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## ON THE SCHUR INDICES OF CERTAIN IRREDUCIBLE CHARACTERS OF REDUCTIVE GROUPS OVER FINITE FIELDS

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**Introduction.** Let  $F_q$  be a finite field with  $q$  elements, of characteristic  $p$ . Let  $G$  be a connected, reductive linear algebraic group defined over  $F_q$ , with Frobenius endomorphism  $F$ , and let  $G^F$  denote the group of  $F$ -fixed points of  $G$ . In [13], we investigated, under the assumption that the centre  $Z$  of  $G$  is connected, the rationality-properties of the characters  $\lambda^{G^F}$  of  $G^F$  induced by certain linear characters  $\lambda$  of a Sylow  $p$ -subgroup of  $G^F$  and, using the results obtained there, proved some propositions concerning the Schur indices of the semisimple or regular irreducible characters of  $G^F$ . In this paper, we shall treat the general case, that is, the case that  $Z$  is not necessarily connected. The main results are stated and proved in § 2. In particular, we get the following (see Corollary 1 to Proposition 1, § 2):

**Theorem.** *Any irreducible Deligne-Lusztig character  $\pm R_T^{\theta}$  of  $G^F$  ([4]) has the Schur index at most two over the field  $\mathbf{Q}$  of rational numbers.*

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**1. Some lemmas.** Let  $G$  and  $F$  be as above. Let  $B$  be an  $F$ -stable Borel subgroup of  $G$  with the unipotent radical  $U$  and  $T$  an  $F$ -stable maximal torus of  $B$ . For a root  $\alpha$  of  $G$  (with respect to  $T$ ), let  $U_{\alpha}$  denote the root subgroup of  $G$  associated with  $\alpha$ . Let  $U$  be the subgroup of  $U$  generated by the non-simple positive root subgroups  $U_{\alpha}$  (the ordering on the roots is the one determined by  $B$ ). Then  $U/U$  is commutative and can be regarded as the direct product  $\prod_{\alpha \in \Delta} U_{\alpha}$ , where  $\Delta$  is the set of simple roots. As  $FU = U$ ,  $F$  acts on  $U/U = \prod_{\alpha \in \Delta} U_{\alpha}$  and this action is the one induced by the maps  $F: U_{\alpha} \rightarrow FU_{\alpha}$ ,  $\alpha \in \Delta$ . Let  $\rho$  be the permutation on the roots  $\alpha$  given by  $FU_{\alpha} = U_{\rho\alpha}$  and let  $I$

be the set of orbits of  $\rho$  on  $\Delta$ . For  $i \in I$ , put  $U_i = \prod_{\alpha \in i} U_\alpha$ . Then  $U/U = \prod_{i \in I} U_i$  and, as each  $U_i$  is  $F$ -stable, we have  $U^F/U^F = \prod_{i \in I} U_i^F$ . For each  $i \in I$ , put  $q_i = q^{|i|}$  and take one simple root  $\gamma_i$  in  $i$ . Then, for each  $i$ , there is an isomorphism  $\phi_i$  of  $U_i^F$  with the additive group of  $\mathbf{F}_{q_i}$  such that  $\phi_i(tut^{-1}) = \gamma_i(t)\phi_i(u)$  for  $u \in U_i^F$  and  $t \in T^F$  (cf. Proof of 11.8 of Steinberg [17] and Carter [3], pp. 76–77). Thus the family  $\phi = (\phi_i)_{i \in I}$  defines an isomorphism

$$(1) \quad \phi: U^F/U^F = \prod_{i \in I} U_i^F \simeq \prod_{i \in I} \mathbf{F}_{q_i}$$

so that, for  $u = \prod_{i \in I} u_i$  with  $u_i \in U_i^F$  for  $i \in I$  and  $t \in T^F$ , we have

$$(2) \quad \phi(tut^{-1}) = \prod_{i \in I} \lambda_i(t)\phi_i(u_i).$$

Now let  $\Lambda$  be the set of characters  $\lambda$  of  $U^F$  such that  $\lambda|U = 1$  and  $\Lambda_0$  the set of characters  $\lambda$  in  $\Lambda$  such that  $\lambda|U_i^F \neq 1$  for all  $i \in I$ . Then we have

**Lemma 1.** *Let  $\lambda \in \Lambda_0$ . Then  $\lambda^{G^F}$  is multiplicity-free (Gel'fand-Graev, Yokonuma, Steinberg) and any irreducible Deligne-Lusztig character  $\pm R_T^0$  of  $G^F$  occurs in  $\lambda^{G^F}$  (Deligne-Lusztig).*

By embedding  $G$  in the connected, reductive group  $G_1 = (G \times T)/\{(z, z^{-1}) \mid z \in Z\}$  ( $Z$  is the centre of  $G$ ) with connected centre and the same derived group ([4], 5.18) and (as to the second assertion) using properties of Green functions (cf. [3], 7.2.8 and 7.7), we are reduced to the case that  $Z$  is connected. In this case the lemma is proved in [4], Theorem 10.7 (or in [3], 8.1.3 and 8.4.5).

Our purpose is to study the rationality of the characters  $\lambda^{G^F}$ ,  $\lambda \in \Lambda$ . Suppose  $p=2$ . Then, by (1),  $U^F/U^F$  is an elementary abelian 2-group, so that, for any  $\lambda \in \Lambda$ ,  $\lambda$ , hence  $\lambda^{G^F}$  is realizable in  $\mathbf{Q}$ . Therefore, from now on, we shall assume that  $p \neq 2$ .

