \quad (5.86)$$

$$I_{1,1}(x) = \frac{\pi^{1-2d} \Gamma(3-d) \Gamma\left(\frac{d}{2} - 1\right)^4 \Gamma\left(d - \frac{5}{2}\right) \Gamma\left(2d - \frac{11}{2}\right)}{256 \Gamma(d-2)^2 |x_{\perp}|^{4d-11}}, \quad (5.87)$$

$$\bar{I}_{1,0}(x) = \frac{\pi^{1-2d} \Gamma\left(\frac{d-2}{2}\right)^4 \Gamma(2d-5)}{256 \Gamma(2d-4) |x_{\perp}|^{4d-10}}, \quad (5.88)$$

$$\bar{I}_{2,0}(x) = \frac{\pi^{2-3d} \Gamma(3-d) \Gamma\left(\frac{d-2}{2}\right)^6 \Gamma(2d-5)^2 \Gamma(3d-8)}{4096 \Gamma(d-2)^2 \Gamma(4d-10) |x_{\perp}|^{6d-16}}. \quad (5.89)$$

5.3.2 Localized RG flows for the single scalar case

By requiring the finiteness of the one-point functions, we can obtain the one-loop corrections to the surface and line couplings. Collecting the relevant pieces, the requirement of the order $O(1/\varepsilon)$ terms to vanish in $\langle \phi^4(x) \rangle$ gives the one-loop correction to the surface coupling

$$\delta h = \frac{3h^2}{32\pi} + \text{higher orders}. \quad (5.90)$$

It is worth noting that taking the parallel coordinate $x_{\parallel} = 0$ makes it easier to obtain the above. By using the fact that the bare coupling h_0 is independent of the renormalization scale M , we can obtain the beta function

$$\beta_h = M \frac{dh}{dM} = -2\varepsilon h - 2\delta h + 2h \frac{\partial \delta h}{\partial h} + \dots = -2\varepsilon h + \frac{3h^2}{16\pi} + \dots. \quad (5.91)$$

Similarly for $\langle \phi^2(x) \rangle$, collecting relevant pieces gives the one-loop correction to the line coupling and then the beta function

$$\delta g = \frac{16g^2 + gh}{32\pi} + \text{higher orders}, \quad (5.92)$$

$$\beta_g = -\varepsilon g - \delta g - \frac{\beta_h}{\varepsilon} \frac{\partial \delta g}{\partial h} + g \frac{\partial \delta g}{\partial g} + \dots = -\varepsilon g + \frac{8g^2 + gh}{16\pi} + \dots, \quad (5.93)$$

where the renormalized surface coupling h enters into the one-loop corrections of the line coupling. The beta functions β_h and β_g determine the RG flows as plotted in Fig. 5.7. When both couplings are turned off, the theory is trivial and has the UV fixed point (black); when either the line or the surface coupling is turned on, the UV theory flows to a non-trivial IR fixed point (red or green); most interestingly, when both couplings are turned on, there exists a non-trivial IR fixed point (magenta), which realizes the composite defect at criticality. In the current situation with a single scalar theory,

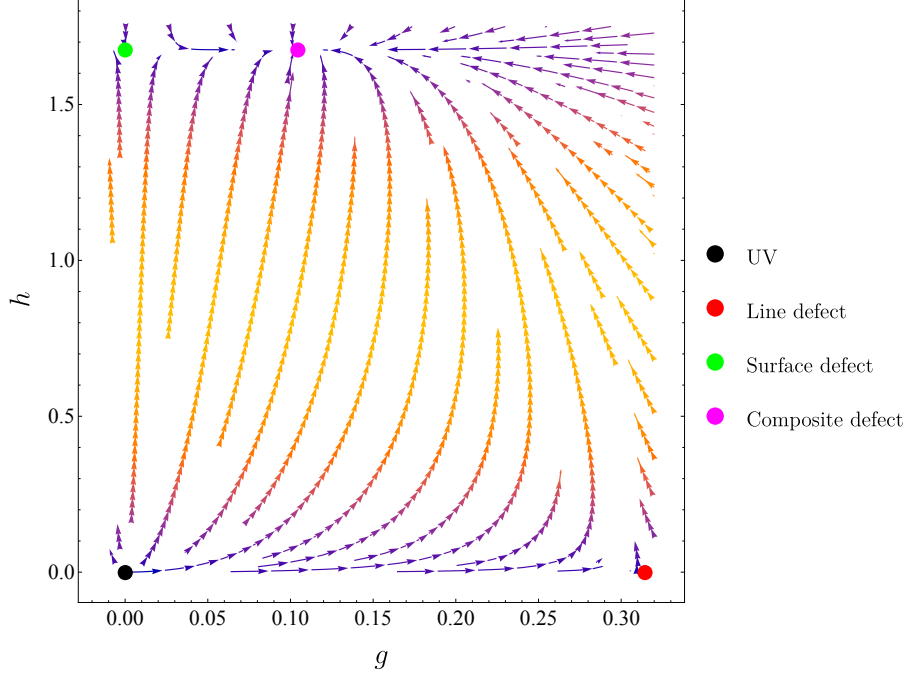


Figure 5.7: RG flows plotted for $\varepsilon = 0.05$. The black point represents the UV fixed point, the red/green point represents the non-trivial IR fixed point with only the line/surface defect, while the magenta point represents the non-trivial IR fixed point with the composite defect.

this composite defect fixed point is the most infrared among the three IR fixed points. Requiring both beta functions (5.91) and (5.93) to vanish gives the explicit expression for the critical surface and line defect couplings:

$$h_* = \frac{32\pi}{3}\varepsilon + O(\varepsilon^2), \quad (5.94)$$

$$g_* = 2\pi\varepsilon - \frac{h_*}{8} + O(\varepsilon^2) = \frac{2\pi}{3}\varepsilon + O(\varepsilon^2). \quad (5.95)$$

Notice that if we set $\varepsilon = 0$, then the composite defect CFT does not exist. This observation is consistent with the fact that there is no non-trivial line or surface defects in a single free scalar theory for three-dimensional bulk [50]. The existence of the critical composite defects is highly non-trivial, even though our discussion is mainly on the free bulk theory. As we show in appendix 5.5.2, such a composite defect fixed point does not exist for all the higher dimensional $d > 4$ free scalar theories. This captures the essence of our main result; in the next section we extend the studies to the $O(N)$ case.

5.4 $O(N)$ vector model with composite defects

In the previous section, we obtained the conformal fixed point which realizes the composite defect CFT in a free scalar theory by focusing on the single scalar case. In this section, we extend the analysis to the $O(N)$ vector model, and explore various features of the model.

5.4.1 Localized RG flows and analysis of fixed points

Clearly, we have a lot of symmetry breaking patterns on both the surface and line defects unlike the single scalar case. For simplicity, we pay our attention to the case where the $O(N)$ symmetry is preserved on the surface, but not necessary on the line. Then, the action of our model in $d = 3 - \epsilon$ can be written as

$$I \equiv \frac{1}{2} \int_{\mathbb{R}^d} d^d x \partial_\mu \phi^I \partial^\mu \phi^I + \frac{h_0}{4!} \int_{\mathbb{R}^2} d^2 \hat{y} (\phi^I \phi^I)^2 + \frac{g_{0,IJ}}{2} \int_{\mathbb{R}} d\tilde{z} \phi^I \phi^J. \quad (5.96)$$

Here ϕ^I is the $O(N)$ vector field, and I, J run from 1 to N . Also, h_0 and $g_{0,IJ}$ are bare couplings for surface and line actions, respectively. In an analogous manner as the previous section, we can compute the beta functions for these couplings at one-loop order,

$$\begin{cases} \beta_h = -2\epsilon h + \frac{N+8}{48\pi} h^2 + (\text{higher order}), \\ \beta_{IJ} = -\epsilon g_{IJ} + \frac{1}{2\pi} (g^2)_{IJ} + \frac{1}{48\pi} (\text{Tr} g \delta_{IJ} + 2g_{IJ}) h + (\text{higher order}), \end{cases} \quad (5.97)$$

where h and g_{IJ} are renormalized couplings, the beta functions β_h and β_{IJ} are defined by

$$\beta_h \equiv M \frac{dh}{dM}, \quad \beta_{IJ} \equiv M \frac{dg_{IJ}}{dM}. \quad (5.98)$$

Conformal fixed points are obtained by solving the equations $\beta_h = \beta_{IJ} = 0$. The former condition leads to the critical surface coupling h_* as follows

$$h_* = \frac{96\pi}{N+8} \epsilon + O(\epsilon^2). \quad (5.99)$$

Substituting this critical value to the equation $\beta_{IJ} = 0$, it turns out that critical sub-defect couplings $(g_*)_{IJ}$ have to satisfy the following matrix quadratic equations

$$(N+8)(g_*^2)_{IJ} - 2(N+4)\pi\epsilon(g_*)_{IJ} + 4\pi\epsilon \text{Tr} g_* \delta_{IJ} = 0. \quad (5.100)$$

From this equation, we can immediately see that $\text{Tr} g_* \neq 0$ for any real symmetric matrix $(g_*)_{IJ}$. We can easily prove this statement by contradiction. Suppose $\text{Tr}(g_*) = 0$. By taking trace with respect to $O(N)$ indices in the quadratic equation (5.100), we then obtain $\text{Tr}(g_*^2) = 0$ in general. Since we now assume that g_* is a real symmetric matrix, this result implies that only trivial solutions are possible. This completes the proof. Solving this quadratic equations (5.100) is, however, a quite hard problem, hence we proceed our analysis by paying our attention to the following symmetry breaking patterns on the sub-defect: $O(N) \rightarrow O(m) \times O(N-m)$. In this case, the coupling constants can be written as

$$g_{IJ} = \text{diag} \left(\underbrace{g, g, \dots, g}_m, \underbrace{g', g', \dots, g'}_{N-m} \right), \quad (5.101)$$

and the beta functions become

$$\beta_g = -\epsilon g + \frac{1}{2\pi} g^2 + \frac{1}{48\pi} (m g + (N-m)g' + 2g)h + \dots, \quad (5.102)$$

$$\beta_{g'} = -\epsilon g' + \frac{1}{2\pi} g'^2 + \frac{1}{48\pi} (m g + (N-m)g' + 2g')h + \dots. \quad (5.103)$$

