Title	On the Schur indices of certain irreducible characters of reductive groups over finite fields
Author(s)	Ohmori, Zyozyu
Citation	Osaka Journal of Mathematics. 1988, 25(1), p. 149-159
Version Type	VoR
URL	https://doi.org/10.18910/10570
rights	
Note	

Ohmori, Z.
Osaka J. Math.
25 (1988), 149–159

ON THE SCHUR INDICES OF CERTAIN IRREDUCIBLE CHARACTERS OF REDUCTIVE GROURS OVER FINITE FIELDS

ZYOZYU OHMORI

(Received September 11, 1986)

Introduction. Let $\boldsymbol{F}_q$ be a finite field with q elements, of characteristic p. Let G be a connected, reductive linear algebraic group defined over $\boldsymbol{F}_q$, with Frobenius endomorphism F, and let G^F denote the group of F-fixed points of G. In [13], we investigated, under the assumption that the centre Z of G is connected, the rationality-properties of the characters λ^{G^F} of G^F induced by certain linear characters λ of a Sylow p-subgroup of G^F and, using the results obtained there, proved some propositions concerning the Schur indices of the semisimple or regular irreducible characters of G^F. In this paper, we shall treat the general case, that is, the case that Z is not necessarily connected. The main results are stated and proved in § 2. In particular, we get the following (see Corollary 1 to Proposition 1, § 2):

Theorem. *Any irreducible Deligne-Lusztig character* $\pm R_T^\theta$ *of* G^F ([4]) *has the Schur index at most two over the field* $\boldsymbol{Q}$ *of rational numbers.*

I wish to thank Profesosr N. Iwahori who kindly taught me properties of the Cartan matrices. I also thank Professor S. Endo for his kind advices during the preparation of the paper. The referee gave me valuable comments for the old version of the paper. Finally, I wish to dedicate this paper to the late Professor T. Miyata.

1. Some lemmas. Let G and F be as above. Let B be an F-stable Borel subgroup of G with the unipotent radical U and T an F-stable maximal torus of B. For a root α of G (with respect to T), let U_α denote the root subgroup of G associated with α. Let $U.$ be the subgroup of U generated by the non-simple positive root subgroups U_α (the ordering on the roots is the one determined by B). Then $U/U.$ is commutative and can be regarded as the direct product $\prod_{\alpha\in\Delta} U_\alpha$, where Δ is the set of simple roots. As $FU.=U.$, F acts on $U/U.=\prod_{\alpha\in\Delta} U_\alpha$ and this action is the one induced by the maps F: $U_\alpha\to FU_\alpha$, $\alpha\in\Delta$. Let ρ be the permutation on the roots α given by $FU_\alpha=U_{\rho\alpha}$ and let I

be the set of orbits of ρ on Δ. For $i\in I$, put $U_i=\prod_{\alpha\in i} U_\alpha$. Then $U/U.=\prod_{i\in I} U_i$ and, as each U_i is F-stable, we have $U^F/U.^F=\prod_{i\in I} U_i^F$. For each $i\in I$, put $q_i=q^{|i|}$ and take one simple root γ_i in i. Then, for each i, there is an isomorphism ϕ_i of U_i^F with the additive group of $\boldsymbol{F}_{q_i}$ such that $\phi_i(tut^{-1})=\gamma_i(t)\phi_i(u)$ for $u\in U_i^F$ and $t\in T^F$ (cf. Proof of 11.8 of Steinberg [17] and Carter [3], pp. 76–77). Thus the family $\phi=(\phi_i)_{i\in I}$ defines an isomorphism

$$(1)\qquad \phi\colon U^F/U.^F=\prod_{i\in I} U_i^F \xrightarrow{\sim} \prod_{i\in I} \boldsymbol{F}_{q_i}$$

so that, for $u=\prod_{i\in I} u_i$ with $u_i\in U_i^F$ for $i\in I$ and $t\in T^F$, we have

$$(2)\qquad \phi(tut^{-1})=\prod_{i\in I} \lambda_i(t)\phi_i(u_i)\,.$$

Now let Λ be the set of characters λ of U^F such that $\lambda\,|\,U.=1$ and Λ_0 the set of characters λ in Λ such that $\lambda\,|\,U_i^F\neq 1$ for all $i\in I$. Then we have

Lemma 1. *Let $\lambda\in\Lambda_0$. Then λ^{G^F} is multiplicity-free (Gel'fand-Graev, Yokonuma, Steinberg) and any irreducible Deligne-Lusztig character $\pm R_T^\theta$ of G^F occurs in λ^{G^F} (Deligne-Lusztig).*

By embedding G in the connected, reductive group $G_1=(G\times T)/\{(z, z^{-1})\,|\,z\in Z\}$ (Z is the centre of G) with connected centre and the same derived group ([4], 5.18) and (as to the second assertion) using properties of Green functions (cf. [3], 7.2.8 and 7.7), we are reduced to the case that Z is connected. In this case the lemma is proved in [4], Theorem 10.7 (or in [3], 8.1.3 and 8.4.5).

Our purpose is to study the rationality of the characters λ^{G^F}, $\lambda\in\Lambda$. Suppose $p=2$. Then, by (1), $U^F/U.^F$ is an elementary abelian 2-group, so that, for any $\lambda\in\Lambda$, λ, hence λ^{G^F} is realiazable in $\boldsymbol{Q}$. Therefore, from now on, we shall assume that $p\neq 2$.

