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ON THE SCHUR INDICES OF CERTAIN IRREDUCIBLE CHARACTERS OF REDUCTIVE GROUPS OVER FINITE FIELDS

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Introduction. Let F_q be a finite field with q elements, of characteristic p . Let G be a connected, reductive linear algebraic group defined over F_q , with Frobenius endomorphism F , and let G^F denote the group of F -fixed points of G . In [13], we investigated, under the assumption that the centre Z of G is connected, the rationality-properties of the characters λ^{G^F} of G^F induced by certain linear characters λ of a Sylow p -subgroup of G^F and, using the results obtained there, proved some propositions concerning the Schur indices of the semisimple or regular irreducible characters of G^F . In this paper, we shall treat the general case, that is, the case that Z is not necessarily connected. The main results are stated and proved in § 2. In particular, we get the following (see Corollary 1 to Proposition 1, § 2):

Theorem. *Any irreducible Deligne-Lusztig character $\pm R_T^{\theta}$ of G^F ([4]) has the Schur index at most two over the field \mathbf{Q} of rational numbers.*

I wish to thank Professor N. Iwahori who kindly taught me properties of the Cartan matrices. I also thank Professor S. Endo for his kind advices during the preparation of the paper. The referee gave me valuable comments for the old version of the paper. Finally, I wish to dedicate this paper to the late Professor T. Miyata.

1. Some lemmas. Let G and F be as above. Let B be an F -stable Borel subgroup of G with the unipotent radical U and T an F -stable maximal torus of B . For a root α of G (with respect to T), let U_{α} denote the root subgroup of G associated with α . Let U be the subgroup of U generated by the non-simple positive root subgroups U_{α} (the ordering on the roots is the one determined by B). Then U/U is commutative and can be regarded as the direct product $\prod_{\alpha \in \Delta} U_{\alpha}$, where Δ is the set of simple roots. As $FU = U$, F acts on $U/U = \prod_{\alpha \in \Delta} U_{\alpha}$ and this action is the one induced by the maps $F: U_{\alpha} \rightarrow FU_{\alpha}$, $\alpha \in \Delta$. Let ρ be the permutation on the roots α given by $FU_{\alpha} = U_{\rho\alpha}$ and let I

be the set of orbits of ρ on Δ . For $i \in I$, put $U_i = \prod_{\alpha \in i} U_\alpha$. Then $U/U = \prod_{i \in I} U_i$ and, as each U_i is F -stable, we have $U^F/U^F = \prod_{i \in I} U_i^F$. For each $i \in I$, put $q_i = q^{|i|}$ and take one simple root γ_i in i . Then, for each i , there is an isomorphism ϕ_i of U_i^F with the additive group of \mathbf{F}_{q_i} such that $\phi_i(tut^{-1}) = \gamma_i(t)\phi_i(u)$ for $u \in U_i^F$ and $t \in T^F$ (cf. Proof of 11.8 of Steinberg [17] and Carter [3], pp. 76–77). Thus the family $\phi = (\phi_i)_{i \in I}$ defines an isomorphism

$$(1) \quad \phi: U^F/U^F = \prod_{i \in I} U_i^F \simeq \prod_{i \in I} \mathbf{F}_{q_i}$$

so that, for $u = \prod_{i \in I} u_i$ with $u_i \in U_i^F$ for $i \in I$ and $t \in T^F$, we have

$$(2) \quad \phi(tut^{-1}) = \prod_{i \in I} \lambda_i(t)\phi_i(u_i).$$

Now let Λ be the set of characters λ of U^F such that $\lambda|U = 1$ and Λ_0 the set of characters λ in Λ such that $\lambda|U_i^F \neq 1$ for all $i \in I$. Then we have

Lemma 1. *Let $\lambda \in \Lambda_0$. Then λ^{G^F} is multiplicity-free (Gel'fand-Graev, Yokonuma, Steinberg) and any irreducible Deligne-Lusztig character $\pm R_T^0$ of G^F occurs in λ^{G^F} (Deligne-Lusztig).*

By embedding G in the connected, reductive group $G_1 = (G \times T)/\{(z, z^{-1}) \mid z \in Z\}$ (Z is the centre of G) with connected centre and the same derived group ([4], 5.18) and (as to the second assertion) using properties of Green functions (cf. [3], 7.2.8 and 7.7), we are reduced to the case that Z is connected. In this case the lemma is proved in [4], Theorem 10.7 (or in [3], 8.1.3 and 8.4.5).

Our purpose is to study the rationality of the characters λ^{G^F} , $\lambda \in \Lambda$. Suppose $p=2$. Then, by (1), U^F/U^F is an elementary abelian 2-group, so that, for any $\lambda \in \Lambda$, λ , hence λ^{G^F} is realizable in \mathbf{Q} . Therefore, from now on, we shall assume that $p \neq 2$.