Notice that without loss of generality, we can set $m \leq N/2$. Under this restriction, the quadratic equation (5.100) is reduced to the following simultaneous equations

$$\begin{aligned} (N+8)g^2 + 4(N-m)\pi\epsilon g' - 2(N-2m+4)\pi\epsilon g &= 0, \\ (N+8)g'^2 + 2(N-2m-4)\pi\epsilon g' + 4m\pi\epsilon g &= 0. \end{aligned} \quad (5.104)$$

In addition to the trivial solution:

$$P_0 : g_* = g'_* = 0, \quad (5.105)$$

there are three non-trivial solutions which are given by

$$P_1 : g_* = g'_* = \frac{8-2N}{N+8}\pi\epsilon + O(\epsilon^2), \quad (5.106)$$

$$P_2 : \begin{cases} g_* = \frac{3N-4m+4+\sqrt{D_{N,m}}}{N+8}\pi\epsilon + O(\epsilon^2), \\ g'_* = \frac{-N+4m+4-\sqrt{D_{N,m}}}{N+8}\pi\epsilon + O(\epsilon^2), \end{cases} \quad (5.107)$$

$$P_3 : \begin{cases} g_* = \frac{3N-4m+4-\sqrt{D_{N,m}}}{N+8}\pi\epsilon + O(\epsilon^2), \\ g'_* = \frac{-N+4m+4+\sqrt{D_{N,m}}}{N+8}\pi\epsilon + O(\epsilon^2), \end{cases} \quad (5.108)$$

where $D_{N,m}$ is the discriminant defined by

$$D_{N,m} \equiv N^2 + 16m^2 - 16Nm - 8N + 16. \quad (5.109)$$

Unlike P_0 and P_1 , two fixed points P_2 and P_3 can be possibly complex depending on the sign of $D_{N,m}$. Indeed, we can easily check that for $N \leq 22$, these two fixed points become complex CFTs regardless of the value of m . Also, when $m = N/2$ for any even integer N , P_2 and P_3 again describe complex CFTs. It is not until $N \geq 23$ that unitary CFTs appear for the following values of m :

$$m = 1, 2, \dots, \left\lfloor \frac{N}{2} - \frac{\sqrt{(3N-4)(N+4)}}{4} \right\rfloor. \quad (5.110)$$

Below, we will see the behaviors of RG flows around unitary and complex CFTs, respectively.

Renormalization group flow around unitary CFTs. We first discuss the sub-defect RG flow in our composite defect system by focusing on the case where both fixed points P_2 and P_3 are unitary (namely, the discriminant $D_{N,m}$ is positive). To analyze the stability around the conformal fixed points, we can perturb the sub-defect couplings around the fixed points:

$$g = g_* + \delta g, \quad g' = g'_* + \delta g', \quad (5.111)$$

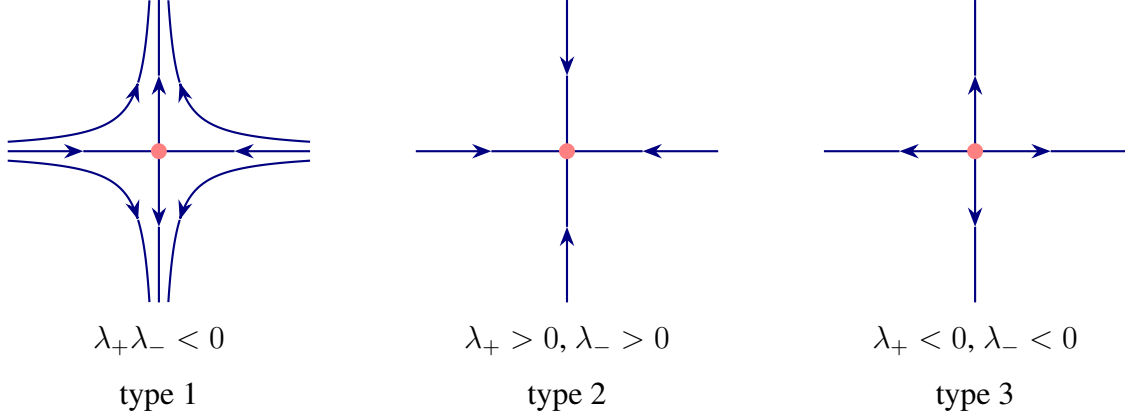


Figure 5.8: Classification of local RG flows around the fixed points (depicted as red points) in the (g_2, g'_2) plane. The blue lines mean the RG flows, and the arrows lying on them point in the IR direction. The local RG behaviors depend on the signs of eigenvalues $\lambda_{\pm}$ of the Jacobi matrix J .

with the surface defect coupling fixed to be h_* . By plugging these expressions into (5.97), we can then obtain the following linear differential equations with respect to δg_2 and $\delta g'_2$:

$$\frac{d\delta \mathbf{g}}{d \log M} = J \delta \mathbf{g}, \quad \delta \mathbf{g} \equiv \begin{pmatrix} \delta g \\ \delta g' \end{pmatrix}, \quad (5.112)$$

where J is the Jacobi matrix with the following form

$$J = \epsilon \begin{pmatrix} -1 + \frac{g_*}{\pi\epsilon} + \frac{2m+4}{N+8} & \frac{2N-2m}{N+8} \\ \frac{2m}{N+8} & -1 + \frac{g'_*}{\pi\epsilon} + \frac{2N-2m+4}{N+8} \end{pmatrix} + O(\epsilon^2). \quad (5.113)$$

As described in Fig. 5.8, the local behaviors of RG flows around the fixed points can be completely characterized by the eigenvalues $\lambda_{\pm}$ of the above Jacobi matrix J , given as

$$\lambda_{\pm}(g_*, g'_*) = \frac{(N+8)(g_* + g'_*) - 8\pi\epsilon \pm \sqrt{D}}{2(N+8)\pi}, \quad (5.114)$$

where D is defined by

$$D \equiv (N+8)^2(g_* - g'_*)^2 - 4(N+8)(N-2m)(g_* - g'_*)\pi\epsilon + 4N^2\pi^2\epsilon^2. \quad (5.115)$$

By substituting the explicit values of fixed points (5.105)–(5.108) to the eigenvalues $\lambda_{\pm}$, we obtain

$$P_0 : \lambda_+ = \frac{N-4}{N+8}\epsilon, \quad \lambda_- = -\frac{N+4}{N+8}\epsilon, \quad (5.116)$$

$$P_1 : \lambda_+ = -\frac{N-4}{N+8}\epsilon, \quad \lambda_- = -\frac{3N-4}{N+8}\epsilon, \quad (5.117)$$

$$P_2 : \lambda_{\pm} = \frac{N \pm \sqrt{2}\sqrt{N^2 + 8m^2 - 8Nm - 4N + 8 + (N-2m)\sqrt{D_{N,m}}}}{N+8}\epsilon, \quad (5.118)$$

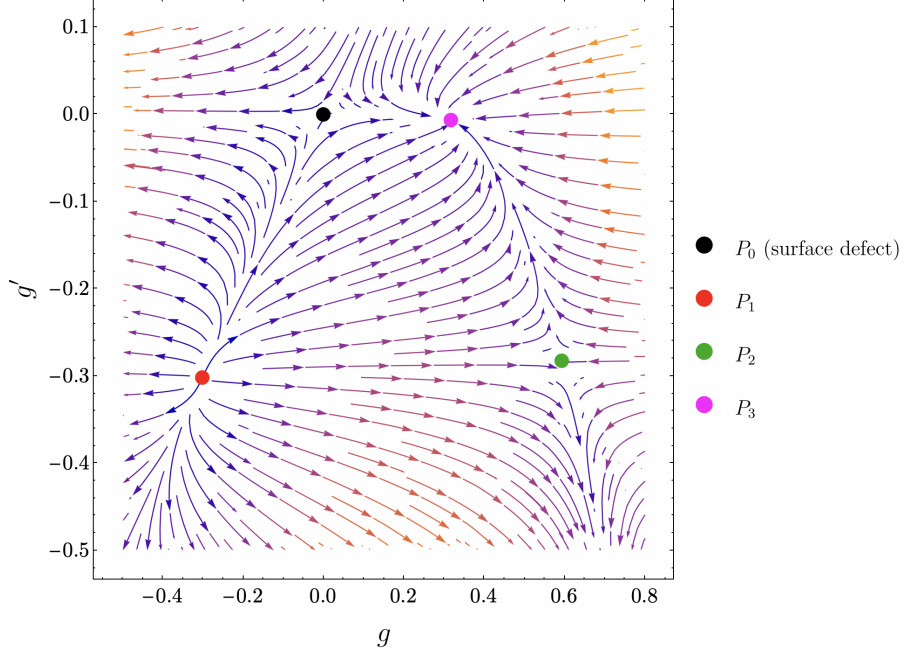


Figure 5.9: Sub-defect RG flow of the composite defect model in the case of $(N, m, \epsilon) = (300, 2, 0.05)$. Black point corresponds to the trivial theory where a conventional surface defect CFT is realized. The other three points are non-trivial conformal fixed points where composite defect CFTs are emergent. Red point (P_1) means the composite defect CFT where the $O(300)$ symmetry is preserved, while green and magenta points (P_2 and P_3) are ones where such a global symmetry is broken to its sub-group $O(298) \times O(2)$ on the sub-defect. There exist some RG flows between $O(300)$ symmetry preserving and breaking phases.

$$P_3 : \lambda_{\pm} = \frac{N \pm \sqrt{2} \sqrt{N^2 + 8m^2 - 8Nm - 4N + 8 - (N - 2m) \sqrt{D_{N,m}}}}{N + 8} \epsilon. \quad (5.119)$$

Notice that the eigenvalues for P_2 and P_3 are all real for unitary theories, namely, $N \geq 23$ and m is in the range of (5.110), although it might be somewhat non-trivial. From these expressions, it turns out that the local RG flows around P_0 and P_2 belong to the type 1 while local flows around P_3 (P_1) do to the type 2 (type 3), respectively in Fig. 5.8. In Fig. 5.9, we also plot the RG flow in the (g, g') -plane with the fixed surface defect coupling $h = h_*$ and setting $(N, m, \epsilon) = (300, 2, 0.05)$. Interestingly, this figure implies that there exist non-trivial RG flows between $O(N)$ symmetry preserving and breaking phases. In Sec. 5.5.2, we see that this obtained RG flow is completely consistent with the sub-defect $\mathcal{C}$ -theorem.