Lemma 2. *Let v be a primitive element of $\boldsymbol{F}_p$ (i.e. $\boldsymbol{F}_p^\times=\langle v\rangle$). Then there exists an element t in T^F such that $t^{p-1}=1$ (possibly $t^{(p-1)/2}=1$) and $\alpha(t)=v^2$ for all simple roots α.*

It suffices to prove the lemma for the derived group G' of G, hence for the simply-connected covering of G'. If G is a simply-connected semisimple group, then we have $G=G_1\times\cdots\times G_m$, where, for $1\leqq i\leqq m$, G_i is an F-stable simply-connected semisimple closed subgroup of G whose simple components are permuted by F cyclically, and the truth of the lemma for each G_i will imply that for G. If $G=G_1\times FG_1\times\cdots\times F^{n-1}G_1$, where G_1 is an F^n-stable simply-connected simple closed subgroup of G for some $n\geqq 1$, then T and B, hence the set of simple roots has the corresponding decomposition, and it is easy to see that the truth of the lemma for G_1 with Frobenius map F^n implies that for

G (cf. [17], 11.2 (b)). Thus we are reduced to the case that G is a simply-connected simple group.

Suppose therefore that G is such a group. Let $X(T)=\mathrm{Hom}(T, \boldsymbol{G}_m)$ and $Y(T)=\mathrm{Hom}(\boldsymbol{G}_m, T)$, and let $\langle\ ,\ \rangle: X(T)\times Y(T)\to \boldsymbol{Z}$ be the natural pairing given by $\langle\chi, \chi^\vee\rangle=$degree of $\chi\circ\chi^\vee$ for $\chi\in X(T)$ and $\chi^\vee\in Y(T)$. Let $\alpha_1, \cdots, \alpha_l$ be the simple roots (as to the numbering of the simple roots, we follow that of Bourbaki [2]) and let $\alpha_1^\vee, \cdots, \alpha_l^\vee$ be the corresponding simple coroots. Then, as G is simply-connected, we have $Y(T)=\langle\alpha_1^\vee, \cdots, \alpha_l^\vee\rangle_Z$, so that the mapping $h: (x_1, \cdots, x_l)\to \prod_{i=1}^{l} \alpha_i^\vee(x_i)$ defines an isomorphism of $(\boldsymbol{G}_m)^l$ with T. Then, for $1\leqq i\leqq l$, we have

$$\alpha_i(h(x_1, \cdots, x_l)) = \prod_{j=1}^{l} x_j^{\langle\alpha_i, \alpha_j^\vee\rangle},$$

where $(\langle\alpha_i, \alpha_j^\vee\rangle)_{1\leqq i,j\leqq l}$ is the Cartan matrix of G. We define an action of F on $Y(T)$ by $F(\chi^\vee)=F\circ\chi^\vee$ for $\chi^\vee\in Y(T)$. Then we have

$$F(\alpha_i^\vee) = q(\rho\alpha_i)^\vee$$

for $1\leqq i\leqq l$ (see [15], 11.4.7). It readily follows that, for $s\in T$, $s=h(x_1, \cdots, x_l)$, we have $Fs=s$ if and only if $x_j=x_i^q$ if $\rho\alpha_i=\alpha_j$. Thus the proof of the lemma has been reduced to solving the following problem:

Find an element $t=h(x_1, \cdots, x_l)$ with $x_i\in \boldsymbol{F}_p^\times$ for $1\leqq i\leqq l$ such taht $\prod_{j=1}^{l} x_j^{\langle\alpha_i, \alpha_j^\vee\rangle}=\nu^2$ for $1\leqq i\leqq l$ and that $x_j=x_i^q$ (hence $x_j=x_i$) if $\rho\alpha_i=\alpha_j$.

When G is adjoint, by the proof of Theorem 1 of [13], there is an element s in T^F of order $p-1$ such that $\alpha(s)=\nu$ for all simple roots α. Hence it suffices to take $t=s^2$. Suppose therefore that G is not adjoint. Then, as $p\neq 2$, G is any one of the following types (Steinberg [17], 11.6; also see [3], 1.19): A_l $(l\geqq1)$, B_l $(l\geqq2)$, C_l $(l\geqq2)$, D_l $(l\geqq3)$, E_6, E_7, ${}^2A_l$ $(l\geqq1)$, ${}^2D_l$ $(l\geqq3)$, ${}^3D_4$, ${}^2E_6$. In each case, an element t of T^F having the property of the lemma (i.e. an solution t of the problem above) can be given as follows (the Cartan matrices are listed up in the appendices of [2]):

Type	t		
A_l ${}^2A_l$	$h(x_1, \cdots, x_l)$	$x_i=\nu^{i(l-i+1)}$	$(1\leqq i\leqq l)$
B_l	$h(x_1, \cdots, x_{l-1}, \nu^{l(l+1)/2})$	$x_i=\nu^{i(2l-i+1)}$	$(1\leqq i\leqq l-1)$
C_l	$h(x_1, \cdots, x_l)$	$x_i=\nu^{i(2l-i)}$	$(1\leqq i\leqq l)$
D_l ${}^2D_l$	$h(x_1, \cdots, x_{l-2}, \nu^{l(l-1)/2}, \nu^{(l-1)/2})$	$x_i=\nu^{i(2l-i-1)}$	$(1\leqq i\leqq l-2)$
E_6 ${}^2E_6$	$h(\nu^{16}, \nu^{22}, \nu^{30}, \nu^{42}, \nu^{30}, \nu^{16})$		
E_7	$h(\nu^{34}, \nu^{49}, \nu^{66}, \nu^{96}, \nu^{75}, \nu^{52}, \nu^{27})$		
${}^3D_4$	$h(\nu^6, \nu^{10}, \nu^6, \nu^6)$		

This completes the proof of Lemma 2.