**Lemma 2.** *Let  $\nu$  be a primitive element of  $\mathbf{F}_p$  (i.e.  $\mathbf{F}_p^\times = \langle \nu \rangle$ ). Then there exists an element  $t$  in  $T^F$  such that  $t^{p-1} = 1$  (possibly  $t^{(p-1)/2} = 1$ ) and  $\alpha(t) = \nu^2$  for all simple roots  $\alpha$ .*

It suffices to prove the lemma for the derived group  $G'$  of  $G$ , hence for the simply-connected covering of  $G'$ . If  $G$  is a simply-connected semisimple group, then we have  $G = G_1 \times \cdots \times G_m$ , where, for  $1 \leq i \leq m$ ,  $G_i$  is an  $F$ -stable simply-connected semisimple closed subgroup of  $G$  whose simple components are permuted by  $F$  cyclically, and the truth of the lemma for each  $G_i$  will imply that for  $G$ . If  $G = G_1 \times FG_1 \times \cdots \times F^{n-1}G_1$ , where  $G_1$  is an  $F^n$ -stable simply-connected simple closed subgroup of  $G$  for some  $n \geq 1$ , then  $T$  and  $B$ , hence the set of simple roots has the corresponding decomposition, and it is easy to see that the truth of the lemma for  $G_1$  with Frobenius map  $F^n$  implies that for

$G$  (cf. [17], 11.2 (b)). Thus we are reduced to the case that  $G$  is a simply-connected simple group.

Suppose therefore that  $G$  is such a group. Let  $X(T) = \text{Hom}(T, \mathbf{G}_m)$  and  $Y(T) = \text{Hom}(\mathbf{G}_m, T)$ , and let  $\langle, \rangle: X(T) \times Y(T) \rightarrow \mathbf{Z}$  be the natural pairing given by  $\langle \chi, \chi^\vee \rangle = \text{degree of } \chi \circ \chi^\vee$  for  $\chi \in X(T)$  and  $\chi^\vee \in Y(T)$ . Let  $\alpha_1, \dots, \alpha_l$  be the simple roots (as to the numbering of the simple roots, we follow that of Bourbaki [2]) and let  $\alpha_1^\vee, \dots, \alpha_l^\vee$  be the corresponding simple coroots. Then, as  $G$  is simply-connected, we have  $Y(T) = \langle \alpha_1^\vee, \dots, \alpha_l^\vee \rangle_{\mathbf{Z}}$ , so that the mapping  $h: (x_1, \dots, x_l) \rightarrow \prod_{i=1}^l \alpha_i^\vee(x_i)$  defines an isomorphism of  $(\mathbf{G}_m)^l$  with  $T$ . Then, for  $1 \leq i \leq l$ , we have

$$\alpha_i(h(x_1, \dots, x_l)) = \prod_{j=1}^l x_j^{\langle \alpha_i, \alpha_j^\vee \rangle},$$

where  $(\langle \alpha_i, \alpha_j^\vee \rangle)_{1 \leq i, j \leq l}$  is the Cartan matrix of  $G$ . We define an action of  $F$  on  $Y(T)$  by  $F(\chi^\vee) = F \circ \chi^\vee$  for  $\chi^\vee \in Y(T)$ . Then we have

$$F(\alpha_i^\vee) = q(\rho \alpha_i)^\vee$$

for  $1 \leq i \leq l$  (see [15], 11.4.7). It readily follows that, for  $s \in T$ ,  $s = h(x_1, \dots, x_l)$ , we have  $Fs = s$  if and only if  $x_j = x_j^q$  if  $\rho \alpha_i = \alpha_j$ . Thus the proof of the lemma has been reduced to solving the following problem:

Find an element  $t = h(x_1, \dots, x_l)$  with  $x_i \in \mathbf{F}_p^\times$  for  $1 \leq i \leq l$  such that  $\prod_{j=1}^l x_j^{\langle \alpha_i, \alpha_j^\vee \rangle} = v^2$  for  $1 \leq i \leq l$  and that  $x_j = x_j^q$  (hence  $x_j = x_i$ ) if  $\rho \alpha_i = \alpha_j$ .

When  $G$  is adjoint, by the proof of Theorem 1 of [13], there is an element  $s$  in  $T^F$  of order  $p-1$  such that  $\alpha(s) = v$  for all simple roots  $\alpha$ . Hence it suffices to take  $t = s^2$ . Suppose therefore that  $G$  is not adjoint. Then, as  $p \neq 2$ ,  $G$  is any one of the following types (Steinberg [17], 11.6; also see [3], 1.19):  $A_l$  ( $l \geq 1$ ),  $B_l$  ( $l \geq 2$ ),  $C_l$  ( $l \geq 2$ ),  $D_l$  ( $l \geq 3$ ),  $E_6$ ,  $E_7$ ,  ${}^2A_l$  ( $l \geq 1$ ),  ${}^2D_l$  ( $l \geq 3$ ),  ${}^3D_4$ ,  ${}^2E_6$ . In each case, an element  $t$  of  $T^F$  having the property of the lemma (i.e. an solution  $t$  of the problem above) can be given as follows (the Cartan matrices are listed up in the appendices of [2]):

Type	$t$	
$A_l$ ${}^2A_l$	$h(x_1, \dots, x_l)$	$x_i = v^{i(l-i+1)}$ ( $1 \leq i \leq l$ )
$B_l$	$h(x_1, \dots, x_{l-1}, v^{l(l+1)/2})$	$x_i = v^{i(2l-i+1)}$ ( $1 \leq i \leq l-1$ )
$C_l$	$h(x_1, \dots, x_l)$	$x_i = v^{i(2l-i)}$ ( $1 \leq i \leq l$ )
$D_l$ ${}^2D_l$	$h(x_1, \dots, x_{l-2}, v^{l(l-1)/2}, v^{(l-1)/2})$	$x_i = v^{i(2l-i-1)}$ ( $1 \leq i \leq l-2$ )
$E_6$ ${}^2E_6$	$h(v^{16}, v^{22}, v^{30}, v^{42}, v^{30}, v^{16})$	
$E_7$	$h(v^{34}, v^{49}, v^{66}, v^{96}, v^{75}, v^{52}, v^{27})$	
${}^3D_4$	$h(v^6, v^{10}, v^6, v^6)$	

This completes the proof of Lemma 2.

