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ON HAEFLIGER'S OBSTRUCTIONS TO EMBEDDINGS AND TRANSFER MAPS

Dedicated to the memory of Professor Katsuo Kawakubo

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1. Introduction and statement of results

Throughout this article, n -manifolds mean compact differentiable (or topological) manifolds of dimension n . The (co-)homology is understood to have \mathbf{Z}_2 for coefficients.

For a manifold V , we denote by $w(V)$ and $\bar{w}(V)(= w(V)^{-1})$, the total Stiefel-Whitney class and the total normal Stiefel-Whitney class of V , respectively. Furthermore, we denote by $U_V \in H^{\dim V}(V \times V)$ the \mathbf{Z}_2 -Thom class (or \mathbf{Z}_2 -diagonal cohomology class) of V [10, p. 125]. For a (continuous) map $f: M^n \rightarrow N^{n+k}$ between closed manifolds M and N , we define the total Stiefel-Whitney class $w(f) = \sum_{i \geq 0} w_i(f)$ by the equation

$$w(f) = \bar{w}(M) f^* w(N).$$

For a map $f: M^n \rightarrow N^{n+k}$, the transfer map (or Umkehr homomorphism) $f_!: H^i(M) \rightarrow H^{i+k}(N)$ is defined by the commutative diagram below:

$$\begin{array}{ccc} H^i(M) & \xrightarrow{f_!} & H^{i+k}(N) \\ \cong \downarrow \cap [M] & & \cong \downarrow \cap [N] \\ H_{n-i}(M) & \xrightarrow{f_*} & H_{n-i}(N). \end{array}$$

Here $[V] \in H_{\dim V}(V)$ denotes the fundamental homology class of a manifold V .

Our main theorem is the following

Theorem 1.1. *For a continuous map $f: M^n \rightarrow N^{n+k}$ between closed topological manifolds, $U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0$ if and only if $f^* f_!(a) = a w_k(f)$ for all $a \in H^*(M)$.*

The cohomology elements, appearing in this theorem, are related to the embeddability of f . A. Haefliger [7, Théorème 5.2] proved the following

Theorem (Haefliger). *If a map $f: M^n \rightarrow N^{n+k}$ between topological manifolds is homotopic to a topological embedding, then $w_i(f) = 0$ for $i > k$ and*

$$U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0 \in H^{n+k}(M \times M).$$

Thus we have immediately the following

Corollary 1.2. *If a map $f: M^n \rightarrow N^{n+k}$ between closed topological manifolds is homotopic to a topological embedding, then $f^* f_i(a) = a w_k(f)$ for all $a \in H^*(M)$.*

REMARK 1. It is well-known, e.g., [4, p. 246], that if f is homotopic to a differentiable embedding then $f^* f_i(a) = a w_k(f)$ for all $a \in H^*(M)$.

REMARK 2. As we will see in §3, the assumption ‘homotopic’ in Haefliger’s theorem or Corollary 1.2 can be weakened to ‘ R -bordant’.

R.L.W. Brown [4] established the conditions that a map $f: M^n \rightarrow N^{n+k}$ is cobordant to a differentiable embedding in the sense of Stong [12]. Here a map $f_1: M_1^n \rightarrow N_1^{n+k}$ between differentiable closed manifolds is said to be cobordant to $f_2: M_2^n \rightarrow N_2^{n+k}$ if there exist two cobordisms (W, M_1^n, M_2^n) , $(V, N_1^{n+k}, N_2^{n+k})$ and a map $F: W \rightarrow V$ such that $F|_{M_i} = f_i$ ($i = 1, 2$).

From Theorem 1.1 and Brown’s theorem [4], we infer immediately a result which means the converse of Haefliger’s theorem up to cobordism of maps in the sense of Stong [12].