Renormalization group flow around complex CFTs. As described previously, when $m = N/2$ for arbitrary even integer N , the fixed points P_2 and P_3 become complex numbers. In particular, one of them is given by

$$\frac{g_*}{\pi\epsilon} = \frac{N+4}{N+8} + i \frac{\sqrt{(3N-4)(N+4)}}{N+8}, \quad \frac{g'_*}{\pi\epsilon} = \frac{N+4}{N+8} - i \frac{\sqrt{(3N-4)(N+4)}}{N+8}. \quad (5.120)$$

We next explore the local RG flow around this complex CFT point. For this purpose, we promote the originally real couplings g and g' to the complex ones as follows:

$$g \equiv g + i\tilde{g} \quad , \quad g' \equiv g' + i\tilde{g}' \quad , \quad (5.121)$$

where $g, \tilde{g}, g'$ and $\tilde{g}'$ are all real couplings. Correspondingly, we can define the beta functions for each coupling as follows:

$$\beta_g \equiv \frac{dg}{d \log M} \quad , \quad \beta_{\tilde{g}} \equiv \frac{d\tilde{g}}{d \log M} \quad , \quad \beta_{g'} \equiv \frac{dg'}{d \log M} \quad , \quad \beta_{\tilde{g}'} \equiv \frac{d\tilde{g}'}{d \log M} \quad . \quad (5.122)$$

We now have the four-dimensional coupling space, and it is hard to visualize RG flows on such a space. Therefore, in the following, we completely fix the values of g' and $\tilde{g}'$ to be the critical points (5.120), namely we set

$$g' = g'_* = \frac{N+4}{N+8} \pi \epsilon \quad , \quad \tilde{g}' = \tilde{g}'_* = -\frac{\sqrt{(3N-4)(N+4)}}{N+8} \pi \epsilon \quad , \quad (5.123)$$

and discuss on the two-dimensional $(g, \tilde{g})$ -plane. To explore RG behaviors at a little away region from the complex CFT point (5.120), we perturb the two couplings g and $\tilde{g}$ in the following way:

$$g = g_* + \delta g \quad , \quad \tilde{g} = \tilde{g}_* + \delta \tilde{g} \quad , \quad (5.124)$$

where g_* and $\tilde{g}_*$ are the fixed point (5.120) defined by

$$g_* = \frac{N+4}{N+8} \pi \epsilon \quad , \quad \tilde{g}_* = \frac{\sqrt{(3N-4)(N+4)}}{N+8} \pi \epsilon \quad . \quad (5.125)$$

By using the beta functions (5.102) and (5.103), we can obtain the following differential equations up to leading term:

$$\frac{d\delta \mathbf{g}}{d \log M} = \mathbf{J} \delta \mathbf{g} \quad , \quad \delta \mathbf{g} \equiv \begin{pmatrix} \delta g \\ \delta \tilde{g} \end{pmatrix} \quad , \quad (5.126)$$

where $\mathbf{J}$ is the Jacobi matrix which is defined by

$$\mathbf{J} = \frac{\epsilon}{N+8} \begin{pmatrix} N & -\sqrt{(3N-4)(N+4)} \\ \sqrt{(3N-4)(N+4)} & N \end{pmatrix} + O(\epsilon^2) \quad . \quad (5.127)$$

As easily checked. unlike the localized RG flows around the real fixed points, the eigenvalues of the matrix $\mathbf{J}$ are complex-valued for $N \geq 2$. We also notice that the real parts of these eigenvalues are positive. This implies that the complex CFT point (5.120) is IR stable in the $(g, \tilde{g})$ -plane, which is illustrated in Fig. 5.10. Interestingly, this RG behavior can be seen as the linear approximation of the equation (41) in the literature [215], and we naturally expect that the higher order contributions to the beta function produce the chaotic RG behavior. We leave more detailed investigations on RG flows around complex CFT points to future studies.

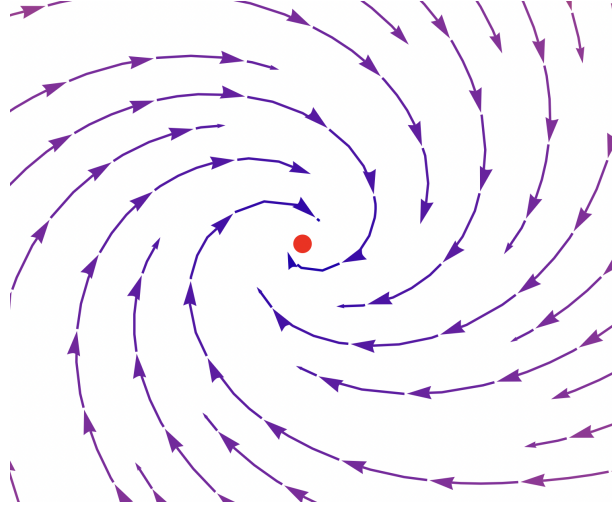


Figure 5.10: The localized RG flow around the complex CFT point in the $(g, \tilde{g})$ -plane. The red point represents the complex fixed point (5.120), and the arrow on the RG flow points to the IR direction. This fixed point is IR stable at least at the leading order.



Figure 5.11: Feynman diagrams contributing to the two-point function of sub-defect fundamental fields. The dashed line represents the propagator and the cross symbols represent the sub-defect fundamental field. Also, the circle represents the interaction localized on the line defect.

5.4.2 Anomalous dimensions of defect and sub-defect local primaries

In the previous section, we investigated the RG structures of composite defect systems, and succeeded in finding conformal fixed points which realize the composite defect CFTs. Composite defects at the critical point can be characterized by conformal data e.g. anomalous dimensions. In this section, hence, we derive the anomalous dimensions of sub-defect local primaries.

Anomalous dimensions of $O(N)$ fundamental fields

As a warm-up, we start with the anomalous dimension of the sub-defect fundamental $O(N)$ vector field $\phi^I(\tilde{z})$. To compute it, we must calculate the sub-defect two-point function $\langle \phi^I(\tilde{z}) \phi^J(0) \rangle$ at the one-loop level. This evaluation can be performed by using the perturbative method as before (see Fig. 5.11):

$$\begin{aligned} \langle \phi^I(\tilde{z}) \phi^J(0) \rangle &= \langle \phi^I(\tilde{z}) \phi^J(0) \rangle_0 - g_{0,KL} \int_{\mathbb{R}} d\tilde{w} \langle \phi^I(\tilde{z}) \phi^K(\tilde{w}) \rangle_0 \langle \phi^J(0) \phi^L(\tilde{w}) \rangle_0 \\ &= \frac{C_{\phi\phi} \delta^{IJ}}{|\tilde{z}|^{1-\epsilon}} - \frac{g_{0,IJ}}{4\pi^2} \left(\frac{1}{\epsilon} + \aleph + O(\epsilon) \right) \frac{1}{|\tilde{z}|^{1-2\epsilon}}, \end{aligned} \quad (5.128)$$

where $\aleph$ is numerical constant which is defined by

$$\aleph \equiv \gamma_E + \log 4\pi . \quad (5.129)$$

Below, we focus on the symmetry breaking case $O(N) \rightarrow O(m) \times O(N-m)$ (see (5.101)). We introduce the wavefunction renormalization factors Z_m and Z_{N-m} :

$$\begin{aligned} \phi^\alpha(\tilde{z}) &= Z_m [\phi^\alpha](\tilde{z}) , & \alpha &= 1 \sim m , \\ \phi^i(\tilde{z}) &= Z_{N-m} [\phi^i](\tilde{z}) , & i &= m+1 \sim N , \end{aligned} \quad (5.130)$$

where $[\phi^\alpha](\tilde{z})$ and $[\phi^i](\tilde{z})$ are renormalized sub-defect local operators. We choose the factors Z_m and Z_{N-m} such that the pole appearing in (5.128) is canceled:

$$Z_m \cong 1 - \frac{g}{2\pi} \frac{1}{\epsilon} , \quad Z_{N-m} \cong 1 - \frac{g'}{2\pi} \frac{1}{\epsilon} . \quad (5.131)$$

From this, we can compute the anomalous dimensions of the fundamental sub-defect operators at the critical point:

$$\gamma_m \equiv \left. \frac{d \log Z_m}{d \log M} \right|_{\text{fixed point}} \cong \frac{g_*}{2\pi} , \quad \gamma_{N-m} \equiv \left. \frac{d \log Z_{N-m}}{d \log M} \right|_{\text{fixed point}} \cong \frac{g'_*}{2\pi} . \quad (5.132)$$

We can then arrive at the following explicit forms of the sub-defect two-point functions:

$$\langle [\phi^\alpha](\tilde{z}) [\phi^\beta](0) \rangle = \delta^{\alpha\beta} \frac{\mathcal{N}_m}{|\tilde{z}|^{2\check{\Delta}_m}} , \quad \mathcal{N}_m \equiv C_{\phi\phi} \left(1 - \frac{g_*}{2\pi} \aleph + O(\epsilon^2) \right) , \quad (5.133)$$

$$\langle [\phi^i](\tilde{z}) [\phi^j](0) \rangle = \delta^{ij} \frac{\mathcal{N}_{N-m}}{|\tilde{z}|^{2\check{\Delta}_{N-m}}} , \quad \mathcal{N}_{N-m} \equiv C_{\phi\phi} \left(1 - \frac{g'_*}{2\pi} \aleph + O(\epsilon^2) \right) , \quad (5.134)$$

where $\check{\Delta}_m$ and $\check{\Delta}_{N-m}$ are conformal dimensions of renormalized fundamental operators $[\phi^\alpha](\tilde{z})$ and $[\phi^j](\tilde{z})$, respectively:

$$\check{\Delta}_m \cong 1 - \frac{\epsilon}{2} + \frac{g_*}{2\pi} + O(\epsilon^2) , \quad \check{\Delta}_{N-m} \cong 1 - \frac{\epsilon}{2} + \frac{g'_*}{2\pi} + O(\epsilon^2) . \quad (5.135)$$