**Lemma 3.** Assume that  $q$  is an even power of  $p$ . Then there exists an element  $t$  in  $T^F$  such that  $t^{2(p-1)}=1$  (possibly  $t^{p-1}=1$ ) and  $\alpha(t)=\nu$  for all simple roots  $\alpha$ .

As in the proof of Lemma 2, we can be reduced to the case that  $G$  is a simply-connected simple group. When  $G$  is adjoint Lemma 3 is proved in the proof of Theorem 1 of [13]. When  $G$  is not adjoint  $t$  can be given by replacing each  $\nu$  in the above table with an element  $\varepsilon \in \mathbf{F}_q$  such that  $\varepsilon^2=\nu$ . (We note that, when  $G$  is a simply-connected simple group, an element  $s=h(x_1, \dots, x_l)$  of  $T$  has the property of Lemma 3 if and only if the  $x_i$  satisfy: (i)  $x_i^{2(p-1)}=1$  for  $1 \leq i \leq l$ , (ii)  $\prod_{j=1}^l x_j^{\langle \alpha_i, \alpha_j^\vee \rangle} = \nu$  for  $1 \leq i \leq l$ , and (iii)  $x_j = x_i^q$  if  $\rho \alpha_i = \alpha_j$ .)

In the following, for an integer  $m$  and a prime number  $r$ ,  $\text{ord}_r m$  denotes the exponent of the  $r$ -part of  $m$ .

**Lemma 4.** Assume that  $G$  is a (non-adjoint) simply-connected simple group of any one of the following types:  $A_l$  with  $2|l$  or  $\text{ord}_2(l+1) > \text{ord}_2(p-1)$ ;  ${}^2A_l$  with  $2|l$ ;  $B_l$  with  $4|l(l+1)$ ;  $D_l$  with either (a)  $4|l(l-1)$  or (b)  $\text{ord}_2(l-1)=1$  and  $p \equiv -1 \pmod{4}$ ;  ${}^2D_l$  with  $4|l(l-1)$ ;  ${}^3D_4$ ;  $E_6$ ;  ${}^2E_6$ . Then there exists an element  $t \in T^F$  such that  $t^{p-1}=1$  and  $\alpha(t)=\nu$  for all simple roots  $\alpha$ .

In fact, for an element  $s=h(x_1, \dots, x_l)$  of  $T$ ,  $s$  satisfies the property of Lemma 4 if and only if the  $x_i$  satisfy: (i)  $x_i \in \mathbf{F}_p^\times$ , (ii)  $\prod x_j^{\langle \alpha_i, \alpha_j^\vee \rangle} = \nu$  for  $1 \leq i \leq l$ , and (iii)  $x_j = x_i^q$  (hence  $x_j = x_i$ ) if  $\rho \alpha_i = \alpha_j$ . By solving these equations, we find that an element  $t$  having the property of the lemma can be given as follows:

Type	$t$
$A_l$ ${}^2A_l$ $2 l$	$h(x_1, \dots, x_l)$
$A_l$ $\text{ord}_2(l+1) > \text{ord}_2(p-1)$	$x_i = \nu^{i(l-i+1)/2} \quad (1 \leq i \leq l)$
	$h(x_1, \dots, x_l)$
	$x_1 = \nu^{(el+p-1)/2e} \left( e = \left( \frac{l+1}{2}, p-1 \right) \right)$
$B_l$ $4 l(l+1)$	$x_i = \nu^{-i(i-1)/2} x_1^i \quad (2 \leq i \leq l)$
$D_l$ ${}^2D_l$ $4 l(l-1)$	$x_i = \nu^{i(2l-i+1)/2} \quad (1 \leq i \leq l-1)$
	$h(x_1, \dots, x_{l-2}, \nu^{l(l-1)/4}, \nu^{l(l-1)/4})$
	$x_i = \nu^{i(2l-i-1)/2} \quad (1 \leq i \leq l-2)$
$D_l$ $\text{ord}_2(l-1)=1$	$h(x_1, \dots, x_{l-2}, \nu^{(l^2-l+p-1)/4}, \nu^{(l^2-l+3p-3)/4})$
$p \equiv -1 \pmod{4}$	$x_i = \nu^{i(2l+p-i-2)/2} \quad (1 \leq i \leq l-2)$
${}^3D_4$	$h(\nu^3, \nu^5, \nu^3, \nu^3)$
$E_6$ ${}^2E_6$	$h(\nu^8, \nu^{11}, \nu^{15}, \nu^{21}, \nu^{15}, \nu^8)$

REMARK. If (at least)  $G$  is split over  $\mathbf{F}_q$ , then Lemmas 2, 4 above are implicit in Lehrer's work [12] where he showed a method to calculate the image  $a(T^F)$  of  $T^F$  under the morphism  $a: T \rightarrow (G_m)^l$  given by  $a(s) = \prod_{i=1}^l \alpha_i(s)$  when  $G$

is a simply-connected simple group (he has carried out the calculation when  $G$  is a classical group). For our purpose, it is essential to know the order of  $t$  (cf. § 2 below).

**2. The main results.** We recall that  $p \neq 2$ . Let  $\zeta_p$  be a primitive  $p$ -th root of unity in the field  $\mathbf{C}$  of complex numbers. Let  $\hat{F}_q = \text{Hom}(F_q, \mathbf{C}^\times)$  (we consider  $F_q$  as an additive group) and fix  $\chi \in \hat{F}_q$ ,  $\chi \neq 1$ . For  $a \in F_q$ , define  $\chi_a \in \hat{F}_q$  by  $\chi_a(x) = \chi(ax)$  for  $x \in F_q$ . Then we have  $\hat{F}_q = \{\chi_a | a \in F_q\}$  and  $\{\chi^\tau | \tau \in \text{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})\} = \{\chi_a | a \in F_q^\times\}$ .