Lemma 2. *Let ν be a primitive element of \mathbf{F}_p (i.e. $\mathbf{F}_p^\times = \langle \nu \rangle$). Then there exists an element t in T^F such that $t^{p-1} = 1$ (possibly $t^{(p-1)/2} = 1$) and $\alpha(t) = \nu^2$ for all simple roots α .*

It suffices to prove the lemma for the derived group G' of G , hence for the simply-connected covering of G' . If G is a simply-connected semisimple group, then we have $G = G_1 \times \cdots \times G_m$, where, for $1 \leq i \leq m$, G_i is an F -stable simply-connected semisimple closed subgroup of G whose simple components are permuted by F cyclically, and the truth of the lemma for each G_i will imply that for G . If $G = G_1 \times FG_1 \times \cdots \times F^{n-1}G_1$, where G_1 is an F^n -stable simply-connected simple closed subgroup of G for some $n \geq 1$, then T and B , hence the set of simple roots has the corresponding decomposition, and it is easy to see that the truth of the lemma for G_1 with Frobenius map F^n implies that for

G (cf. [17], 11.2 (b)). Thus we are reduced to the case that G is a simply-connected simple group.

Suppose therefore that G is such a group. Let $X(T) = \text{Hom}(T, \mathbf{G}_m)$ and $Y(T) = \text{Hom}(\mathbf{G}_m, T)$, and let $\langle, \rangle: X(T) \times Y(T) \rightarrow \mathbf{Z}$ be the natural pairing given by $\langle \chi, \chi^\vee \rangle = \text{degree of } \chi \circ \chi^\vee$ for $\chi \in X(T)$ and $\chi^\vee \in Y(T)$. Let $\alpha_1, \dots, \alpha_l$ be the simple roots (as to the numbering of the simple roots, we follow that of Bourbaki [2]) and let $\alpha_1^\vee, \dots, \alpha_l^\vee$ be the corresponding simple coroots. Then, as G is simply-connected, we have $Y(T) = \langle \alpha_1^\vee, \dots, \alpha_l^\vee \rangle_{\mathbf{Z}}$, so that the mapping $h: (x_1, \dots, x_l) \rightarrow \prod_{i=1}^l \alpha_i^\vee(x_i)$ defines an isomorphism of $(\mathbf{G}_m)^l$ with T . Then, for $1 \leq i \leq l$, we have

$$\alpha_i(h(x_1, \dots, x_l)) = \prod_{j=1}^l x_j^{\langle \alpha_i, \alpha_j^\vee \rangle},$$

where $(\langle \alpha_i, \alpha_j^\vee \rangle)_{1 \leq i, j \leq l}$ is the Cartan matrix of G . We define an action of F on $Y(T)$ by $F(\chi^\vee) = F \circ \chi^\vee$ for $\chi^\vee \in Y(T)$. Then we have

$$F(\alpha_i^\vee) = q(\rho \alpha_i)^\vee$$

for $1 \leq i \leq l$ (see [15], 11.4.7). It readily follows that, for $s \in T$, $s = h(x_1, \dots, x_l)$, we have $Fs = s$ if and only if $x_j = x_j^q$ if $\rho \alpha_i = \alpha_j$. Thus the proof of the lemma has been reduced to solving the following problem:

Find an element $t = h(x_1, \dots, x_l)$ with $x_i \in \mathbf{F}_p^\times$ for $1 \leq i \leq l$ such that $\prod_{j=1}^l x_j^{\langle \alpha_i, \alpha_j^\vee \rangle} = v^2$ for $1 \leq i \leq l$ and that $x_j = x_j^q$ (hence $x_j = x_i$) if $\rho \alpha_i = \alpha_j$.

When G is adjoint, by the proof of Theorem 1 of [13], there is an element s in T^F of order $p-1$ such that $\alpha(s) = v$ for all simple roots α . Hence it suffices to take $t = s^2$. Suppose therefore that G is not adjoint. Then, as $p \neq 2$, G is any one of the following types (Steinberg [17], 11.6; also see [3], 1.19): A_l ($l \geq 1$), B_l ($l \geq 2$), C_l ($l \geq 2$), D_l ($l \geq 3$), E_6 , E_7 , 2A_l ($l \geq 1$), 2D_l ($l \geq 3$), 3D_4 , 2E_6 . In each case, an element t of T^F having the property of the lemma (i.e. an solution t of the problem above) can be given as follows (the Cartan matrices are listed up in the appendices of [2]):

Type	t		
A_l 2A_l	$h(x_1, \dots, x_l)$	$x_i = v^{i(l-i+1)}$	$(1 \leq i \leq l)$
B_l	$h(x_1, \dots, x_{l-1}, v^{l(l+1)/2})$	$x_i = v^{i(2l-i+1)}$	$(1 \leq i \leq l-1)$
C_l	$h(x_1, \dots, x_l)$	$x_i = v^{i(2l-i)}$	$(1 \leq i \leq l)$
D_l 2D_l	$h(x_1, \dots, x_{l-2}, v^{l(l-1)/2}, v^{(l-1)/2})$	$x_i = v^{i(2l-i-1)}$	$(1 \leq i \leq l-2)$
E_6 2E_6	$h(v^{16}, v^{22}, v^{30}, v^{42}, v^{30}, v^{16})$		
E_7	$h(v^{34}, v^{49}, v^{66}, v^{96}, v^{75}, v^{52}, v^{27})$		
3D_4	$h(v^6, v^{10}, v^6, v^6)$		

This completes the proof of Lemma 2.