Lemma 3. *Assume that q is an even power of p. Then there exists an element t in T^F such that $t^{2(p-1)}=1$ (possibly $t^{p-1}=1$) and $\alpha(t)=\nu$ for all simple roots α.*

As in the proof of Lemma 2, we can be reduced to the case that G is a simply-connected simple group. When G is adjoint Lemma 3 is proved in the proof of Theorem 1 of [13]. When G is not adjoint t can be given by replacing each ν in the above table with an element $\varepsilon\in\boldsymbol{F}_q$ such that $\varepsilon^2=\nu$. (We note that, when G is a simply-connected simple group, an element $s=h(x_1, \cdots, x_l)$ of T has the property of Lemma 3 if and only if the x_i satisfy: (i) $x_i^{2(p-1)}=1$ for $1\leqq i\leqq l$, (ii) $\prod_{j=1}^{l} x_j^{\langle\alpha_i,\alpha_j^\vee\rangle}=\nu$ for $1\leqq i\leqq l$, and (iii) $x_j=x_i^q$ if $\rho\alpha_i=\alpha_j$.)

In the following, for an integer m and a prime number r, $\mathrm{ord}_r m$ denotes the exponent of the r-part of m.

Lemma 4. *Assume that G is a (non-adjoint) simply-connected simple group of any one of the following types: A_l with $2|l$ or $\mathrm{ord}_2(l+1)>\mathrm{ord}_2(p-1)$; ${}^2A_l$ with $2|l$; B_l with $4|l(l+1)$; D_l with either (a) $4|l(l-1)$ or (b) $\mathrm{ord}_2(l-1)=1$ and $p\equiv-1 \pmod 4$; ${}^2D_l$ with $4|l(l-1)$; ${}^3D_4$; E_6; ${}^2E_6$. Then there exists an element $t\in T^F$ such that $t^{p-1}=1$ and $\alpha(t)=\nu$ for all simple roots α.*

In fact, for an element $s=h(x_1, \cdots, x_l)$ of T, s satisfies the property of Lemma 4 if and only if the x_i satisfy: (i) $x_i\in F_p^\times$, (ii) $\prod x_j^{\langle\alpha_i,\alpha_j^\vee\rangle}=\nu$ for $1\leqq i\leqq l$, and (iii) $x_j=x_i^q$ (hence $x_j=x_i$) if $\rho\alpha_i=\alpha_j$. By solving these equations, we find that an element t having the property of the lemma can be given as follows:

Type	t		
A_l ${}^2A_l$ $2\|l$	$h(x_1, \cdots, x_l)$	$x_i=\nu^{i(l-i+1)/2}$	$(1\leqq i\leqq l)$
A_l $\mathrm{ord}_2(l+1)>\mathrm{ord}_2(p-1)$			
	$h(x_1, \cdots, x_l)$	$x_1=\nu^{(el+p-1)/2e}$	$\left(e=\left(\frac{l+1}{2}, p-1\right)\right)$
		$x_i=\nu^{-i(i-1)/2}x_1^i$	$(2\leqq i\leqq l)$
B_l $4\|l(l+1)$	$h(x_1, \cdots, x_{l-1}, \nu^{l(l+1)/4})$	$x_i=\nu^{i(2l-i+1)/2}$	$(1\leqq i\leqq l-1)$
D_l ${}^2D_l$ $4\|l(l-1)$	$h(x_1, \cdots, x_{l-2}, \nu^{l(l-1)/4}, \nu^{l(l-1)/4})$		
		$x_i=\nu^{i(2l-i-1)/2}$	$(1\leqq i\leqq l-2)$
D_l $\mathrm{ord}_2(l-1)=1$	$h(x_1, \cdots, x_{l-2}, \nu^{(l^2-l+p-1)/4}, \nu^{(l^2-l+3p-3)/4})$		
$p\equiv-1 \pmod 4$		$x_i=\nu^{i(2l+p-i-2)/2}$	$(1\leqq i\leqq l-2)$
${}^3D_4$	$h(\nu^3, \nu^5, \nu^3, \nu^3)$		
E_6 ${}^2E_6$	$h(\nu^8, \nu^{11}, \nu^{15}, \nu^{21}, \nu^{15}, \nu^8)$		

REMARK. If (at least) G is split over $\boldsymbol{F}_q$, then Lemmas 2, 4 above are implicit in Lehrer's work [12] where he showed a method to calculate the image $a(T^F)$ of T^F under the morphism a: $T\to(\boldsymbol{G}_m)^l$ given by $a(s)=\prod_{i=1}^{l}\alpha_i(s)$ when G

is a simply-connected simple group (he has carried out the calculation when G is a classical group). For our purpose, it is essential to know the order of t (cf. § 2 below).

2. The main results. We recall that $p \neq 2$. Let ζ_p be a primitive p-th root of unity in the field $\boldsymbol{C}$ of complex numbers. Let $\hat{\boldsymbol{F}}_q = \mathrm{Hom}(\boldsymbol{F}_q, \boldsymbol{C}^\times)$ (we consider $\boldsymbol{F}_q$ as an additive group) and fix $\chi \in \hat{\boldsymbol{F}}_q$, $\chi \neq 1$. For $a \in \boldsymbol{F}_q$, define $\chi_a \in \hat{\boldsymbol{F}}_q$ by $\chi_a(x) = \chi(ax)$ for $x \in \boldsymbol{F}_q$. Then we have $\hat{\boldsymbol{F}}_q = \{\chi_a \mid a \in \boldsymbol{F}_q\}$ and $\{\chi^\tau \mid \tau \in \mathrm{Gal}(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q})\} = \{\chi_a \mid a \in \boldsymbol{F}_p^\times\}$.