In the following, if  $\chi$  is a character of a finite group and  $L$  is a field of characteristic zero,  $L(\chi)$  is the field generated over  $L$  by the values of  $\chi$ . If  $\chi$  is irreducible, then  $m_L(\chi)$  denotes the Schur index of  $\chi$  with respect to  $L$ . If  $L$  is an algebraic number field and  $v$  is a place of  $L$ , then  $L_v$  is the completion of  $L$  at  $v$ . Now let  $k$  be the quadratic subfield  $\mathbf{Q}(\sqrt{\varepsilon p})$ ,  $\varepsilon = (-1)^{(p-1)/2}$ , of  $\mathbf{Q}(\zeta_p)$ .

**Proposition 1.** *Let  $G, F$  be as in Introduction. Let  $\lambda \in \Lambda$ ,  $\lambda \neq 1$ . Then we have the following:*

- (i)  $\lambda^{G^F}$  takes all its values in  $k$ ; if  $p \equiv -1 \pmod{4}$ ,  $\lambda^{G^F}$  is realizable in  $k$ ; if  $p \equiv 1 \pmod{4}$ , then, for any finite place  $v$  of  $k$ ,  $\lambda^{G^F}$  is realizable in  $k_v$ .
- (ii) Assume that  $q$  is an even power of  $p$ . Then  $\lambda^{G^F}$  takes all its values in  $\mathbf{Q}$  and, for any prime number  $r \neq p$ ,  $\lambda^{G^F}$  is realizable in  $\mathbf{Q}_r$ .
- (iii) If  $G$  is an adjoint semisimple group or any one of the groups described in Lemma 4, then  $\lambda^{G^F}$  is realizable in  $\mathbf{Q}_r$ .

*Proof of (i).* Let  $t$  be an element of  $T^F$  having the property of Lemma 2. Then  $z = t^{(p-1)/2}$  lies in the centre  $Z^F$  of  $G^F$  since  $\alpha(z) = 1$  for all simple roots  $\alpha$ . Put  $c = |\langle z \rangle|$  ( $c = 1$  or  $2$ ). Let  $M = \langle t \rangle U^F$ . Then  $M$  acts on  $\Lambda$  by  $\lambda^m(u) = \lambda(mum^{-1})$  ( $\lambda \in \Lambda$ ,  $m \in M$ ,  $u \in U^F$ ). Let  $\lambda \in \Lambda$ ,  $\lambda \neq 1$ . Then, by (1),  $\lambda$  can be expressed as  $\lambda = (\lambda_i)_{i \in I}$  with  $\lambda_i \in \hat{F}_{q_i}$  for  $i \in I$ . And, by (2), we have

$$\lambda^t = ((\lambda_i)_{\gamma_i(t)})_{i \in I} = ((\lambda_i)_{\gamma_i^2})_{i \in I} = (\lambda_i^{\sigma^2})_{i \in I} = \lambda^{\sigma^2},$$

where  $\sigma$  is a suitable generator of  $\text{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$ . Thus, on  $U^F$ , we have

$$\lambda^M = c \sum_{j=1}^{(p-1)/2} \lambda^{t^j} = c \sum_{j=1}^{(p-1)/2} \lambda^{\sigma^{2j}},$$

hence  $\mathbf{Q}(\lambda^M) = \mathbf{Q}(\zeta_p)^{\langle \sigma^2 \rangle} = k$ . Therefore the values of  $\lambda^{G^F} = (\lambda^M)^{G^F}$  lie in  $k$ .

Suppose  $t^{(p-1)/2} = 1$ . Then  $\lambda^M$  is irreducible. By Gow's argument [7], p. 104, we have  $m_k(\lambda^M) = 1: \lambda^M | \langle t \rangle =$  the character of the regular representation of  $\langle t \rangle$ , hence  $\langle \lambda^M, 1_{\langle t \rangle} \rangle_{\langle t \rangle} = 1$ ; hence, by Schur's theorem (see e.g. Feit [5], 11.4),  $m_k(\lambda^M) = 1$ . Thus  $\lambda^M$ , hence  $\lambda^{G^F} = (\lambda^M)^{G^F}$  is realizable in  $k$ .

Assume that  $t^{(p-1)/2} \neq 1$ . Then  $\lambda^M$  is reducible and is equal to the sum  $\mu_0 + \mu_1$  where, for  $i = 0, 1$ ,  $\mu_i$  is the irreducible character of  $M$  induced by the

linear character of  $\langle z \rangle^{U^F}$  given by  $z^j u \rightarrow (-1)^{ji} \lambda(u)$  ( $j=0, 1$ ). We have  $\mathbf{Q}(\mu_0) = \mathbf{Q}(\mu_1) = k$ . For  $i=0, 1$ , the simple direct summand  $A_i$  of the group algebra  $k[M]$  of  $M$  over  $k$  corresponding to  $\mu_i$  is isomorphic over  $k$  to the cyclic algebra  $((k(\zeta_p)/k, \sigma^2, (-1)^i)$  over  $k$  (cf. Proof of Proposition 3.5 of Yamada [18]).  $A_0$  clearly splits over  $k$ , hence  $m_k(\mu_0)=1$  and  $\mu_0$  is realizable in  $k$ . If  $p \equiv -1 \pmod{4}$ , then  $-1$  is a norm in  $k(\zeta_p)/k$ , hence  $A_1$  splits over  $k$ . Thus, in this case,  $\mu_1$ , hence  $\lambda^M = \mu_0 + \mu_1$  is realizable in  $k$ . Suppose  $p \equiv 1 \pmod{4}$ . Then  $A_1$  has non-zero invariants only at two real places of  $k$  (see Janusz [10], Proposition 3). Thus, for any finite place  $v$  of  $k$ ,  $\mu_1$ , hence  $\lambda^M = \mu_0 + \mu_1$  is realizable in  $k_v$ .

