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THE EQUIVARIANT WHITEHEAD GROUPS OF SEMIALGEBRAIC G-SETS

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1. Introduction

Throughout this paper, the base field is the real numbers $\mathbb{R}$ and all semialgebraic maps are assumed to be continuous. For general terminology and the theory of semialgebraic sets we refer the reader to [1].

Let G be a compact semialgebraic group. One can see easily that every compact semialgebraic group has a Lie group structure, and conversely, every compact Lie group has a semialgebraic group structure. A *semialgebraic representation* of G is, by definition, a semialgebraic homomorphism $\rho\colon G \to \mathrm{GL}(n, \mathbb{R})$ for some n. In this case $\mathbb{R}^n$ equipped with the linear action of G via ρ is denoted by $\mathbb{R}^n(\rho)$ and called a *semialgebraic representation space* of G. A *semialgebraic G-set* is a G-invariant semialgebraic set in some finite dimensional semialgebraic representation space of G. One may define a semialgebraic G-set as a semialgebraic set with a semialgebraic action of G, but two definitions are equivalent when G is semialgebraically isomorphic to a semialgebraic subgroup of some $\mathrm{GL}(k, \mathbb{R})$, see [17, Thoerem 1.1]. Note that $\mathrm{GL}(k, \mathbb{R})$ is a semialgebraic set in $M_k(\mathbb{R}) \cong \mathbb{R}^{k^2}$ where $M_k(\mathbb{R})$ denotes the set of all $k \times k$ real matrices. A G-equivariant semialgebraic map between semialgebraic G-sets is called a *semialgebraic G-map*.

The simple homotopy theory and the theory of Whitehead torsions have equivariant generalizations in the topological category, see e.g. [6]. In this paper we consider the equivariant generalizations of them to the semialgebraic category. Namely, we define the equivariant Whitehead group of a semialgebraic G-set and the Whitehead torsion of a G-homotopy equivalence between semialgebraic G-sets. Moreover, we prove the semialgebraic invariance of the equivariant Whitehead torsion.

The basic ingredients for the development are the existence of an equivariant semialgebraic G-CW complex structure of a semialgebraic G-set (Proposition 2.2) and equivariant semialgebraic homotopy theory in [16]. We remark that the (equivariant) Whitehead group is defined on a complete (G-) CW complex [6, 13]. However, in general, a semialgebraic G-set has a finite open G-CW complex structure which is not

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necessarily complete. See Definition 2.1 for the definitions of “open” and “complete” G-CW complexes.

The *core* of X, denoted by $\mathrm{co}(X)$, is defined to be the maximal complete (and thus compact) G-CW subcomplex of X. It is shown, in [16, p.166], that there exists a semialgebraic G-retract $r_X\colon X \to \mathrm{co}(X)$ such that the inclusion map $i_X\colon \mathrm{co}(X) \hookrightarrow X$ is the semialgebraic G-homotopy inverse of r_X. Since $\mathrm{co}(X)$ is a complete G-CW complex, the equivariant Whitehead group $\mathrm{Wh}_G(\mathrm{co}(X))$ is defined as in [6]. We define the equivariant Whitehead group of a semialgebraic G-set M by

$$\mathrm{Wh}_G(M) := \mathrm{Wh}_G(\mathrm{co}(X_0))$$

where X_0 is a preferred semialgebraic G-CW complex structure on M. Let X be another semialgebraic G-CW complex structure on M. We may assume that both X and X_0 have the same underlying topological space M. Let $\lambda^X_{X_0}$ denote the composition

$$\lambda^X_{X_0}\colon\ \mathrm{co}(X) \overset{i_X}{\hookrightarrow} X = X_0 \overset{r_{X_0}}{\to} \mathrm{co}(X_0).$$

Then we show in Section 3 that $\lambda^X_{X_0}$ induces an isomorphism

$$(\lambda^X_{X_0})_*\colon\ \mathrm{Wh}_G(\mathrm{co}(X)) \to \mathrm{Wh}_G(\mathrm{co}(X_0)),$$

which implies that the definition of $\mathrm{Wh}_G(M)$ is independent of the choice of a semialgebraic G-CW complex structure on M.

For a G-homotopy equivalence $f\colon M \to N$ between two semialgebraic G-sets we define the Whitehead torsion $\tau_G(f)$ of f to be an element in $\mathrm{Wh}_G(M)$ as follows. Choose any semialgebraic G-CW complex structures X and Y on M and N, respectively. Put $\tilde{f} = r_Y \circ f \circ i_X\colon \mathrm{co}(X) \to \mathrm{co}(Y)$. Then $\tau_G(\tilde{f}) \in \mathrm{Wh}_G(\mathrm{co}(X))$. We define $\tau_G(f)$ by

$$\tau_G(f) = (\lambda^X_{X_0})_*(\tau_G(\tilde{f})) \in \mathrm{Wh}_G(M).$$

Our main theorem asserts that such defined Whitehead torsion is well-defined, i.e., independent of the choice of semialgebraic G-CW complex structures, and is an equivariant semialgebraic invariant. Namely we have the following theorem.

Theorem 1.1. *Let G be a compact semialgebraic group, and let M and N be semialgebraic G-sets. For a G-homotopy equivalence $f\colon M \to N$ there is a well-defined Whitehead torsion $\tau_G(f) \in \mathrm{Wh}_G(M)$, and if f is a semialgebraic G-homeomorphism then $\tau_G(f) = 0$.*

Notice that the topological invariance of the Whitehead torsion does not hold in the equivariant topological category, see Examples I.4.25 and I.4.26 in [13].

Recently, S. Illman proved a similar result in [10, 11] for (not necessarily compact) Lie groups acting smoothly and properly on smooth G-manifolds with the compact orbit spaces, which are not necessarily compact. Our results in this paper are motivated by those in [10, 11]. Indeed, Theorem 1.1 (and Theorem 1.2 below) is modelled on [10, Theorem III] and [11, Theorem III] (resp. [11, Theorem IV]). But the differences of Theorem 1.1 (and Theorem 1.2) and the S. Illman's are as follows: The stability under the finite union is one of many wide differences between the semialgebraic category and the other (subanalytic or smooth) category. For example, the infinite union of locally finite semialgebraic sets is not a semialgebraic set. Thus the attaching map of infinite semialgebraic maps, which are well-defined in the intersections of the domains, is not semialgebraic. So, in general, we must prove semialgebraic results in finite steps. From this reason, in the semialgebraic category, we only consider finite G-CW complexes.

The assumption that the orbit spaces are compact is used essentially in [10, 11]. It is used in the construction of a complete finite G-CW complex structure of a proper G-manifold. Thus every G-CW complex in [11] is complete. However, in Theorem 1.1 (and Theorem 1.2), the acting groups are compact but the semialgebraic G-sets and their orbit spaces are not necessarily compact. Thus the (semialgebraic) G-CW complex is not complete in general. Therefore the completeness of the G-CW complexes is another wide difference between this paper and [10, 11].

When a semialgebraic G-set M is compact (equivalently, M/G is compact since G is compact), we have Theorem 1.1 (and Theorem 1.2) from the results in [10, 11]. So the focus of this paper is to generalize the results in [11] on finite complete G-CW complexes to the finite open G-CW complex case.

In this paper we also discuss the restriction homomorphism between Whitehead groups in the semialgebraic category. We first discuss the operation of restricting the compact (topological) group G to a closed subgroup H of G. Let X be a compact G-CW complex. The underlying topological space X with the restricted action of H does not have an H-CW complex structure which is compatible with the given G-CW complex X in a canonical way, see [8, Section 2].

In the case when X is a semialgebraic G-CW complex structure of a semialgebraic G-set M, one can always construct a semialgebraic H-CW complex structure $\mathrm{I}_H X$ on the H-space X such that each G-equivariant cell of X is an H-subcomplex of $\mathrm{I}_H X$, see Lemma 4.1. We call $\mathrm{I}_H X$ an *identity H-reduction of X*, following Illman [11]. Let Y denote the collection of (open) H-cells of $\mathrm{I}_H X$ which are contained in $|\operatorname{co}(X)|$. Then Y is a compact (and thus complete) H-CW subcomplex of $\mathrm{I}_H X$ with $|\operatorname{co}(X)|$ as the underlying space, and hence Y is a semialgebraic H-CW structure of $|\operatorname{co}(X)| = |\operatorname{co}(X) \cap \mathrm{I}_H X|$. Moreover $\operatorname{co}(X)$ and $\operatorname{co}(\mathrm{I}_H X)$ have the same H-homotopy type. In particular we may consider Y as a preferred H-reduction $\mathrm{R}_H(\operatorname{co}(X))$ of the G-CW complex $\operatorname{co}(X)$ in [8, 9, 10, 11] (Proposition 4.6). From this reason we denote Y by $\mathrm{R}_H(\operatorname{co}(X))$, and call $\mathrm{R}_H(\operatorname{co}(X))$ a *preferred H-reduction of* $\operatorname{co}(X)$, following Illman [8, 9]. See Section 4 and [8, 9] for more details. Because $\mathrm{co}(X)$ is compact (and hence complete), there is the restriction homomorphism $\mathrm{Res}_H^G\colon \mathrm{Wh}_G(\mathrm{co}(X)) \to \mathrm{Wh}_H(\mathrm{R}_H(\mathrm{co}(X)))$ by [8, 9]. We are now able to define the restriction homomorphism

$$\mathcal{R}es_H^G\colon \mathrm{Wh}_G(\mathrm{co}(X)) \overset{\mathrm{Res}_H^G}{\to} \mathrm{Wh}_H(\mathrm{R}_H(\mathrm{co}(X))) \overset{(i_X)_*}{\to} \mathrm{Wh}_H(\mathrm{co}(I_H X))$$

by $\mathcal{R}es_H^G = (i_X)_* \circ \mathrm{Res}_H^G$, where $i_X\colon \mathrm{R}_H(\mathrm{co}(X)) = \mathrm{co}(X) \hookrightarrow \mathrm{co}(I_H X)$ is the inclusion map. By using the properties of Res_H^G with the fact that we define the Whitehead group $\mathrm{Wh}_G(M)$ of a semialgebraic G-set M by $\mathrm{Wh}_G(\mathrm{co}(X))$ for arbitrarily semialgebraic G-CW complex structure X of M, we prove the following theorem in Section 4, 5 and 6.

