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## *On the Behaviour of Analytic Functions on Abstract Riemann Surfaces*

By Zenjiro KURAMOCHI

In this article we shall study mainly the structure of the covering surface, over the  $w$ -plane, of a function which is meromorphic on an abstract Riemann surface  $F$ . As a theorem of representation, we shall prove

**Theorem 1.** *Let  $F'$  be the remaining surface complementary to a compact subset  $F_0$  of  $F$ . Then, if  $F \notin O_G$ <sup>1)</sup> and  $\in O_{HB}$  ( $O_{HD} = O_{HBD}$ ), we have  $F' \in O_{AB}(O_{AD})$ .*

Proof of the former part. Since  $F \notin O_G$ , there exists a positive bounded harmonic function  $\omega(p)$  on  $F'$  such that  $\omega(p) = 0$  on  $\partial F_0$ <sup>2)</sup>. Let  $F'^\infty$  be the universal covering surface of  $F'$ . We map  $F'^\infty$  conformally onto  $|z| < 1$  by  $z = \varphi(p)$ . Assume  $F' \notin O_{AB}$ , then there exists a bounded analytic function  $A(p)$  on  $F'$ . Consider  $\omega(z) = \omega(\varphi^{-1}(z))$  on  $|z| < 1$ . Then there exists a set  $E$  of positive linear measure on  $|z| = 1$  such that  $\omega(z)$  has angular limits larger than  $\delta$  ( $\delta > 0$ ) on  $E$ . Let  $\{\mathfrak{S}_n\}$  be a sequence of triangulation of the  $w$ -plane such that  $\mathfrak{S}_{n+1}$  is a subdivision of  $\mathfrak{S}_n$  and becomes as fine as we please when  $n \rightarrow \infty$ . Denote by  $\{\Delta_n^i\}$ <sup>3)</sup> ( $i = 1, 2, \dots$ ) the triangles of  $\mathfrak{S}_n$ . On account of Fatou's theorem  $A(p(z))$  has angular limits almost everywhere on  $E$ . The subset of  $E$ , where  $A(z)$  has angular limits contained in  $\Delta_n^i$  will be denoted by  $E_n^i$ . Then every  $E_n^i$  is linearly measurable. There exists at least two  $E_n^i, E_n^{i'}$  such that  $E_n^i \cap E_n^{i'} = 0$  and both  $\text{mes } E_n^i$  and  $\text{mes } E_n^{i'}$  are positive. On the contrary, suppose for every  $n$  there exists  $i(n)$  such that  $\text{mes } E_n^{i(n)} = \text{mes } E$ .  $A(z)$  must be a constant contained in  $\bigcap_{n=1}^{\infty} \Delta_n^{i(n)}$ . Let  $U(z)$  be a harmonic function in  $|z| < 1$  such that  $U(z) = 1$

1)  $O_G, O_{HP}, O_{HB}, O_{HD}, O_{AB}$  and  $O_{AD}$  are the classes of Riemann surfaces on which the Green's function, non-constant positive, bounded, Dirichlet-bounded, bounded analytic and Dirichlet-bounded analytic function does not exist respectively.

2) We denote by  $\partial S$  the relative boundary of  $S$  with respecto to  $F$ .

3)  $\Delta_n^i$  are made half open so that they are mutually disjoint for fixed  $n$ .

4) M. Tsuji: Theory of meromorphic function in the neighbourhood of a closed set of capacity zero. Jap. Journ. 1944.

on  $E'_n$ ,  $U(z) = 0$  on the image of  $\partial F_0$  on  $|z| = 1$  and  $U(z) = -1$  on  $E''_n$ .  $U(z)$  can be considered on  $F'$ , since it is automorphic. Let  $\{F_n\}$ <sup>5)</sup> be an exhaustion of  $F$  and put  $F^+ = \varepsilon \left\{ p : U(p) \geq \frac{1}{2} \right\}$ ,  $F^- = \varepsilon \left\{ p : U(p) \leq -\frac{1}{2} \right\}$ . Then neither  $\partial F^+$  nor  $\partial F^-$  intersects  $\partial F_0$ . Let  $\{V_n(p)\}$  be a sequence of harmonic functions in  $F_n$  such that  $V_n(p) = 1$  on  $\partial F_n \cap F^+$  and  $V_n(p) = -1$  on  $\partial F_n - (\partial F_n \cap F^+)$ . Then we can<sup>6)</sup> easily define a non-constant bounded harmonic function  $V(p)$  from a subsequence  $\{V_{n'}(p)\}$  which converges uniformly in  $F$ . Hence  $F \notin O_{HB}$ .

Proof of the latter part. Assume, there exists a Dirichlet-bounded analytic function  $A(p)$  on  $F'$ .