Corollary 1.3. *Let $k > 0$. Then a map $f: M^n \rightarrow N^{n+k}$ between differentiable manifolds is cobordant to a differentiable embedding if $w_i(f) = 0$ ($i > k$) and $U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0$.*

For an n -manifold M , we use the same symbol M as the generator of $H^n(M) \cong \mathbb{Z}_2$, i.e., $H^n(M) = \mathbb{Z}_2 \langle M \rangle$, and denote the $H^p(M) \times H^q(M)$ -component of $u \in H^{p+q}(M \times M)$ by $[u]_{p,q}$. To prove Theorem 1.1 we use the following

Proposition 1.4. *For a map $f: M^n \rightarrow N^{n+k}$ and two elements $x, y \in H^*(M)$ with $\dim x + \dim y = r \leq n - k$,*

$$[(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(x \times y)]_{n,k+r} = M \times (x w_k(f) + f^* f_i(x)) y.$$

Using this proposition, we can reformulate Brown’s theorem [4] in case $k > n/2$.

Theorem 1.5. *Let $k > n/2$. Then a differentiable map $f: M^n \rightarrow N^{n+k}$ is cobordant to a differentiable embedding if and only if the following two conditions hold:*

- (1) $\langle w_I(M)w_J(f)w_i(f), [M] \rangle = 0$ for any integer i ($i > k$) and sequences I, J of non-negative integers such that $|I| + |J| + i = n$.
- (2) $(U_M(1 \times w_k(f)) + (f \times f)^*U_N)(w_I(M) \times f^*(w_J(N))w_K(M)) = 0$ for any sequences I, J, K of non-negative integers such that $|I| + |J| + |K| = n - k$.

Here, $w_I(M) = w_{i_1}(M) \cdots w_{i_r}(M)$ and $|I| = \sum_{1 \leq j \leq r} i_j$ for a finite sequence $I = (i_1, \dots, i_r)$ of non-negative integers.

The rest of this article is organized as follows: In §2, we will prove Theorem 1.1, Proposition 1.4 and Theorem 1.5. §3 will be devoted to the study of the relation between R -bordism and Haefliger's obstruction. In §4, we will give some examples of maps $f: M^n \rightarrow N^{n+k}$, e.g., a map which is cobordant to a differentiable embedding but not R -bordant to a topological embedding.

2. Proofs

To prove Theorem 1.1 and Proposition 1.4, we use the following two lemmas, the first of which is a slight generalization of [8, Lemma 2].

Lemma 2.1. *For a map $f: M^n \rightarrow N^{n+k}$ and an element $x \in H^r(M)$, we have*

$$[(f \times f)^*U_N(x \times 1)]_{n,k+r} = M \times f^*f_!(x).$$

Proof. We can choose bases $\{u_i \mid i \in I\}$ and $\{v_i \mid i \in I\}$ for $H^*(N)$ such that $\langle u_i v_j, [N] \rangle = \delta_{ij}$. Then the Thom class U_N of N can be described as $U_N = \sum_{i \in I} u_i \times v_i$ by, e.g., [10, Theorem 11.11]. The element $f_!(x)$ can be described as $f_!(x) = \sum_{i \in I} \alpha_i v_i$ ($\alpha_i \in \mathbb{Z}_2$). Let $I_0 = \{i \in I \mid f^*(u_i)x = M\}$. Then

$$\begin{aligned} \alpha_i &= \langle \alpha_i u_i v_i, [N] \rangle = \left\langle u_i \sum_{i \in I} \alpha_i v_i, [N] \right\rangle = \langle u_i f_!(x), [N] \rangle \\ &= \langle u_i, f_!(x) \cap [N] \rangle = \langle u_i, f_*(x \cap [M]) \rangle = \langle f^*(u_i)x, [M] \rangle \\ &= \begin{cases} 1 & i \in I_0 \\ 0 & i \notin I_0. \end{cases} \end{aligned}$$

Thus, $f_!(x) = \sum_{i \in I_0} v_i$ and so $f^*f_!(x) = \sum_{i \in I_0} f^*(v_i)$. Hence, we have

$$\begin{aligned} [(f \times f)^*U_N \cdot (x \times 1)]_{n,k+r} &= \left[\left(\sum_{i \in I} f^*(u_i) \times f^*(v_i) \right) (x \times 1) \right]_{n,k+r} \\ &= \left[\sum_{i \in I} f^*(u_i)x \times f^*(v_i) \right]_{n,k+r} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in I_0} M \times f^*(v_i) = M \times \sum_{i \in I_0} f^*(v_i) \\
&= M \times f^* f_!(x).
\end{aligned}$$