Anomalous dimensions of sub-defect composite operators

We next move on to the anomalous dimensions of sub-defect composite operators. In particular, we concentrate on the following scalar primaries with the canonical scaling dimension one:

$$\Phi \equiv \frac{1}{\sqrt{m}} \sum_{\alpha=1}^m (\phi^\alpha)^2 , \quad \Psi \equiv \frac{1}{\sqrt{N-m}} \sum_{i=m+1}^N (\phi^i)^2 , \quad (5.136)$$

which are the interacting terms localized on the line defect (5.96). Also, the normalization factors are chosen for later convenience. We remark that these operators have the same canonical dimensions,

hence we can expect that Φ and Ψ should mix to each other due to the quantum corrections. The renormalized fields $[\Phi]$ and $[\Psi]$ are related to the bare fields Φ and Ψ as

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = Z_S \begin{pmatrix} [\Phi] \\ [\Psi] \end{pmatrix}, \quad (5.137)$$

where Z_S is the wavefunction renormalization matrix whose size is two. By using this matrix, we can compute the anomalous dimension matrix γ which is defined by

$$\gamma \equiv Z_S^{-1} \frac{d}{d \log M} Z_S \Big|_{\text{fixed point}} \quad (5.138)$$

From the general grounds, we can write this matrix in the following manner:

$$Z_S = \mathbf{1}_{2 \times 2} + \delta Z_S, \quad (5.139)$$

where $\mathbf{1}_{2 \times 2}$ is the unit matrix. By requiring that the following three-point functions:

$$\langle [\phi^I](z_1) [\phi^J](z_2) [\Phi](0) \rangle, \quad \langle [\phi^I](z_1) [\phi^J](z_2) [\Psi](0) \rangle, \quad (5.140)$$

should have no poles, we can fix the renormalization matrix as follows:

$$\delta Z_S = -\frac{1}{96\pi\epsilon} \begin{pmatrix} (m+2)h + 96g & \sqrt{m(N-m)}h \\ \sqrt{m(N-m)}h & (N-m+2)h + 96g' \end{pmatrix}. \quad (5.141)$$

(See appendix 5.5.2 for more details.) By plugging this into (5.138), we can obtain the anomalous dimension matrix at the leading order:

$$\gamma = \frac{1}{48\pi} \begin{pmatrix} (m+2)h_* + 48g_* & \sqrt{m(N-m)}h_* \\ \sqrt{m(N-m)}h_* & (N-m+2)h_* + 48g'_* \end{pmatrix} + O(\epsilon^2), \quad (5.142)$$

where h_* , g_* and g'_* are fixed points which are given in Sec. 5.4.1. Notice that the above anomalous dimension matrix is symmetric. This is the reason why we introduced the factors $\sqrt{m}$ and $\sqrt{N-m}$ in (5.136).⁷¹ The eigenvalues $\gamma_{\pm}$ of this matrix gives rise to the anomalous dimensions of the two operators $S_{\pm}$ resulting from the mixing of operators Φ and Ψ , which are given by

$$\gamma_{\pm} = \frac{48(g_* + g'_*) + (N+4)h_* \pm \sqrt{192\Delta_g(12\Delta_g + mh_*) - 96N\Delta_g h_* + N^2 h_*^2}}{96\pi}, \quad (5.143)$$

where $\Delta_g \equiv g_* - g'_*$. One notes that for the $O(N)$ preserving case, $\Delta_g = 0$, the anomalous dimensions for the two operators are independent on the partition m . Substituting the fixed point values for P_1 , the anomalous dimensions gives

$$\gamma_{\pm} \Big|_{P_1} = \frac{12 - N \pm N}{8 + N} \epsilon. \quad (5.144)$$

⁷¹Equivalently, one can introduce these factors into the renormalization matrix factors instead of including these factors in the definitions of Φ and Ψ .

5.5 Towards a $\mathcal{C}$ -theorem in composite defect CFTs

For a theory with a defect $\mathcal{D}^{(p)}$ of p dimensions, one can trigger an RG flow localized on $\mathcal{D}^{(p)}$. In this circumstance, the defect $\mathcal{C}$ -theorem is conjectured to hold, which states the existence of a function called the $\mathcal{C}$ -function that decreases under any localized RG flow [126]. Motivated by the defect $\mathcal{C}$ -theorem conjecture, we extend it to the case with a composite defect and propose the following:

Conjecture. In a unitary CFT_d with a composite defect $\mathcal{D}^{(p_1, \dots, p_n)} = \cup_{i=1}^n \mathcal{D}^{(p_i)}$ consisting of n sub-defects of p_i dimensions satisfying $0 < p_1 < p_2 < \dots < p_n < d$ and $\mathcal{D}^{(p_1)} \subset \mathcal{D}^{(p_2)} \subset \dots \subset \mathcal{D}^{(p_n)}$, let $Z^{(p_1, \dots, p_n)} \equiv \langle \mathcal{D}^{(p_1, \dots, p_n)} \rangle$ be the partition function on a d -sphere. Then, the function $\mathcal{C}$ defined by

$$\mathcal{C} \equiv \sin\left(\frac{\pi p_1}{2}\right) \log \left| \frac{Z^{(p_1, p_2, \dots, p_n)}}{Z^{(p_2, \dots, p_n)}} \right| \quad (5.145)$$

does not increase under any localized RG flow on the sub-defect $\mathcal{D}^{(p_1)}$ of the lowest dimension,

$$\mathcal{C}_{\text{UV}} \geq \mathcal{C}_{\text{IR}}. \quad (5.146)$$

Note that our conjecture is a weak $\mathcal{C}$ -theorem which compares the values of the sub-defect $\mathcal{C}$ -function (5.145) at the UV and IR fixed points connected by a localized RG flow. While the $\mathcal{C}$ -function (5.145) is defined for the theories away from the fixed points, it may not be monotonically decreasing function under the flow, hence be a weak $\mathcal{C}$ -function.⁷²

5.5.1 Conformal perturbation theory on sub-defect

We examine the validity of our conjecture in the conformal perturbation theory with slightly relevant perturbation. The following argument is a slight modification of the previous works in [126, 218–221].

Let $I_{\text{DCFT}}^{(p_1, \dots, p_n)}$ be the action of a CFT with a composite defect $\mathcal{D}^{(p_1, \dots, p_n)}$, and consider an RG flow localized on the sub-defect $\mathcal{D}^{(p_1)}$ of lowest dimensions:

$$I^{(p_1, \dots, p_n)} = I_{\text{DCFT}}^{(p_1, \dots, p_n)} + \check{\lambda}_0 \int d^{p_1} \check{x} \sqrt{\check{g}} \check{\mathcal{O}}(\check{x}), \quad (5.147)$$

where $\check{\mathcal{O}}(\check{x})$ is a slightly relevant primary operator of dimension $\check{\Delta} = p_i - \varepsilon$ with small ε on $\mathcal{D}^{(p_1)}$, and $\check{\lambda}_0$ is the bare coupling at the UV cutoff scale μ_{UV} .⁷³ The two-point function at the unperturbed DCFT is normalized on $\mathbb{R}^d$ as

$$\langle \check{\mathcal{O}}(\check{x}_1) \check{\mathcal{O}}(\check{x}_2) \rangle_0 = \frac{1}{|\check{x}_1 - \check{x}_2|^{2\check{\Delta}}}, \quad (5.148)$$

⁷²There are three types of $\mathcal{C}$ -functions: weak, strong, and strongest ones [216, 217]. The function $\mathcal{C}$ satisfying (5.146) is called the weak $\mathcal{C}$ -function while it is called the strong $\mathcal{C}$ -function when $\mathcal{C}$ is a monotonically decreasing function under an RG flow. The function $\mathcal{C}$ is called the strongest $\mathcal{C}$ -function if an RG flow is its gradient flow.

⁷³We use ε for the slightly relevant perturbations to distinguish from ϵ for the ϵ -expansion.

while the three-point function is fixed by the residual conformal symmetry on $\mathcal{D}^{(p_1)}$ as⁷⁴

$$\langle \check{\mathcal{O}}(\check{x}_1) \check{\mathcal{O}}(\check{x}_2) \check{\mathcal{O}}(\check{x}_3) \rangle_0 = \frac{\check{C}}{|\check{x}_1 - \check{x}_2|^{\check{\Delta}} |\check{x}_2 - \check{x}_3|^{\check{\Delta}} |\check{x}_3 - \check{x}_1|^{\check{\Delta}}}, \quad (5.149)$$

where $\check{C}$ is a constant. It follows from the correlation functions that the OPE of a pair of the local operator $\check{\mathcal{O}}$ is

$$\check{\mathcal{O}}(\check{x}_1) \check{\mathcal{O}}(\check{x}_2) \underset{x_1 \rightarrow x_2}{=} \frac{1}{|\check{x}_1 - \check{x}_2|^{2\check{\Delta}}} + \check{C} \frac{\check{\mathcal{O}}(\check{x}_1)}{|\check{x}_1 - \check{x}_2|^{\check{\Delta}}} + \dots. \quad (5.150)$$

Now consider a correlation function $\langle \dots \rangle$ of the perturbed DCFT on $\mathbb{R}^d$ and expand it as a function of $\check{\lambda}_0$ as

$$\begin{aligned} \langle \dots \rangle &= \langle \dots \rangle_0 - \check{\lambda}_0 \int d^{p_1} \check{x}_1 \langle \check{\mathcal{O}}(\check{x}_1) \dots \rangle_0 \\ &\quad + \frac{\check{\lambda}_0^2}{2} \int d^{p_1} \check{x}_1 \int d^{p_1} \check{x}_2 \langle \check{\mathcal{O}}(\check{x}_1) \check{\mathcal{O}}(\check{x}_2) \dots \rangle_0 + \dots \\ &= \langle \dots \rangle_0 - \check{\lambda}_0 \int d^{p_1} \check{x}_1 \langle \check{\mathcal{O}}(\check{x}_1) \dots \rangle_0 \\ &\quad + \frac{\check{\lambda}_0^2}{2} \int d^{p_1} \check{x}_1 \int d^{p_1} \check{x}_2 \frac{\check{C}}{|\check{x}_1 - \check{x}_2|^{\check{\Delta}}} \langle \check{\mathcal{O}}(\check{x}_1) \dots \rangle_0 + \dots \\ &= \langle \dots \rangle_0 - \check{\lambda}(\mu) \int d^{p_1} \check{x}_1 \langle \check{\mathcal{O}}(\check{x}_1) \dots \rangle_0 + \dots \end{aligned} \quad (5.151)$$

where $\check{\lambda}(\mu)$ is the effective coupling at the energy scale μ :

$$\begin{aligned} \check{\lambda}(\mu) &= \check{\lambda}_0 - \frac{\check{\lambda}_0^2}{2} \int_{\mu_{UV}^{-1} < |\check{x}_1 - \check{x}_2| < \mu^{-1}} d^{p_1} \check{x}_2 \frac{\check{C}}{|\check{x}_1 - \check{x}_2|^{\check{\Delta}}} + \dots \\ &= \check{\lambda}_0 - \check{\lambda}_0^2 \frac{\pi^{\frac{p_1}{2}} \check{C}}{\varepsilon \Gamma(\frac{p_1}{2})} [\mu^{-\varepsilon} - \mu_{UV}^{-\varepsilon}] + \dots. \end{aligned} \quad (5.152)$$