Case 1. The domain which is covered by  $w = A(p)$  is dense in the  $w$ -plane. Since  $\partial F_0$  is compact,  $w = A(p)$  ( $p \in \partial F_0$ ) is bounded. Let  $M$  and  $N$  be the maximum and minimum of  $\operatorname{Re} A(p)$  ( $p \in \partial F_0$ ). Then there exist at least two components of  $F'$  on which  $\operatorname{Re} A(p) \geq M'$  and  $\operatorname{Re} A(p) \leq N'$  respectively, where  $M' > M$ ,  $N' > N$ . We denote them by  $F^+$  and  $F^-$ . Then neither  $\partial F^+$  nor  $\partial F^-$  intersects  $\partial F_0$ . Consider  $\operatorname{Re} A(p) - M'$  and  $\operatorname{Re} A(p) - N'$  on  $F^+$  and  $F^-$ . Then  $\operatorname{Re} A(p) - M' = 0$  on  $\partial F^+$ ,  $\operatorname{Re} A(p) - N' = 0$  on  $\partial F^-$  and  $D_{F^+}(\operatorname{Re} A(p)) < \infty$ ,  $D_{F^-}(\operatorname{Re} A(p)) < \infty$ . Let  $\{F_n\}$  be an exhaustion of  $F$  and  $\{V_n(p)\}$  be a sequence of harmonic function in  $F_n$  such that  $V_n(p) = \operatorname{Re} A(p) - M'$  on  $\partial F_n \cap F^+$  and  $= 0$  on  $\partial F_n - (\partial F_n \cap F^+)$ . Then we can define<sup>7)</sup> a non-constant Dirichlet-bounded function on  $F$  from uniformly convergent subsequence  $\{V_{n'}(p)\}$ .

Case 2. If  $A(p)$  does not cover a domain, take a point  $w_0$  in it. Then we see easily  $D_{F'}\left(\frac{1}{A(p) - w_0}\right) < \infty$  and  $\left|\frac{1}{A(p) - w_0}\right| < \infty$ . We can suppose without loss of generality that  $D_{F'}(A(p)) < M_1 < +\infty$  and  $|A(p)| < M_2 < +\infty$ . We map the universal covering surface  $F'^\infty$  of  $F'$  conformally onto  $|z| < 1$  by  $z = \varphi(p)$ . Denote by  $E_0$  the image of  $\partial F_0$  on  $|z| = 1$  and by  $E_I$  the complementary set of  $E_0$  on  $|z| = 1$ . Since  $F \notin O_G$ , there exists a bounded harmonic function  $\omega(p)$  on  $F'$  such that  $\omega(p) = 0$  on  $\partial F_0$  and  $\omega(\varphi^{-1}(z)) = 1$  almost everywhere on  $E_I$ . In the same manner as above, we can find triangles  $\Delta_i$  and two subset  $E_i$  ( $i = 1, 2$ ) of positive measure of  $E_I$  such that  $A(p) = A(p(z))$  has angular limits, on  $E_i$ , contained in  $\Delta_i$ . By a suitable choice of the coordinate axes we can suppose without loss of generality that  $\delta_1 \leq \operatorname{Re}(w) \leq \delta_2$

5) In this article we assume  $\{F_n\}$  has a compact relative boundary  $\{\partial F_n\}$ .

6) A. Mori: On the existence of harmonic functions on a Riemann surfaces, Journal of the Fac. Univ. Tokyo, 1951, 247-257.

M. Parrean: Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann, Annales de Fourier. 1952, 1-95.

7) See 6)

( $w \in \Delta_1$ ),  $\delta_3 \leq \operatorname{Re}(w) \leq \delta_4$  ( $w \in \Delta_2$ ) ( $\delta_2 < \delta_3$ ) and  $\operatorname{mes} E_1 > 0$ ,  $\operatorname{mes} E_2 > 0$ . Let  $U_1(p)$  be a harmonic function on  $F'$  such that  $U_1(p) = \operatorname{Re} A(p)$  on  $\partial F_0$  and  $U_1(\varphi^{-1}(p)) = N_1$  on  $E_1$ , where  $N_1 < N \leq \operatorname{Re} A(p) \leq M_2$ . Then  $D_{F'}(U_1(p)) < M_3 < \infty$ . Put  $U_2(p) = \operatorname{Re} A(p) - U_1(p)$  ( $\equiv$  constant) and  $F^+ = \varepsilon \left\{ p: U_2(p) > \frac{\delta_2 + \delta_3}{2} - N_1 \right\}$  and  $F^- = \varepsilon \left\{ p: U_2(p) < \frac{\delta_2 + \delta_3}{2} - N_1 \right\}$ . Then  $\partial F^+$  does not intersect  $\partial F_0$  and  $D_{F^+}(U_2(p)) < M_4 < \infty$ . Denote by  $C_\rho$  the ring:  $\rho < |z| < 1$  ( $\rho < 1$ ). Since  $U_2(p)$  has angular limits between  $\delta_3 - N_1$  and  $\delta_4 - N_1$  almost everywhere on  $E_2$ , we can construct an angular domain  $D$  which contains an end part of every  $A(\theta): |\arg(1 - e^{-i\theta}z)| < \frac{\pi}{4}$  ( $e^{i\theta} \in E_2' \subset E_2$ ,  $\operatorname{mes} E_2' > 0$ ) and find  $\rho$  such that  $\delta_3 - N_1 + \varepsilon \geq U_2(z) \geq \delta_4 - N_1 - \varepsilon$  in  $D \cap C_\rho$ , where  $\varepsilon < \frac{\delta_3 - \delta_2}{4}$ . Now  $D \cap C_\rho$  consists of a finite number of domains, then there exists a domain  $D'$ , such that  $D'$  has a subset  $E_2'' \subset E_2'$  of linear measure positive on its boundary and that the boundary of  $D'$  is rectifiable. Hence there exists a non-constant bounded harmonic function  $\omega'(z)$  in  $D'$  such that  $\omega'(z) = 0$  on the boundary of  $D'$  except  $E_2''$  and  $0 \leq \omega'(z) \leq 1$ . Let  $\{F_n\}$  be an exhaustion of  $F$  and  $\omega_n(p)$  be a sequence of harmonic function in  $F_n \cap (F' - F^+)$  such that  $\omega_n(p) = 0$  on  $\partial F_0 + (\partial F^+ \cap F)$  and  $= 1$  on  $\partial F_n \cap (F' - F^+)$ . Then for sufficiently large  $n$  the image of  $\partial F_n$  is contained in  $C_\rho$  and the image of  $\partial F_0 + (\partial F^+ \cap F)$  does not fall in  $D' (\subset C_\rho)$ . Consider  $\omega_n(p)$  in  $D'$ . Then we see  $\omega_n(p) \geq \omega'(z)$ , whence  $\omega(p) = \lim_n \omega_n(p) \equiv 0$ . Let  $\{V_n(p)\}$  be a sequence of harmonic functions such that  $V_n(p)$  is harmonic in  $F_n$ ,  $V_n(p) = U_2(p)$  on  $\partial F_n \cap F^+$  and  $V_n(p) = \frac{\delta_2 + \delta_3}{2} - N_1$  on  $\partial F_n - (\partial F_n \cap F^+)$ . Then we can define a non-constant Dirichlet-bounded harmonic function<sup>8)</sup> as above. Hence  $F \notin O_{HD} = O_{HBD}$ .