Lemma 3. Assume that q is an even power of p . Then there exists an element t in T^F such that $t^{2(p-1)}=1$ (possibly $t^{p-1}=1$) and $\alpha(t)=\nu$ for all simple roots α .

As in the proof of Lemma 2, we can be reduced to the case that G is a simply-connected simple group. When G is adjoint Lemma 3 is proved in the proof of Theorem 1 of [13]. When G is not adjoint t can be given by replacing each ν in the above table with an element $\varepsilon \in \mathbf{F}_q$ such that $\varepsilon^2=\nu$. (We note that, when G is a simply-connected simple group, an element $s=h(x_1, \dots, x_l)$ of T has the property of Lemma 3 if and only if the x_i satisfy: (i) $x_i^{2(p-1)}=1$ for $1 \leq i \leq l$, (ii) $\prod_{j=1}^l x_j^{\langle \alpha_i, \alpha_j^\vee \rangle} = \nu$ for $1 \leq i \leq l$, and (iii) $x_j = x_i^q$ if $\rho \alpha_i = \alpha_j$.)

In the following, for an integer m and a prime number r , $\text{ord}_r m$ denotes the exponent of the r -part of m .

Lemma 4. Assume that G is a (non-adjoint) simply-connected simple group of any one of the following types: A_l with $2|l$ or $\text{ord}_2(l+1) > \text{ord}_2(p-1)$; 2A_l with $2|l$; B_l with $4|l(l+1)$; D_l with either (a) $4|l(l-1)$ or (b) $\text{ord}_2(l-1)=1$ and $p \equiv -1 \pmod{4}$; 2D_l with $4|l(l-1)$; 3D_4 ; E_6 ; 2E_6 . Then there exists an element $t \in T^F$ such that $t^{p-1}=1$ and $\alpha(t)=\nu$ for all simple roots α .

In fact, for an element $s=h(x_1, \dots, x_l)$ of T , s satisfies the property of Lemma 4 if and only if the x_i satisfy: (i) $x_i \in \mathbf{F}_p^\times$, (ii) $\prod x_j^{\langle \alpha_i, \alpha_j^\vee \rangle} = \nu$ for $1 \leq i \leq l$, and (iii) $x_j = x_i^q$ (hence $x_j = x_i$) if $\rho \alpha_i = \alpha_j$. By solving these equations, we find that an element t having the property of the lemma can be given as follows:

Type	t
A_l 2A_l $2 l$	$h(x_1, \dots, x_l)$
A_l $\text{ord}_2(l+1) > \text{ord}_2(p-1)$	$x_i = \nu^{i(l-i+1)/2} \quad (1 \leq i \leq l)$
	$h(x_1, \dots, x_l)$
	$x_1 = \nu^{(el+p-1)/2e} \left(e = \left(\frac{l+1}{2}, p-1 \right) \right)$
B_l $4 l(l+1)$	$x_i = \nu^{-i(i-1)/2} x_1^i \quad (2 \leq i \leq l)$
D_l 2D_l $4 l(l-1)$	$x_i = \nu^{i(2l-i+1)/2} \quad (1 \leq i \leq l-1)$
	$h(x_1, \dots, x_{l-2}, \nu^{l(l-1)/4}, \nu^{l(l-1)/4})$
	$x_i = \nu^{i(2l-i-1)/2} \quad (1 \leq i \leq l-2)$
D_l $\text{ord}_2(l-1)=1$	$h(x_1, \dots, x_{l-2}, \nu^{(l^2-l+p-1)/4}, \nu^{(l^2-l+3p-3)/4})$
$p \equiv -1 \pmod{4}$	$x_i = \nu^{i(2l+p-i-2)/2} \quad (1 \leq i \leq l-2)$
3D_4	$h(\nu^3, \nu^5, \nu^3, \nu^3)$
E_6 2E_6	$h(\nu^8, \nu^{11}, \nu^{15}, \nu^{21}, \nu^{15}, \nu^8)$

REMARK. If (at least) G is split over \mathbf{F}_q , then Lemmas 2, 4 above are implicit in Lehrer's work [12] where he showed a method to calculate the image $a(T^F)$ of T^F under the morphism $a: T \rightarrow (G_m)^l$ given by $a(s) = \prod_{i=1}^l \alpha_i(s)$ when G

is a simply-connected simple group (he has carried out the calculation when G is a classical group). For our purpose, it is essential to know the order of t (cf. § 2 below).