This completes the proof. \square

Lemma 2.2. *For an n -manifold M^n , and an element $x \in H^r(M)$, we have*

$$[U_M(x \times 1)]_{n,r} = M \times x.$$

Proof. The Thom class U_M can be described as $U_M = M \times 1 + \sum_j a_j \times b_j$, ($\dim a_j < n$) and there is a relation $U_M(x \times 1) = U_M(1 \times x)$ (e.g., [10, Lemma 11.8]). Thus the lemma follows immediately. \square

Proof of Proposition 1.4. Let $x, y \in H^*(M)$ with $\dim x + \dim y = r$. Then, we have

$$\begin{aligned}
&[(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(x \times y)]_{n,k+r} \\
&= [U_M(1 \times w_k(f))(x \times 1) + (f \times f)^* U_N(x \times 1)]_{n,k+\dim x}(1 \times y) \\
&= M \times x w_k(f) y + M \times f^* f_!(x) y \quad \text{by Lemmas 2.1–2.2} \\
&= M \times (x w_k(f) + f^* f_!(x)) y.
\end{aligned}$$

Thus, the proposition follows. \square

Proof of Theorem 1.1. First we assume that $U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0$. Take any $a \in H^r(M)$. Then

$$\begin{aligned}
0 &= [(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(a \times 1)]_{n,k+r} \\
&= M \times (a w_k(f) + f^* f_!(a)) \quad \text{by Proposition 1.4}
\end{aligned}$$

Thus we get $f^* f_!(a) = a w_k(f)$ for all $a \in H^*(M)$.

Conversely, suppose that $f^* f_!(a) = a w_k(f)$ for all $a \in H^*(M)$. Since $U_M(1 \times w_k(f)) + (f \times f)^* U_N \in H^{n+k}(M \times M)$, it is sufficient for our purpose to show that $(U_M(1 \times w_k(f)) + (f \times f)^* U_N)u = 0$ for all $u \in H^{n-k}(M \times M)$. By the Künneth formula, we may assume that $u = a \times b$ with $\dim a + \dim b = n - k$. Then by Proposition 1.4, we have

$$\begin{aligned}
&(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(a \times b) \\
&= [(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(a \times b)]_{n,n} \\
&= M \times (a w_k(f) + f^* f_!(a)) b = 0.
\end{aligned}$$

Hence we get $U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0$. \square

Proof of Theorem 1.5. The condition (1) of Theorem 1.5 is just a restatement of the condition (i) of Brown's theorem. On the other hand, by the assumption that $k > n/2$, we have only to consider the case $r = 2$ in the condition (ii) of Brown's theorem, which is reduced to

$$\langle f^*(w_J(N))f^*f_I(w_I(M))w_K(M), [M] \rangle = \langle f^*(w_J(N))w_I(M)w_K(M)w_k(f), [M] \rangle.$$

Applying Proposition 1.4 for $x = w_I(M)$ and $y = f^*(w_J(N))w_K(M)$, we see that this equality is equivalent to the condition (2) of Theorem 1.5. \square

3. Relations between R -bordisms and Haefliger's obstructions

The concept of R -bordism of maps is introduced in [3, §3]. Let $f_i: M_i^n \rightarrow N^{n+k}$ ($i = 1, 2$) be maps between topological manifolds, where M_i 's are closed (while N is not necessarily closed). The two maps are said to be R -bordant if there exist a topological cobordism (W, M_1, M_2) and a continuous map $F: W \rightarrow N$ such that (1) $F|_{M_i} = f_i$ ($i = 1, 2$) and (2) there exist retractions $r_i: W \rightarrow M_i$ ($i = 1, 2$).