Theorem 1.2. *Let G be a compact semialgebraic group, and let $H < G$ be a closed semialgebraic subgroup of G. Let M be a semialgebraic G-set. Then there exists a well-defined restriction homomorphism*

$$\mathcal{R}es_H^G\colon \mathrm{Wh}_G(M) \to \mathrm{Wh}_H(M),$$

such that for a G-homotopy equivalence $f\colon M \to N$ between semialgebraic G-sets, and for the induced H-homotopy equivalence $f_H\colon M_H \to N_H$ obtained from the restriction of the acting group G to H, we have

$$\tau_H(f_H) = \mathcal{R}es_H^G(\tau_G(f)) \in \mathrm{Wh}_H(M).$$

Furthermore, for closed semialgebraic subgroups $K < H < G$, we have

$$\mathcal{R}es_K^G = \mathcal{R}es_K^H \circ \mathcal{R}es_H^G.$$

Remark that $\mathcal{R}es_H^G = \mathrm{Res}_H^G$ when M is compact. Theorem 1.2 is modelled on Theorem VI of [11]. However, as is mentioned in the remark after Theorem 1.1, there exist several wide differences between semialgebraic and the other categories, which make our theorem meaningful and independent from the results in [11].

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2. *G*-CW complex structures on semialgebraic *G*-sets

In this section we discuss the semialgebraic G-CW complex structures on a given semialgebraic G-set and list some of its basic properties. We begin with basic definitions.

DEFINITION 2.1. For the definition of a G-CW complex we refer the reader to [6]. A G-CW complex X is said to be *finite* if X has only a finite number of G-cells. A

G-CW *complex pair* (X, Y) consists of a G-CW complex X and a G-CW subcomplex Y of X.

A G-CW complex X is called *straight* if each closed G-cell $c \in X$ has a cross section $s\colon \pi_X(c) \to c$ of the restricted orbit map $\pi_X|_c$ as in [11, Definition 11.2].

Let X be a G-CW complex such that the orbit space X/G is a simplicial complex and the orbit map $\pi_X\colon X \to X/G$ is a cellular map, i.e., for each closed G-cell c, $\pi_X(c)$ is a simplex of X/G. For a G-CW complex X, an equivariant subdivision X^* of X induces a subdivision X^*/G of X/G. Conversely, if X is a straight G-CW complex then any subdivision $(X/G)^*$ of the orbit space X/G induces an equivariant subdivision $X^* := \{\pi^{-1}(\sigma) \mid \sigma \text{ is a simplex of } (X/G)^*\}$ of X such that $X^*/G = (X/G)^*$. In this case the *n-th barycentric subdivision* of X is the induced subdivision of X by the n-th barycentric subdivision of X/G.

Recall that an open G-cell is the image of the restriction of a characteristic G-map $\varphi_c\colon G/H_c \times D^n \to c$ to $G/H_c \times \mathrm{int}(D^n)$, where $\mathrm{int}(D^n)$ is the interior of D^n. Note that, when $n = 0$, we let $\mathrm{int}(D^0) = D^0$ as usual.

A *finite open G-CW complex* is defined to be a G-invariant subspace of some finite G-CW complex X by removing some open G-cells of X. Classical G-CW complexes are called *complete* G-CW complexes here. Let X be a finite open G-CW complex. Recall that the *core of X*, denoted by $\mathrm{co}(X)$, is the maximal complete G-CW subcomplex of X. Hence $\mathrm{co}(X)$ is compact. If X is complete, then clearly $\mathrm{co}(X) = X$.

From now on G denotes a compact semialgebraic group. In the equivariant semialgebraic category, the following proposition shows that any semialgebraic G-set has a straight semialgebraic finite open G-CW complex structure. This proposition is proved in [15, 17], but the proof has a minor mistake with an incomplete proof. So we proof it here. Moreover, by a similar way of the following proof, we can prove this proposition for semialgebraic proper actions of noncompact semialgebraic groups (see, [2, Theorem 4.4]).

Proposition 2.2. *Let M be a semialgebraic G-set and C a semialgebraic G-subset of M. Then there exist a pair (X, A) of finite open G-CW complexes such that*

(1) *the underlying spaces of X and A are equal to M and C, respectively*;

(2) *X/G is a finite open simplicial complex which is compatible with the orbit types and C. Moreover the orbit map π_X is a semialgebraic cellular map*;

(3) *each open G-cell c of X is a semialgebraic G-set, and thus its closure $\overline{c}$ in M is a semialgebraic G-set*;

(4) *each characteristic G-map $f_c\colon G/H_c \times \mathrm{int}(D^n) \to c$ is a semialgebraic G-homeomorphism*;

(5) *each open G-cell c of X has a semialgebraic cross section $s_c\colon \pi_X(\overline{c}) \to \overline{c}$ of $\pi_X|_{\overline{c}}$, where $\overline{c}$ denotes the closure of c in $X = M$.*

Proof. Since every semialgebraic G-set has only finitely many orbit types [17, Theorem 3.2], M has finite orbit types, say $(G/H_1), \dots, (G/H_l)$. Let $M_{(H_i)}$ denote the set of points on orbit of type (G/H_i), i.e.,

$$M_{(H_i)} = \{x \in M \mid G_x = gH_ig^{-1} \quad \text{for some } g \in G\}.$$

Then $M_{(H_i)}$ is a semialgebraic G-subset of M. Let $\pi \colon M \to M/G$ be the semialgebraic orbit map. By the semialgebraic triangulation theorem [5, 12], there are a finite open simplicial complex K and a semialgebraic homeomorphism $\tau \colon |K| \to M/G$ which is compatible with $\{\pi(M_{(H_1)}), \dots, \pi(M_{(H_l)})\}$ and $\pi(C)$. Let K' denote the first barycentric subdivision of K. Let L denote the subcomplex $\tau^{-1}(\pi(C))$ of K'. For simplicity, identify $|K'|$ and $|L|$ with M/G and C/G, respectively.

We claim that

$$\begin{aligned} X &= \{c = \pi^{-1}(\sigma) \mid \sigma \quad \text{is an open simplex of } K'\} \\ A &= \{c = \pi^{-1}(\sigma) \mid \sigma \quad \text{is an open simplex of } L\} \end{aligned}$$

are desired semialgebraic G-CW complexes.

Let $v_0, \dots, v_n$ be generically independent points of $\mathbb{R}^m$. The n-simplex $\langle v_0, \dots, v_n \rangle$ spanned by $v_0, \dots, v_n$ is defined by

$$\langle v_0, \dots, v_n \rangle = \left\{ \sum_{i=0}^{n} t_i v_i \in \mathbb{R}^m \;\middle|\; \sum_{i=0}^{n} t_i = 1, \ t_i \geq 0 \right\}.$$

The open n-simplex $(v_0, \dots, v_n)$ spanned by $v_0, \dots, v_n$ is defined by

$$(v_0, \dots, v_n) = \left\{ \sum_{i=0}^{n} t_i v_i \in \mathbb{R}^m \;\middle|\; \sum_{i=0}^{n} t_i = 1, \ t_i > 0 \right\}.$$

Note that the open 0-simplex (v) spanned by v is not empty but the vertex set $\{v\}$. Let $\Delta_k = \langle v_0, \dots, v_k \rangle$ for $k = 0, \dots, n$. We filter by $\Delta_0 \subset \cdots \subset \Delta_k \subset \cdots \subset \Delta_n$, where Δ_k is considered as a subset of Δ_n through the (semialgebraic) inclusion $i \colon \Delta_k \hookrightarrow \Delta_n$.

For each open simplex σ of K', we can view $\sigma = (v_0, \dots, v_n)$ and let δ denote the closure of σ in $|K'| = M/G$. Then δ is obtained from Δ_n by deleting some lower dimensional open faces. A straight filtration of δ is a filtration

$$\varnothing = \delta_{-1} \subset \delta_0 \subset \cdots \subset \delta_n = \delta$$

where $\delta_k = \delta \cap \Delta_k$ for all $k = 0, \dots, n$. If $\delta_0 = \delta_1 = \cdots = \delta_{p-1} = \varnothing$ but $\delta_p \neq \varnothing$, then $\dim \delta_i = i$ for all $i \geq p$. Moreover, the vertices $v_0, \dots, v_n$ can be ordered in such a way that

(i) $\delta_k - \delta_{k-1}$ has a constant isotropy type, say (H_k), for $p \leq k \leq n$.

(ii) $v_n \in \delta = \delta_n$.
(iii) $(H_n) \leq (H_{n-1}) \leq \cdots \leq (H_p)$.

Since each n-simplex of M/G is of the form δ we restrict our attention to the orbit map $\pi \colon N = \pi^{-1}(\delta) \to N/G = \delta$.

CLAIM 1. There exists a semialgebraic global H_p-slice of N.

Proof of Claim 1. It is enough to construct a semialgebraic G-map $N \to G/H_p$. The construction is done by induction on dimensions of δ. If $\dim \delta = 0$, δ is a point and N is an orbit, and hence the semialgebraic G-map $\pi^{-1}(\delta_0) \to G/H_p$ is the identity map. Assume that there exists a semialgebraic G-map $f_{n-1} \colon \pi^{-1}(\delta_{n-1}) \to G/H_p$. Then it is sufficient to extend the G-map f_{n-1} to $\pi^{-1}(\delta_n) = N \to G/H_p$.

Since $\delta_n - \delta_{n-1}$ has one orbit type and contractible, there is a semialgebraic cross section $s \colon \delta_n - \delta_{n-1} \to N^{H_n} \subset N$. Let Y denote the closure of $s(\delta_n - \delta_{n-1})$ in N. Let $B = \pi^{-1}(\delta_{n-1}) \cap Y$, then the orbit map π maps (Y, B) onto (δ_n, δ_{n-1}).

We now claim that there exists a semialgebraic retraction $\varphi \colon Y \to B$. Let L be a semialgebraic triangulation of Y which is compatible with B. Replace L by its barycentric subdivision. Take a regular open semialgebraic neighborhood $U' = \mathrm{St}_L(B)$ of B in L. Then, by [4, Theorem 1, Theorem 2.7], there is a semialgebraic retraction $h \colon U' \to B$ since B is closed.

On the other hand, the set $U = \pi(U')$ is a semialgebraic neighborhood of δ_{n-1}. Let T be a semialgebraic triangulation of δ_n which is compatible with $\{\delta_{n-1}, U, \delta_n\}$ and replace T by its barycentric subdivision. Let V be the regular neighborhood $\mathrm{St}_T(\delta_{n-1})$ of δ_{n-1} in T. Then $\delta_{n-1} \subset V \subset U$. Since $V - \delta_{n-1}$ is contractible, by [4, Theorem 3], there is a semialgebraic retraction $r \colon \delta_n - \delta_{n-1} \to V - \delta_{n-1}$. Let $V' = \pi^{-1}(V) \cap Y \subset U'$. Then the cross section s induces the semialgebraic retraction

$$r' = s \circ r \circ \pi \colon Y - B \to V' - B.$$

By composing h and r', we get a semialgebraic retraction

$$\varphi = \begin{cases} h \circ r' & \text{on} \quad Y - B \\ \mathrm{id} & \text{on} \quad B. \end{cases}$$

It extends to a semialgebraic G-retraction $\phi \colon N = \pi^{-1}(\delta_n) \to \pi^{-1}(\delta_{n-1})$ by $\phi(gx) = g\varphi(x)$ for all $g \in G$ and $x \in Y$. (Note that ϕ is a G-map because $s(\delta_n - \delta_{n-1}) \subset N^{H_n}$.)