2. The main results. We recall that $p \neq 2$. Let ζ_p be a primitive p -th root of unity in the field \mathbf{C} of complex numbers. Let $\hat{F}_q = \text{Hom}(F_q, \mathbf{C}^\times)$ (we consider F_q as an additive group) and fix $\chi \in \hat{F}_q$, $\chi \neq 1$. For $a \in F_q$, define $\chi_a \in \hat{F}_q$ by $\chi_a(x) = \chi(ax)$ for $x \in F_q$. Then we have $\hat{F}_q = \{\chi_a | a \in F_q\}$ and $\{\chi^\tau | \tau \in \text{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})\} = \{\chi_a | a \in F_q^\times\}$.

In the following, if χ is a character of a finite group and L is a field of characteristic zero, $L(\chi)$ is the field generated over L by the values of χ . If χ is irreducible, then $m_L(\chi)$ denotes the Schur index of χ with respect to L . If L is an algebraic number field and v is a place of L , then L_v is the completion of L at v . Now let k be the quadratic subfield $\mathbf{Q}(\sqrt{\varepsilon p})$, $\varepsilon = (-1)^{(p-1)/2}$, of $\mathbf{Q}(\zeta_p)$.

Proposition 1. *Let G, F be as in Introduction. Let $\lambda \in \Lambda$, $\lambda \neq 1$. Then we have the following:*

- (i) λ^{G^F} takes all its values in k ; if $p \equiv -1 \pmod{4}$, λ^{G^F} is realizable in k ; if $p \equiv 1 \pmod{4}$, then, for any finite place v of k , λ^{G^F} is realizable in k_v .
- (ii) Assume that q is an even power of p . Then λ^{G^F} takes all its values in \mathbf{Q} and, for any prime number $r \neq p$, λ^{G^F} is realizable in \mathbf{Q}_r .
- (iii) If G is an adjoint semisimple group or any one of the groups described in Lemma 4, then λ^{G^F} is realizable in \mathbf{Q}_r .

Proof of (i). Let t be an element of T^F having the property of Lemma 2. Then $z = t^{(p-1)/2}$ lies in the centre Z^F of G^F since $\alpha(z) = 1$ for all simple roots α . Put $c = |\langle z \rangle|$ ($c = 1$ or 2). Let $M = \langle t \rangle U^F$. Then M acts on Λ by $\lambda^m(u) = \lambda(mum^{-1})$ ($\lambda \in \Lambda$, $m \in M$, $u \in U^F$). Let $\lambda \in \Lambda$, $\lambda \neq 1$. Then, by (1), λ can be expressed as $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i \in \hat{F}_{q_i}$ for $i \in I$. And, by (2), we have

$$\lambda^t = ((\lambda_i)_{\gamma_i(t)})_{i \in I} = ((\lambda_i)_{\gamma_i^2})_{i \in I} = (\lambda_i^{\sigma^2})_{i \in I} = \lambda^{\sigma^2},$$

where σ is a suitable generator of $\text{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$. Thus, on U^F , we have

$$\lambda^M = c \sum_{j=1}^{(p-1)/2} \lambda^{t^j} = c \sum_{j=1}^{(p-1)/2} \lambda^{\sigma^{2j}},$$

hence $\mathbf{Q}(\lambda^M) = \mathbf{Q}(\zeta_p)^{\langle \sigma^2 \rangle} = k$. Therefore the values of $\lambda^{G^F} = (\lambda^M)^{G^F}$ lie in k .

Suppose $t^{(p-1)/2} = 1$. Then λ^M is irreducible. By Gow's argument [7], p. 104, we have $m_k(\lambda^M) = 1: \lambda^M | \langle t \rangle =$ the character of the regular representation of $\langle t \rangle$, hence $\langle \lambda^M, 1_{\langle t \rangle} \rangle_{\langle t \rangle} = 1$; hence, by Schur's theorem (see e.g. Feit [5], 11.4), $m_k(\lambda^M) = 1$. Thus λ^M , hence $\lambda^{G^F} = (\lambda^M)^{G^F}$ is realizable in k .

Assume that $t^{(p-1)/2} \neq 1$. Then λ^M is reducible and is equal to the sum $\mu_0 + \mu_1$ where, for $i = 0, 1$, μ_i is the irreducible character of M induced by the

linear character of $\langle z \rangle U^F$ given by $z^j u \rightarrow (-1)^{ji} \lambda(u)$ ($j=0, 1$). We have $\mathbf{Q}(\mu_0) = \mathbf{Q}(\mu_1) = k$. For $i=0, 1$, the simple direct summand A_i of the group algebra $k[M]$ of M over k corresponding to μ_i is isomorphic over k to the cyclic algebra $((k(\zeta_p)/k, \sigma^2, (-1)^i)$ over k (cf. Proof of Proposition 3.5 of Yamada [18]). A_0 clearly splits over k , hence $m_k(\mu_0)=1$ and μ_0 is realizable in k . If $p \equiv -1 \pmod{4}$, then -1 is a norm in $k(\zeta_p)/k$, hence A_1 splits over k . Thus, in this case, μ_1 , hence $\lambda^M = \mu_0 + \mu_1$ is realizable in k . Suppose $p \equiv 1 \pmod{4}$. Then A_1 has non-zero invariants only at two real places of k (see Janusz [10], Proposition 3). Thus, for any finite place v of k , μ_1 , hence $\lambda^M = \mu_0 + \mu_1$ is realizable in k_v .