**IVERSEN'S PROPERTY.** Let  $F$  be an abstract Riemann surface,  $\{F_n\}$  be its exhaustion and  $\omega_n(p)$  be the harmonic measure of  $\partial F_n$  with respect to  $F_n - F_0$ , i. e.,  $\omega_n(p) = 0$  on  $\partial F_0$  and  $\omega_n(p) = 1$  on  $\partial F_n$ . Denote by  ${}^\rho C_n$  the niveau curve of  $\omega_n(p)$  with height  $\rho$ .  ${}^\rho C_n$  consists of a finite number of analytic curves  ${}^\rho l_n^1, {}^\rho l_n^2, \dots, {}^\rho l_n^{K_n}$ . Put  ${}^\rho L_n^i = \int_{{}^\rho l_n^i} \frac{\partial \omega_n}{\partial n} ds$  and  $\Lambda_n(\rho) = \max_i {}^\rho L_n^i$ .

**Theorem 2.** *If*

$$\lim_{n \rightarrow \infty} \int_0^1 e^{-4 \int_0^{\rho_n} \frac{d\rho}{\Lambda_n(\rho)}} d\rho_n = \infty,$$

8) A. Mori: A remark on the class of  $O_{HD}$  of Riemann surfaces, Kōdai Math. Semi. Report, No. 2 June 1952, 57-58.

then every connected piece of  $F$  over  $|w-w_0|<S$  covers every point except possibly a null set of  $E_{AB}$ <sup>9)</sup>.

We can prove the theorem similarly as in the previous<sup>10)</sup>.

REMARK. Pfluger<sup>11)</sup> proved, if  $\lim_{n \rightarrow \infty} \int_0^1 e^{\int_0^{\rho_n} \frac{d\rho}{A_n(\rho)}} d\rho_n = \infty$ ,  $F \in O_{AB}$ .

Let  $F$  be a Riemann surface of finite genus. Then  $F$  can be mapped conformally onto a subsurface  $F$  of another closed Riemann surface  $F^*$  of the same genus. Suppose  $F \in O_{AB}$  is represented as a covering surface  $F_w$  over the  $w$ -plane by a mapping function  $w=f(p)$ .

Let  $V_\rho(w_0)$  be a connected piece of  $F_w$  over the circle  $|w-w_0|<\rho$ . Then  $V_\rho(w)$  has a finite or enumerably infinite number of analytic curves  $\alpha_n$  lying on  $|w-w_0|=\rho$  as its relative boundary. Let  $f^{-1}(\alpha_n)$  be the image of  $\alpha_n$  on  $F^*$ . Then  $\sum f^{-1}(\alpha_n)$  and a subset of  $(F^*-F)$  will be a finite or infinite number of continua which are denoted by  $b_i (i=1, 2, \dots)$ .  $b_i$  and their limit points enclose a domain  $V$  such that  $V \supset f^{-1}(V_\rho(w_0))$ .

**Theorem 3.** *Let  $F$  be a Riemann surface of finite genus and  $F^*-F (F \in O_{AB})$  be a set of linear measure zero. If the number of continuum boundary components of  $V$  is finite, the connected piece  $V_\rho(w_0)$  covers  $|w-w_0|<\rho$  except possibly a null set of  $E_{AB}$ .*