Hence we obtain a semialgebraic G-map $f_n = f_{n-1} \circ \phi \colon N \to G/H_p$, and a global semialgebraic H_p-slice $S = f_n^{-1}(eH_p)$. □

CLAIM 2. There exists a semialgebraic cross section $\tilde{s} \colon \delta \to N$ of π such that any point of $\tilde{s}(\delta_k - \delta_{k-1})$ has the constant isotropy subgroup for each $0 \leq k \leq n$.

Proof of Claim 2. We prove the claim by the double induction on the dimension of G, and the number of components of G.

By Claim 1, there exists a semialgebraic retraction $f\colon N \to G/H_p = \pi^{-1}(v_p)$. If $H_p \neq G$, consider the slice $S = f^{-1}(eH_p)$. Then S is an H_p-space with the orbit space $S/H_p \cong N/G$. By the induction hypothesis, we can find a semialgebraic section $s\colon \delta \to S \subset \pi^{-1}(\delta) = N$ of the orbit map $S \to S/H_p$, and this section is a desired section.

On the other hand if $H_p = G$, then $N^G \neq \emptyset$. Let $N' = N - N^G$. Then $\delta' = \delta - \pi(N^G)$ is again a simplex, and by the previous argument of the case when $H_p \neq G$, we can find a semialgebraic section $s'\colon \delta' \to N'$ of the orbit map $N' \to N'/G$. We now define a semialgebraic section $s\colon \delta \to N$ by

$$s(x) = \begin{cases} s'(x), & x \in \delta' \\ \pi^{-1}(x), & x \in \pi(N^G). \end{cases}$$

Such defined s is continuous because G is compact. □

Let $c = \pi^{-1}(\sigma)$ and $\bar{c}$ the closure of c in M. Then $\bar{c} = \pi^{-1}(\delta)$. By Claim 2, there exists a semialgebraic cross section $s_\delta\colon \delta \to \bar{c}$ of the orbit map. We now define the semialgebraic characteristic G-map $f_c\colon G/H_c \times \delta \to \bar{c}$ by $(gH_c, x) \mapsto gs_\delta(x)$, where H_c is the isotropy subgroup of $s_\delta(\delta^n - \delta^{n-1})$. The properties (1)–(5) follows easily from the construction. This completes the proof. □

Note that the property (5) in Proposition 2.2 is the straightness condition of X.

Proposition 2.3 ([16, p.166]). *Let X be a finite open G-*CW *complex structure which satisfies* (1)–(5) *in* Proposition 2.2. *Then there exists a semialgebraic strong G-deformation retract $R\colon X' \times I \to X'$ such that $R_0 = R(\cdot, 0) = \mathrm{id}_{X'}$, $R(a, t) = a$ for all $(a, t) \in \mathrm{co}(X') \times I$ and $R_1 = R(\cdot, 1) = r\colon X' \to \mathrm{co}(X')$ is a semialgebraic G-retraction, where X' is the first barycentric subdivision of X.*

DEFINITION 2.4. Let M be a semialgebraic G-set. A *semialgebraic G-*CW *complex structure on M* is, by definition, a finite open G-CW complex (for M) which satisfies the properties (1)–(5) in Proposition 2.2. If we replace X by its barycentric subdivision, then there is a semialgebraic strong G-deformation retraction $r\colon X \to \mathrm{co}(X)$ by Proposition 2.3. Hence, without loss of generality we may assume that *every semialgebraic G-*CW *complex structure X on a semialgebraic G-set admits a semialgebraic strong G-deformation retraction $r_X\colon X \to \mathrm{co}(X)$.* In particular, every semialgebraic G-CW complex structure is finite.

Note that the underlying space of X is equal to M. Therefore, for a semialgebraic G-set M, there exist a compact semialgebraic G-subset $A = \mathrm{co}(X)$ of M and a semialgebraic strong G-deformation retract $R\colon M\times I\to M$ such that $R_0=\mathrm{id}_M$, $R(a,t)=a$ for all $(a,t)\in A\times I$ and $R_1=r\colon M\to A$ is a semialgebraic G-retraction. Moreover, the inclusion $i\colon A\hookrightarrow M$ is a semialgebraic G-homotopy inverse of r. In particular, r is a semialgebraic G-homotopy equivalence.

REMARK 2.5. Proposition 2.3 is proved in [16, Theorem 3.7 and p.166]. But since the construction of the G-retraction r is used in Section 3, we sketch the proof of it here for readers convenience. Let X be a semialgebraic G-CW complex structure on a semialgebraic G-set M in a G-representation space Ω. We replace X by its barycentric subdivision X'. For the semialgebraic characteristic G-map $f_c\colon G/H\times \mathrm{int}(D^n)\to c$ of an open G-cell c of X, let σ denote the set $f_c(eH\times\mathrm{int}(D^n))$ and let $G\sigma$ (resp. f_σ) denote c (resp. f_c). Then for each open G-n-cell $G\sigma$, there exists a semialgebraic characteristic G-map $f_\sigma\colon G/H\times\delta\to G\overline{\sigma}\subset X$, where δ is a subset of a compact standard n-simplex Δ^n by removing some finite open lower-dimensional faces of Δ^n. Thus $\sigma=f_\sigma(eH\times\mathrm{int}(\delta))$ and $\overline{\sigma}=f_\sigma(eH\times\delta)$.

Put $A=\mathrm{co}(X)$. Then A is the union of all open G-cells $G\sigma$ of X such that $cl(G\sigma)\subset M$, where $cl(G\sigma)$ denotes the closure of $G\sigma$ in Ω. Namely, A is the union of open G-cells which have semialgebraic characteristic G-maps $f_\sigma\colon G/H\times\delta\to G\overline{\sigma}=\overline{G\sigma}$ such that δ is some compact standard n-simplex Δ^n. Then $G\overline{\sigma}\cap A\neq\varnothing$ for all open G-cells $G\sigma$ of X.

Let $\mathcal{C}_n$ be the set of open G-n-cells $G\sigma$ of X such that $G\sigma\cap A=\varnothing$. Clearly $\mathcal{C}_n$ is a finite set and $\mathcal{C}_0=\varnothing$.

Let $X_0=A$ and $X_n=A\cup X^{(n)}$ for $n\geq 1$, where $X^{(n)}$ is the n-skeleton of X. Clearly $X_n=A\cup\{G\sigma \mid G\sigma\in\mathcal{C}_k,\ 0\leq k\leq n\}$.

For each open G-n-cell $G\sigma\in\mathcal{C}_n$, by the nonequivariant result in [4], there exists a nonequivariant semialgebraic strong deformation retraction $F^n_{\overline{\sigma}}\colon\overline{\sigma}\times I\to\overline{\sigma}$ from $\overline{\sigma}$ to $\partial\sigma=\overline{\sigma}-\mathrm{int}(\sigma)$. Hence we have a semialgebraic strong G-deformation retraction

$$R^n_{G\overline{\sigma}}\colon G\overline{\sigma}\times I\to G\overline{\sigma}\qquad (gx,t)\mapsto gF^n_{\overline{\sigma}}(x,t)$$

from $G\overline{\sigma}$ to $G\partial\sigma(\subset X_{n-1})$.

Put $R^n=\cup\{R^n_{G\overline{\sigma}}\mid G\sigma\in\mathcal{C}_n\}$. Then $R^n\colon X_n\times I\to X_n$ is a semialgebraic strong deformation retraction from X_n to X_{n-1}. We denote $R^n(\cdot,1)$ and $R^n_{G\overline{\sigma}}(\cdot,1)$ by r^n and $r^n_{G\overline{\sigma}}$, respectively. Clearly $r^n=\cup\{r^n_{G\overline{\sigma}}\mid G\sigma\in\mathcal{C}_n\}$.

Then we can define $R^{i+1}\circledast R^i\colon X_{i+1}\times I\to X$ for $1\leq i$,

$$R^{i+1}\circledast R^i(x,t)=\begin{cases} R^{i+1}(x,2t) & \text{if } 0\leq t\leq 1/2\\ R^i(R^{i+1}(x,1),2t-1) & \text{if } 1/2\leq t\leq 1.\end{cases}$$

The map R in Proposition 2.3 is obtained by

$$R=R^m\circledast R^{m-1}\circledast\cdots\circledast R^2\circledast R^1\colon X\times I\to X$$

which is the desired semialgebraic strong G-deformation retraction from X to A, where $m = \min\{n \in \mathbb{N} \mid X = X_n\}$. In particular $r = R_1 = r^1 \circ r^2 \circ \cdots \circ r^{m-1} \circ r^m$ is a semialgebraic G-retraction from X to A.

Now we study some elementary properties of semialgebraic G-CW complex structures on semialgebraic G-sets, which are useful in the equivariant CW category. The following three lemmas and proofs of them are semialgebraic versions of Corollary 11.7, Lemma 11.8 and 11.9 given in [11].