Proof of (ii). Let t be an element of T^F having the property of Lemma 3, and put $M = \langle t \rangle U^F$. Then, as $\lambda^t = \lambda^\sigma$ ($\lambda \neq 1$), on U^F , we have

$$\lambda^M = c \sum_{j=1}^{p-1} \lambda^{t^j} = c \sum_{j=1}^{p-1} \lambda^{\sigma^j} \quad (c = |\langle t^{p-1} \rangle|).$$

Thus $\mathbf{Q}(\lambda^M) = \mathbf{Q}(\zeta_p)^{\langle \sigma \rangle} = \mathbf{Q}$.

If $t^{p-1} = 1$, then λ^M is irreducible and Gow's argument shows that $m_{\mathbf{Q}}(\lambda^M) = 1$, hence λ^{G^F} is realizable in \mathbf{Q} . Suppose $t^{p-1} \neq 1$. Then λ^M is reducible and is equal to the sum $\mu_0 + \mu_1$, where, for $i=0, 1$, μ_i is the irreducible character of M induced by the linear character of $\langle t^{p-1} \rangle U^F$ given by $(t^{p-1})^j u \rightarrow u(-1)^{ji} \lambda(u)$. We have $\mathbf{Q}(\mu_0) = \mathbf{Q}(\mu_1) = \mathbf{Q}$. For $i=0, 1$, the simple direct summand A_i of $\mathbf{Q}[M]$ corresponding to μ_i is isomorphic over \mathbf{Q} to $(\mathbf{Q}(\zeta_p)/\mathbf{Q}, \sigma, (-1)^i)$. A_0 splits, hence μ_0 is realizable in \mathbf{Q} . A_1 has the invariants $\frac{1}{2} \pmod{1}$ at ∞, p and $0 \pmod{1}$ at any other place of \mathbf{Q} . Thus, for any prime number $r \neq p$, μ_1 , hence $\lambda^M = \mu_0 + \mu_1$ is realizable in \mathbf{Q}_r .

Proof of (iii). When G is adjoint the assertion is contained in Theorem 1 of [13]. Assume that G is not adjoint. Let t be an element of T^F having the property of Lemma 4 and put $M = \langle t \rangle U^F$. Then λ^M is irreducible and $\mathbf{Q}(\lambda^M) = \mathbf{Q}$. And, by Gow's argument, we have $m_{\mathbf{Q}}(\lambda^M) = 1$. Thus λ^M , hence $\lambda^{G^F} = (\lambda^M)^{G^F}$ is realizable in \mathbf{Q} .

We note that, for $G = SL_n, Sp_{2n}$, Proposition 1 is proved by Gow [7], [8].

Corollary 1. *Let G, F be as in Proposition 1. Recall that $p \neq 2$. Let χ be an irreducible character of G^F such that $\langle \chi, \lambda^{G^F} \rangle_{G^F} = 1$ for some $\lambda \in \Lambda$ (any irreducible component of λ^{G^F} for $\lambda \in \Lambda_0$ has this property (see Lemma 1)). Then we have $m_{\mathbf{Q}}(\chi) \leq 2$. Thus, in particular, we have $m_{\mathbf{R}}(\chi) \leq 2$ for any irreducible Deligne-Lusztig character $\chi = \pm R_T^{\theta}$ of G^F . If $\lambda = 1$, then λ^{G^F} is realizable in \mathbf{Q} , hence we have $m_{\mathbf{Q}}(\chi) = 1$. Assume that $\lambda \neq 1$. Let r be any prime number and v a place of k lying above r . Then, by Proposition 1, we have $m_{k_v}(\chi) = 1$, hence $m_{\mathbf{Q}_r}(\chi) \leq 2$ as $[k_r(\chi) : \mathbf{Q}_r(\chi)] \leq 2$. We also have $m_{\mathbf{R}}(\chi) \leq 2$. Thus, $m_{\mathbf{Q}}(\chi)$, being the least*

common multiple of the $m_{Q_w}(\chi)$ with w running over all places of \mathbf{Q} , is at most two. The last assertion follows from this fact and Lemma 1.

Corollary 2. *Assume that q is an even power of p . Let χ be an irreducible character of G^F such that $\langle \chi, \lambda^G \rangle_{G^F} = 1$ for some $\lambda \in \Lambda$. Then, for any prime number $r \neq p$, we have $m_{Q_r}(\chi) = 1$.*

This follows at once from Proposition 1, (ii).

Corollary 3. *Assume that G is an adjoint semisimple group or any one of the groups described in Lemma 4. Let χ be an irreducible character of G^F such that $\langle \chi, \lambda^{G^F} \rangle_{G^F} = 1$ for some $\lambda \in \Lambda$. Then we have $m_{\mathbf{Q}}(\chi) = 1$.*

This follows from Proposition 1, (iii).