In the following, if χ is a character of a finite group and L is a field of characteristic zero, $L(\chi)$ is the field generated over L by the values of χ. If χ is irreducible, then $m_L(\chi)$ denotes the Schur index of χ with respect to L. If L is an algebraic number field and v is a place of L, then L_v is the completion of L at v. Now let k be the quadratic subfield $\boldsymbol{Q}(\sqrt{\varepsilon p})$, $\varepsilon = (-1)^{(p-1)/2}$, of $\boldsymbol{Q}(\zeta_p)$.

Proposition 1. *Let G, F be as in Introduction. Let $\lambda \in \Lambda$, $\lambda \neq 1$. Then we have the following:*

(i) *λ^{G^F} takes all its values in k; if $p \equiv -1$ (mod 4), λ^{G^F} is realizable in k; if $p \equiv 1$ (mod 4), then, for any finite place v of k, λ^{G^F} is realizable in k_v.*

(ii) *Assume that q is an even power of p. Then λ^{G^F} takes all its values in $\boldsymbol{Q}$ and, for any prime number $r \neq p$, λ^{G^F} is realizable in $\boldsymbol{Q}_r$.*

(iii) *If G is an adjoint semisimple group or any one of the groups described in Lemma 4, then λ^{G^F} is realizable in $\boldsymbol{Q}_r$.*

Proof of (i). Let t be an element of T^F having the property of Lemma 2. Then $z = t^{(p-1)/2}$ lies in the centre Z^F of G^F since $\alpha(z) = 1$ for all simple roots α. Put $c = |\langle z \rangle|$ ($c = 1$ or 2). Let $M = \langle t \rangle U^F$. Then M acts on Λ by $\lambda^m(u) = \lambda(mum^{-1})$ ($\lambda \in \Lambda$, $m \in M$, $u \in U^F$). Let $\lambda \in \Lambda$, $\lambda \neq 1$. Then, by (1), λ can be expressed as $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i \in \hat{F}_{q_i}$ for $i \in I$. And, by (2), we have

$$\lambda^t = ((\lambda_i)_{\gamma_i(t)})_{i \in I} = ((\lambda_i)_{\nu^2})_{i \in I} = (\lambda_i^{\sigma^2})_{i \in I} = \lambda^{\sigma^2},$$

where σ is a suitable generator of $\mathrm{Gal}(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q})$. Thus, on U^F, we have

$$\lambda^M = c \sum_{j=1}^{(p-1)/2} \lambda^{t^j} = c \sum_{j=1}^{(p-1)/2} \lambda^{\sigma^{2j}},$$

hence $\boldsymbol{Q}(\lambda^M) = \boldsymbol{Q}(\zeta_p)^{\langle \sigma^2 \rangle} = k$. Therefore the values of $\lambda^{G^F} = (\lambda^M)^{G^F}$ lie in k.

Suppose $t^{(p-1)/2} = 1$. Then λ^M is irreducible. By Gow's argument [7], p. 104, we have $m_k(\lambda^M) = 1$: $\lambda^M \mid \langle t \rangle =$ the character of the regular representation of $\langle t \rangle$, hence $\langle \lambda^M, 1_{\langle t \rangle} \rangle_{\langle t \rangle} = 1$; hence, by Schur's theorem (see e.g. Feit [5], 11.4), $m_k(\lambda_M) = 1$. Thus λ^M, hence $\lambda^{G^F} = (\lambda^M)^{G^F}$ is realizable in k.

Assume that $t^{(p-1)/2} \neq 1$. Then λ^M is reducible and is equal to the sum $\mu_0 + \mu_1$ where, for $i = 0, 1$, μ_i is the irreducible character of M induced by the linear character of $\langle z\rangle U^F$ given by $z^j u \to (-1)^{ji}\lambda(u)$ ($j=0, 1$). We have $\boldsymbol{Q}(\mu_0)=\boldsymbol{Q}(\mu_1)=k$. For $i=0, 1$, the simple direct summand A_i of the group algebra $k[M]$ of M over k corresponding to μ_i is isomorphic over k to the cyclic algebra $((k(\zeta_p)/k, \sigma^2, (-1)^i)$ over k (cf. Proof of Proposition 3.5 of Yamada [18]). A_0 clearly splits over k, hence $m_k(\mu_0)=1$ and μ_0 is realizable in k. If $p\equiv -1 \pmod 4$, then -1 is a norm in $k(\zeta_p)/k$, hence A_1 splits over k. Thus, in this case, μ_1, hence $\lambda^M=\mu_0+\mu_1$ is realizable in k. Suppose $p\equiv 1 \pmod 4$. Then A_1 has non-zero invariants only at two real places of k (see Janusz [10], Proposition 3). Thus, for any finite place v of k, μ_1, hence $\lambda^M=\mu_0+\mu_1$ is realizable in k_v.

Proof of (ii). Let t be an element of T^F having the property of Lemma 3, and put $M=\langle t\rangle U^F$. Then, as $\lambda^t=\lambda^\sigma$ ($\lambda\neq 1$), on U^F, we have

$$\lambda^M = c\sum_{j=1}^{p-1}\lambda^{t^j} = c\sum_{j=1}^{p-1}\lambda^{\sigma j} \qquad (c=|\langle t^{p-1}|\rangle) .$$

Thus $\boldsymbol{Q}(\lambda^M)=\boldsymbol{Q}(\zeta_p)^{\langle\sigma\rangle}=\boldsymbol{Q}$.