Proof. We see that every  $b_i$  consists of finite or an enumerably infinite number of  $f^{-1}(\alpha_n)$  and a subset of  $F^*-F$ . Let  $\beta$  be a subarc of  $b_i$  and let us draw a rectifiable curve  $\gamma$  connecting two endpoints of  $\beta$  such that  $\beta$  and  $\gamma$  encloses a simply connected subdomain  $N$  of  $V$ . Let  $G$  be a simply connected domain such that  $G \supset N$  and the distance between  $\partial N$  and  $\partial G$  is positive. We map conformally  $G$  onto  $|z|<1$ , and  $N$  onto  $|\xi|<1$  by  $z=\varphi(p)$  and  $\xi=\psi(p)$  respectively. Since the composed function  $z=\varphi(\psi^{-1}(p))=z(\xi)$  is bounded in  $|\xi|<1$ ,  $z(\xi)$  has angular limits and angular derivatives (containing) infinity. Denote by  $E$  the set on  $|\xi|=1$ , where  $\frac{dz(\xi)}{d\xi}=\infty$  and the angular domain:  $|\arg(1-e^{-i\theta\xi})|<\frac{\pi}{4}$  at  $e^{i\theta}$  by  $A(e)$ . If  $E$  is a set of positive linear measure, we can find a closed subset  $E'(\subset E)$  of positive measure such that  $\frac{dz(\xi)}{d\xi}$  tends to the angular limit  $\infty$  uniformly as  $\xi \rightarrow e^{i\theta} \in E'$  from the

9)  $E_{AB}$  is the boundary of a domain  $\in O_{AB}$  on the  $w$ -plane.

10) Z. Kuramochi: On covering property of abstract Riemann surfaces, Osaka Math. Journ., 6 (1954).

11) A. Pfluger: Über das Anwachsen eindeutiger analytischen Funktionen auf offene Riemannschen Flächen, Annales Acad. Fenn., 1948.

inside of  $A(\theta)$ . In usual manner we get a domain  $D(|\xi| < 1)$ , which contains an end part of every  $A(\theta)$  for  $e^{i\theta} \in E'$ , and is bounded by a rectifiable curve  $C$  consisting of  $E'$  and segments lying on the boundary of  $A(\theta)$  ( $e^{i\theta} \in E'$ ) and further an analytic curve. Denote by  $\xi_i$  points in  $|\xi| < 1$  where  $\frac{dz}{d\xi} = \infty$ . We can suppose  $\left| \frac{dz}{d\xi} \right| > 1$  ( $\xi \in D$ ), therefore the characteristic function  $T\left(\frac{dz}{d\xi}\right)$  of  $\frac{dz}{d\xi}$  is bounded, which implies that  $\sum_i G(\xi, \xi_i) < \infty$ , where  $G(\xi, \xi_i)$  is the Green's function of  $D$  with its pole at  $\xi_i$ . We map conformally  $D$  onto  $|\zeta| < 1$  by  $\xi = \xi(\zeta)$ . Then  $E'$  is transformed onto a set  $E'_\zeta$  of positive linear measure on  $|\zeta| = 1$ . Since  $D$  has a rectifiable boundary, we can construct a domain  $D'$  containing an end part  $|\arg(1 - e^{-i\theta}\zeta)| < \frac{\pi}{4}$  for  $e^{i\theta} \in E'_\zeta$ , where  $E'_\zeta$  is a set of positive linear measure on which  $\log \left| \frac{dz}{d\xi} \right| - \sum_i G(\zeta, \zeta_i) = U(\zeta)$  tends uniformly to  $\infty$ , when  $\zeta$  tends to  $E'$  inside  $D'$ . It follows that  $U(\zeta) \equiv \infty$ . Hence  $E$  is a set of linear measure zero.

If  $\beta \cap (F^* - F)$  is mapped onto a set  $E_1$  of a positive measure on  $|\xi| = 1$ , we can find a set  $E'_1 (\subset E_1)$  of positive measure and we construct an angular domain  $D_1$  containing the end part  $A(\theta)$  for  $e^{i\theta} \in E'_1$  and having a rectifiable boundary  $C$  such that  $\left| \frac{dz}{d\xi} \right| < M < \infty$  on the boundary of  $D_1$ . We see at once  $\varphi^{-1}(C + M_1')$  is rectifiable. On account of Riesz's theorem  $E'_1$  corresponds to a set of linear measure positive of  $\beta$ . This contradicts our assumption. Since  $b_i$  is covered by a finite number of subarcs, we observe that  $\sum b_i \cap (F^* - F)$  is a set of harmonic measure zero with respect to  $V$ .

If  $V_\rho(w_0)$  the connected piece, on  $|w - w_0| < \rho$ , does not cover a set larger than  $E_{AB}$ , there would exist a non constant bounded analytic function  $A(w) = U(w) + iV(w)$  such that  $U(w) = 0$  on  $|w - w_0| = \rho$ . If  $A(p) = A(f^{-1}(p))$  regular through  $V$ ,  $A(p)$  must be a constant, because  $U(p) \equiv 0$ , whence there exists a closed set  $E^*$  where  $A(p)$  is not regular. Since  $E^* (\subset F^* - F)$  is totally disconnected, we can find a domain  $G (\subset V)$  containing  $E^{**} (\subset E^*)$  such that  $\partial G$  has a positive distance from  $\sum b_i + E^{**}$ . We can find a non constant harmonic function  $\tilde{U}(p)$  by Neumann's method such that  $\tilde{U}(p)$  is bounded in  $F^* - G$ ,  $U(p) - \tilde{U}(p)$  is bounded and the conjugate function of  $\tilde{U}(p)$  is one valued and bounded in  $G - E^{**}$ . Since the genus of  $F^*$  is finite, we can construct a non constant bounded analytic function on  $F$  from  $\tilde{U}(p)$  with a linear form of Abel's first kind integrals. This contradicts the fact  $F \in O_{AB}$ . Hence  $V_\rho(w_0)$  covers  $|w - w_0| < \rho$  except possibly  $E_{AB}$ .