Let $j_i: M_i \rightarrow W$ be the natural inclusion ($i = 1, 2$). Then by [6, Theorem 1.2],

$$(r_2 j_1)_*: H_*(M_1) \rightarrow H_*(M_2)$$

is an isomorphism, and by [3, §3]

$$f_{1*} = f_{2*}(r_2 j_1)_*: H_*(M_1) \rightarrow H_*(N).$$

In this section, we will prove

Theorem 3.1. *Let $f: M^n \rightarrow N^{n+k}$ be a map between closed topological manifolds. If f is R -bordant to a topological embedding, then $w_i(f) = 0$ ($i > k$) and*

$$U_M(1 \times w_k(f)) + (f \times f)^* U_N = 0.$$

This theorem, together with Corollary 1.3, leads to the following

Corollary 3.2. *Let $f: M^n \rightarrow N^{n+k}$ be a map between closed differentiable manifolds. If f is R -bordant to a topological embedding, then f is cobordant to a differentiable embedding.*

REMARK 3. If we consider *cobordism* and *embeddings* in topological category, the conclusion of this corollary is rather trivial.

Theorem 3.1 follows from Proposition 3.3 (or Corollary 3.4) below and Haefliger's theorem.

Proposition 3.3. *Let $f_i: M_i^n \rightarrow N^{n+k}$ ($i = 1, 2$) and $g: M_1^n \rightarrow M_2^n$ be maps such that $g_*: H_*(M_1) \rightarrow H_*(M_2)$ is an isomorphism and $f_{1*} = f_{2*}g_*: H_*(M_1) \rightarrow H_*(N)$. Then $w(f_1) = g^*w(f_2)$ and*

$$\begin{aligned} U_{M_1}(1 \times w_k(f_1)) + (f_1 \times f_1)^*U_N \\ = (g \times g)^*(U_{M_2}(1 \times w_k(f_2)) + (f_2 \times f_2)^*U_N). \end{aligned}$$

Proof. Let $\{u_i \mid i \in I\}$ and $\{v_i \mid i \in I\}$ be two bases for $H^*(M_2)$ such that $\langle u_i v_j, [M_2] \rangle = \delta_{ij}$. Then the Thom class U_{M_2} of M_2 can be described as $U_{M_2} = \sum_{i \in I} u_i \times v_i$ (see [10, Theorem 11.11]). Since $g_*[M_1] = [M_2]$ and g^* is an isomorphism, because so is g_* , we have the two bases $\{g^*u_i \mid i \in I\}$ and $\{g^*v_i \mid i \in I\}$ for $H^*(M_1)$ with $\langle (g^*u_i)(g^*v_j), [M_1] \rangle = \delta_{ij}$. Hence,

$$U_{M_1} = \sum_{i \in I} g^*u_i \times g^*v_i = (g \times g)^* \sum_{i \in I} u_i \times v_i = (g \times g)^*U_{M_2}.$$

Since $f_{1*} = f_{2*}g_*$, we have $f_1^* = g^*f_2^*$ and $w(f_1) = g^*w(f_2)$ by [3, Theorem 4.2]. Hence we have

$$\begin{aligned} U_{M_1}(1 \times w_k(f_1)) + (f_1 \times f_1)^*U_N \\ = (g \times g)^*U_{M_2}(1 \times g^*w_k(f_2)) + (g \times g)^*(f_2 \times f_2)^*U_N \\ = (g \times g)^*(U_{M_2}(1 \times w_k(f_2)) + (f_2 \times f_2)^*U_N). \end{aligned}$$

This completes the proof. \square

Corollary 3.4. *Let $f_i: M_i^n \rightarrow N^{n+k}$ ($i = 1, 2$) be maps between closed topological manifolds. If f_1 is R -bordant to f_2 , then, $w_i(f_1)$ ($i \geq 0$) and $U_{M_1}(1 \times w_k(f_1)) + (f_1 \times f_1)^*U_N$ correspond to $w_i(f_2)$ ($i \geq 0$) and $U_{M_2}(1 \times w_k(f_2)) + (f_2 \times f_2)^*U_N$, respectively, by the canonical isomorphisms.*

REMARK 4. By virtue of Proposition 1.4 and the fact that for $f: M^n \rightarrow N^{n+k}$, $w_k(f) + f^*f_1(1)$ is the Poincaré dual to the element $\theta(f) \in H_{n-k}(M)$ in [3], the results in Theorem 3.1, Proposition 3.3 and Corollary 3.4 are, respectively, somewhat stronger than those in [3, Corollary 4.4, Theorem 4.2 and Corollary 4.3] in case N is a closed manifold.