If $t^{p-1}=1$, then λ^M is irreducible and Gow's argument shows that $m_{\boldsymbol{Q}}(\lambda^M)=1$, hence λ^{G^F} is realizable in $\boldsymbol{Q}$. Suppose $t^{p-1}\neq 1$. Then λ^M is reducible and is equal to the sum $\mu_0+\mu_1$, where, for $i=0, 1$, μ_i is the irreducible character of M induced by the linear character of $\langle t^{p-1}\rangle U^F$ given by $(t^{p-1})^j u \to u(-1)^{ji}\lambda(u)$. We have $\boldsymbol{Q}(\mu_0)=\boldsymbol{Q}(\mu_1)=\boldsymbol{Q}$. For $i=0, 1$, the simple direct summand A_i of $\boldsymbol{Q}[M]$ corresponding to μ_i is isomorphic over $\boldsymbol{Q}$ to $(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q}, \sigma, (-1)^i)$. A_0 splits, hence μ_0 is realizable in $\boldsymbol{Q}$. A_1 has the invariants $\frac{1}{2} \bmod 1$ at ∞, p and $0 \bmod 1$ at any other place of $\boldsymbol{Q}$. Thus, for any prime number $r\neq p$, μ_1, hence $\lambda^M=\mu_0+\mu_1$ is realizable in $\boldsymbol{Q}_r$.

Proof of (iii). When G is adjoint the assertion is contained in Theorem 1 of [13]. Assume that G is not adjoint. Let t be an element of T^F having the property of Lemma 4 and put $M=\langle t\rangle U^F$. Then λ^M is irreducible and $\boldsymbol{Q}(\lambda^M)=\boldsymbol{Q}$. And, by Gow's argument, we have $m_{\boldsymbol{Q}}(\lambda^M)=1$. Thus λ^M, hence $\lambda^{G^F}=(\lambda^M)^{G^F}$ is realizable in $\boldsymbol{Q}$.

We note that, for $G=SL_n$, Sp_{2n}, Proposition 1 is proved by Gow [7], [8].

Corollary 1. *Let G, F be as in Proposition 1. Recall that $p\neq 2$. Let be χ an irreducible character of G^F such that $\langle\chi, \lambda^{G^F}\rangle_{G^F}=1$ for some $\lambda\in\Lambda$ (any irreducible component of λ^{G^F} for $\lambda\in\Lambda_0$ has this property (see Lemma 1)). Then we have $m_{\boldsymbol{Q}}(\chi)\leqq 2$. Thus, in particular, we have $m_{\boldsymbol{R}}(\chi)\leqq 2$ for any irreducible Deligne-Lusztig character $\chi=\pm R_{T'}^{\theta}$ of G^F.* If $\lambda=1$, then λ^{G^F} is realizable in $\boldsymbol{Q}$, hence we have $m_{\boldsymbol{Q}}(\chi)=1$. Assume that $\lambda\neq 1$. Let r be any prime number and v a place of k lying above r. Then, by Proposition 1, we have $m_{k_v}(\chi)=1$, hence $m_{\boldsymbol{Q}_r}(\chi)\leqq 2$ as $[k_v(\chi): \boldsymbol{Q}_r(\chi)]\leqq 2$. We also have $m_{\boldsymbol{R}}(\chi)\leqq 2$. Thus, $m_{\boldsymbol{Q}}(\chi)$, being the least common multiple of the $m_{Q_w}(\chi)$ with w running over all places of $\boldsymbol{Q}$, is at most two. The last assertion follows from this fact and Lemma 1.

Corollary 2. *Assume that q is an even power of p. Let χ be an irreducible character of G^F such that $\langle\chi, \lambda^G\rangle_{G^F}=1$ for some $\lambda\in\Lambda$. Then, for any prime number $r\neq p$, we have $m_{Q_r}(\chi)=1$.*

This follows at once from Proposition 1, (ii).

Corollary 3. *Assume that G is an adjoint semisimple group or any one of the groups described in Lemma 4. Let χ be an irreducible character of G^F such that $\langle\chi, \lambda^{G^F}\rangle_{G^F}=1$ for some $\lambda\in\Lambda$. Then we have $m_Q(\chi)=1$.*

This follows from Proposition 1, (iii).

Corollary 4. *Let G, F be as in Proposition* 1. *Assume that p is a good prime for G* ([16], I, 4.1). *Let χ be an irreducible character of G^F and let u be a regular unipotent element in G^F. Then $\chi(u)$ is an algebraic integer in k, and if $p\nmid\chi(1)$, we have $m_Q(\chi)\leqq 2$.*

We first note that, as p is good for G, $U_\cdot^F$ is equal to the derived group of U^F, hence Λ is the set of linear characters of U^F (Howlett [9], Lehrer [11]), and that, if $u\in U^F$, then $\mu(u)=0$ for any non-linear irreducible character μ of U^F (Lehrer [11]).

Let $\mathcal{O}_k$ be the ring of integers in k. We show that $\chi(u)$ belongs to $\mathcal{O}_k$. We may assume that $u\in U^F$ as u is conjugate to an element of U^F. Let t be an element of T^F having the property of Lemma 2, and let $\Lambda_1, \cdots, \Lambda_r$ be the orbits of $\langle t\rangle$ on Λ. Thus, as $\chi^t=\chi$, if we put $a_\lambda=\langle\chi, \lambda\rangle_{U^F}$ for $\lambda\in\Lambda$, a_λ is constant on each Λ_i. Hence we have

$$\chi(u)=\sum_{\lambda\in\Lambda}a_\lambda\lambda(u)=\sum_{i=1}^{r}a_i\Big(\sum_{\lambda\in\Lambda_i}\lambda(u)\Big),$$

where $a_i=a_\lambda$ on Λ_i. Each $\sum_{\lambda\in\Lambda_i}\lambda(u)$ is stable under the action of $\langle t\rangle$, hence under the action of $\langle\sigma^2\rangle$. Thus $\chi(u)\in\mathcal{O}_k$.