Remark. We constructed a covering surface  $F^{(12)}$  over the  $z$ -plane such that  $F$  satisfies the conditions of theorem 3, i. e., 1°) harmonic measure of  $(F^* - F)$  is zero with respect to  $V$  (above defined) of every connected piece. 2°)  $F$  is mapped conformally onto a domain on the  $w$ -plane, whose boundary is a set of linear measure zero on the real axis, and we proved that  $F$  has not Gross's property. Hence the conditions of the theorem are sufficient to have Iversen's property but not Gross's property for  $F$ .

**GROSS'S PROPERTY.** It is well known

**Theorem 4.**<sup>13)</sup> *Let  $z = z(p)$  be a meromorphic function on an abstract Riemann surface  $\in O_G$ . If we denote by  $p = p(z)$  its inverse function and if  $p(z)$  is regular at  $z_0$ , we can continue  $z(p)$  analytically on half lines:  $z = z_0 + re^{i\theta}$  ( $0 \leq r < \infty$ ) except for  $\theta$  of angular measure zero.*

We have, however,

**Theorem 5.** *A Riemann surface of  $O_{HP}$  has not necessarily Gross's property.*

In order to prove the theorem, it is sufficient to construct a Riemann surface which has not Gross's property but on which no non-constant single-valued positive harmonic function exists. As preparations, we shall prove some lemmas and define notations as follows.

**Lemma 1.** *Let  $G$  be a curvilinear rectangle on the  $z$ - (=  $x + iy$ ) plane whose sides are  $C_1: -a \leq x \leq a, y = 0$ ,  $C_2: x - a = \varphi(y), (0 \leq y \leq b), y = y$ ,  $C_3: \varphi(b) + a \geq x \geq \varphi(b) - a, y = b$  and  $C_4: x + a = \varphi(y), y = y (b \geq y \geq 0)$ , where  $\varphi(y)$  is a continuous function such that  $\varphi(0) = 0$ . Suppose a positive harmonic function  $U(z)$  on  $G$  such that  $U(z) \geq M$  on  $C_2 + C_4$  and  $U(z) \geq 0$  on  $C_1 + C_3$ , then there exists a curve connecting  $C_2$  with  $C_4$  on which  $U(z)$  is larger than  $M\omega(a, b)$ , where  $\omega(a, b)$  is the value at  $w = \frac{ib}{2}$  of the harmonic measure of sides  $(\overleftrightarrow{p_1 p_4}) + (\overleftrightarrow{p_3 p_2})$  with respect to the rectangle, on the  $w$ -plane with vertices such that  $p_1: w = -a, p_2: w = a, p_3: w = a + ib$  and  $p_4: w = -a + ib$ .*

Proof. We map  $G$  conformally by the function  $z = f(w)$  onto a rectangle with vertices  $q_1: w = -a, q_2: w = a, q_3: w = a + ib'$  and  $q_4: w = -a + ib'$ . Then  $2a \leq \int_{-a+\varphi}^{a+\varphi} \left| \frac{\partial f}{\partial u} \right| du$  ( $w = u + iv$ ), and by Schwarz's inequality we have

12) See 10).

13) R. Nevanlinna: *Eindeutige Analytische Funktionen*, 1936.

Z. Yujobo: On the Riemann surfaces, no Green's functions of which exists, *Mathematica Japonica*, No. 2 (1951), 61-68,

$$4a^2b \leq 2a \int_0^{b'} \int_{-a+\varphi}^{a+\varphi} \left| \frac{\partial f}{\partial u} \right|^2 dudv = 4a^2b'.$$

It follows  $b \leq b'$ . Consider  $U(z)$  in the  $w$ -plane, then we see that  $U(f(w)) \geq M\omega(a, b') \geq M\omega(a, b)$  on the segment  $w = u + \frac{ib'}{2}$  ( $-a \leq u \leq a$ ). If we denote by  $l$  in the  $z$ -plane the image of the segment,  $l$  is the required curve.

NOTATION 1. The number  $P_{n-1} (n=2, 3, \dots; \lim P_{n-1} = \infty)$ .

Put  $r_n = \frac{2^{1+\frac{1}{2}+\dots+\frac{1}{2^{n-1}}}}{4}$ ,  $s_n = \frac{r_{n+1}-r_n}{6}$  and let  $R_n$  be a ring such that  $r_{n-1} + \frac{6}{5}s_{n-1} \leq |z| \leq r_n - \frac{6}{5}s_n$  and  $M_n$  be the module of  $R_n$ :  $M_n = \log \left( \frac{11-6 \cdot 2^{2^n}}{6-2^{2^{n-1}}} \right)$ . The transformations:  $R_n \rightarrow$  the rectangle with vertices  $(-\pi, 0), (\pi, 0), (\pi, iM_n), (-\pi, iM_n)$  in the  $\xi$ -plane  $\rightarrow$  the upper  $\eta$ -half plane  $\left( A = -\frac{1}{K}, B = -1, D = 1, E = \frac{1}{K} \right) \rightarrow$  the unit circle of  $\zeta$ -plane are carried by  $\xi = \log z$ ,

$$\xi = \frac{1}{h} \int_0^\eta \frac{d\eta}{\sqrt{(1-\eta^2)(1-K^2\eta^2)}}, \quad \zeta = \frac{(1+i)\eta + \sqrt{\frac{1}{K}}(1-i)}{(1-i)\eta + \sqrt{\frac{1}{K}}(1+i)}$$

respectively. We have  $\omega(\pi, M_n) = e^{-\pi^2 \left( \frac{1}{M_n} \right) / 32}$  by some calculations. Put  $P_{n-1} = \frac{1}{\omega(\pi, M_n)}$ .