Corollary 4. *Let G, F be as in Proposition 1. Assume that p is a good prime for G ([16], I, 4.1). Let χ be an irreducible character of G^F and let u be a regular unipotent element in G^F . Then $\chi(u)$ is an algebraic integer in k , and if $p \nmid \chi(1)$, we have $m_{\mathbf{Q}}(\chi) \leq 2$.*

We first note that, as p is good for G , U^F is equal to the derived group of U^F , hence Λ is the set of linear characters of U^F (Howlett [9], Lehrer [11]), and that, if $u \in U^F$, then $\mu(u) = 0$ for any non-linear irreducible character μ of U^F (Lehrer [11]).

Let \mathcal{O}_k be the ring of integers in k . We show that $\chi(u)$ belongs to \mathcal{O}_k . We may assume that $u \in U^F$ as u is conjugate to an element of U^F . Let t be an element of T^F having the property of Lemma 2, and let $\Lambda_1, \dots, \Lambda_r$ be the orbits of $\langle t \rangle$ on Λ . Thus, as $\chi^t = \chi$, if we put $a_\lambda = \langle \chi, \lambda \rangle_{U^F}$ for $\lambda \in \Lambda$, a_λ is constant on each Λ_i . Hence we have

$$\chi(u) = \sum_{\lambda \in \Lambda} a_\lambda \lambda(u) = \sum_{i=1}^r a_i \left(\sum_{\lambda \in \Lambda_i} \lambda(u) \right),$$

where $a_i = a_\lambda$ on Λ_i . Each $\sum_{\lambda \in \Lambda_i} \lambda(u)$ is stable under the action of $\langle t \rangle$, hence under the action of $\langle \sigma^2 \rangle$. Thus $\chi(u) \in \mathcal{O}_k$.

To prove the second assertion, we embed G in G_1 as in the proof of Lemma 1. Assume that $p \nmid \chi(1)$ and take an irreducible character χ_1 of G_1^F such that $\langle \chi, \chi_1|_{G^F} \rangle_{G^F} \neq 0$. Then, by the Clifford theory, we have $\chi_1|_{G^F} = e(\chi^{(1)} + \chi^{(2)} + \dots + \chi^{(s)})$, where e is a positive integer dividing $(G_1^F : G^F)$ and $\chi^{(1)}, \chi^{(2)}, \dots, \chi^{(s)}$ are the G_1^F -conjugates of $\chi = \chi^{(1)}(s|(G_1^F : G^F))$. Let r be any prime number and v a place of k lying above r . Put $m_v = m_{k_v}(\chi^{(1)}) = \dots = m_{k_v}(\chi^{(s)})$. For $1 \leq i \leq s$ and for $\lambda \in \Lambda$, put $a_v^{(i)} = \langle \chi^{(i)}, \lambda \rangle_{U^F}$. Then, by Proposition 1 (i), m_v divides the $a_v^{(i)}$, $1 \leq i \leq s$, $\lambda \in \Lambda$. As $p \nmid (G_1^F : G^F)$, $p \nmid \chi_1(1)$, so that, by a theorem of Green-Lehrer-Lusztig (see [3], 8.3.6), we have $\chi_1(u) = \pm 1$. Therefore we have the expression

$$\pm 1/m_v = \chi_1(u)/m_v = \{e \cdot \sum_{i=1}^s \chi^{(i)}(u)\}/m_v = e \cdot \sum_{i=1}^s \sum_{\lambda \in \Lambda} (a_\lambda^{(i)}/m_v) \cdot \lambda(u),$$

where the right-hand side is an algebraic integer and the left-hand side is a rational number. Hence $m_v=1$, and $m_{\mathbf{Q}_r}(\chi) \leq 2$. As r is an arbitrary prime number, we hence have $m_{\mathbf{Q}}(\chi) \leq 2$. This completes the proof of Corollary 4.

Corollary 5. *Assume that q is an even power of p and that p is good for G . Let u be a regular unipotent element in G^F . Then, for any irreducible character χ of G^F , $\chi(u)$ is a rational integer, and if $p \nmid \chi(u)$, we have $m_{\mathbf{Q}_r}(\chi)=1$ for any prime number $r \neq p$.*

The proof is similar to the proof of Corollary 4 (we use Proposition 1, (ii)).

Corollary 6. *Let G be an adjoint semisimple group or any one of the groups described in Lemma 4. Assume that p is good for G . Let u be a regular unipotent element in G^F and let χ be an irreducible character of G^F . Then $\chi(u)$ is a rational integer and if $p \nmid \chi(u)$, we have $m_{\mathbf{Q}}(\chi)=1$.*

REMARK. Lehrer [12] has calculated the values of the cuspidal irreducible characters of G^F at the regular unipotent elements of G^F when G is a semisimple group. As to the upper bound of the indices of the characters of related finite groups, we refer to Gow [8] for classical finite groups and Benard [1] and Feit [6] for the sporadic simple groups.