Proof of (ii). Let  $t$  be an element of  $T^F$  having the property of Lemma 3, and put  $M = \langle t \rangle^{U^F}$ . Then, as  $\lambda^t = \lambda^\sigma$  ( $\lambda \neq 1$ ), on  $U^F$ , we have

$$\lambda^M = c \sum_{j=1}^{p-1} \lambda^{t^j} = c \sum_{j=1}^{p-1} \lambda^{\sigma^j} \quad (c = |\langle t^{p-1} \rangle|).$$

Thus  $\mathbf{Q}(\lambda^M) = \mathbf{Q}(\zeta_p)^{\langle \sigma \rangle} = \mathbf{Q}$ .

If  $t^{p-1} = 1$ , then  $\lambda^M$  is irreducible and Gow's argument shows that  $m_{\mathbf{Q}}(\lambda^M) = 1$ , hence  $\lambda^{G^F}$  is realizable in  $\mathbf{Q}$ . Suppose  $t^{p-1} \neq 1$ . Then  $\lambda^M$  is reducible and is equal to the sum  $\mu_0 + \mu_1$ , where, for  $i=0, 1$ ,  $\mu_i$  is the irreducible character of  $M$  induced by the linear character of  $\langle t^{p-1} \rangle^{U^F}$  given by  $(t^{p-1})^j u \rightarrow u(-1)^{ji} \lambda(u)$ . We have  $\mathbf{Q}(\mu_0) = \mathbf{Q}(\mu_1) = \mathbf{Q}$ . For  $i=0, 1$ , the simple direct summand  $A_i$  of  $\mathbf{Q}[M]$  corresponding to  $\mu_i$  is isomorphic over  $\mathbf{Q}$  to  $(\mathbf{Q}(\zeta_p)/\mathbf{Q}, \sigma, (-1)^i)$ .  $A_0$  splits, hence  $\mu_0$  is realizable in  $\mathbf{Q}$ .  $A_1$  has the invariants  $\frac{1}{2} \pmod{1}$  at  $\infty, p$  and  $0 \pmod{1}$  at any other place of  $\mathbf{Q}$ . Thus, for any prime number  $r \neq p$ ,  $\mu_1$ , hence  $\lambda^M = \mu_0 + \mu_1$  is realizable in  $\mathbf{Q}_r$ .

Proof of (iii). When  $G$  is adjoint the assertion is contained in Theorem 1 of [13]. Assume that  $G$  is not adjoint. Let  $t$  be an element of  $T^F$  having the property of Lemma 4 and put  $M = \langle t \rangle^{U^F}$ . Then  $\lambda^M$  is irreducible and  $\mathbf{Q}(\lambda^M) = \mathbf{Q}$ . And, by Gow's argument, we have  $m_{\mathbf{Q}}(\lambda^M) = 1$ . Thus  $\lambda^M$ , hence  $\lambda^{G^F} = (\lambda^M)^{G^F}$  is realizable in  $\mathbf{Q}$ .

We note that, for  $G = SL_n, Sp_{2n}$ , Proposition 1 is proved by Gow [7], [8].

**Corollary 1.** *Let  $G, F$  be as in Proposition 1. Recall that  $p \neq 2$ . Let  $\chi$  be an irreducible character of  $G^F$  such that  $\langle \chi, \lambda^{G^F} \rangle_{G^F} = 1$  for some  $\lambda \in \Lambda$  (any irreducible component of  $\lambda^{G^F}$  for  $\lambda \in \Lambda_0$  has this property (see Lemma 1)). Then we have  $m_{\mathbf{Q}}(\chi) \leq 2$ . Thus, in particular, we have  $m_{\mathbf{R}}(\chi) \leq 2$  for any irreducible Deligne-Lusztig character  $\chi = \pm R_T^0$  of  $G^F$ . If  $\lambda = 1$ , then  $\lambda^{G^F}$  is realizable in  $\mathbf{Q}$ , hence we have  $m_{\mathbf{Q}}(\chi) = 1$ . Assume that  $\lambda \neq 1$ . Let  $r$  be any prime number and  $v$  a place of  $k$  lying above  $r$ . Then, by Proposition 1, we have  $m_{k_v}(\chi) = 1$ , hence  $m_{\mathbf{Q}_r}(\chi) \leq 2$  as  $[k_r(\chi) : \mathbf{Q}_r(\chi)] \leq 2$ . We also have  $m_{\mathbf{R}}(\chi) \leq 2$ . Thus,  $m_{\mathbf{Q}}(\chi)$ , being the least*

common multiple of the  $m_{Q_w}(\chi)$  with  $w$  running over all places of  $\mathbf{Q}$ , is at most two. The last assertion follows from this fact and Lemma 1.

**Corollary 2.** *Assume that  $q$  is an even power of  $p$ . Let  $\chi$  be an irreducible character of  $G^F$  such that  $\langle \chi, \lambda^G \rangle_{G^F} = 1$  for some  $\lambda \in \Lambda$ . Then, for any prime number  $r \neq p$ , we have  $m_{Q_r}(\chi) = 1$ .*

This follows at once from Proposition 1, (ii).

**Corollary 3.** *Assume that  $G$  is an adjoint semisimple group or any one of the groups described in Lemma 4. Let  $\chi$  be an irreducible character of  $G^F$  such that  $\langle \chi, \lambda^{G^F} \rangle_{G^F} = 1$  for some  $\lambda \in \Lambda$ . Then we have  $m_{\mathbf{Q}}(\chi) = 1$ .*

This follows from Proposition 1, (iii).