NOTATION 2.  $\mu_n'$  ( $\mu_n'$  is an integer) ( $n=1, 2, \dots; \lim \mu_n' = \infty$ ).

Let  $\{I_{n\nu}\}$  ( $n=1, 2, \dots; \nu=1, 2, 3, \dots, 2^{\mu_n}$ ) be slits such that  $r_n - s_n \leq |z| \leq r_n + s_n$ ,  $\arg z = \frac{2\pi\nu}{2^{\mu_n}}$  and denote by  $R_n'$  the ring such that  $r_{n-1} - \frac{11}{10}s_{n-1} \leq |z| \leq r_n + \frac{11}{10}s_n$  and  $U(z)$  be a harmonic function in  $R_n'$  such that  $0 \leq U(z) \leq P_{n+1}^{\delta} (\delta > 0)$  and  $U(z) = 0$  on  $\sum_{\nu=1}^{2^{\mu_n}} I_{n\nu} + \sum_{\nu=1}^{2^{\mu_{n-1}}} I_{n-1,\nu}$ . Then there exist  $\{\mu_n'\}$  such that maximum of  $U(z)$  in  $R_n$  is smaller than  $\frac{1}{n}$  for  $\mu_n \geq \mu_n', \mu_{n-1} \geq \mu_{n-1}'$ , (Fig. 1).

NOTATION 3.  $\mu_{n-1}'$  and  $\mu_n$  ( $\mu_n''$  and  $\mu_n$  are integers) ( $n=2, 3, \dots$ ;



$\lim_n \mu''_{n-1} = \infty$ .

Let  $U(z)$  be a harmonic function as above. Then  $U(z) = \frac{1}{2\pi} \int_{\partial R_n} U(\zeta) \frac{\partial g(z, \zeta)}{\partial n(\zeta)} ds(\zeta)$ , where  $g(z, \zeta)$  is the Green's function of  $R_n$  with pole at  $z$ . Since  $\frac{\partial g(z, \zeta)}{\partial n(\zeta)}$  is continuous function of  $z$  for fixed  $\zeta$ , there exist  $\{\mu''_{n-1}\}$  such that  $|U(z_1) - U(z_2)| < \frac{1}{n}$  for every pair  $z_1$  and  $z_2$  lying on the circle  $|z| = \frac{r_n + r_{n-1}}{2}$  such that  $|\arg z_1 - \arg z_2| < \frac{2\pi}{2^{\mu_{n-1}}}$  for  $\mu_{n-1} \geq \mu''_{n-1}$ . Put  $\mu_n = \text{Max}(\mu'_n, \mu''_n)$ .

NOTATION 4. The number  $N_n$  ( $N_n$  is an integer) ( $n=1, 2, 3, \dots$ ).

Let  $U(z)$  be a harmonic function on  $R'_n$  such that  $0 \leq U(z) \leq P_{n+1}^{1+\delta}$  on  $R'_n$ ,  $U(z) \leq \frac{3}{n}$  on  $\sum_\nu I_{n-1, \nu}$ , and  $U(z) \leq \frac{3}{n+1}$  on  $\sum_\nu I_{n\nu}$  except possibly a set of measure smaller than length of  $I_{n-1\nu}/N'$ , length of  $I_{n\nu}/N''$  respectively. Since, if  $N'$  and  $N'' = \infty$ , we have  $U(z) \leq \frac{1}{n}$  on  $R_n$  by Notation 1, there exist  $\{N_n\}$  such that  $U(z) \leq \frac{3}{n}$  on  $R_n$  for  $N' \geq N_{n-1}$ , and  $N'' \geq N_n$ , (Fig. 1).