Let G be a connected, reductive algebraic group over an algebraically closed field K of characteristic $p > 0$ and F a surjective endomorphism of G such that G^F is finite. Then Lemma 2 still holds for such G^F , so that the statements in Proposition 1, (i) and in Corollary 1 (except for the comment for Lemma 1) hold for G^F . Assume that K is an algebraic closure of \mathbf{F}_p and that some power of F is the Frobenius endomorphism relative to a rational structure on G over a finite subfield of K . Then Lemma 1 holds for G^F (cf. Carter [3], 8.1.3 and 8.4.5), so that all the statements in Corollary 1, hence the theorem in Introduction holds for G^F . If p is good for G , then the theorem of Green-Lehrer-Lusztig holds for G^F (if Z is connected: see [3], 8.3.6), so that Corollary 4 holds for G^F .

3. Example. We calculate all the local indices of the cuspidal irreducible Deligne-Lusztig characters $\pm R_{T'}^{\theta}$ of $SL_n(\mathbf{F}_q)$ when q is an even power of p ($\neq 2$).

Let G be SL_n and F the endomorphism $(g_{ij}) \rightarrow (g_{ij}^q)$ (q may be any power of any prime p). Let T' be a minisotropic maximal torus of G and let $W = N_G(T')^F/T'^F$ (T' is unique up to G^F -conjugate). Then, taking an element γ of order $(q^n - 1)/(q - 1)$ in $\mathbf{F}_q^{\times n}$, we have $T'^F = \langle t_0 \rangle$, where t_0 is G -conjugate to

$\text{diag}(\gamma, \gamma^q, \dots, \gamma^{q^{n-1}})$, and $W = \langle w_0 \rangle \simeq \mathbf{Z}/n\mathbf{Z}$, where w_0 is defined by $t_0^w = w_0 t_0 w_0^{-1} = t_0^q$ ($w_0 \in N_G(T')^F$ represents w_0). (All these statements can be easily checked by using [16], II, 1.3, 1.10 and 1.14.) W acts on $\hat{T}'^F = \text{Hom}(T'^F, \mathbf{C}^\times)$ by $\theta^w(s) = \theta(s^w)$ for $w \in W$, $\theta \in \hat{T}'^F$ and $s \in T'^F$. If θ is in general position, i.e., no non-identity element of W fixes θ , then $(-1)^{n-1} R_{T'}^\theta$ is a cuspidal irreducible character of $G^F = SL_n(\mathbf{F}_q)$ ([4], 7.4, 8.3).

Let $\theta \in \hat{T}'^F$. Then, by [4], 4.2, for $g \in G^F$, if $g = su = us$ (s semisimple, u unipotent) is its Jordan decomposition, we have

$$(3) \quad R_T^\theta(g) = \frac{1}{|Z_G(s)^F|} \sum_{\substack{h \in G^F \\ h^{-1}sh \in T'}} Q_{hT'h^{-1}, Z_G(s)}(u) \cdot \theta(h^{-1}sh),$$

where the $Q_{hT'h^{-1}, Z_G(s)}$ are Green functions of $Z_G(s)$ (which is connected since G is simply-connected). It follows that, if s is not conjugate in G^F to any element of T'^F , we have $R_{T'}^\theta(g) = 0$, and if $s \in T'^F$, we have

$$(4) \quad R_{T'}^\theta(g) = Q_{T', Z_G(s)}(u) \frac{1}{|W(s)|} \sum_{w \in W} \theta^w(s),$$

where $W(s) = \{w \in W \mid s^w = s\}$ (we note that the minisotropic maximal tori of $Z_G(s)$ form a single $Z_G(s)^F$ -conjugacy class (cf. [16], II, 1.3, 1.10 and 1.14) and that any two elements of T' that are conjugate in G^F are conjugate under the action of W). Thus, as the Green functions take integral values, by putting $\theta(t_0) = \zeta$, we get from (4):

$$(5) \quad Q(R_{T'}^\theta) = Q\left(\sum_{w \in W} \theta^w\right) = Q(\zeta + \zeta^q + \dots + \zeta^{q^{n-1}}).$$

Lemma 5. *Assume that θ is in general position. Let $q = p^m$. We further assume that n is even. Then we have*

$$\text{ord}_2[\mathbf{Q}_p(R_{T'}^\theta): \mathbf{Q}_p] = \text{ord}_2 m.$$

Let ϕ be the automorphism of $\mathbf{Q}_p(\zeta)$ defined by $\zeta^\phi = \zeta^q$. Then ϕ has order n (by assumption) and we have $\mathbf{Q}_p(\zeta)^{\langle \phi \rangle} = \mathbf{Q}_p(R_{T'}^\theta)$ (cf. (5)). Let $f = [\mathbf{Q}_p(\zeta): \mathbf{Q}_p]$ and $e = |\langle \zeta \rangle|$. Then f is equal to the least integer $h \geq 1$ subject for the condition: $p^h \equiv 1 \pmod{e}$ (see Serre [14], p. 85). As $\phi^n = 1$ and $\phi^i \neq 1$ for $1 \leq i \leq n-1$, we find that $f \mid mn$ but $f \nmid mi$ for $1 \leq i \leq n-1$ [in fact, if $f \mid mi$, then $p^f - 1 \mid p^{mi} - 1$, hence $e \mid p^{mi} - 1$, hence $\phi^i = 1$]. This shows that $\text{ord}_r f = \text{ord}_r m + \text{ord}_r n$ for any prime divisor r of n . Thus, in particular, we have $\text{ord}_2 f = \text{ord}_2 m + \text{ord}_2 n$. As $[\mathbf{Q}_p(\zeta): \mathbf{Q}_p(R_{T'}^\theta)] = [\mathbf{Q}_p(\zeta): \mathbf{Q}_p(\zeta)^{\langle \phi \rangle}] = n$, we hence have $\text{ord}_2[\mathbf{Q}_p(R_{T'}^\theta): \mathbf{Q}_p] = \text{ord}_2 m$, as desired.