**Corollary 4.** *Let  $G, F$  be as in Proposition 1. Assume that  $p$  is a good prime for  $G$  ([16], I, 4.1). Let  $\chi$  be an irreducible character of  $G^F$  and let  $u$  be a regular unipotent element in  $G^F$ . Then  $\chi(u)$  is an algebraic integer in  $k$ , and if  $p \nmid \chi(1)$ , we have  $m_{\mathbf{Q}}(\chi) \leq 2$ .*

We first note that, as  $p$  is good for  $G$ ,  $U^F$  is equal to the derived group of  $U^F$ , hence  $\Lambda$  is the set of linear characters of  $U^F$  (Howlett [9], Lehrer [11]), and that, if  $u \in U^F$ , then  $\mu(u) = 0$  for any non-linear irreducible character  $\mu$  of  $U^F$  (Lehrer [11]).

Let  $\mathcal{O}_k$  be the ring of integers in  $k$ . We show that  $\chi(u)$  belongs to  $\mathcal{O}_k$ . We may assume that  $u \in U^F$  as  $u$  is conjugate to an element of  $U^F$ . Let  $t$  be an element of  $T^F$  having the property of Lemma 2, and let  $\Lambda_1, \dots, \Lambda_r$  be the orbits of  $\langle t \rangle$  on  $\Lambda$ . Thus, as  $\chi^t = \chi$ , if we put  $a_\lambda = \langle \chi, \lambda \rangle_{U^F}$  for  $\lambda \in \Lambda$ ,  $a_\lambda$  is constant on each  $\Lambda_i$ . Hence we have

$$\chi(u) = \sum_{\lambda \in \Lambda} a_\lambda \lambda(u) = \sum_{i=1}^r a_i \left( \sum_{\lambda \in \Lambda_i} \lambda(u) \right),$$

where  $a_i = a_\lambda$  on  $\Lambda_i$ . Each  $\sum_{\lambda \in \Lambda_i} \lambda(u)$  is stable under the action of  $\langle t \rangle$ , hence under the action of  $\langle \sigma^2 \rangle$ . Thus  $\chi(u) \in \mathcal{O}_k$ .

To prove the second assertion, we embed  $G$  in  $G_1$  as in the proof of Lemma 1. Assume that  $p \nmid \chi(1)$  and take an irreducible character  $\chi_1$  of  $G_1^F$  such that  $\langle \chi, \chi_1 | G^F \rangle_{G^F} \neq 0$ . Then, by the Clifford theory, we have  $\chi_1 | G^F = e(\chi^{(1)} + \chi^{(2)} + \dots + \chi^{(s)})$ , where  $e$  is a positive integer dividing  $(G_1^F : G^F)$  and  $\chi^{(1)}, \chi^{(2)}, \dots, \chi^{(s)}$  are the  $G_1^F$ -conjugates of  $\chi = \chi^{(1)}(s | (G_1^F : G^F))$ . Let  $r$  be any prime number and  $v$  a place of  $k$  lying above  $r$ . Put  $m_v = m_{k_v}(\chi^{(1)}) = \dots = m_{k_v}(\chi^{(s)})$ . For  $1 \leq i \leq s$  and for  $\lambda \in \Lambda$ , put  $a_v^{(i)} = \langle \chi^{(i)}, \lambda \rangle_{U^F}$ . Then, by Proposition 1, (i),  $m_v$  divides the  $a_v^{(i)}$ ,  $1 \leq i \leq s$ ,  $\lambda \in \Lambda$ . As  $p \nmid (G_1^F : G^F)$ ,  $p \nmid \chi_1(1)$ , so that, by a theorem of Green-Lehrer-Lusztig (see [3], 8.3.6), we have  $\chi_1(u) = \pm 1$ . Therefore we have the expression



$$\pm 1/m_v = \chi_1(u)/m_v = \{e \cdot \sum_{i=1}^s \chi^{(i)}(u)\}/m_v = e \cdot \sum_{i=1}^s \sum_{\lambda \in \Lambda} (a_\lambda^{(i)}/m_v) \cdot \lambda(u),$$

where the right-hand side is an algebraic integer and the left-hand side is a rational number. Hence  $m_v=1$ , and  $m_{\mathbf{Q}_r}(\chi) \leq 2$ . As  $r$  is an arbitrary prime number, we hence have  $m_{\mathbf{Q}}(\chi) \leq 2$ . This completes the proof of Corollary 4.

**Corollary 5.** *Assume that  $q$  is an even power of  $p$  and that  $p$  is good for  $G$ . Let  $u$  be a regular unipotent element in  $G^F$ . Then, for any irreducible character  $\chi$  of  $G^F$ ,  $\chi(u)$  is a rational integer, and if  $p \nmid \chi(u)$ , we have  $m_{\mathbf{Q}_r}(\chi)=1$  for any prime number  $r \neq p$ .*

The proof is similar to the proof of Corollary 4 (we use Proposition 1, (ii)).

**Corollary 6.** *Let  $G$  be an adjoint semisimple group or any one of the groups described in Lemma 4. Assume that  $p$  is good for  $G$ . Let  $u$  be a regular unipotent element in  $G^F$  and let  $\chi$  be an irreducible character of  $G^F$ . Then  $\chi(u)$  is a rational integer and if  $p \nmid \chi(u)$ , we have  $m_{\mathbf{Q}}(\chi)=1$ .*

REMARK. Lehrer [12] has calculated the values of the cuspidal irreducible characters of  $G^F$  at the regular unipotent elements of  $G^F$  when  $G$  is a semisimple group. As to the upper bound of the indices of the characters of related finite groups, we refer to Gow [8] for classical finite groups and Benard [1] and Feit [6] for the sporadic simple groups.