Lemma 2.6. *If X_1 and X_2 are two semialgebraic G-*CW *complex structures on a semialgebraic G-set M, then there exists a semialgebraic G-*CW *complex structure X^* on M such that X^* is a common G-subdivision of X_1 and X_2.*

Proof. For each $i(= 1, 2)$ the orbit space X_i/G is a finite open simplicial complex structure of M/G with underlying space M/G such that all $\pi(c)$ are semialgebraic subsets of M/G for all open G-cells c of X_i, where $\pi\colon M \to M/G$ is the orbit map. Let $\mathcal{C}$ be the set of subsets $\pi(c)$ for all open G-cells c of X_1 or X_2. Then $\mathcal{C}$ is a collection of finite semialgebraic subsets of a semialgebraic set M/G. By the nonequivariant triangulation theorem [5], there is a finite semialgebraic open simplicial complex structure $(X/G)^*$ of M/G which is compatible with $\mathcal{C}$. By the straightness of X_i, $(X/G)^*$ induces a semialgebraic G-CW complex structure X^* on M which is a common G-subdivision of X_1 and X_2. □

Lemma 2.7. *Let X and Y be semialgebraic G-*CW *complex structures on some semialgebraic G-sets, respectively. If $f\colon X \to Y$ is a semialgebraic G-map then there exists a semialgebraic G-subdivision Y^* of Y such that $f\colon X \to Y^*$ is skeletal, and each G-cell c of X the image $f(c)$ is a G-*CW *subcomplex of Y^*.*

Proof. Each open G-cell c of X is a semialgebraic G-subset of X, and hence $f(c)$ is a semialgebraic G-subset of Y. Set

$$\mathcal{A} = \{f(c) \mid c \text{ is an open } G\text{-cell of } X\},$$
$$\mathcal{B} = \{d \mid d \text{ is an open } G\text{-cell of } Y\}.$$

Then $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is a family of finite semialgebraic G-subsets of Y. Thus the set

$$\pi(\mathcal{C}) = \{\pi(\delta) \mid \delta \in \mathcal{C}\}$$

is also a family of finite semialgebraic subsets of Y/G, where π is the orbit map of Y. By the semialgebraic triangulation theorem [5], there exists a semialgebraic finite open simplicial complex structure K on Y/G which is compatible with $\pi(\mathcal{C})$. Then K is a subdivision of Y/G. Since Y is straight, K induces a semialgebraic G-CW complex Y^* on Y which is compatible with $\mathcal{C}$. Let $\overline{f}: X/G \to Y^*/G$ be the induced semialgebraic map by f. Then $\overline{f}$ is skeletal because every semialgebraic map can not be dimension increasing, see [1, Theorem 2.2.8]. Hence $f: X \to Y^*$ is also skeletal. □

It is shown in [16] that any topological G-homotopy class of a continuous G-map between two semialgebraic G-sets can be represented by a semialgebraic G-map. From this and the above lemma, we can see that any continuous G-map between semialgebraic G-sets is G-homotopic to a G-skeletal map.

Lemma 2.8. *Let $f: X \to Y$ be a semialgebraic G-map between semialgebraic G-*CW *complex structures on some semialgebraic G-sets, respectively. Then there exists a semialgebraic G-subdivision X^* of X such that f maps each open G-cell of X^* into an open G-cell of Y.*

Proof. Since Y is a semialgebraic G-CW complex structure,

$$\mathcal{A} = \{f^{-1}(d) \mid d \text{ is an open } G\text{-cell of } Y\}$$

is a family of finite semialgebraic G-subsets of X. Set

$$\mathcal{B} = \{c \mid c \text{ is an open } G\text{-cell of } X\}.$$

Then $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is also a family of finite semialgebraic G-subsets of X. By the similar argument as in the proof of Lemma 2.7, there exists a semialgebraic G-subdivision X^* of X which is compatible with $\mathcal{C}$. Clearly X^* is the desired G-CW complex. □

We shall use the following lemma to prove Theorem 1.1 in Section 3.

Lemma 2.9. *Let $f: M \to N$ be a semialgebraic G-homeomorphism between semialgebraic G-sets. Then there are semialgebraic G-*CW *complexes X and Y on M and N, respectively, such that $f: X \to Y$ is a G-isomorphism of G-*CW *complexes.*

Proof. From Proposition 2.2 we take a semialgebraic G-CW complex structure X on M. Set

$$Y = f_*(X) = \{f(c) \mid c \text{ is an open } G\text{-cell of } X\}.$$

Since f is a semialgebraic homeomorphism, Y is a semialgebraic G-CW complex structure on N. Moreover $f: X \to Y$ is a G-isomorphism of G-CW complexes. □

3. The Whitehead group of a semialgebraic G-set

In this section we will prove Theorem 1.1. We first discuss some basic properties of Whitehead groups and Whitehead torsions. For details of the equivariant Whitehead group and the torsion we refer the reader to [6], [7] and [13].

Let G be a compact Lie group. For a finite complete G-CW complex X, we denote its equivariant Whitehead group by $\mathrm{Wh}_G(X)$. Note that $\mathrm{Wh}_G(X)$ is an abelian group. Remember that each element of $\mathrm{Wh}_G(X)$ is an equivalence class $s_G(V, X)$ of a G-CW complex pair (V, X) such that X is a strong G-deformation retract of V, where two pairs (V, X) and (W, X) are equivalent if there is an equivariant formal deformation from V to W *rel* X.

A G-map $f: X \to Y$ between finite complete G-CW complexes induces a group homomorphism $f_*: \mathrm{Wh}_G(X) \to \mathrm{Wh}_G(Y)$ defined by $f_*(s_G(V, X)) = s_G(V \underset{f}{\cup} Y, Y)$. Therefore

$$\mathrm{Wh}_G\colon \{\text{ finite complete } G\text{-CW complex }\} \to \{\text{ abelian group }\}$$

is a covariant functor. For a G-homotopy equivalence $f: X \to Y$ its *Whitehead torsion* $\tau_G(f) \in \mathrm{Wh}_G(X)$ is the class $s_G(Z(\bar{f}), X)$ where $Z(\bar{f})$ is the mapping cylinder of any equivariant skeletal approximation $\bar{f}$ of f. This definition of the Whitehead torsion of f is the same as the one defined in [6, 7]. Note that the Whitehead torsion of f is defined to be $f_*(\tau_G(f)) \in \mathrm{Wh}_G(Y)$ in [13].

We now discuss basic properties of Whitehead groups and Whitehead torsions.

Proposition 3.1 ([6, Lemma 2.1, 2.2 and Proposition 3.8], [7, Theorem B]). *Let X, Y and Z be finite complete G-*CW *complexes.*

(1) *If $f, g: X \to Y$ are G-maps which are G-homotopic, then $f_* = g_*:$* $\mathrm{Wh}_G(X) \to \mathrm{Wh}_G(Y)$.

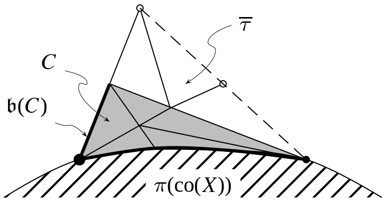
(2) *Let X^* be an equivariant subdivision of X. Then the identity map $f: X^* \to X$ is a simple G-homotopy equivalence, and thus $\tau_G(f) = 0$.*

(3) *Let $f: X \to Y$ and $g: Y \to Z$ be G-homotopy equivalences. Then we have*
$$\tau_G(g \circ f) = \tau_G(f) + f_*^{-1}\tau_G(g).$$

(4) *Let $X \subset Y \subset Z$. If X is a strong G-deformation retract of Y and Y is a strong G-deformation retract of Z. Then we have*
$$s_G(Z, X) = r_* s_G(Z, Y) + s_G(Y, X)$$
where $r: Y \to X$ is a skeletal G-retract.

In nonequivariant case, we have the following proposition.

Lemma 3.2. *If X is a simply connected finite complete* CW *complex then* $\mathrm{Wh}(X) = 0$.



$\pi(\mathrm{co}(X)) \cup \overline{\tau}$

Fig. 3.1.

The above lemma is obtained from the isomorphism $\mathrm{Wh}(X) \cong \mathrm{Wh}(\pi_1(X))$ for a connected space X, see [3, Section 21], [6, Corollary I.2.8], [19, p.105].

From now on G denotes a compact semialgebraic group. Let M be a semialgebraic G-set and X a semialgebraic G-CW complex structure on M. Then there is a semialgebraic strong G-deformation retraction $r_X\colon X \to \mathrm{co}(X)$. Let X^* be a semialgebraic G-subdivision of X. Then the underlying space of $\mathrm{co}(X)$ is contained in the underlying space of $\mathrm{co}(X^*)$. Recall that $\mathrm{co}(X)$ and $\mathrm{co}(X^*)$ are complete semialgebraic G-CW complexes. Let $j\colon \mathrm{co}(X) \to \mathrm{co}(X^*)$ denote the inclusion map between the underlying spaces. Then j is skeletal and $j(\mathrm{co}(X))$ can be viewed as a G-CW subcomplex of $\mathrm{co}(X^*)$. Moreover, the restriction map $r_X|_{\mathrm{co}(X^*)}$ of r_X is the semialgebraic G-homotopy inverse of j.

Lemma 3.3. *The inclusion map* $j\colon \mathrm{co}(X) \to \mathrm{co}(X^*)$ *is a simple G-homotopy equivalence and thus* $\tau_G(j) = 0 \in \mathrm{Wh}_G(\mathrm{co}(X))$.

Proof. It suffices to construct a formal G-deformation from $\mathrm{co}(X^*)$ to $\mathrm{co}(X)$. With the same notation as in Remark 2.5, let n be the maximal integer such that $\mathcal{C}_n \neq \varnothing$. Let $G\sigma \in \mathcal{C}_n$, i.e., $G\sigma$ is an open G-n-cell of X such that $G\sigma \cap \mathrm{co}(X) = \varnothing$ and $G\overline{\sigma} \cap \mathrm{co}(X) \neq \varnothing$. Note that the orbit spaces X/G and X^*/G are semialgebraic finite open simplicial complex structures on M/G which are induced by the orbit map $\pi\colon M \to M/G$. We denote $\pi(G\overline{\sigma})$ by $\overline{\tau}$. Set $GB := \mathrm{co}(X^*) \cap G\overline{\sigma}$, $G\mathfrak{b}(B) := GB \cap X_{n-1}$ and $C := \pi(GB)$, $\mathfrak{b}(C) := \pi(G\mathfrak{b}(B))$, see Fig. 3.1. Then C and $\mathfrak{b}(C)$ are simplicial subcomplexes of $\mathrm{co}(X^*/G)$. In particular, $\mathfrak{b}(C)$ (and thus C) is simply connected because $\overline{\tau}$ (and thus C) is strong deformation retract to $\overline{\tau} \cap \pi(\mathrm{co}(X)) = C \cap \pi(\mathrm{co}(X)) \cong \Delta^{n-1}$. The inclusion $i\colon \mathfrak{b}(C) \to C$ is a homotopically equivalent skeletal map. By Lemma 3.2, we have $\mathrm{Wh}(\mathfrak{b}(C)) = 0$, and thus i is a simple homotopy equivalence. Hence there is a formal deformation from C to $\mathfrak{b}(C)$. Using the cross section from $\overline{\tau} = \pi(G\overline{\sigma})$ to $G\overline{\sigma}$ and the given G-action, we have a formal G-deformation from GC to $G\mathfrak{b}(C)$. By the finiteness of the number of elements of $\mathcal{C}_k$ and the induction argument on k, we get a formal G-deformation from $\mathrm{co}(X^*) \to \mathrm{co}(X)$. ☐

Since $r_X|_{\mathrm{co}(X^*)} \circ j = \mathrm{id}_{\mathrm{co}(X)}$, we have $\tau_G(r_X|_{\mathrm{co}(X^*)}) = -j_*\tau_G(j) = 0 \in \mathrm{Wh}_G(\mathrm{co}(X^*))$

from Proposition 3.1 and Lemma 3.3.