By introducing the dimensionless couplings $\check{g}(\mu) = \check{\lambda}(\mu) \mu^{-\varepsilon}$ and $\check{g}_0 = \check{\lambda}_0 \mu_{UV}^{-\varepsilon}$, we obtain

$$\check{g}(\mu) = \check{g}_0 \left(\frac{\mu_{UV}}{\mu} \right)^{\varepsilon} - \check{g}_0^2 \frac{\pi^{\frac{p_1}{2}} \check{C}}{\varepsilon \Gamma(\frac{p_1}{2})} \left[\left(\frac{\mu_{UV}}{\mu} \right)^{2\varepsilon} - \left(\frac{\mu_{UV}}{\mu} \right)^{\varepsilon} \right] + \dots. \quad (5.153)$$

The beta function for the dimensionless coupling becomes

$$\beta(\check{g}) = \mu \frac{d\check{g}}{d\mu} = -\varepsilon \check{g} + \frac{\pi^{\frac{p_1}{2}} \check{C}}{\Gamma(\frac{p_1}{2})} \check{g}^2 + \dots, \quad (5.154)$$

which, if $\check{C} > 0$, has the IR fixed point at

$$\check{g}_* = \frac{\Gamma(\frac{p_1}{2})}{\pi^{\frac{p_1}{2}} \check{C}} \varepsilon + O(\varepsilon^2). \quad (5.155)$$

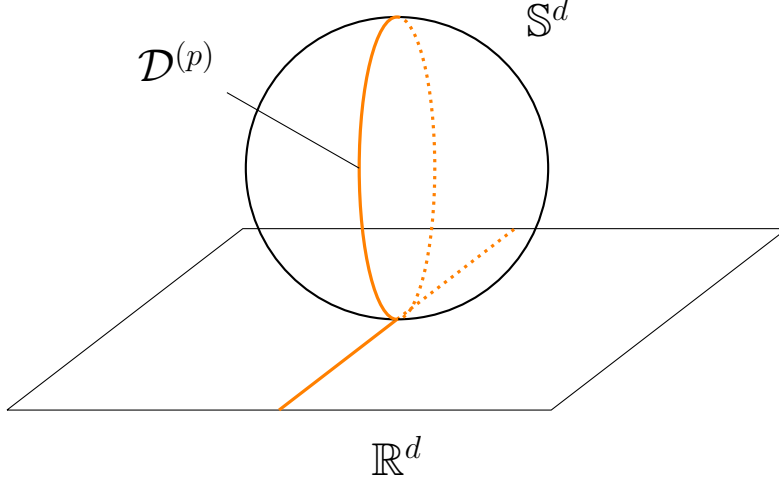


Figure 5.12: The stereographic projection of $\mathbb{S}^d$ to $\mathbb{R}^d$. A p -dimensional spherical defect $\mathcal{D}^{(p)}$ is mapped to a planar defect on $\mathbb{R}^d$.

By mapping the perturbed DCFT (5.147) on $\mathbb{R}^d$ to a d -sphere $\mathbb{S}^d$ of radius R (see figure (5.12)) and expanding the sphere partition function $Z^{(p_1, \dots, p_n)}(\check{\lambda}_0)$ around $\check{\lambda}_0 = 0$, we obtain

$$\log Z^{(p_1, \dots, p_n)}(\check{\lambda}_0) = \log Z^{(p_1, \dots, p_n)}(0) + \frac{\check{\lambda}_0^2}{2} \mathcal{I}_2 - \frac{\check{\lambda}_0^3}{6} \mathcal{I}_3 + O(\check{\lambda}_0^4), \quad (5.156)$$

where $\mathcal{I}_2$ and $\mathcal{I}_3$ are the two- and three-point connected functions in the unperturbed theory integrated over the sub-defect $\mathcal{D}^{(p_1)}$,

$$\mathcal{I}_2 = \int d^{p_1} \check{x}_1 \sqrt{\check{g}} \int d^{p_1} \check{x}_2 \sqrt{\check{g}} \langle \check{\mathcal{O}}(\check{x}_1) \check{\mathcal{O}}(\check{x}_2) \rangle_0, \quad (5.157)$$

$$\mathcal{I}_3 = \int d^{p_1} \check{x}_1 \sqrt{\check{g}} \int d^{p_1} \check{x}_2 \sqrt{\check{g}} \int d^{p_1} \check{x}_3 \sqrt{\check{g}} \langle \check{\mathcal{O}}(\check{x}_1) \check{\mathcal{O}}(\check{x}_2) \check{\mathcal{O}}(\check{x}_3) \rangle_0. \quad (5.158)$$

To evaluate $\mathcal{I}_2$ and $\mathcal{I}_3$, it is convenient to use the stereographic coordinates of the d -sphere:

$$ds_{\mathbb{S}^d}^2 = (2R)^2 \frac{\sum_{\mu=1}^d (dx^\mu)^2}{(1 + |x|^2)^2}. \quad (5.159)$$

The stereographic projection can be seen as a conformal mapping from $\mathbb{R}^d$ to $\mathbb{S}^d$, where a planar defect $\mathcal{D}^{(p_1)}$ located at $x^i = 0$ ($i = p_1 + 1, \dots, d$) is mapped to a spherical defect of radius R . The two- and three-point functions after the conformal map take the same forms as (5.148) and (5.149), respectively, with $|\check{x}_1 - \check{x}_2|$ replaced with the chordal distance $s(\check{x}_1, \check{x}_2) = 2R |\check{x}_1 - \check{x}_2| / \sqrt{(1 + |\check{x}_1|^2)(1 + |\check{x}_2|^2)}$ on $\mathbb{S}^{p_1}$. Substituting them into (5.157) and (5.158) and performing

⁷⁴ $\langle \check{\mathcal{O}}(\check{x}) \rangle_0$ stands for the vev of the local operator $\check{\mathcal{O}}(\check{x})$ at the unperturbed DCFT.

the integration on $\mathbb{S}^{p_1}$ of radius R , we obtain

$$\begin{aligned}\mathcal{I}_2 &= 2^{2\varepsilon} \pi^{p_1} R^{2\varepsilon} \frac{\Gamma(\varepsilon - \frac{p_1}{2}) \Gamma(\frac{p_1}{2})}{\Gamma(p_1) \Gamma(\varepsilon)} = -\frac{2^{2\varepsilon+1} \pi^{p_1+1} R^{2\varepsilon}}{\sin(\frac{\pi p_1}{2}) \Gamma(p_1+1)} \frac{\Gamma(\varepsilon - \frac{p_1}{2})}{\Gamma(-\frac{p_1}{2}) \Gamma(\varepsilon)}, \\ \mathcal{I}_3 &= 8 \pi^{\frac{3(p_1+1)}{2}} R^{3\varepsilon} \frac{\Gamma(\frac{3\varepsilon-p_1}{2})}{\Gamma(p_1) \Gamma(\frac{\varepsilon+1}{2})^3} \check{C}.\end{aligned}\tag{5.160}$$

On the other hand, the bare dimensionless couplings are related by

$$\check{\lambda}_0 (2R)^\varepsilon = \check{g} + \frac{\pi^{\frac{p_1}{2}} \check{C}}{\varepsilon \Gamma(\frac{p_1}{2})} \check{g}^2 + O(\check{g}),\tag{5.161}$$

where $\check{g} = \check{g}(\mu)$ and we choose the energy scale as $\mu = 1/(2R)$ while letting $\mu_{\text{UV}} \rightarrow \infty$ [219]. With this relation, it follows from (5.156) that the deviation of the sphere partition function from that at the UV fixed point ($\check{g} = 0$) is

$$\begin{aligned}\delta \log Z^{(p_1, \dots, p_n)}(\check{g}) &\equiv \log Z^{(p_1, \dots, p_n)}(\check{g}) - \log Z^{(p_1, \dots, p_n)}(0) \\ &= \frac{2 \pi^{p_1+1}}{\sin(\frac{\pi p_1}{2}) \Gamma(p_1+1)} \left[-\frac{\varepsilon}{2} \check{g}^2 + \frac{\pi^{\frac{p_1}{2}} \check{C}}{3 \Gamma(\frac{p_1}{2})} \check{g}^3 + O(\check{g}^4) \right].\end{aligned}\tag{5.162}$$

Substituting the IR value of the dimensionless coupling (5.155) and recalling the definition of the $\mathcal{C}$ -function (5.145), we find

$$\mathcal{C}(\check{g}_*) - \mathcal{C}(\check{g} = 0) = -\frac{\pi \Gamma(\frac{p_1}{2})^2}{3 \Gamma(p_1+1)} \frac{\varepsilon^3}{\check{C}^2} + O(\varepsilon^4),\tag{5.163}$$

which is negative as is consistent with our sub-defect $\mathcal{C}$ -theorem conjecture.