To prove the second assertion, we embed G in G_1 as in the proof of Lemma 1. Assume that $p\nmid\chi(1)$ and take an irreducible character χ_1 of G_1^F such that $\langle\chi, \chi_1|G^F\rangle_{G^F}\neq 0$. Then, by the Clifford theory, we have $\chi_1|G^F=e(\chi^{(1)}+\chi^{(2)}+\cdots+\chi^{(s)})$, where e is a positive integer dividing $(G_1^F\colon G^F)$ and $\chi^{(1)}, \chi^{(2)}, \cdots, \chi^{(s)}$ are the G_1^F-conjugates of $\chi=\chi^{(1)}(s\,|(G_1^F\colon G^F))$. Let r be any prime number and v a place of k lying above r. Put $m_v=m_{k_v}(\chi^{(1)})=\cdots=m_{k_v}(\chi^{(s)})$. For $1\leqq i\leqq s$ and for $\lambda\in\Lambda$, put $a_\lambda^{(i)}=\langle\chi^{(i)}, \lambda\rangle_{U^F}$. Then, by Proposition 1, (i), m_v divides the $a_\lambda^{(i)}$, $1\leqq i\leqq s$, $\lambda\in\Lambda$. As $p\nmid(G_1^F\colon G^F)$, $p\nmid\chi_1(1)$, so that, by a theorem of Green-Lehrer-Lusztig (see [3], 8.3.6), we have $\chi_1(u)=\pm 1$. Therefore we have the expression

$$\pm 1/m_v = \chi_1(u)/m_v = \{e\cdot \sum_{i=1}^{s} \chi^{(i)}(u)\}/m_v = e\cdot\sum_{i=1}^{s}\sum_{\lambda\in\Lambda} (a_\lambda^{(i)}/m_v)\cdot\lambda(u)\,,$$

where the right-hand side is an algebraic integer and the left-hand side is a rational number. Hence $m_v=1$, and $m_{Q_r}(\chi)\leqq 2$. As r is an arbitrary prime number, we hence have $m_Q(\chi)\leqq 2$. This completes the proof of Corollary 4.

Corollary 5. *Assume that q is an even power of p and that p is good for G. Let u be a regular unipotent element in G^F. Then, for any irreducible character χ of G^F, $\chi(u)$ is a rational integer, and if $p\not|\chi(u)$, we have $m_{Q_r}(\chi)=1$ for any prime number $r\neq p$.*

The proof is similar to the proof of Corollary 4 (we use Proposition 1, (ii)).

Corollary 6. *Let G be an adjoint semisimple group or any one of the groups described in Lemma 4. Assume that p is good for G. Let u be a regular unipotent element in G^F and let χ be an irreducible charactre of G^F. Then $\chi(u)$ is a rational integer and if $p\not|\chi(u)$, we have $m_Q(\chi)=1$.*

Remark. Lehrer [12] has calculated the values of the cuspidal irreducible characters of G^F at the regular unipotent elements of G^F when G is a semisimple group. As to the upper bound of the indices of the characters of related finite groups, we reffer to Gow [8] for classical finite groups and Benard [1] and Feit [6] for the sporadic simple groups.

Let G be a connected, reductive algebraic group over an algebraically closed field K of characteristic $p>0$ and F a surjective endomorphism of G such that G^F is finite. Then Lemma 2 still holds for such G^F, so that the statements in Proposition 1, (i) and in Corollary 1 (except for the comment for Lemma 1) hold for G^F. Assume that K is an algebraic closure of $\boldsymbol{F}_p$ and that some power of F is the Frobenius endomorphism relative to a rational structure on G over a finite subfield of K. Then Lemma 1 holds for G^F (cf. Carter [3], 8.1.3 and 8.4.5), so that all the statements in Corollary 1, hence the theorem in Introduction holds for G^F. If p is good for G, then the theorem of Green-Lehrer-Lusztig holds for G^F (if Z is connected: see [3], 8.3.6), so that Corollary 4 holds for G^F.

3. Example. We calculate all the local indices of the cuspidal irreducible Deligne-Lusztig characters $\pm R_{T'}^{\theta}$ of $SL_n(\boldsymbol{F}_q)$ when q is an even power of p $(\neq 2)$.

Let G be SL_n and F the endomorphism $(g_{ij})\rightarrow(g_{ij}^q)$ (q may be any power of any prime p). Let T' be a minisotropic maximal torus of G and let $W=N_G(T')^F/T'^F$ (T' is unique up to G^F-con conjugate). Then, taking an element γ of order $(q^n-1)/(q-1)$ in $\boldsymbol{F}_{q^n}^{\times}$, we have $T'^F=\langle t_0\rangle$, where t_0 is G-conjugate to diag $(\gamma, \gamma^q, \cdots, \gamma^{q^{n-1}})$, and $W=\langle w_0\rangle \simeq \boldsymbol{Z}/n\boldsymbol{Z}$, where w_0 is defined by $t_0^{w_0}=\dot{w}_0 t_0 \dot{w}_0^{-1} = t_0^q$ ($\dot{w}_0 \in N_G(T')^F$ represents w_0). (All these statements can be easily checked by using [16], II, 1.3, 1.10 and 1.14.) W acts on $\hat{T}'^F=\mathrm{Hom}(T'^F, \boldsymbol{C}^\times)$ by $\theta^w(s)=\theta(s^w)$ for $w\in W$, $\theta\in\hat{T}'^F$ and $s\in T'^F$. If θ is in general position, i.e., no non-identity element of W fixes θ, then $(-1)^{n-1}R_{T'}^\theta$ is a cuspidal irreducible character of $G^F=SL_n(\boldsymbol{F}_q)$ ([4], 7.4, 8.3).