NOTATION. Let X be a semialgebraic G-CW complex structure on a semialgebraic G-set M. The map $i_X\colon \operatorname{co}(X) \to M = X$ denotes the inclusion from $\operatorname{co}(X)$ to M and $r_X\colon M = X \to \operatorname{co}(X)$ denotes the semialgebraic strong G-deformation retract as in Proposition 2.3. Let Y be another semialgebraic G-CW complex structure on M. Then we denote $r_Y \circ i_X = r_Y|_{\operatorname{co}(X)}\colon \operatorname{co}(X) \to \operatorname{co}(Y)$ by λ_Y^X. Note that λ_Y^X has a semialgebraic G-homotopy inverse $\lambda_X^Y\colon \operatorname{co}(Y) \to \operatorname{co}(X)$. So we denote λ_X^Y by $(\lambda_Y^X)^{[-1]}$.

Now we prove that any semialgebraic G-set has a well-defined simple G-homotopy type.

Lemma 3.4. *Let M be a semialgebraic G-set. Let X and Y be semialgebraic G-*CW *complex structures on M. Then $\tau_G(r_Y \circ i_X) = 0 \in \mathrm{Wh}_G(\operatorname{co}(X))$, and thus $\lambda_Y^X = r_Y \circ i_X\colon \operatorname{co}(X) \to \operatorname{co}(Y)$ is a simple G-homotopy equivalence.*

Proof. By Lemma 2.6, there exists a semialgebraic G-CW complex structure Z on M which is a common G-subdivision of X and Y. Let $j_X\colon \operatorname{co}(X) \to \operatorname{co}(Z)$ denote the inclusion map and thus the restriction map $r_X|_{\operatorname{co}(Z)}$ of r_X is the G-homotopy inverse of j_X. By Lemma 3.3, j_X is a simple G-homotopy equivalence, i.e., $\tau_G(j_X) = 0 \in \mathrm{Wh}_G(\operatorname{co}(X))$, and thus $\tau_G(r_X|_{\operatorname{co}(Z)}) = -(j_X)_*\tau_G(j_X) = 0 \in \mathrm{Wh}_G(\operatorname{co}(Z))$. Similarly we have $\tau_G(j_Y) = 0 \in \mathrm{Wh}_G(\operatorname{co}(Y))$ and $\tau_G(r_Y|_{\operatorname{co}(Z)}) = 0 \in \mathrm{Wh}_G(\operatorname{co}(Z))$. By Proposition 3.1,

$$\tau_G(r_Y \circ i_X) = \tau_G(r_Y|_{\operatorname{co}(Z)} \circ j_X) = \tau_G(j_X) + (j_X)_*^{-1}\tau_G(r_Y|_{\operatorname{co}(Z)}) = 0$$

in $\mathrm{Wh}_G(X)$. Therefore $r_Y \circ i_X\colon \operatorname{co}(X) \to \operatorname{co}(Y)$ is a simple G-homotopy equivalence. □

It follows from Lemma 3.4 that the simple G-homotopy type on M, defined in this way, is independent of the choice of the semialgebraic G-CW complex structure on M.

Now we define the equivariant Whitehead group for a given semialgebraic G-set M. Let X and Y be semialgebraic G-CW complex structures on M. Then the map $\lambda_Y^X = r_Y \circ i_X\colon \operatorname{co}(X) \to \operatorname{co}(Y)$ has a G-homotopy inverse $\lambda_X^Y\colon \operatorname{co}(Y) \to \operatorname{co}(X)$. By Proposition 3.1, we have

$$(\lambda_X^Y)_* \circ (\lambda_Y^X)_* = (\lambda_X^Y \circ \lambda_Y^X)_* = (\mathrm{id}_{\operatorname{co}(X)})_* = \mathrm{id}_{\mathrm{Wh}_G(\operatorname{co}(X))}$$

and $(\lambda_Y^X)_* \circ (\lambda_X^Y)_* = \mathrm{id}_{\mathrm{Wh}_G(\operatorname{co}(Y))}$. Accordingly, the map λ_Y^X induces an isomorphism

$$(\lambda_Y^X)_*\colon \mathrm{Wh}_G(\operatorname{co}(X)) \xrightarrow{\cong} \mathrm{Wh}_G(\operatorname{co}(Y))$$

with the inverse homomorphism

$$(\lambda_X^Y)_*\colon \mathrm{Wh}_G(\mathrm{co}(Y)) \to \mathrm{Wh}_G(\mathrm{co}(X)).$$

Let us fix a semialgebraic G-CW complex structure X_0 on M. We define the Whitehead group of M by

$$\mathrm{Wh}_G(M) := \mathrm{Wh}_G(\mathrm{co}(X_0)).$$

Thus we obtain the Whitehead group $\mathrm{Wh}_G(M)$ such that, for each semialgebraic G-CW complex structure X on M, there is an isomorphism

$$(\lambda_X)_*\colon \mathrm{Wh}_G(\mathrm{co}(X)) \xrightarrow{\cong} \mathrm{Wh}_G(\mathrm{co}(X_0)) = \mathrm{Wh}_G(M)$$

where $\lambda_X = \lambda_{X_0}^X$.

Now we prove Theorem 1.1.

Theorem 3.5 (Theorem 1.1). *Let M and N be semialgebraic G-sets.*
(1) *Any G-homotopy equivalence $f\colon M \to N$ has a well-defined equivariant Whitehead torsion $\tau_G(f) \in \mathrm{Wh}_G(M)$.*
(2) *If $f\colon M \to N$ is a semialgebraic G-homeomorphism then $\tau_G(f) = 0 \in \mathrm{Wh}_G(M)$, and thus f is a simple G-homotopy equivalence.*

Proof. (1) We choose a semialgebraic G-CW complex structures X and Y on M and N, respectively. Then the composite map

$$\tilde{f} = r_Y \circ f \circ i_X\colon \mathrm{co}(X) \to \mathrm{co}(Y)$$

is a G-homotopy equivalence. We define $\tau_G(f) = (\lambda_X)_*(\tau_G(\tilde{f})) \in \mathrm{Wh}_G(M)$ where $\tau_G(\tilde{f}) \in \mathrm{Wh}_G(\mathrm{co}(X))$.

It remains to prove that $\tau_G(f)$ is independent of the choice of semialgebraic G-CW complex structures on M and N. Suppose X' and Y' are another semialgebraic G-CW complex structures on M and N, respectively. Put

$$\tilde{f}' = r_{Y'} \circ f \circ i_{X'}\colon \mathrm{co}(X') \to \mathrm{co}(Y').$$

Let $\lambda_{X'}^X = r_{X'} \circ i_X\colon \mathrm{co}(X) \to \mathrm{co}(X')$ and $\lambda_{Y'}^Y = r_{Y'} \circ i_Y\colon \mathrm{co}(Y) \to \mathrm{co}(Y')$, then $\lambda_{Y'}^Y \circ \tilde{f}$, $\tilde{f}' \circ \lambda_{X'}^X\colon \mathrm{co}(X) \to \mathrm{co}(Y')$ are G-homotopy equivalences which are G-homotopic, and thus $\tau_G(\lambda_{Y'}^Y \circ \tilde{f}) = \tau_G(\tilde{f}' \circ \lambda_{X'}^X)$. By Proposition 3.1, we have

$$\tau_G(\tilde{f}) + \tilde{f}_*^{-1}\tau_G(\lambda_{Y'}^Y) = \tau_G(\lambda_{X'}^X) + (\lambda_{X'}^X)_*^{-1}\tau_G(\tilde{f}').$$

By Lemma 3.4, $\tau_G(\lambda_{X'}^X) = 0$ and $\tau_G(\lambda_{Y'}^Y) = 0$, and thus $\tau_G(\tilde{f}) = (\lambda_{X'}^X)_*^{-1}\tau_G(\tilde{f}')$.

On the other hand, $\lambda_{X'}^X = r_{X'} \circ i_X$ is G-homotopic to $r_{X'} \circ i_{X_0} \circ r_{X_0} \circ i_X = (\lambda_{X'})^{-1} \circ \lambda_X$ because $i_{X_0} \circ r_{X_0} = r_{X_0}$ is G-homotopic to the identity map id_{X_0} on X_0. This implies $(\lambda_{X'}^X)_* = ((\lambda_{X'})^{[-1]})_* \circ (\lambda_X)_*$. Hence we have

$$(\lambda_X)_*(\tau_G(\tilde{f})) = (\lambda_{X'})_*(\tau_G(\tilde{f}')).$$

This completes the proof.

(2) By Lemma 2.9, there exist semialgebraic G-CW complex structures X and Y on M and N, respectively, such that $f\colon X \to Y$ is a G-isomorphism of open G-CW complexes X and Y. Moreover, $\tilde{f} = r_Y \circ f \circ i_X = f|_{\mathrm{co}(X)}\colon \mathrm{co}(X) \to \mathrm{co}(Y)$ is also a G-isomorphism of complete G-CW complexes $\mathrm{co}(X)$ and $\mathrm{co}(Y)$. Thus $\tau_G(\tilde{f}) = 0 \in \mathrm{Wh}_G(\mathrm{co}(X))$. Therefore $\tau_G(f) = (\lambda_X)_*(\tau_G(\tilde{f})) = 0 \in \mathrm{Wh}_G(M)$. □

4. *H*-reductions of semialgebraic *G*-CW complexes

In this section we introduce the notion of the identity H-reduction and the preferred H-reduction of a G-CW complexes, following [8, 9, 10, 11].

Note that, in this paper, we only treat a semialgebraic G-set which is contained in a semialgebraic G-representation space $\mathbb{R}^n(\rho)$ for some semialgebraic G-representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$. Hence we can replace G by $\rho(G)(\subset \mathrm{GL}_n(\mathbb{R}))$. In this case we know that every compact subgroup of $\mathrm{GL}_n(\mathbb{R})$ is algebraic (see [14, 17]), and thus semialgebraic. If H is a closed subgroup of a compact semialgebraic group $G(\subset \mathrm{GL}_n(\mathbb{R}))$, then H is a compact subgroup of $\mathrm{GL}_n(\mathbb{R})$, and thus H is a semialgebraic subgroup of G. Conversely, every semialgebraic subgroup of a semialgebraic group is closed ([18, Corollary 2.8], [2]).

Let G denote a compact semialgebraic group, and H a closed semialgebraic subgroup of G. Let M be a semialgebraic G-set. We consider the induced actions of H on M by restricting the acting group G to H, and denote this semialgebraic H-set by M_H. Let X be a semialgebraic G-CW complex structure on M, and consider the induced action of H on X. In this case one can always give the H-space X a semialgebraic H-CW complex structure as follows.