5.5.2 Perturbative test in $O(N)$ vector model with composite defects

In the last subsection, we gave a general argument for the validity of the conjectured sub-defect $\mathcal{C}$ -theorem in terms of the conformal perturbation theory with a one-parameter deformation. In the current subsection, we examine that by considering the $O(N)$ vector model studied in Sec. 5.4, where a line defect is embedded inside a surface defect, i.e. $p_2 = 2$ and $p_1 = 1$. In particular, we study the case where two line deformations which preserves the $O(m) \times O(N-m)$ internal symmetry are turned on, giving rise to three non-trivial IR fixed points. Using the usual perturbative approach, we can compute the $\mathcal{C}$ -functions (or the modified free energy) at all the IR fixed points. Adapting the definition of the $\mathcal{C}$ -function, in (5.145), we have

$$\mathcal{C} = \ln \frac{Z(h_0, g_0, g'_0)}{Z(h_0, 0, 0)} = C_2 + C_3 + C'_3 + (\text{higher order}),\tag{5.164}$$

where we keep the terms up to third order in the couplings, C_2 and C_3 denote the quadratic and cubic terms in the line couplings, while C'_3 denotes the mixed term between the surface and the line coupling. The contributions purely from the surface defect is subtracted from the partition function in the denominator with only the surface defect turned on.

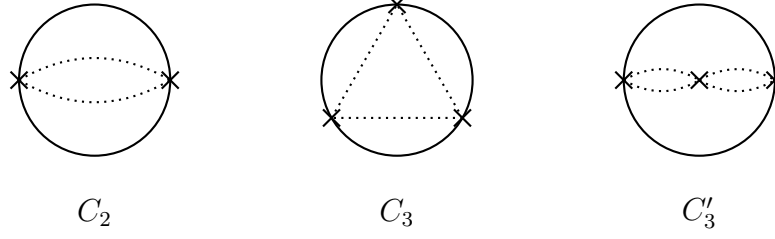


Figure 5.13: The three diagrams for the contributions to the sub-defect $\mathcal{C}$ function up to the third order of the couplings, C_2 and C_3 denote the quadratic and cubic terms in the line couplings, while C'_3 denotes the mixed term between the surface and the line coupling.

Normally, we evaluate the partition function on $\mathbb{S}^d$ to get rid of the IR divergences. However, the ratio between the partition functions gives the same result if we map $\mathbb{S}^d$ to $\mathbb{R}^d$, as the IR divergences and the conformal anomaly induced by the surface defect cancel. In order to understand the localized RG flow on the line defect, a further map is required for the evaluation. The line defect is mapped to a circle $\mathbb{S}^1$ of radius R while the surface defect is kept as the plane $\mathbb{R}^2$, then separately each term gives

$$C_2 = \frac{C_{\phi\phi}^2}{4} (m g_0^2 + (N - m) g_0'^2) I_2, \quad (5.165)$$

$$C_3 = -\frac{C_{\phi\phi}^3}{6} (m g_0^3 + (N - m) g_0'^3) I_3, \quad (5.166)$$

$$C'_3 = -\frac{C_{\phi\phi}^4}{4!} h_0 [m(m + 2) g_0^2 + (N - m)(N - m + 2) g_0'^2 + 2m(N - m) g_0 g_0'] I'_3, \quad (5.167)$$

where I_2, I_3, I'_3 are given by

$$I_2 = \int d^2 z_1 d^2 z_2 \frac{\delta(|z_1| - R) \delta(|z_2| - R)}{|z_1 - z_2|^{2(d-2)}} = 4^\varepsilon \pi^{\frac{3}{2}} R^{2\varepsilon} \frac{\Gamma(\varepsilon - \frac{1}{2})}{\Gamma(\varepsilon)}, \quad (5.168)$$

$$I_3 = \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{\delta(|z_1| - R) \delta(|z_2| - R) \delta(|z_3| - R)}{|z_1 - z_2|^{(d-2)} |z_1 - z_3|^{(d-2)} |z_2 - z_3|^{(d-2)}} = 8\pi^3 R^{3\varepsilon} \frac{\Gamma(\frac{3\varepsilon-1}{2})}{\Gamma(\frac{1+\varepsilon}{2})^3}, \quad (5.169)$$

$$I'_3 = \int d^2 z_1 d^2 z_2 d^2 y \frac{\delta(|z_1| - R) \delta(|z_2| - R)}{|z_1 - y|^{2(d-2)} |z_2 - y|^{2(d-2)}} = 4^{1-\varepsilon} \pi^3 R^{4\varepsilon} \frac{\Gamma(2\varepsilon - \frac{1}{2}) \Gamma(\frac{1}{2} - \varepsilon)}{\Gamma(\frac{1}{2} + \varepsilon)^2 \Gamma(1 - \varepsilon)}, \quad (5.170)$$

Gathering all the terms and keeping up to third orders in the coupling, we can write the perturbative result for the sub-defect $\mathcal{C}$ -function using the beta functions:

$$\mathcal{C} = -\frac{m g^3 + (N - m) g'^3}{192 \pi} + \frac{m g \beta_g + (N - m) g' \beta_{g'}}{32}, \quad (5.171)$$

where the beta functions β_g and $\beta_{g'}$ are defined as in (5.102) and (5.103) respectively. One can check to this order that the $\mathcal{C}$ -function is stationary at the conformal fixed points in the following sense that the variations with respect to the renormalized couplings are proportional to the corresponding beta functions,

$$\frac{\partial \mathcal{C}}{\partial g} = -\frac{m}{16} \beta_g, \quad \frac{\partial \mathcal{C}}{\partial g'} = -\frac{N - m}{16} \beta_{g'}. \quad (5.172)$$

Now we can compare the different fixed points using the modified sub-defect free energy. For the $O(N)$ preserving phase, the difference between the trivial (P_0) and non-trivial (P_1) fixed points is

$$\mathcal{C}(P_1) - \mathcal{C}(P_0) = \frac{\pi^2 N (N-4)^3}{24 (N+8)^3} \epsilon^3 + O(\epsilon^4), \quad (5.173)$$

of which the leading order is positive for $N > 4$. This can be compared with the general analysis in (5.163) using the conformal perturbation theory. For that we need to take $\epsilon = (4 - N) \epsilon / (N + 8)$ and the three point function (normalized by the two-point function normalization) $\check{C} = 8NC_{\phi\phi}^3 / (2NC_{\phi\phi}^2)^{3/2} = \sqrt{8/N}$. For the DCFT model to be unitary, as put before, $N \geq 23$, this suggests that the non-trivial fixed point is always more UV than the trivial one, which is consistent with the local RG flow analysis. For the $O(N)$ broken phase, the difference between the two fixed points gives

$$\mathcal{C}(P_2) - \mathcal{C}(P_3) = \frac{\pi^2 (N - 2m) (D_{N,m})^{\frac{3}{2}}}{24 (N + 8)^3} \epsilon^3 + O(\epsilon^4), \quad (5.174)$$

which indicates that the UV/IR properties of the two $O(N)$ fixed points exchanges depending on whether $m > N/2$. One can also compare the fixed points between the $O(N)$ symmetric and $O(N)$ broken phases, it turns out that the non-trivial $O(N)$ symmetric fixed point is always the most UV among the four fixed points.

5.A Some useful formulas

In this section, we collect a few formulas that are useful for the one-loop calculation,

$$\int d^d p \frac{p^{2\alpha}}{(p^2 + M^2)^\beta} = \pi^{\frac{d}{2}} M^{d+2\alpha-2\beta} \frac{\Gamma(\alpha + \frac{d}{2}) \Gamma(\beta - \alpha - \frac{d}{2})}{\Gamma(\frac{d}{2}) \Gamma(\beta)}, \quad (5.175)$$

$$\int d^d p \frac{1}{(k^2 + p^2)^\alpha p^{2\beta}} = \frac{\pi^{d/2} \Gamma(\frac{d}{2} - \beta) \Gamma(-\frac{d}{2} + \alpha + \beta)}{\Gamma(\alpha) \Gamma(\frac{d}{2})} \left(\frac{1}{k^2}\right)^{\alpha+\beta-\frac{d}{2}}, \quad (5.176)$$

$$\begin{aligned} & \iint \frac{d^d k d^d l}{(k^2 + m^2)^\alpha [(k+l)^2]^\beta (l^2 + m^2)^\gamma} \\ &= \frac{\pi^d \Gamma(\alpha + \beta - d/2) \Gamma(\beta + \gamma - d/2) \Gamma(d/2 - \beta) \Gamma(\alpha + \beta + \gamma - d)}{\Gamma(\alpha) \Gamma(\gamma) \Gamma(\alpha + 2\beta + \gamma - d) \Gamma(d/2) (m^2)^{\alpha+\beta+\gamma-d}}. \end{aligned} \quad (5.177)$$

5.B Free model in even dimensions $d = 2n - \epsilon$: no composite DCFT

We consider the defect RG flow in even spacetime dimensions $d = 2n - \epsilon$ for a single scalar. The action including the two localized deformations ($r = n - 1$) is given as

$$I = \frac{1}{2} \int d^d x (\partial\phi)^2 + \frac{h_0}{2} \int_{\mathbb{R}^{2r}} d^{2r} \hat{y} \phi^2 + g_0 \int_{\mathbb{R}^r} d^r \tilde{z} \phi, \quad (5.178)$$

where the lower dimensional defect is placed at $x^r = x^{r+1} = \dots = x^{2n-1} = 0$ and the higher dimensional defect is placed at $x^{2r} = x^{2r+1} = \dots = x^{2n-1} = 0$. The hatted coordinates cover the higher dimensional defect and the tilde coordinates cover the lower dimensional defect. We work in the minimal subtraction scheme, we can write the bare couplings in terms of the renormalized ones as

$$h_0 = M^\varepsilon \left(h + \frac{\delta h}{\varepsilon} + \frac{\delta_2 h}{\varepsilon^2} + \dots \right), \quad (5.179)$$

$$g_0 = M^{\varepsilon/2} \left(h + \frac{\delta g}{\varepsilon} + \frac{\delta_2 g}{\varepsilon^2} + \dots \right). \quad (5.180)$$