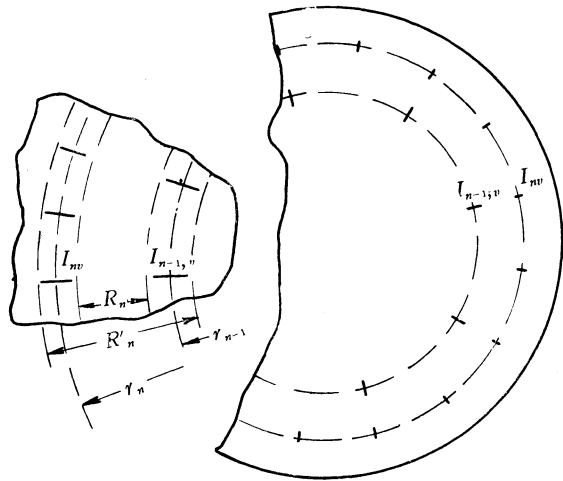


Fig. 1

**Lemma 2.** Let  $G$  be a domain in the  $z$ -plane with boundaries consisting of analytic curves  $\gamma_1, \gamma_2, \dots, \gamma_{n-1}, \gamma_n$ . Map  $G$  conformally onto a ring  $R_\zeta$  in the  $\zeta$ -plane such that  $1 \leq |\zeta| \leq e^{2\pi M}$  so that  $\gamma_1, \gamma_n, \gamma_2, \dots, \gamma_{n-1}$  may correspond to  $|\zeta|=1, |\zeta|=e^{2\pi M}$  and radial slits respectively in this ring. Let  $U(z)$  be harmonic in  $G$  with boundary value  $\varphi_1(z)$  on  $\gamma_1$  and  $\varphi_2(z)$  on  $\gamma_n$  respectively, where  $\varphi_i(z)$  ( $i=1, 2$ ) is a continuous function of  $z$ . Then

$$D_G(U(z)) = D_{R_\zeta}(U(\zeta)) \geq \frac{1}{2\pi M} \int_0^{2\pi} |\varphi_1(\theta) - \varphi_2(\theta)|^2 d\theta,$$

where  $U(\zeta)$  and  $\varphi_i(\theta)$  ( $i=1, 2$ ) are transformed functions from  $U(z)$  and  $\varphi_i(z)$ .

Proof. Let  $\tau(\zeta)$  be a harmonic function on  $R_\zeta$  such that  $\tau(\zeta) = \varphi_1(\theta)$

on  $|\zeta|=1$ ,  $\tau(\zeta)=\varphi_2(\theta)$  on  $|\zeta|=e^{\mathfrak{M}}$  and  $\frac{\partial\tau}{\partial n}=0$  on radial slits. We can prove easily  $D(U(\zeta))\geq D(\tau(\zeta))$ .

Now we divide the ring into sufficiently narrow circular rectangles  $A_j: 1\leq|\zeta|\leq e^{\mathfrak{M}}$ ,  $\theta_j\leq\arg\zeta<\theta_{j+1}$  ( $j=1, 2, \dots, m$ ) such that  $\text{Max}\varphi_1^i(\theta)-\text{Min}\varphi_1^i(\theta)\leq\frac{1}{n}$  and  $\text{Max}\varphi_2^i(\theta)-\text{Min}\varphi_2^i(\theta)\leq\frac{1}{n}$ , where  $\text{Max}\varphi_i^i(\theta)$ ,  $\text{Min}\varphi_i^i(\theta)$  is the maximum and minimum of  $\varphi_i(\theta)$  ( $i=1, 2$ ,  $\theta_j\leq\theta\leq\theta_{j+1}$ ) respectively.

Let  $\{A_j'\}$  and  $\{A_j''\}$  be  $A_j$  such that  $\text{Max}\varphi_1^i(\theta)\leq\text{Min}\varphi_2^i(\theta)$  and  $\text{Min}_i\varphi^i(\theta)\geq\text{Max}\varphi_1^i(\theta)$  respectively and  $\{A_j'''\}$  be rectangles contained neither in  $\{A_j'\}$  nor in  $\{A_j''\}$ . If  $A_j\in\{A_j'\}$ , let  $\tilde{U}_j(\zeta)$  be a harmonic function in  $A_j$  such that  $\tilde{U}_j(\zeta)=\varphi_1(\theta)$  on  $\gamma_1$ ,  $\tilde{U}_j(\zeta)=\varphi_2(\theta)$  on  $\gamma_n$  and  $\frac{\partial\tilde{U}_j(\zeta)}{\partial n}=0$  on two segments:  $1\leq|\zeta|\leq e^{\mathfrak{M}}$ ,  $\arg\zeta=\theta_j$ ,  $1\leq|\zeta|\leq e^{\mathfrak{M}}$ ,  $\arg\zeta=\theta_{j+1}$  and on radial slits in  $A_j$ . Let  $U_j^*(\zeta)$  be a harmonic function in  $A_j$  such that  $U_j^*(\zeta)=\text{Max}\varphi_1^i(\theta)$  on  $\gamma_1$ ,  $U_j^*(\zeta)=\text{Min}\varphi_2^i(\theta)$  on  $\gamma_n$  and  $\frac{\partial U_j^*(\zeta)}{\partial n}=0$  on boundary segments and radial slits. Since  $\tilde{U}_j(\zeta)-U_j^*(\zeta)\leq 0$  and  $\frac{\partial U_j^*(\zeta)}{\partial n}\leq 0$  on  $\gamma_1$  and  $\tilde{U}_j(\zeta)-U_j^*(\zeta)\geq 0$  and  $\frac{\partial U_j^*(\zeta)}{\partial n}\geq 0$  on  $\gamma_n$ , we have  $D(U_j^*(\zeta), \tilde{U}_j(\zeta)-U_j^*(\zeta))\leq 0$ . Clearly  $D_{A_j}(U_j^*(\zeta))=\frac{(\theta_{j+1}-\theta_j)}{2\pi\mathfrak{M}l}|\text{Max}\varphi_2^i(\theta)-\text{Min}\varphi_1^i(\theta)|^2$  and  $D_{A_j}(U(\zeta))\geq D_{A_j}(\tilde{U}_j(\zeta))\geq D_{A_j}(U_j^*(\zeta))+D_{A_j}(\tilde{U}_j(\zeta)-U_j^*(\zeta))+2D_{A_j}(U_j^*(\zeta), (\tilde{U}_j(\zeta)-U_j^*(\zeta))\geq D_{A_j}(U_j^*(\zeta))\geq\frac{1}{2\pi\mathfrak{M}l}\int_{\theta_j\leq\theta<\theta_{j+1}}|\varphi_2(\theta)-\varphi_1(\theta)|^2d\theta-\frac{4(\theta_{j+1}-\theta_j)}{2\pi n^2}$ . If  $A_j\in\{A_j''\}$ , we can prove similarly the above inequality. If  $A_j\in\{A_j'''\}$ , let  $U_j^*(\zeta)\equiv 0$ . Then