4. Relations among obstructions to embeddings

For a map $f: M^n \rightarrow N^{n+k}$, we describe conditions (0)–(3) below:

- (0) $w_i(f) = 0$ for $i > k$.
 - (1) $f^*f_1(a) + aw_k(f) = 0$ for all $a \in H^*(M)$.
- (or equivalently, $U_M(1 \times w_k(f)) + (f \times f)^*U_N = 0$ by Theorem 1.1.)

(2) $f^* f_!(w_I(M)) + w_I(M)w_k(f) = 0$ for all sequences I of non-negative integers, where $w_I(M) = w_{i_1}(M) \cdots w_{i_r}(M)$ if $I = (i_1, \dots, i_r)$.

(3) $f^* f_!(1) + w_k(f) = 0$.

So far, for a map $f: M^n \rightarrow N^{n+k}$ between closed differentiable manifolds, we know

$$\begin{array}{lcl}
 f \text{ is homotopic to a topological embedding} & & \\
 \Downarrow & & \\
 f \text{ is } R\text{-bordant to a topological embedding} & \Rightarrow & (0) + (1) \\
 & & \Downarrow \\
 f \text{ is cobordant to a differentiable embedding} & \Leftarrow & (0) + (2) \\
 & & \Downarrow \\
 f \text{ is cobordant to a differentiable embedding} & \nRightarrow & (0) + (3)
 \end{array}$$

REMARK 5. If $k \geq n - 4$, $2k > n$ and if f satisfies the conditions (0) and (3), then f is cobordant to a differentiable embedding ([1, Theorems (3.6) and (3.9)] and [9, Corollary 1.3]).

REMARK 6. Even if f is cobordant to an embedding, the conditions (0) and (3) do not necessarily hold ([8, Remark 2]).

In this section, we will show that

- (a) even if f is R -bordant to an embedding, f is not necessarily homotopic to an embedding (see Example 1 below),
- (b) the conditions (0) and (2) do not imply the conditions (1) (see Example 2),
- (c) the condition (3) does not lead to the condition (2) (see Example 3), and
- (d) the conditions (0) and (3) induce the relation (see Proposition 4.1)

$$f^* f_!(v_i(M)) = v_i(M)w_k(f),$$

where $v_i(M)$ stands for the i -th Wu class of M defined by $Sq(\sum_{0 \leq i} v_i(M)) = w(M)$.

EXAMPLE 1. Let $S^1 = \{z \in \mathbb{C}^1 \mid |z| = 1\}$ be the circle, and let $f: S^1 \rightarrow S^1 \times S^1$ be a map defined by $f(z) = (f_1(z), f_2(z)) = (z^2, 1)$. Then f is not homotopic to an embedding. But f is R -bordant to an embedding.

REMARK 7. This example is a modification of an example appearing in earlier versions of [3], but omitted in the final one.

Proof. Suppose that f is homotopic to a topological embedding $g = (g_1, g_2): S^1 \rightarrow S^1 \times S^1$. Then g_2 is homotopic to the constant map f_2 . Hence, g_2 has a lifting $g'_2: S^1 \rightarrow \mathbb{R}^1$. If we put $g' = (g_1, g'_2): S^1 \rightarrow S^1 \times \mathbb{R}^1$, then g' is also an embedding.