Let  $G$  be a connected, reductive algebraic group over an algebraically closed field  $K$  of characteristic  $p > 0$  and  $F$  a surjective endomorphism of  $G$  such that  $G^F$  is finite. Then Lemma 2 still holds for such  $G^F$ , so that the statements in Proposition 1, (i) and in Corollary 1 (except for the comment for Lemma 1) hold for  $G^F$ . Assume that  $K$  is an algebraic closure of  $\mathbf{F}_p$  and that some power of  $F$  is the Frobenius endomorphism relative to a rational structure on  $G$  over a finite subfield of  $K$ . Then Lemma 1 holds for  $G^F$  (cf. Carter [3], 8.1.3 and 8.4.5), so that all the statements in Corollary 1, hence the theorem in Introduction holds for  $G^F$ . If  $p$  is good for  $G$ , then the theorem of Green-Lehrer-Lusztig holds for  $G^F$  (if  $Z$  is connected: see [3], 8.3.6), so that Corollary 4 holds for  $G^F$ .

**3. Example.** We calculate all the local indices of the cuspidal irreducible Deligne-Lusztig characters  $\pm R_{T'}^{\theta}$  of  $SL_n(\mathbf{F}_q)$  when  $q$  is an even power of  $p$  ( $\neq 2$ ).

Let  $G$  be  $SL_n$  and  $F$  the endomorphism  $(g_{ij}) \rightarrow (g_{ij}^q)$  ( $q$  may be any power of any prime  $p$ ). Let  $T'$  be a minisotropic maximal torus of  $G$  and let  $W = N_G(T')^F/T'^F$  ( $T'$  is unique up to  $G^F$ -conjugate). Then, taking an element  $\gamma$  of order  $(q^n - 1)/(q - 1)$  in  $\mathbf{F}_q^{\times n}$ , we have  $T'^F = \langle t_0 \rangle$ , where  $t_0$  is  $G$ -conjugate to

$\text{diag}(\gamma, \gamma^q, \dots, \gamma^{q^{n-1}})$ , and  $W = \langle w_0 \rangle \simeq \mathbf{Z}/n\mathbf{Z}$ , where  $w_0$  is defined by  $t_0^w = w_0 t_0 w_0^{-1} = t_0^q$  ( $w_0 \in N_G(T')^F$  represents  $w_0$ ). (All these statements can be easily checked by using [16], II, 1.3, 1.10 and 1.14.)  $W$  acts on  $\hat{T}'^F = \text{Hom}(T'^F, \mathbf{C}^\times)$  by  $\theta^w(s) = \theta(s^w)$  for  $w \in W$ ,  $\theta \in \hat{T}'^F$  and  $s \in T'^F$ . If  $\theta$  is in general position, i.e., no non-identity element of  $W$  fixes  $\theta$ , then  $(-1)^{n-1} R_{T'}^\theta$  is a cuspidal irreducible character of  $G^F = SL_n(\mathbf{F}_q)$  ([4], 7.4, 8.3).

Let  $\theta \in \hat{T}'^F$ . Then, by [4], 4.2, for  $g \in G^F$ , if  $g = su = us$  ( $s$  semisimple,  $u$  unipotent) is its Jordan decomposition, we have

$$(3) \quad R_T^\theta(g) = \frac{1}{|Z_G(s)^F|} \sum_{\substack{h \in G^F \\ h^{-1}sh \in T'}} Q_{hT'h^{-1}, Z_G(s)}(u) \cdot \theta(h^{-1}sh),$$

where the  $Q_{hT'h^{-1}, Z_G(s)}$  are Green functions of  $Z_G(s)$  (which is connected since  $G$  is simply-connected). It follows that, if  $s$  is not conjugate in  $G^F$  to any element of  $T'^F$ , we have  $R_{T'}^\theta(g) = 0$ , and if  $s \in T'^F$ , we have

$$(4) \quad R_{T'}^\theta(g) = Q_{T', Z_G(s)}(u) \frac{1}{|W(s)|} \sum_{w \in W} \theta^w(s),$$

where  $W(s) = \{w \in W \mid s^w = s\}$  (we note that the minisotropic maximal tori of  $Z_G(s)$  form a single  $Z_G(s)^F$ -conjugacy class (cf. [16], II, 1.3, 1.10 and 1.14) and that any two elements of  $T'$  that are conjugate in  $G^F$  are conjugate under the action of  $W$ ). Thus, as the Green functions take integral values, by putting  $\theta(t_0) = \zeta$ , we get from (4):

$$(5) \quad Q(R_{T'}^\theta) = Q\left(\sum_{w \in W} \theta^w\right) = Q(\zeta + \zeta^q + \dots + \zeta^{q^{n-1}}).$$

**Lemma 5.** *Assume that  $\theta$  is in general position. Let  $q = p^m$ . We further assume that  $n$  is even. Then we have*