Lemma 4.1. *Let G be a compact semialgebraic group and H a closed semialgebraic subgroup of G. Let M be a semialgebraic G-set and X a semialgebraic G-CW complex structure on M. Then there is a semialgebraic H-CW complex structure $\mathrm{I}_H X$ on M_H such that each G-cell of X is an H-subcomplex of $\mathrm{I}_H X$.*

Proof. We recall that the orbit space M/G and the orbit map $\pi\colon M \to M/G$ are semialgebraic. The G-CW complex structure X induces a semialgebraic finite open simplicial complex structure X/G of M/G by π. Note that each (open) G-cell of X is an H-invariant semialgebraic set. Take a semialgebraic finite open simplicial complex structure L of M/H which is compatible with open simplices of X/G and orbit types $\pi_H((M_H)_{(K)})$ of M_H where $\pi_H\colon M_H \to (M_H)/H$ is the orbit map. We replace L by its barycentric subdivision. Then L induces a desired semialgebraic H-CW complex structure $\mathrm{I}_H X$, see Proposition 2.2. □

We call a semialgebraic H-CW complex structure $\mathrm{I}_H X$, as in Lemma 4.1, an *identity H-reduction of X* following Illman in [11]. From Lemma 4.1 we have

Corollary 4.2. *Let X be a semialgebraic G-*CW *complex structure on some semialgebraic G-set M. If $\mathrm{I}_H X$ and $\mathrm{I}'_H X$ are two identity H-reductions of a semialgebraic G-*CW *complex X then there exists an identity H-reduction $\mathrm{I}^*_H X$ of X which is a common H-subdivision of $\mathrm{I}_H X$ and $\mathrm{I}'_H X$.*

Proof. By Lemma 2.6, there exists a semialgebraic H-CW complex structure $\mathrm{I}^*_H X$ which is a common H-subdivision of $\mathrm{I}_H X$ and $\mathrm{I}'_H X$. Then $\mathrm{I}^*_H X$ also has the property that each G-cell of X is a semialgebraic (open) H-CW subcomplex of $\mathrm{I}^*_H X$, and thus $\mathrm{I}^*_H X$ is an identity H-reduction of X. $\square$

By considering the collection of all H-cells in $\mathrm{I}_H X$ contained in $|\operatorname{co}(X)|$, we can view $\operatorname{co}(X)$ as a semialgebraic H-CW complex structure on $|\operatorname{co}(X)|$.

Corollary 4.3. *Let M be a semialgebraic G-set and X a semialgebraic G-*CW *complex structure on M. Let $\eta = i_X\colon \operatorname{co}(X) \hookrightarrow \operatorname{co}(\mathrm{I}_H X)$ be the inclusion map. Then*
(1) η is a semialgebraic H-homotopy equivalence, and
(2) if $\mathrm{I}'_H X$ is another identity H-reduction of X and $\eta'\colon \operatorname{co}(X) \to \operatorname{co}(\mathrm{I}'_H X)$ is the inclusion map, then $\eta' \circ \eta^{[-1]}\colon \operatorname{co}(\mathrm{I}_H X) \to \operatorname{co}(\mathrm{I}'_H X)$ is a simple H-homotopy equivalence, where $\eta^{[-1]}$ denotes an H-homotopy inverse of η.

Proof. (1) Since $r_X\colon X \to \operatorname{co}(X)$ is the semialgebraic G-homotopy inverse of the inclusion map $i_X\colon \operatorname{co}(X) \hookrightarrow X$, the restriction $r_X|_{\operatorname{co}(\mathrm{I}_H X)}\colon \operatorname{co}(\mathrm{I}_H X) \to \operatorname{co}(X)$ is the semialgebraic H-homotopy inverse of η.
(2) We choose another identity H-reduction $\mathrm{I}'_H X$ of X and the inclusion $\eta'\colon \operatorname{co}(X) \hookrightarrow \operatorname{co}(\mathrm{I}'_H X)$. Since $|\mathrm{I}_H X| = |\mathrm{I}'_H X| = |X| = M$, we have the following diagram commute up to H-homotopies:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\operatorname{co}(X) \arrow[r, shift left, "\eta'"] \arrow[d, shift left, "\eta"] \arrow[dr, shift left, "i_X"] & \operatorname{co}(\mathrm{I}'_H X) \arrow[l, shift left, "r_X|_{\operatorname{co}(\mathrm{I}'_H X)}"] \arrow[d, shift left, "i_{\mathrm{I}'_H X}"] \\
\operatorname{co}(\mathrm{I}_H X) \arrow[u, shift left, "r_X|_{\operatorname{co}(\mathrm{I}_H X)}"] \arrow[r, shift left, "i_{\mathrm{I}_H X}"] & {|X|} \arrow[l, shift left, "r_{\mathrm{I}_H X}"] \arrow[u, shift left, "r_{\mathrm{I}'_H X}"] \arrow[ul, shift left, "r_X"]
\end{tikzcd}$$

Thus $\eta' \circ \eta^{[-1]}$ and $r_{\mathrm{I}'_H X} \circ i_{\mathrm{I}_H X}$ are H-homotopic. Hence, by Lemma 3.4,

$$\eta' \circ \eta^{[-1]}\colon \operatorname{co}(\mathrm{I}_H X) \to \operatorname{co}(\mathrm{I}'_H X)$$

is a simple H-homotopy equivalence. $\square$

When M is a compact semialgebraic G-set, every semialgebraic G-CW complex structure X on M is finite and complete. Thus $\mathrm{co}(X) = X$, $\mathrm{co}(\mathrm{I}_H X) = \mathrm{I}_H(X)$ and $|X| = |\mathrm{I}_H X| = M$. From this we obtain the following corollary as a consequence of Corollary 4.2 and 4.3.

Corollary 4.4. *Let X be a semialgebraic G-CW complex structure on a compact semialgebraic G-set M, and let $\mathrm{I}_H X$ be an identity H-reduction of X. Then $\mathrm{I}_H X$ represents the simple H-homotopy type of the semialgebraic H-set M_H.*

In [8, 9], Illman proves the following proposition for the finite complete G-CW complexes when G is compact Lie group.

Proposition 4.5 ([8, 9]). *Let G be a compact Lie group. For a finite complete G-CW complex X there exist a finite complete H-CW complex $\mathrm{R}_H X$ and an H-homotopy equivalence $\eta\colon X \to \mathrm{R}_H X$ such that*
(1) $\dim(\mathrm{R}_H X)^K = \dim X^K$, *for each closed subgroup K of H, and*
(2) *the H-isotropy types occurring in $\mathrm{R}_H X$ and in the H-space X are equal.*
Moreover, $\eta\colon X \to \mathrm{R}_H X$ is unique up to a simple H-homotopy type, i.e., if $\eta'\colon X \to \mathrm{R}'_H X$ is another such choice then $\eta' \circ \eta^{[-1]}\colon \mathrm{R}_H X \to \mathrm{R}'_H X$ is a simple H-homotopy equivalence.

We call $\eta\colon X \to \mathrm{R}_H X$ a *preferred H-reduction* of X following [8, 9, 11].

Suppose that (V, X) is a finite complete G-CW complex pair and that $\eta\colon X \to \mathrm{R}_H X$ is a preferred H-reduction of X. Then there exists a preferred H-reduction $\overline{\eta}\colon V \to \mathrm{R}_H V$ of V, which extends η, and this construction is unique up to a simple H-homotopy equivalence *rel* $\mathrm{R}_H X$, see [9, Theorem 6.3].

Moreover, suppose that $f\colon X \to Y$ is a G-map between two finite complete G-CW complexes. Let $\theta\colon X \to \mathrm{R}_H X$ and $\eta\colon Y \to \mathrm{R}_H Y$ be preferred H-reductions of X and Y, respectively. Then we obtain an induced H-map

$$\mathrm{R}_H f\colon \mathrm{R}_H X \to \mathrm{R}_H Y$$

by defining $\mathrm{R}_H f = \eta \circ f \circ \theta^{[-1]}$. If f is a G-homotopy equivalence then $\mathrm{R}_H f$ is an H-homotopy equivalence.

The construction of a preferred H-reduction $\eta\colon X \to \mathrm{R}_H X$ and an identity H-reduction $\mathrm{I}_H X$ of X are quite different. However, the following proposition says that they agree for all semialgebraic G-CW complex structures on a compact semialgebraic G-set. In fact, Illman [11] proves the following proposition in the equivariant subanalytic category by using similar techniques used in the proof of Theorem 6.1 in [9]. If we follow exactly the same argument as in the proof of Theorem 14.2 in [11], we can prove the following proposition. The details are left to the reader.

Remember that every semialgebraic G-CW complex structure on a semialgebraic set is (automatically) complete when M is compact.

Proposition 4.6 (cf. [11, Theorem 14.2]). *Let M be a compact semialgebraic G-set and X a complete semialgebraic G-*CW *complex structure on M. Let $\mathrm{I}_H X$ be an identity H-reduction. Then the identity map*

$$\theta = \mathrm{id}_X : X \to \mathrm{I}_H X$$

is a preferred H-reduction of X.

From Proposition 4.6, we have the following. Let X and Y be complete semialgebraic G-CW complex structures on some compact semialgebraic G-sets. If $f: X \to Y$ is a G-map then $\mathrm{R}_H f = \mathrm{id}_Y \circ f \circ \mathrm{id}_X^{[-1]} = \mathrm{id}_Y \circ f \circ \mathrm{id}_X = f = f_H = \mathrm{I}_H f$.

Generally, let X be a semialgebraic G-CW complex structure on a semialgebraic G-set M, and let $\mathrm{I}_H X$ be an identity H-reduction of X. Recall that $\mathrm{co}(X) \cap \mathrm{co}(\mathrm{I}_H X)$ denotes the H-CW complex structure on $\mathrm{co}(X)$ which is the collection of H-cells of $\mathrm{co}(\mathrm{I}_H X)$ lying in the underlying space of $\mathrm{co}(X)$. Then $\mathrm{co}(X) \cap \mathrm{co}(\mathrm{I}_H X)$ and $\mathrm{co}(\mathrm{I}_H X)$ are complete and the set $\mathrm{co}(X) \cap \mathrm{co}(\mathrm{I}_H X) = \mathrm{co}(X) \cap \mathrm{I}_H X$ is a H-CW subcomplex of $\mathrm{co}(\mathrm{I}_H X)$ with $|\mathrm{co}(X) \cap \mathrm{co}(\mathrm{I}_H X)| = |\mathrm{co}(X)|$. Moreover we can identify $\mathrm{co}(X) \cap \mathrm{co}(\mathrm{I}_H X)$ with $\mathrm{I}_H(\mathrm{co}(X))$. Thus by the above proposition

$$\theta = \mathrm{id}_{\mathrm{co}(X)} : \mathrm{co}(X) \to \mathrm{co}(X) \cap \mathrm{co}(\mathrm{I}_H X)$$

is a preferred H-reduction of $\mathrm{co}(X)$. From this reason we denote $\mathrm{co}(X) \cap \mathrm{co}(\mathrm{I}_H X)$ by $\mathrm{R}_H(\mathrm{co}(X))$.