Identifying $S^1 \times \mathbf{R}^1$ with $\mathbf{C}^1 - \{0\}$, we have a topological embedding $g': S^1 \rightarrow \mathbf{C}^1 - \{0\}$. From now on, the authors owe C. Biasi, J. Daccach and O. Saeki for the proof. Note that $g'_*: H_1(S^1, \mathbf{Z})(\cong \mathbf{Z}) \rightarrow H_1(\mathbf{C}^1 - \{0\}, \mathbf{Z})(\cong \mathbf{Z})$ maps $a \in \mathbf{Z}$ to $2a$. By the Schoenflies theorem, $g'(S^1)$ bounds a region U in \mathbf{C}^1 homeomorphic to the closed 2-dimensional disk. If $0 \notin U$, then g' is null-homotopic in $\mathbf{C}^1 - \{0\}$, which is a contradiction. If $0 \in U$, then g' represents a generator of $H_1(\mathbf{C}^1 - \{0\})$, which is also a contradiction. Thus f is not homotopic to an embedding. On the other hand, f is R -bordant to an embedding by [3, Example 4.8]. \square

EXAMPLE 2. We denote by P^m the real projective m -space. Furthermore, $\pi: P^3 \rightarrow P^3/P^2 = S^3$ and $j: P^l \subset P^{l+k}$ stand for the natural projection and inclusion, respectively. Let $M^n = P^3 \times P^l$, $N = S^3 \times P^{l+k}$ and let $f = \pi \times j: M^n \rightarrow N^{n+k}$. Then f satisfies (0) and (2), but f does not satisfy (1).

Proof. Put

$$H^1(P^3) = \mathbf{Z}_2\langle x_1 \rangle, \quad H^1(P^l) = \mathbf{Z}_2\langle x_2 \rangle, \quad H^3(S^3) = \mathbf{Z}_2\langle s \rangle, \quad H^1(P^{l+k}) = \mathbf{Z}_2\langle y \rangle.$$

Then

$$f^*(s) = x_1^3, \quad f^*(y) = x_2, \quad w(f) = (1 + x_2)^{-l-1}(1 + x_2)^{l+k+1} = (1 + x_2)^k.$$

Therefore

$$w_i(f) = 0 \quad \text{for } i > k, \quad w_k(f) = x_2^k.$$

The Thom classes of M and N are given by

$$\begin{aligned} U_M &= \sum_{0 \leq i \leq l} x_1^3 x_2^i \times x_2^{l-i} + \sum_{0 \leq i \leq l} x_1^2 x_2^i \times x_1 x_2^{l-i} \\ &\quad + \sum_{0 \leq i \leq l} x_1 x_2^i \times x_1^2 x_2^{l-i} + \sum_{0 \leq i \leq l} x_2^i \times x_1^3 x_2^{l-i}, \\ U_N &= \sum_{0 \leq i \leq l+k} s y^i \times y^{l+k-i} + \sum_{0 \leq i \leq l+k} y^i \times s y^{l+k-i}. \end{aligned}$$

Hence, and because $f^*(y^{l+1}) = x_2^{l+1} = 0$, we have

$$[U_M(1 \times w_k(f)) + (f \times f)^* U_N]_{n,k} = M \times (w_k(f) + f^* f_!(1)) = 0,$$

$$[U_M(1 \times w_k(f)) + (f \times f)^* U_N]_{n-1,k+1} = x_1^2 x_2^l \times x_1 x_2^k,$$

$$M \times f^* f_!(x_2^i) = [(f \times f)^* U_N](x_2^i \times 1)_{n,k+i} = M \times x_2^{k+i}.$$

Thus f does not satisfy the condition (1). But f satisfies (2), because $w_i(M) = \binom{l+1}{i} x_2^i$ and $f^* f_!(x_2^r) = x_2^{r+k} = x_2^r w_k(f)$. \square

REMARK 8. The above example shows that a map f satisfying the conditions (0) and (2) is not necessarily R -bordant to an embedding, in particular that a map which is cobordant to a differentiable embedding is not necessarily R -bordant to a topological embedding.

EXAMPLE 3. Let $\pi: P^2 \rightarrow P^2/P^1 = S^2$ and $j: P^l \subset P^{l+k}$ be the natural projection and inclusion, respectively and let $f = \pi \times j: M = P^2 \times P^l \rightarrow S^2 \times P^{l+k}$. Then, if k is even, the relation $f^* f_!(1) = w_k(f)$ holds, however (2) does not hold.