$$\text{ord}_2[\mathbf{Q}_p(R_{T'}^\theta): \mathbf{Q}_p] = \text{ord}_2 m.$$

Let  $\phi$  be the automorphism of  $\mathbf{Q}_p(\zeta)$  defined by  $\zeta^\phi = \zeta^q$ . Then  $\phi$  has order  $n$  (by assumption) and we have  $\mathbf{Q}_p(\zeta)^{\langle \phi \rangle} = \mathbf{Q}_p(R_{T'}^\theta)$  (cf. (5)). Let  $f = [\mathbf{Q}_p(\zeta): \mathbf{Q}_p]$  and  $e = |\langle \zeta \rangle|$ . Then  $f$  is equal to the least integer  $h \geq 1$  subject for the condition:  $p^h \equiv 1 \pmod{e}$  (see Serre [14], p. 85). As  $\phi^n = 1$  and  $\phi^i \neq 1$  for  $1 \leq i \leq n-1$ , we find that  $f \mid mn$  but  $f \nmid mi$  for  $1 \leq i \leq n-1$  [in fact, if  $f \mid mi$ , then  $p^f - 1 \mid p^{mi} - 1$ , hence  $e \mid p^{mi} - 1$ , hence  $\phi^i = 1$ ]. This shows that  $\text{ord}_r f = \text{ord}_r m + \text{ord}_r n$  for any prime divisor  $r$  of  $n$ . Thus, in particular, we have  $\text{ord}_2 f = \text{ord}_2 m + \text{ord}_2 n$ . As  $[\mathbf{Q}_p(\zeta): \mathbf{Q}_p(R_{T'}^\theta)] = [\mathbf{Q}_p(\zeta): \mathbf{Q}_p(\zeta)^{\langle \phi \rangle}] = n$ , we hence have  $\text{ord}_2[\mathbf{Q}_p(R_{T'}^\theta): \mathbf{Q}_p] = \text{ord}_2 m$ , as desired.

**REMARK.** Professor K. Iimura showed to the author (by an elementary proof) that  $n = f/(m, f)$  and  $[\mathbf{Q}_p(\zeta)^{\langle \phi \rangle}: \mathbf{Q}_p] = (m, f)$ .

**Proposition 2.** *Let  $\chi$  be any cuspidal irreducible Deligne-Lusztig character  $(-1)^{n-1}R_{T'}^q$  of  $G^F = SL_n(\mathbb{F}_q)$ , where we assume that  $q$  is an even power of  $p \neq 2$ . Then, if  $n$  is odd or  $\text{ord}_2 n \geq 2$ , we have  $m_Q(\chi) = 1$ . Assume that  $\text{ord}_2 n = 1$ . Then we have  $m_{Q_r}(\chi) = 1$  for any prime number  $r$  and  $m_Q(\chi) = m_R(\chi) \leq 2$ . And we have  $m_R(\chi) = 2$  if and only if  $\chi$  is real and  $\chi(-1_n) = -\chi(1_n)$  (i.e.  $\theta(-1_n) = -1$ ).*

REMARK. Let  $\chi$  be as above. Assume that  $n$  is even and let  $n = 2m$ . Fixing a generator  $\theta_0$  of  $\hat{T}'^F$ , put  $\theta = \theta_0^i$ . Then the following can be shown:

(i)  $\chi$  is real if and only if  $\frac{q^m - 1}{q - 1} \mid i$ .

(ii) Assume that  $\text{ord}_2 n = 1$  and let  $i = \frac{q^m - 1}{q - 1} i'$  with  $i' \in \mathbb{Z}$  (hence  $\chi$  is real).

Then  $\theta(-1_n) = 1$  if and only if  $i'$  is even, and the latter condition is equivalent to the condition that  $\theta \mid Z^F = 1$ .

Proof of Proposition 2. Let  $\lambda \in \Lambda_0$ . Then, by Lemma 1, we have  $\langle \chi, \lambda^{G^F} \rangle_{G^F} = 1$ . Thus, if  $n$  is odd or  $\text{ord}_2 n > \text{ord}_2(p-1)$ , by Proposition 1, (iii), we have  $m_Q(\chi) = 1$ . Assume that  $1 \leq \text{ord}_2 n \leq \text{ord}_2(p-1)$ . Let  $t$  be an element of  $T^F$  having the property of Lemma 3. Then, under our assumption, we have  $t^{p-1} = -1_n$  (cf. Proof of Lemma 4 and Proof of Lemma 3.3 (a) of Gow [8]). Let us use the notation of the proof of Proposition 1, (ii). Then  $\lambda^M = \mu_0 + \mu_1$ . As  $\mu_i(-1_n) = (-1)^i \mu_i(1_n)$  for  $i = 0, 1$ , by Schur's lemma, we have  $\langle \chi, \mu_0 \rangle_M = 1$  if  $\chi(-1_n) = \chi(1_n)$ , and  $\langle \chi, \mu_1 \rangle_M = 1$  if  $\chi(-1_n) = -\chi(1_n)$ . As  $\mu_0$  is realizable in  $\mathbf{Q}$ , we have  $m_Q(\chi) = 1$  in the first case. Assume that  $\chi(-1_n) = -\chi(1_n)$ . If  $r$  is any prime number  $\neq p$ , then  $\mu_1$  is realizable in  $\mathbf{Q}_r$ , hence we have  $m_{Q_r}(\chi) = 1$ . As  $q$  is an even power of  $p$ , by Lemma 5, we have  $2 \mid [\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$ . Hence  $A_1 \otimes_{\mathbf{Q}} \mathbf{Q}_p(\chi)$  splits (see [14], Chap. XIII, § 3, Prop. 7), hence  $\mu_1$  is realizable in  $\mathbf{Q}_p(\chi)$ . Hence we have  $m_{Q_p}(\chi) = m_{Q_p(\chi)}(\chi) = 1$ . Thus we have  $m_Q(\chi) = m_R(\chi)$ . If  $\chi$  is real, we must have  $m_R(\chi) = 2$  since otherwise  $\chi$  will be realizable in  $\mathbf{R}$ , so that, by Schur's theorem, we have  $(2 =) m_R(\chi_1) \mid \langle \chi, \mu_1 \rangle_M = 1$ , a contradiction. If  $\text{ord}_2 n \geq 2$ , then  $\chi$  cannot be real since  $G^F$  contains a central element  $z$  of order 4 such that  $z^2 = -1_n$  and  $\chi(z) = \pm \sqrt{-1} \chi(1_n)$  ([7], p. 107). Finally, we note that, by [4], 1.22, we have  $\chi(-1_n) = -\chi(1_n)$  if and only if  $\theta(-1_n) = -1$ . This completes the proof of Proposition 2.

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