5. The restriction homomorphism

In this section we construct the restriction homomorphism $\mathcal{R}es_H^G : \mathrm{Wh}_G(M) \to \mathrm{Wh}_H(M)$ for a semialgebraic G-set M where H is a closed semialgebraic subgroup of G. Illman proves the following proposition.

Proposition 5.1 ([8, 9, 11]). *Let G be a compact Lie group and H a closed subgroup of G. Let X be a finite complete G-*CW *complex and $\mathrm{R}_H X$ a preferred H-reduction of X. Then there exists a well-defined restriction homomorphism $\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \to \mathrm{Wh}_H(X)$ as follows; choose a preferred H-reduction $\theta: X \to \mathrm{R}_H X$, and define $\mathrm{Res}_H^G(s_G(V, X)) = s_H(\mathrm{R}_H V, \mathrm{R}_H X) \in \mathrm{Wh}_H(X)$. Moreover,*
(R1) *suppose that $f: X \to Y$ is a G-homotopy equivalence between finite complete G-*CW *complexes and let $\mathrm{R}_H f : \mathrm{R}_H(X) \to \mathrm{R}_H(Y)$ be the H-homotopy equivalence induced from f. Then $\mathrm{Res}_H^G(\tau_G(f)) = \tau_H(\mathrm{R}_H f) \in \mathrm{Wh}_H(X)$.*
(R2) *Let $K < H < G$. Then the K-*CW *complexes $\mathrm{R}_K(\mathrm{R}_H X)$ and $\mathrm{R}_K X$ have the same simple K-homotopy type. Furthermore, $\mathrm{Res}_K^H \circ \mathrm{Res}_H^G = \mathrm{Res}_K^G$, for each finite complete G-*CW *complex X.*

From this and Proposition 4.6, we have the following lemma when M is compact.

Lemma 5.2. *Let M be a compact semialgebraic G-set and X a semialgebraic G-*CW *complex structure on M. Then there exists a well-defined restriction homomorphism* $\mathrm{Res}^G_H\colon \mathrm{Wh}_G(X) \to \mathrm{Wh}_H(X)$ *which satisfies the above conditions* (R1) *and* (R2).

We now consider the case when M is not necessarily a compact semialgebraic G-set. Let X be a finite (open) semialgebraic G-CW complex structure on M. As we have mentioned in the previous section, $\mathrm{R}_H(\mathrm{co}(X)) = \mathrm{co}(\mathrm{I}_H X) \cap \mathrm{co}(X)$ is a preferred H-reduction of $\mathrm{co}(X)$. Note that $\mathrm{co}(X)$ is complete and $|\mathrm{R}_H(\mathrm{co}(X))| = |\mathrm{co}(X)|$. Thus, by Lemma 5.2, there exists a restriction homomorphism

$$\mathrm{Res}^G_H\colon \mathrm{Wh}_G(\mathrm{co}(X)) \to \mathrm{Wh}_H(\mathrm{co}(X)) = \mathrm{Wh}_H(\mathrm{R}_H(\mathrm{co}(X)))$$

which satisfies the conditions (R1) and (R2). Let $i_X\colon \mathrm{R}_H(\mathrm{co}(X)) \hookrightarrow \mathrm{co}(\mathrm{I}_H X)$ be the inclusion map and let $r_X\colon X \to \mathrm{co}(X)$ be the (semialgebraic) G-restriction map. In particular, i_X is an (semialgebraic) H-homotopy equivalence with $(i_X)^{[-1]} = r_X|_{\mathrm{co}(\mathrm{I}_H X)}$, and hence

$$(i_X)_*\colon \mathrm{Wh}_H(\mathrm{R}_H(\mathrm{co}(X))) \xrightarrow{\cong} \mathrm{Wh}_H(\mathrm{co}(\mathrm{I}_H X))$$

is an isomorphism.

We are now able to define the restriction homomorphism

$$\mathcal{R}es^G_H\colon \mathrm{Wh}_G(\mathrm{co}(X)) \to \mathrm{Wh}_H(\mathrm{co}(\mathrm{I}_H X))$$

by $\mathcal{R}es^G_H = (i_X)_* \circ \mathrm{Res}^G_H$. Then, by the definition of $\mathcal{R}es^G_H$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Wh}_G(\mathrm{co}(X)) \arrow[rr, "\mathcal{R}es^G_H"] \arrow[dr, "\mathrm{Res}^G_H"'] & & \mathrm{Wh}_H(\mathrm{co}(\mathrm{I}_H X)) \\
& \mathrm{Wh}_H(\mathrm{R}_H(\mathrm{co}(X))) \arrow[ur, "(i_X)_*"'] &
\end{tikzcd}$$

Theorem 5.3. *Let M and N be semialgebraic G-sets, and let X and Y be semialgebraic G-*CW *complex structures on M and N, respectively. Suppose that $f\colon X \to Y$ is a G-homotopy equivalence and let $\mathrm{I}_H f\colon \mathrm{co}(\mathrm{I}_H X) \to \mathrm{co}(\mathrm{I}_H Y)$ be the H-homotopy equivalence induced from f, i.e., $\mathrm{I}_H f = r_{I_H Y} \circ f \circ i_{I_H X}$. Then* $\mathcal{R}es^G_H(\tau_G(f)) = \tau_H(\mathrm{I}_H f) \in \mathrm{Wh}_H(\mathrm{co}(\mathrm{I}_H X))$.

Proof. Since $\mathrm{R}_H(\mathrm{co}(X)) \subset \mathrm{co}(\mathrm{I}_H X) \subset G\,\mathrm{co}(\mathrm{I}_H X) \subset X$, we can take a semialgebraic G-subdivision X^* of X which is compatible with the semialgebraic G-set

$G\,\mathrm{co}(\mathrm{I}_H X)$. Then $\mathrm{co}(X^*) \supset \mathrm{co}(\mathrm{I}_H X)$ because $G\,\mathrm{co}(\mathrm{I}_H X)$ is compact. Let the H-maps

$$\begin{aligned} i_1&\colon\ \mathrm{R}_H\,\mathrm{co}(X) \hookrightarrow \mathrm{R}_H\,\mathrm{co}(X^*),\\ i_2&\colon\ \mathrm{R}_H\,\mathrm{co}(X) \hookrightarrow \mathrm{co}(\mathrm{I}_H X), \quad \text{and}\\ i_3&\colon\ \mathrm{co}(\mathrm{I}_H X) \hookrightarrow \mathrm{R}_H\,\mathrm{co}(X^*)\end{aligned}$$

denote the inclusions so that $i_1 = i_3 \circ i_2$. By Lemma 3.3, the inclusion $i_X\colon \mathrm{co}(X) \to \mathrm{co}(X^*)$ is a simple G-homotopy equivalence, i.e., $\tau_G(i_X) = 0 \in \mathrm{Wh}_G(\mathrm{co}(X))$. It implies that $\tau_H(i_1) = \mathrm{Res}^G_H(\tau_G(i_X)) = 0 \in \mathrm{Wh}_H(\mathrm{R}_H(\mathrm{co}(X)))$ since $\mathrm{R}_H\, i_X = i_1$. Hence

$$s_H(\mathrm{R}_H\,\mathrm{co}(X^*), \mathrm{R}_H\,\mathrm{co}(X)) = s_H(Z(i_1), \mathrm{R}_H\,\mathrm{co}(X)) = 0 \in \mathrm{Wh}_H(\mathrm{R}_H(\mathrm{co}(X)))$$

where $Z(i_1)$ denotes the mapping cylinder of i_1.

We now consider the following semialgebraic strong H-deformation retractions;

$$\begin{aligned} r_1 &= \mathrm{R}_H\, r_X|_{\mathrm{R}_H\,\mathrm{co}(X^*)}\colon\ \mathrm{R}_H\,\mathrm{co}(X^*) \to \mathrm{R}_H\,\mathrm{co}(X),\\ r_2 &= \mathrm{R}_H\, r_X|_{\mathrm{co}(\mathrm{I}_H X)}\colon\ \mathrm{co}(\mathrm{I}_H X) \to \mathrm{R}_H\,\mathrm{co}(X), \quad \text{and}\\ r_3 &= r_{\mathrm{I}_H X}|_{\mathrm{R}_H\,\mathrm{co}(X^*)}\colon\ \mathrm{R}_H\,\mathrm{co}(X^*) \to \mathrm{co}(\mathrm{I}_H X)\end{aligned}$$

with the H-homotopy inverses i_k for $k = 1, 2, 3$, respectively. Note that r_1 and $r_2 \circ r_3$ are H-homotopic and $\tau_H(r_1) = 0 \in \mathrm{Wh}_H(\mathrm{R}_H\,\mathrm{co}(X^*))$. By Proposition 3.1 (4), we have

$$\begin{aligned}&(r_2)_* s_H(\mathrm{R}_H\,\mathrm{co}(X^*), \mathrm{co}(\mathrm{I}_H X)) + s_H(\mathrm{co}(\mathrm{I}_H X), \mathrm{R}_H\,\mathrm{co}(X))\\ &= s_H(\mathrm{R}_H\,\mathrm{co}(X^*), \mathrm{R}_H\,\mathrm{co}(X)) = 0.\end{aligned}$$

From this we have

$$s_H(\mathrm{R}_H\,\mathrm{co}(X^*), \mathrm{co}(\mathrm{I}_H X)) + (i_2)_* s_H(\mathrm{co}(\mathrm{I}_H X), \mathrm{R}_H\,\mathrm{co}(X)) = 0.$$

But from the definition of $(i_2)_*$ we have

$$(i_2)_* s_H(\mathrm{co}(\mathrm{I}_H X), \mathrm{R}_H\,\mathrm{co}(X)) = s_H(\mathrm{co}(\mathrm{I}_H X), \mathrm{co}(\mathrm{I}_H X)) = 0$$

in $\mathrm{Wh}_H(\mathrm{co}(\mathrm{I}_H X))$. Hence

$$\tau_H(i_3) = s_H(\mathrm{R}_H\,\mathrm{co}(X^*), \mathrm{co}(\mathrm{I}_H X)) = 0 \in \mathrm{Wh}_H(\mathrm{co}(\mathrm{I}_H X)).$$

By Proposition 3.1, we have $\tau_H(i_k) = 0$ and $\tau_H(r_k) = 0$ for all $k = 1, 2, 3$.