Let $\theta\in\hat{T}'^F$. Then, by [4], 4.2, for $g\in G^F$, if $g=su=us$ (s semisimple, u unipotent) is its Jordan decomposition, we have

$$R_T^\theta(g)=\frac{1}{|Z_G(s)^F|}\sum_{\substack{h\in G^F\\ h^{-1}sh\in T'}} Q_{hT'h^{-1},Z_G(s)}(u)\cdot\theta(h^{-1}sh)\,, \tag{3}$$

where the $Q_{hT'h^{-1},Z_G(s)}$ are Green functions of $Z_G(s)$ (which is connected since G is simply-connected). It follows that, if s is not conjugate in G^F to any element of T'^F, we have $R_{T'}^\theta(g)=0$, and if $s\in T^{F'}$, we have

$$R_{T'}^\theta(g)=Q_{T',Z_G(s)}(u)\frac{1}{|W(s)|}\sum_{w\in W}\theta^w(s)\,, \tag{4}$$

where $W(s)=\{w\in W\,|\,s^w=s\}$ (we note that the minisotropic maximal tori of $Z_G(s)$ form a single $Z_G(s)^F$-conjugacy class (cf. [16], II, 1.3, 1.10 and 1.14) and that any two elements of T' that are conjugate in G^F are conjugate under the action of W). Thus, as the Green functions take integeral values, by putting $\theta(t_0)=\zeta$, we get from (4):

$$Q(R_{T'}^\theta)=Q(\sum_{w\in W}\theta^w)=Q(\zeta+\zeta^q+\cdots+\zeta^{q^{n-1}})\,. \tag{5}$$

Lemma 5. *Assume that θ is in general position. Let $q=p^m$. We further assume that n is even. Then we have*

$$\mathrm{ord}_2[\boldsymbol{Q}_p(R_{T'}^\theta):\boldsymbol{Q}_p]=\mathrm{ord}_2 m\,.$$

Let ϕ be the automorphism of $\boldsymbol{Q}_p(\zeta)$ defined by $\zeta^\phi=\zeta^q$. Then ϕ has order n (by assumption) and we have $\boldsymbol{Q}_p(\zeta)^{\langle\phi\rangle}=\boldsymbol{Q}_p(R_{T'}^\theta)$ (cf. (5)). Let $f=[\boldsymbol{Q}_p(\zeta):\boldsymbol{Q}_p]$ and $e=|\langle\zeta\rangle|$. Then f is equal to the least integer $h\geqq 1$ subject for the condition: $p^h\equiv 1 \pmod e$ (see Serre [14], p. 85). As $\phi^n=1$ and $\phi^i\neq 1$ for $1\leqq i\leqq n-1$, we find that $f\,|\,mn$ but $f\nmid mi$ for $1\leqq i\leqq n-1$ [in fact, if $f\,|\,mi$, then $p^f-1\,|\,p^{mi}-1$, hence $e\,|\,p^{mi}-1$, hence $\phi^i=1$]. This shows that $\mathrm{ord}_r f=\mathrm{ord}_r m+\mathrm{ord}_r n$ for any prime divisor r of n. Thus, in particular, we have $\mathrm{ord}_2 f=\mathrm{ord}_2 m+\mathrm{ord}_2 n$. As $[\boldsymbol{Q}_p(\zeta):\boldsymbol{Q}_p(R_{T'}^\theta)]=[\boldsymbol{Q}_p(\zeta):\boldsymbol{Q}_p(\zeta)^{\langle\phi\rangle}]=n$, we hence have $\mathrm{ord}_2[\boldsymbol{Q}_p(R_{T'}^\theta):\boldsymbol{Q}_p]=\mathrm{ord}_2 m$, as desired.

REMARK. Professor K. Iimura showed to the author (by an elementary proof) that $n=f/(m,f)$ and $[\boldsymbol{Q}_p(\zeta)^{\langle\phi\rangle}:\boldsymbol{Q}_p]=(m,f)$.

Proposition 2. *Let χ be any cuspidal irreducible Deligne-Lusztig character $(-1)^{n-1}R_{T'}^{\theta}$ of $G^F=SL_n(\boldsymbol{F}_q)$, where we assume that q is an even power of $p\neq 2$. Then, if n is odd or $ord_2 n\geqq 2$, we have $m_{\boldsymbol{Q}}(\chi)=1$. Assume that $ord_2 n=1$. Then we have $m_{\boldsymbol{Q}_r}(\chi)=1$ for any prime number r and $m_{\boldsymbol{Q}}(\chi)=m_{\boldsymbol{R}}(\chi)\leqq 2$. And we have $m_{\boldsymbol{R}}(\chi)=2$ if an only if χ is real and $\chi(-1_n)=-\chi(1_n)$ (i,.e. $\theta(-1_n)=-1$).*

REMARK. Let χ be as above. Assume that n is even and let $n=2m$. Fixing a generator θ_0 of $\hat{T}'^F$, put $\theta=\theta_0^i$. Then the following can be shown:

(i) χ is real if and only if $\frac{q^m-1}{q-1}\,|\,i$.