REMARK. Professor K. Iimura showed to the author (by an elementary proof) that $n = f/(m, f)$ and $[\mathbf{Q}_p(\zeta)^{\langle \phi \rangle}: \mathbf{Q}_p] = (m, f)$.

Proposition 2. *Let χ be any cuspidal irreducible Deligne-Lusztig character $(-1)^{n-1}R_{T'}^q$ of $G^F = SL_n(\mathbb{F}_q)$, where we assume that q is an even power of $p \neq 2$. Then, if n is odd or $\text{ord}_2 n \geq 2$, we have $m_Q(\chi) = 1$. Assume that $\text{ord}_2 n = 1$. Then we have $m_{Q_r}(\chi) = 1$ for any prime number r and $m_Q(\chi) = m_R(\chi) \leq 2$. And we have $m_R(\chi) = 2$ if and only if χ is real and $\chi(-1_n) = -\chi(1_n)$ (i.e. $\theta(-1_n) = -1$).*

REMARK. Let χ be as above. Assume that n is even and let $n = 2m$. Fixing a generator θ_0 of \hat{T}'^F , put $\theta = \theta_0^i$. Then the following can be shown:

(i) χ is real if and only if $\frac{q^m - 1}{q - 1} \mid i$.

(ii) Assume that $\text{ord}_2 n = 1$ and let $i = \frac{q^m - 1}{q - 1} i'$ with $i' \in \mathbb{Z}$ (hence χ is real).

Then $\theta(-1_n) = 1$ if and only if i' is even, and the latter condition is equivalent to the condition that $\theta \mid Z^F = 1$.

Proof of Proposition 2. Let $\lambda \in \Lambda_0$. Then, by Lemma 1, we have $\langle \chi, \lambda^{G^F} \rangle_{G^F} = 1$. Thus, if n is odd or $\text{ord}_2 n > \text{ord}_2(p - 1)$, by Proposition 1, (iii), we have $m_Q(\chi) = 1$. Assume that $1 \leq \text{ord}_2 n \leq \text{ord}_2(p - 1)$. Let t be an element of T^F having the property of Lemma 3. Then, under our assumption, we have $t^{p-1} = -1_n$ (cf. Proof of Lemma 4 and Proof of Lemma 3.3 (a) of Gow [8]). Let us use the notation of the proof of Proposition 1, (ii). Then $\lambda^M = \mu_0 + \mu_1$. As $\mu_i(-1_n) = (-1)^i \mu_i(1_n)$ for $i = 0, 1$, by Schur's lemma, we have $\langle \chi, \mu_0 \rangle_M = 1$ if $\chi(-1_n) = \chi(1_n)$, and $\langle \chi, \mu_1 \rangle_M = 1$ if $\chi(-1_n) = -\chi(1_n)$. As μ_0 is realizable in \mathbf{Q} , we have $m_Q(\chi) = 1$ in the first case. Assume that $\chi(-1_n) = -\chi(1_n)$. If r is any prime number $\neq p$, then μ_1 is realizable in \mathbf{Q}_r , hence we have $m_{Q_r}(\chi) = 1$. As q is an even power of p , by Lemma 5, we have $2 \mid [\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$. Hence $A_1 \otimes_{\mathbf{Q}} \mathbf{Q}_p(\chi)$ splits (see [14], Chap. XIII, § 3, Prop. 7), hence μ_1 is realizable in $\mathbf{Q}_p(\chi)$. Hence we have $m_{Q_p}(\chi) = m_{Q_p(\chi)}(\chi) = 1$. Thus we have $m_Q(\chi) = m_R(\chi)$. If χ is real, we must have $m_R(\chi) = 2$ since otherwise χ will be realizable in \mathbf{R} , so that, by Schur's theorem, we have $(2 =) m_R(\chi_1) \mid \langle \chi, \mu_1 \rangle_M = 1$, a contradiction. If $\text{ord}_2 n \geq 2$, then χ cannot be real since G^F contains a central element z of order 4 such that $z^2 = -1_n$ and $\chi(z) = \pm \sqrt{-1} \chi(1_n)$ ([7], p. 107). Finally, we note that, by [4], 1.22, we have $\chi(-1_n) = -\chi(1_n)$ if and only if $\theta(-1_n) = -1$. This completes the proof of Proposition 2.

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