Let $\mathrm{R}_H f\colon \mathrm{R}_H(\mathrm{co}(X)) \to \mathrm{R}_H(\mathrm{co}(Y))$ be the H-homotopy equivalence induced from f, i.e., $\mathrm{R}_H f = \mathrm{R}_H(r_Y \circ f|_{\mathrm{co}(X)})$. By (R1) we have $\mathrm{Res}^G_H(\tau_G(f)) = \tau_H(\mathrm{R}_H f)$, and so it remains to show $(i_2)_*(\tau_H(\mathrm{R}_H f)) = \tau_H(\mathrm{I}_H f)$. Let $i'_2\colon \mathrm{R}_H\,\mathrm{co}(Y) \hookrightarrow \mathrm{co}(\mathrm{I}_H Y)$ denote the inclusion. Note that $i'_2 \circ \mathrm{R}_H f$ and $\mathrm{I}_H f \circ i_2$ are H-homotopic and thus $\tau_H(i'_2 \circ \mathrm{R}_H f) = \tau_H(\mathrm{I}_H f \circ i_2)$. But

$$\tau_H(i'_2 \circ \mathrm{R}_H f) = \tau_H(\mathrm{R}_H f) + (\mathrm{R}_H f)_*^{-1}\tau_H(i'_2) = \tau_H(\mathrm{R}_H f)$$

and

$$\tau_H(\mathrm{I}_H f \circ i_2) = \tau(i_2) + ((i_2)^{[-1]})_*(\tau_H(\mathrm{I}_H f)) = ((i_2)^{[-1]})_*(\tau_H(\mathrm{I}_H f)).$$

Thus $(i_2)_*(\tau_H(\mathrm{R}_H f)) = \tau_H(\mathrm{I}_H f)$. Hence we have

$$\mathcal{R}es^G_H(\tau_G(f)) = \tau_H(\mathrm{I}_H f) \in \mathrm{Wh}_H(\mathrm{co}(\mathrm{I}_H X)). \qquad \square$$

Recall that the Whitehead group $\mathrm{Wh}_G(M)$ of a semialgebraic G-set is defined by $\mathrm{Wh}_G(\mathrm{co}(X))$ for arbitrarily semialgebraic G-CW complex structure X on M, and from the definitions of $\tau_H(f_H)$ and $\mathrm{I}_H f$ we get $\tau_H(f_H) = \tau_H(r_{\mathrm{I}_H Y} \circ f \circ i_{\mathrm{I}_H X}) = \tau_H(\mathrm{I}_H f)$. Thus we have the following.

Corollary 5.4. *Let G be a compact semialgebraic group and let H be a closed semialgebraic subgroup of G. Let M and N be semialgebraic G-sets. Then there exists a well-defined restriction homomorphism $\mathcal{R}es^G_H \colon \mathrm{Wh}_G(M) \to \mathrm{Wh}_H(M)$ such that if $f \colon M \to N$ is a G-homotopy equivalence, then $\mathcal{R}es^G_H(\tau_G(f)) = \tau_H(f_H) \in \mathrm{Wh}_H(M)$, where $f_H \colon M_H \to N_H$ is the H-homotopy equivalence obtained from the restriction of the acting group G to H.*

As a consequence of Corollary 5.4, we obtain the following.

Corollary 5.5. *Let M and N be semialgebraic G-sets. If $f \colon M \to N$ is a simple G-homotopy equivalence then the induced H-map $f_H \colon M_H \to N_H$ by f is also a simple H-homotopy equivalence.*

Remark that if M is compact semialgebraic then $\mathcal{R}es^G_H = \mathrm{Res}^G_H$ and $\mathrm{R}_H f = I_H f = f_H$.

6. The transitivity property of the restriction homomorphism

In this section G denotes a compact semialgebraic group, and $K < H < G$ denote closed semialgebraic subgroups of G. We shall prove the transitivity property in condition (R2) of the restriction homomorphism.

Lemma 6.1. *Let M be a semialgebraic G-set. Let X be a semialgebraic G-CW complex structure on M. Then for $K < H < G$*
(1) $\mathrm{co}(\mathrm{I}_K(\mathrm{I}_H X))$ *and* $\mathrm{co}(\mathrm{I}_K X)$ *have the same simple K-homotopy type.*
(2) $\mathrm{R}_K(\mathrm{R}_H(\mathrm{co}(X)))$ *and* $\mathrm{R}_K(\mathrm{co}(X))$ *have the same simple K-homotopy type.*

Proof. (1) Since $\mathrm{I}_K(\mathrm{I}_H X)$ and $\mathrm{I}_K X$ are semialgebraic K-CW complex structures on M_K, $\mathrm{co}(\mathrm{I}_K(\mathrm{I}_H X))$ and $\mathrm{co}(\mathrm{I}_K X)$ have the same simple K-homotopy type by Lemma 3.4.

(2) Because $\mathrm{R}_K(\mathrm{R}_H(\mathrm{co}(X)))$, $\mathrm{R}_K(\mathrm{co}(X))$ have the same underlying space $|\mathrm{co}(X)|$ and $\mathrm{co}(X)$ is complete (and thus compact), $\mathrm{R}_K(\mathrm{R}_H(\mathrm{co}(X)))$ and $\mathrm{R}_K(\mathrm{co}(X))$ have the same simple K-homotopy type by Lemma 3.4. □

Remark that Proposition 5.1 (R2) implies Lemma 6.1 (2) directly because $\mathrm{co}(X)$ is a complete G-CW complex. Note that we can take $\mathrm{I}_K\, X$ by $\mathrm{I}_K(\mathrm{I}_H\, X)$.

Theorem 6.2. *Let X be a semialgebraic G-*CW *complex structure on some semialgebraic G-set. Then for $K < H < G$ the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Wh}_G(\mathrm{co}(X)) \arrow[rr, "\mathcal{R}es_K^G"] \arrow[dr, "\mathcal{R}es_H^G"'] & & \mathrm{Wh}_K(\mathrm{co}(\mathrm{I}_K\, X)) \\
& \mathrm{Wh}_H(\mathrm{co}(\mathrm{I}_H\, X)) \arrow[ur, "\mathcal{R}es_K^H"'] &
\end{tikzcd}$$

Proof. By the definition of the restriction homomorphism, $\mathcal{R}es_H^G = (i_1)_* \circ \mathrm{Res}_H^G$, $\mathcal{R}es_K^H = (i_2)_* \circ \mathrm{Res}_K^H$ and $\mathcal{R}es_K^G = (i_3)_* \circ \mathrm{Res}_K^G$ where

$$\begin{aligned}
&i_1 \colon\ \mathrm{R}_H(\mathrm{co}(X)) \hookrightarrow \mathrm{co}(\mathrm{I}_H\, X),\\
&i_2 \colon\ \mathrm{R}_K(\mathrm{co}(\mathrm{I}_H\, X)) \hookrightarrow \mathrm{co}(\mathrm{I}_K\, X) \text{ and}\\
&i_3 \colon\ \mathrm{co}(X) \hookrightarrow \mathrm{co}(\mathrm{I}_K\, X)
\end{aligned}$$

are the inclusions such that $i_3 = i_2 \circ i_1$.

CLAIM. $\mathrm{Res}_K^H \circ (i_1)_* = j_* \circ \mathrm{Res}_K^H$ where $j \colon \mathrm{R}_K\, \mathrm{R}_H(\mathrm{co}(X)) \hookrightarrow \mathrm{R}_K(\mathrm{co}(\mathrm{I}_H\, X))$ is the inclusion.

Note that $i_2 \circ j = i_3$ because $|\mathrm{R}_K\, \mathrm{R}_H(\mathrm{co}(X))| = |\mathrm{co}(X)|$. Hence we have

$$\begin{aligned}
\mathcal{R}es_K^H \circ \mathcal{R}es_H^G &= (i_2)_* \circ \mathrm{Res}_K^H \circ (i_1)_* \circ \mathrm{Res}_H^G\\
&= (i_2)_* \circ j_* \circ \mathrm{Res}_K^H \circ \mathrm{Res}_H^G\\
&= (i_2 \circ j)_* \circ \mathrm{Res}_K^G\\
&= (i_3)_* \circ \mathrm{Res}_K^G\\
&= \mathcal{R}es_K^G.
\end{aligned}$$

So it remains to show the above claim.

Proof of Claim. This follows from Lemma 6.1 and the definition of Res_K^H as follows:

$$\begin{aligned}
&j_* \circ \mathrm{Res}_K^H(s_H(V, \mathrm{R}_H(\mathrm{co}(X))))\\
&= j_*(s_K(\mathrm{R}_K\, V, \mathrm{R}_K\, \mathrm{R}_H(\mathrm{co}(X))))\\
&= s_K(\mathrm{R}_K\, V \cup_{\mathrm{R}_K\, \mathrm{R}_H(\mathrm{co}(X))} \mathrm{R}_K(\mathrm{co}(\mathrm{I}_H\, X)), \mathrm{R}_K(\mathrm{co}(\mathrm{I}_H\, X))).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\mathrm{Res}_K^H \circ (i_1)_*(s_H(V, \mathrm{R}_H(\mathrm{co}(X))))\\
&= \mathrm{Res}_K^H(s_H(V \cup_{\mathrm{R}_H(\mathrm{co}(X))} \mathrm{co}(\mathrm{I}_H X), \mathrm{co}(\mathrm{I}_H X)))\\
&= s_K(\mathrm{R}_K(V \cup_{\mathrm{R}_H(\mathrm{co}(X))} \mathrm{co}(\mathrm{I}_H X)), \mathrm{R}_K(\mathrm{co}(\mathrm{I}_H X)))\\
&= s_K(\mathrm{R}_K V \cup_{\mathrm{R}_K \mathrm{R}_H(\mathrm{co}(X))} \mathrm{R}_K(\mathrm{co}(\mathrm{I}_H X)), \mathrm{R}_K(\mathrm{co}(\mathrm{I}_H X))).
\end{aligned}$$

Hence $\mathrm{Res}_K^H \circ (i_1)_* = j_* \circ \mathrm{Res}_K^H$. □

This completes the proof. □

As a corollary of Theorem 6.2, we have the following.

Corollary 6.3. *Let G be a compact semialgebraic group, and $K < H < G$ be closed semialgebraic subgroups of G. Let M be a semialgebraic G-set, then the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Wh}_G(M) \arrow[rr, "\mathcal{R}es_K^G"] \arrow[dr, "\mathcal{R}es_H^G"'] & & \mathrm{Wh}_K(M) \\
& \mathrm{Wh}_H(M) \arrow[ur, "\mathcal{R}es_K^H"'] &
\end{tikzcd}$$

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