To obtain the beta functions of the two couplings, it is enough to consider two types of bulk fields, the elementary one $\phi(x)$ and the composite one $\phi^2(x)$. The bare fields and the renormalized ones are related as

$$\phi(x) = Z_\phi [\phi](x), \quad \phi^2(x) = Z_{\phi^2} [\phi^2](x), \quad (5.181)$$

with Z_ϕ and Z_{ϕ^2} being the wave-function renormalizations, however, for free bulk theory $Z_\phi = Z_{\phi^2} = 1$. The physical condition requires that the correlation of the renormalized fields to give finite answer as $\varepsilon \rightarrow 0$, in the case of one-point functions, that is

$$\langle [\phi](x) \rangle = \text{finite}, \quad \langle [\phi^2](x) \rangle = \text{finite}. \quad (5.182)$$

In the current situation, the above correlations can be evaluated to all loop order diagrammatically. Noting that only the completely connected diagrams contribute.⁷⁵ Let us start by considering the first one-point function, for the completely connected ones, it is of the chain type Γ_m as depicted on the left side in Fig. 5.14. It has m insertions of the higher-dimensional defects and ends on the lower-dimensional defect. For simplicity, let us denote the field on the surface defect as $\hat{\phi}(\hat{x})$ and on the line defect as $\check{\phi}(\check{x})$. The chain diagram can be evaluated using the Wick contraction with multiplicity $2^m m!$,

$$\begin{aligned} \Gamma_m(x) &= (-g_0) \frac{(-h_0/2)^m}{m!} \int d^r \tilde{z} \int \prod_{i=0}^m d^{2r} \hat{y}_i \langle \phi(x) \prod_{j=0}^m \hat{\phi}^2(\hat{y}_j) \check{\phi}(\check{z}) \rangle \\ &= -g_0 (-h_0)^m \int \frac{d^{d-r} \check{p}}{(2\pi)^{d-r}} f_r^m(|\check{p}_{\parallel,H}|) \frac{e^{-i\check{p}x}}{\check{p}^2} \end{aligned} \quad (5.184)$$

where the function f_r only depends on the momenta parallel to the higher-dimensional defect while orthogonal to the lower-dimensional one, more explicitly

$$f_r(|\check{p}_{\parallel,H}|) = (4\pi)^{\frac{\varepsilon-2}{2}} |\check{p}_{\parallel,H}|^{-\varepsilon} \Gamma\left(\frac{\varepsilon}{2}\right), \quad (5.185)$$

⁷⁵More precisely, the correlation function in presence of the defects is defined as

$$\langle \phi(x) \rangle \equiv \frac{\langle \phi(x) D_L D_H \rangle}{\langle D_L D_H \rangle}, \quad \langle \phi^2(x) \rangle \equiv \frac{\langle \phi^2(x) D_L D_H \rangle}{\langle D_L D_H \rangle}, \quad (5.183)$$

where $D_{L,H} = e^{-I_{L,H}}$ with $I_{L,H}$ being the action of the lower/higher dimensional deformation. In this way, it is clear that the partially connected diagrams are excluded.

which is actually independent of the spacetime dimension. Summing up all the diagrams gives

$$\langle \phi(x) \rangle = (-g_0) \int \frac{d^{d-r} \tilde{p}}{(2\pi)^{d-r}} \frac{1}{1 + h_0 f(|\tilde{p}_{\parallel, H}|)} \frac{e^{-i \tilde{p} x}}{\tilde{p}^2}. \quad (5.186)$$

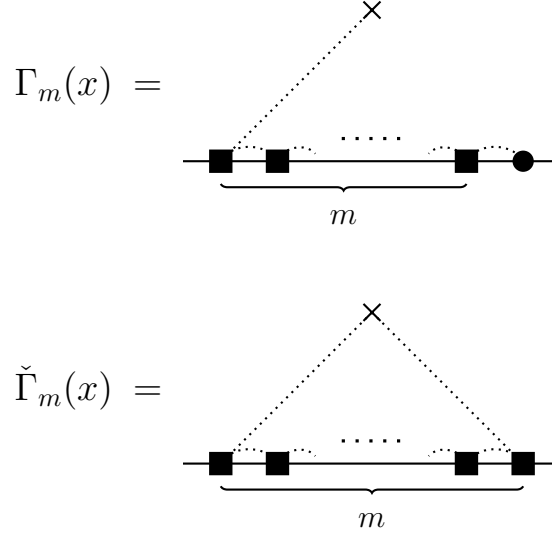


Figure 5.14: The basic diagrams for the evaluation of the one-point functions to all-loop order, with the left one for $\langle \phi(x) \rangle$ and right one for the non-factorizable part of $\langle \phi^2(x) \rangle$.

For the evaluation of $\langle \phi^2(x) \rangle$, there are two types of completely connected diagrams, with either zero or two insertions of the lower-dimensional defect. The diagram with two insertions is essentially a product of two chain-type diagrams coming from the one-point function $\langle \phi(x) \rangle$, summing up all such diagrams gives

$$\frac{(-g_0)^2}{2} \sum_{m=0}^{\infty} \frac{(-h_0/2)^m}{m!} \sum_{i=0}^m 2 \binom{m}{i} \Gamma_i(x) \Gamma_{m-i}(x) = (\langle \phi(x) \rangle)^2, \quad (5.187)$$

where the factor of 2 in the second sum comes from the pairing of the bulk field and the two insertions of the lower-dimensional defects. The non-factorizable contribution comes from the loop-type diagram $\tilde{\Gamma}_m(x)$ depicted on the right side of Fig. 5.14. They can be evaluated similar to the chain-type diagram with the same multiplicity factor $2^m m!$,

$$\begin{aligned} \tilde{\Gamma}_m(x) &= \frac{(-h_0/2)^m}{m!} \int \prod_{i=1}^m d^{2r} \hat{y}_i \langle \phi^2(x) \prod_{j=1}^m \hat{\phi}^2(\hat{y}_j) \rangle \\ &= (-h_0)^m \int \frac{d^{d-2r} \hat{p}}{(2\pi)^{d-2r}} \frac{d^d p}{(2\pi)^d} \frac{e^{-i x \hat{p}}}{(\hat{p} + p)^2 p^2} f_r^{m-1}(|p_{\parallel, H}|) \end{aligned} \quad (5.188)$$

where the function f_r has the same form as in eq. (5.185). Summing up over all the contributions gives

$$\langle \phi^2(x) \rangle = (\langle \phi(x) \rangle)^2 + \int \frac{d^{d-2r} \hat{p}}{(2\pi)^{d-2r}} \frac{d^d p}{(2\pi)^d} \frac{e^{-i x \hat{p}}}{(\hat{p} + p)^2 p^2} \frac{-h_0}{1 + h_0 f_r(|p_{\parallel, H}|)}. \quad (5.189)$$

The finiteness condition (5.182) requires that both eq. (5.186) and (5.189) are finite on its own,

$$\frac{M^{\varepsilon/2}}{g_0} + \frac{h_0}{g_0} \frac{M^{-\varepsilon/2}}{2\pi\varepsilon} = \frac{1}{g}, \quad \frac{M^\varepsilon}{h_0} + \frac{1}{2\pi\varepsilon} = \frac{1}{h} \quad (5.190)$$

solving them gives

$$h_0 = \frac{hM^\varepsilon}{1 - \frac{h}{2\pi\varepsilon}}, \quad g_0 = \frac{gM^{\varepsilon/2}}{1 - \frac{h}{2\pi\varepsilon}}. \quad (5.191)$$

Taking the derivatives with respect to the renormalization scale M for the above equations gives the beta functions for the renormalized couplings, one also has to use that fact that the bare coupling is independent of the renormalization scale

$$\beta_h = \frac{dh}{d \ln M} = \frac{h(h - 2\pi\varepsilon)}{2\pi}, \quad \beta_g = \frac{dg}{d \ln M} = \frac{g(h - \pi\varepsilon)}{2\pi}. \quad (5.192)$$

There is still an exact IR fixed point for the higher-dimensional defect, i.e. at $h_* = 2\pi\varepsilon$, while there is no IR fixed point for the lower-dimensional defect.

5.C Computations of anomalous dimensions

In this appendix, we present the detailed computations of anomalous dimensions of composite operators, which were skipped in the main text. We particularly fix the renormalized matrix Z_S which is parameterized by

$$Z_S = \mathbf{1}_{2 \times 2} + \begin{pmatrix} \delta P & \delta Q \\ \delta R & \delta S \end{pmatrix}. \quad (5.193)$$

by requiring the renormalization conditions explained around (5.140). For this purpose, we first need to evaluate the following three-point function:

$$\begin{aligned} & \langle \phi^I(\tilde{z}_1) \phi^J(\tilde{z}_2) \phi^K \phi^L(0) \rangle \\ &= \langle \phi^I(\tilde{z}_1) \phi^J(\tilde{z}_2) \phi^K \phi^L(0) \rangle_0 - \frac{h_0}{4!} \int d^2 \hat{y} \langle \phi^I(\tilde{z}_1) \phi^J(\tilde{z}_2) \phi^K \phi^L(0) |\vec{\phi}(\hat{y})|^4 \rangle_0 \\ & \quad - \frac{g_{0,PQ}}{2} \int d\tilde{z} \langle \phi^I(\tilde{z}_1) \phi^J(\tilde{z}_2) \phi^K \phi^L(0) \phi^P \phi^Q(\tilde{z}) \rangle_0 + (\text{higher order terms}). \end{aligned} \quad (5.194)$$

(See Fig. 5.15 for the corresponding Feynman diagrams.) By using the following integral formulas:

$$\begin{aligned} \int d^2 \hat{y} \frac{1}{|\tilde{z}_1 - \hat{y}|^{1-\epsilon} |\tilde{z}_2 - \hat{y}|^{1-\epsilon} |\hat{y}|^{2-2\epsilon}} &= \frac{\pi}{\epsilon} \frac{1}{|\tilde{z}_1| |\tilde{z}_2|} + O(\epsilon^0), \\ \int d\tilde{z} \frac{1}{|\tilde{z}|^{1-\epsilon} |\tilde{z} - \tilde{z}_i|^{1-\epsilon}} &= \frac{4}{\epsilon} \frac{1}{|\tilde{z}_i|} + O(\epsilon^0), \end{aligned} \quad (5.195)$$