(ii) Assume that $\mathrm{ord}_2 n=1$ and let $i=\frac{q^m-1}{q-1}i'$ with $i'\in\boldsymbol{Z}$ (hence χ is real). Then $\theta(-1_n)=1$ if and only if i' is even, and the latter condition is equivalent to the condition that $\theta\,|\,Z^F=1$.

Proof of Proposition 2. Let $\lambda\in\Lambda_0$. Then, by Lemma 1, we have $\langle\chi, \lambda^{G^F}\rangle_{G^F}=1$. Thus, if n is odd or $\mathrm{ord}_2 n>\mathrm{ord}_2(p-1)$, by Proposition 1, (iii), we have $m_{\boldsymbol{Q}}(\chi)=1$. Assume that $1\leqq\mathrm{ord}_2 n\leqq\mathrm{ord}_2(p-1)$. Let t be an element of T^F having the property of Lemma 3. Then, under our assumption, we have $t^{p-1}=-1_n$ (cf. Proof of Lemma 4 and Proof of Lemma 3.3 (a) of Gow [8]). Let us use the notation of the proof of Proposition 1, (ii). Then $\lambda^M=\mu_0+\mu_1$. As $\mu_i(-1_n)=(-1)^i\mu_i(1_n)$ for $i=0$, 1, by Schur's lemma, we have $\langle\chi, \mu_0\rangle_M=1$ if $\chi(-1_n)=\chi(1_n)$, and $\langle\chi, \mu_1\rangle_M=1$ if $\chi(-1_n)=-\chi(1_n)$. As μ_0 is realizable in $\boldsymbol{Q}$, we have $m_{\boldsymbol{Q}}(\chi)=1$ in the first case. Assume that $\chi(-1_n)=-\chi(1_n)$. If r is any prime number $\neq p$, then μ_1 is realizable in $\boldsymbol{Q}_r$, hence we have $m_{\boldsymbol{Q}_r}(\chi)=1$. As q is an even power of p, by Lemma 5, we have $2\,|\,[\boldsymbol{Q}_p(\chi)\colon \boldsymbol{Q}_p]$. Hence $A_1\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p(\chi)$ splits (see [14], Chap. XIII, § 3, Prop. 7), hence μ_1 is realizable in $\boldsymbol{Q}_p(\chi)$. Hence we have $m_{\boldsymbol{Q}_p}(\chi)=m_{\boldsymbol{Q}_p(\chi)}(\chi)=1$. Thus we have $m_{\boldsymbol{Q}}(\chi)=m_{\boldsymbol{R}}(\chi)$. If χ is real, we must have $m_{\boldsymbol{R}}(\chi)=2$ since otherwise χ will be realizable in $\boldsymbol{R}$, so that, by Schur's theorem, we have $(2=)m_{\boldsymbol{R}}(\chi_1)\,|\,\langle\chi, \mu_1\rangle_M=1$, a contradiction. If $\mathrm{ord}_2 n\geqq 2$, then χ cannot be real since G^F contains a central element z of order 4 such that $z^2=-1_n$ and $\chi(z)=\pm\sqrt{-1}\chi(1_n)$ ([7], p. 107). Finally, we note that, by [4], 1.22, we have $\chi(-1_n)=-\chi(1_n)$ if and only if $\theta(-1_n)=-1$. This completes the proof of Proposition 2.

References

[1] M. Benard: *Schur indexes of sporadic simple groups,* J. Algebra **58** (1979), 508–522.

[2] N. Bourbaki: Groupes et algèbres de Lie, chapitres 4, 5 et 6, Hermann, Paris, 1968.

[3] R.W. Carter: Finite groups of Lie type: conugacy classes and complex characters, John Wiley and Sons, Chechester, 1985.

[4] P. Deligne and G. Lusztig: *Representations of reductive groups over finite fields,* Ann. of Math. **103** (1976), 103–161.

[5] W. Feit: Characters of finite groups, W.A. Benjamin, Inc, New York, 1967.

[6] W. Feit: *The computations of some Schur indices,* Israel J. Math. **46** (1983), 274–300.

[7] R. Gow: *Schur indices of some groups of Lie type*, J. Algebra **42** (1976), 102–120.

[8] R. Gow: *On the Schur indices of characters of finite classical groups*, J. London Math. Soc. (2) **24** (1981), 135–147.

[9] R.B. Howlett: *On the degrees of Steinberg characters of Chevalley groups*, Math. Z. **135** (1974), 125–135.

[10] G.J. Janusz: *Simple components of* $\boldsymbol{Q}[SL(2,q)]$, Comm. Algebra **1** (1974), 1–22.

[11] G.I. Lehrer, *Adjoint groups, regular unipotent elements and discrete series characters,* Trans. Amer. Math. Soc. **214** (1975), 249–260.

[12] G.I. Lehrer: *On the values of characters of semisimple groups over finite fields,* Osaka J. Math. **15** (1978), 77–99.

[13] Z. Ohmori: *On the Schur indices of reductive groups* II, Quart. J. Math. Oxford Ser. (2) **32** (1981), 443–452.

[14] J.P. Serre: Crops locaux, deuxieme edition, Hermann, Paris, 1968.

[15] T.A. Springer: Linear algebraic groups, Birkhäuser, Boston, 1981.

[16] T.A. Springer and R. Steinberg: *Congugacy classes,* in Seminar on Algebraic Groups and Related Finite Groups, by A. Borel et al., Lecture Notes in Math. 131, Springer, Berlin-Heidelberg-New York, 1970.

[17] R. Steinberg: Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc. 80 (1968).

[18] T. Yamada: Schur subgroup of the Brauer group, Lecutre Notes in Math. 397, Springer, Berlin-Heidelberg-New York, 1974.

Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya-ku
Tokyo, 158 Japan