Proof. As in Example 2, put

$$H^1(P^2) = \mathbf{Z}_2 \langle x_1 \rangle, \quad H^1(P^l) = \mathbf{Z}_2 \langle x_2 \rangle, \quad H^2(S^2) = \mathbf{Z}_2 \langle s \rangle, \quad H^1(P^{l+k}) = \mathbf{Z}_2 \langle y \rangle.$$

Then

$$w_1(M) = x_1 + (l+1)x_2, \quad f^*(s) = x_1^2, \quad f^*(y) = x_2, \quad w_k(f) = x_2^k.$$

Just as in Example 2, we have

$$M \times (w_k(f) + f^* f_!(1)) = [U_M(1 \times w_k(f)) + (f \times f)^* U_N]_{n,k} = 0,$$

$$\begin{aligned} M \times (w_1(M)w_k(f) + f^* f_!(w_1(M))) \\ = [(U_M(1 \times w_k(f)) + (f \times f)^* U_N)(w_1(M) \times 1)]_{n,k+1} \\ = M \times x_1 x_2^k. \end{aligned}$$

Thus the relation $f^* f_!(1) = w_k(f)$ holds, however $f^* f_!(w_1(M)) \neq w_1(M)w_k(f)$. \square

Proposition 4.1. Assume that $f: M^n \rightarrow N^{n+k}$ satisfies the conditions that $w_i(f) = 0$ ($k < i$) and $f^* f_!(1) = w_k(f)$, then

$$f^* f_!(v_i(M)) = v_i(M)w_k(f) \quad (0 < i).$$

Proof. For each $x \in H^{n-k-i}(M)$, we have

$$\begin{aligned} x f^* f_!(v_i(M)) &= v_i(M) f^* f_!(x) \quad \text{by, e.g., [9, Lemma 2.1, (4)]} \\ &= Sq^i f^* f_!(x) \quad \text{because } \dim f^* f_!(x) = n - i \\ &= [Sq f^* f_!(x)]_n \end{aligned}$$

$$\begin{aligned}
&= [f^* f_!(Sq(x)w(f))]_n \quad \text{by, e.g., [9, Lemma 2.1, (2)]} \\
&= f^* f_! \left(\sum_{0 \leq j} Sq^j(x)w_{i-j}(f) \right) \\
&= \sum_{0 \leq j} Sq^j(x)w_{i-j}(f)f^* f_!(1) \quad \text{by, e.g., [9, Lemma 2.1, (4)]} \\
&= \sum_{0 \leq j} Sq^j(x)w_{i-j}(f)w_k(f) \quad \text{because } f^* f_!(1) = w_k(f) \\
&= \sum_{0 \leq j} Sq^j(x)Sq^{i-j}w_k(f) \quad \text{because } w_i(f) = 0 \ (k < i) \\
&= Sq^i(xw_k(f)) = v_i(M)xw_k(f).
\end{aligned}$$

Here, $[y]_j$ for $y \in \sum_{0 \leq i} H^i(M)$ means the j -dimensional component of y . Thus $xf^* f_!(v_i(M)) = xv_i(M)w_k(f)$ for all $x \in H^{n-k-i}(M)$. Hence $f^* f_!(v_i(M)) = v_i(M)w_k(f)$ by the Poincaré duality. \square

For $k = 1$, the conditions (0) and (3) imply the condition (2), i.e. we have

Proposition 4.2. *Assume that $f: M^n \rightarrow N^{n+1}$ satisfies the conditions that $w_i(f) = 0$ ($1 < i$) and $f^* f_!(1) = w_1(f)$, then for all sequences I of non-negative integers, we have*

$$f^* f_!(w_I(M)) = w_I(M)w_1(f).$$

Proof. By the assumption we have $\bar{w}(M)f^*w(N) = w(f) = 1 + w_1(f) = 1 + f^* f_!(1)$. Hence $w(M) = f^*w(N)(1 + f^* f_!(1))^{-1} = f^*(w(N)(1 + f_!(1))^{-1}) \in f^*H^*(N)$. Thus $w_I(M) \in f^*H^*(N)$ for all I , and therefore we obtain the result since $f_!(f^*y) = yf_!(1)$ for all $y \in H^*(N)$. \square

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