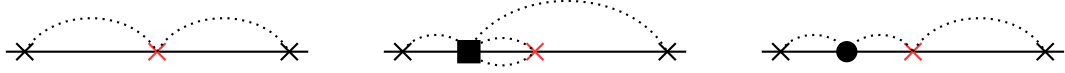


Figure 5.15: Feynman diagrams contributing to the three-point function of sub-defect local operators $\langle \phi^I(\tilde{z}_1) \phi^J(\tilde{z}_2) \phi^K \phi^L(0) \rangle$. The dashed line represents the propagator. The black and red cross symbols represent the sub-defect local fundamental field ϕ^I and composite one $\phi^K \phi^L$, respectively. Also, the circle and square represent the interaction localized on the line and surface defects, respectively.

we can compute the above three-point function as follows:

$$\langle \phi^I(\tilde{z}_1) \phi^J(\tilde{z}_2) \phi^K \phi^L(0) \rangle = \frac{C_{\phi\phi}^2}{|\tilde{z}_1| |\tilde{z}_2|} \left(A^{IJKL} - \frac{h_0}{48\pi\epsilon} B^{IJKL} - \frac{1}{\pi\epsilon} C^{IJKL}(g_0) \right), \quad (5.196)$$

where A^{IJKL} , B^{IJKL} and C^{IJKL} are defined by

$$\begin{aligned} A^{IJKL} &\equiv \delta^{IK} \delta^{JL} + \delta^{IL} \delta^{JK}, \\ B^{IJKL} &\equiv \delta^{IJ} \delta^{KL} + \delta^{IK} \delta^{JL} + \delta^{IL} \delta^{JK}, \\ C^{IJKL}(g_0) &\equiv g_{0,JL} \delta_{IK} + g_{0,JK} \delta_{IL} + g_{0,IL} \delta_{JK} + g_{0,IK} \delta_{JL}. \end{aligned} \quad (5.197)$$

From this, we can immediately deduce all three-point functions which are necessary to derive anomalous dimensions of composite operators:

$$\begin{aligned} \langle \phi^\alpha(\tilde{z}_1) \phi^\beta(\tilde{z}_2) \Phi(0) \rangle &= \frac{1}{16\pi^2 \sqrt{m}} \left(2 - \frac{m+2}{48\pi\epsilon} h_0 - \frac{4}{\pi\epsilon} g_0 \right) \frac{\delta^{\alpha\beta}}{|\tilde{z}_1| |\tilde{z}_2|} + O(\epsilon^0), \\ \langle \phi^\alpha(\tilde{z}_1) \phi^\beta(\tilde{z}_2) \Psi(0) \rangle &= -\frac{\sqrt{N-m} h_0}{768\pi^3 \epsilon} \frac{\delta^{\alpha\beta}}{|\tilde{z}_1| |\tilde{z}_2|} + O(\epsilon^0), \\ \langle \phi^i(\tilde{z}_1) \phi^j(\tilde{z}_2) \Phi(0) \rangle &= -\frac{\sqrt{m} h_0}{768\pi^3 \epsilon} \frac{\delta^{ij}}{|\tilde{z}_1| |\tilde{z}_2|} + O(\epsilon^0), \\ \langle \phi^i(\tilde{z}_1) \phi^j(\tilde{z}_2) \Psi(0) \rangle &= \frac{1}{16\pi^2 \sqrt{N-m}} \left(2 - \frac{N-m+2}{48\pi\epsilon} h_0 - \frac{4}{\pi\epsilon} g'_0 \right) \frac{\delta^{ij}}{|\tilde{z}_1| |\tilde{z}_2|} + O(\epsilon^0), \\ \langle \phi^i(\tilde{z}_1) \phi^\alpha(\tilde{z}_2) \Phi(0) \rangle &= \langle \phi^i(\tilde{z}_1) \phi^\alpha(\tilde{z}_2) \Psi(0) \rangle = 0. \end{aligned} \quad (5.198)$$

Furthermore, by using the results (5.131), we can obtain the three-point functions consisting of renormalized fields at leading order:

$$\begin{aligned} \langle [\phi^\alpha](\tilde{z}_1) [\phi^\beta](\tilde{z}_2) [\Phi](0) \rangle &= \frac{1}{768\sqrt{m}\pi^3\epsilon} (96\pi\epsilon \delta P + (m+2)h + 96g) + O(\epsilon), \\ \langle [\phi^\alpha](\tilde{z}_1) [\phi^\beta](\tilde{z}_2) [\Psi](0) \rangle &= -\frac{1}{768\sqrt{m}\pi^3\epsilon} (96\pi\epsilon \delta R + \sqrt{m(N-m)}h) + O(\epsilon), \\ \langle [\phi^i](\tilde{z}_1) [\phi^j](\tilde{z}_2) [\Phi](0) \rangle &= \frac{1}{768\sqrt{N-m}\pi^3\epsilon} (96\pi\epsilon \delta Q + \sqrt{m(N-m)}h) + O(\epsilon), \\ \langle [\phi^i](\tilde{z}_1) [\phi^j](\tilde{z}_2) [\Psi](0) \rangle &= -\frac{1}{768\sqrt{N-m}\pi^3\epsilon} (96\pi\epsilon \delta S + (N-m+2)h + 96g') + O(\epsilon). \end{aligned} \quad (5.199)$$

Here, we omit the three-point functions $\langle \phi^i(\tilde{z}_1) \phi^\alpha(\tilde{z}_2) \Phi(0) \rangle$ and $\langle \phi^i(\tilde{z}_1) \phi^\alpha(\tilde{z}_2) \Psi(0) \rangle$ since these both of them do not include divergences. The renormalization conditions, therefore, force the matrix Z_S to be fixed as (5.141) in the $\overline{\text{MS}}$ scheme.

Chapter 6

Conclusion and Future Directions

In this thesis, entitled *Many Faces of Defects in Conformal Field Theories*, we have explored conformal defects from multiple complementary perspectives, emphasizing their universal, dynamical, and organizational roles within conformal field theories. By systematically studying defects of different origins—ranging from purely single defect and disorder-induced defects to genuinely composite defects—we have demonstrated that defects are not merely passive probes of bulk CFTs, but rather active structures that give rise to rich and sometimes unexpected infrared physics. In what follows, we briefly revisit the main results of each chapter and, building on these findings, outline several future directions that naturally emerge from our analysis.

In Chap. 2, we laid the conceptual and technical groundwork for the entire thesis by reviewing conformal field theories and their generalization in the presence of defects. We highlighted how defects modify symmetry constraints, introduce new operator sectors, and necessitate refined notions of conformal data such as defect operator expansions and defect conformal blocks. This chapter established a unified language that allowed us to treat diverse defect phenomena within a single CFT framework and served as a reference point for all subsequent analyses.

Chap. 3 focused on a paradigmatic example of a dynamical defect: a line defect in the $O(N)$ vector model in $d = 4 - \epsilon$ dimensions. By combining the axiomatic approach of Rychkov and Tan with defect operator expansions, we demonstrated that defect conformal symmetry with bulk dynamics is powerful enough to fix the critical defect coupling at the Wilson–Fisher fixed point at leading order. Furthermore, we computed leading anomalous dimensions of defect local operators, including composite operators, without relying on explicit Feynman diagrammatics. The agreement with known perturbative results not only validated our approach but also illustrated the effectiveness of symmetry-based and analyticity-based methods for extracting defect CFT data. Looking ahead, we expect that the underlying logic of our analysis is not restricted to the $O(N)$ vector model with a line defect. Similar symmetry-based arguments should be applicable to a broader class of CFTs with line defects, e.g. [25, 26] and explicitly demonstrated in [46].

In Chap. 4, we introduced a qualitatively new class of defects generated by quenched disorder localized on lower-dimensional subspaces. Assuming conformal invariance in the infrared, we developed a general framework for logarithmic conformal defects, in which bulk operators remain ordinary conformal primaries while defect operators organize into logarithmic representations.

Using representation theory and symmetry constraints, we derived general results for correlation functions, conformal block decompositions, and logarithmic defect operator expansions. Explicit realizations in free scalar and $O(N)$ vector models with Gaussian disorder revealed that nontrivial logarithmic defect CFTs emerge only for codimension-two defects. These findings provided concrete and previously unexplored examples of defect criticality induced purely by localized disorder and clarified the structural origin of logarithmic behavior in defect settings. Several interesting directions emerge from this analysis. A natural extension is to study disorder-induced logarithmic defects in interacting CFTs, such as the $O(N)$ model in $d = 4 - \epsilon$, where localized random-field disorder may generate new perturbative defect fixed points. It would also be important to test the predicted logarithmic behaviors numerically, for instance in the 3d Ising model with random pinning defects. Beyond Gaussian disorder, exploring alternative disorder distributions and their possible defect analogs of the Nishimori line [83] presents another intriguing direction. More generally, understanding constraints on RG flows of disordered defects, including monotonicity theorems and the role of symmetries and anomalies after disorder averaging, remains an open and promising avenue for future work.

Chap. 5 addressed yet another face of defects by introducing composite defect CFTs, in which a conformal defect itself hosts a lower-dimensional sub-defect. We developed a general formalism suitable for such nested structures, generalizing known results for wedge CFTs to composite defects of arbitrary codimensions. As an explicit realization, we studied the $O(N)$ vector model in $d = 3 - \epsilon$ dimensions with a surface defect intersected by a line defect. Remarkably, we found that even with a free bulk theory, nontrivial infrared composite defect fixed points can arise due to the interplay of localized RG flows on the surface and the line. The resulting pattern of symmetry breaking and the existence of multiple infrared fixed points demonstrated that composite defects lead to genuinely new universality classes, substantially enriching the landscape of defect CFTs. An intriguing future direction is to understand composite defects from a holographic perspective. Recently, bottom-up holographic duals of defect CFTs have been proposed in [222]. It would be particularly interesting to develop a holographic description of composite defect CFTs, where sub-defect operator expansions may give rise to dynamical degrees of freedom localized on nested AdS submanifolds. Such a construction could clarify how multiple localized RG flows and symmetry-breaking patterns are geometrized in the bulk. More generally, extending composite defect CFTs to interacting, supersymmetric, and gauge theories may offer a fertile ground for uncovering new holographic dualities and for constraining defect and sub-defect CFT data using integrability and symmetry.

Taken together, the results of this thesis highlight the role of conformal symmetry in organizing a wide variety of defect phenomena. Despite their diverse physical origins—ranging from symmetry breaking and disorder to nested defect structures—the defects studied here can all be understood within a simple conformal framework.

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