$$D_{A_j}(U(\zeta))\geq 0\geq\frac{1}{2\pi\mathfrak{M}l}\int_{\theta_j\leq\theta<\theta_{j+1}}|\varphi_2(\theta)-\varphi_1(\theta)|^2d\theta-\frac{4(\theta_{j+1}-\theta_j)}{2\pi n^2}.$$

Hence

$$D_{R_\zeta}(U(\zeta))\geq\sum_{j=1}^m\left(\frac{1}{2\pi\mathfrak{M}l}\int_{\theta_j\leq\theta<\theta_{j+1}}|\varphi_2(\theta)-\varphi_1(\theta)|^2d\theta\right)-\frac{4}{n^2}.$$

Let  $n\rightarrow\infty$ . We have the lemma.

**Lemma 3.** Let  $R_1$  and  $R_2$  be rings  $1\leq|\zeta|\leq e^\beta$  with the same slits  $\{S^k\}$  ( $i=1, 2$ ) such that

$${}^1S^k: e^{\frac{\beta}{6}}\leq|\zeta|\leq e^{\frac{2\beta}{6}}, \arg\zeta=\frac{2k\pi}{l} \quad (k=1, 2, \dots, l)$$

$${}^2S^k: e^{\frac{5\beta}{6}}\leq|\zeta|\leq e^{\frac{5\beta}{6}}, \arg\zeta=\frac{2k\pi}{l} \quad ( \quad , \quad ).$$

Connect  $R_1$  with  $R_2$  on  $\{^i S^k\}$ , ( $i=1, 2$ ;  $k=1, 2, \dots, l$ ) crosswise. Then we have two-sheeted Riemann surface  $R$ . Denote by  $\tilde{\zeta}$  a point such that  $\tilde{\zeta}$  has the same projection as  $\zeta$  and let  $U(\zeta)$  be a single-valued positive harmonic function on  $R$  such that  $U(\zeta) \leq P$  on  $R$ . Then

$$|V(\zeta)| = |U(\zeta) - U(\tilde{\zeta})| < \lambda P$$

at every point whose projection lies on  $|\zeta| = e^{\frac{\beta}{l}}$ , where  $\lambda (< 1)$  depends continuously on the ratio  $\frac{\beta}{l}$  only.

Proof. Map  $R$  conformally onto a strip  $R_\eta$  by  $\eta = \log \zeta$  and consider  $V(\zeta) = V(\log \eta)$  on this strip. Then  $V(\eta)$  is harmonic and vanishes at every end points of the image  $\{\hat{\eta} S^k\}$  of  $\{^i S^k\}$ , and has the same absolute value with opposite sign on two sides of every  $\{\hat{\eta} S^k\}$ . Now we fix  $\eta$ . Let  $q, \hat{q}$  be two points facing each other on two sides of  $\{\hat{\eta} S^k\}$  and let  $\{\hat{\eta} \hat{S}^k\}$  be a set of point  $\hat{q}$  on  $\{\hat{\eta} S^k\}$  (one of  $q$  and  $\hat{q}$ ) such that  $\frac{\partial g(\hat{q}, \eta)}{\partial n(q)} \geq \frac{\partial g(q, \eta)}{\partial n(q)} \geq 0$ .

Let  $V^*(\eta)$  be a harmonic function on  $R_\eta$  such that  $V^*(\eta) = P$  on  $\text{Re } \eta = 1$ ,  $\text{Re } \eta = \beta$  and  $\{\hat{\eta} \hat{S}^k\}$  and that  $V^*(\zeta) = 0$  on the remaining boundary of  $R_\eta$ . Since  $\int_{\partial R_\eta} \frac{\partial g(q, \eta)}{\partial n} ds(q) = 2\pi$ , we have  $V^*(\zeta) = \frac{1}{2\pi} \int_{\partial R_\eta - \{\hat{S}\}} P \frac{\partial g(q, \zeta)}{\partial n} ds < \lambda(\eta)P$ , ( $\lambda(\eta) < 1$ ). On the other hand,  $V^*(\eta) = V^*\left(\eta + \frac{2\pi}{l}\right)$  and  $\lambda(\eta)$  is continuous with respect to  $\eta$ . Then we have  $V^*(\eta) \leq \lambda P(\lambda)$  on  $\text{Re } \eta = \frac{\beta}{2}$ . By maximum and minimum principle, we have  $|V(\eta)| \leq |V^*(\eta)| < \lambda P$ , where  $\lambda$  depends continuously on the ratio  $\frac{\beta}{l}$ .

In the following we fix two bounds  $M_1$  and  $M_2$  ( $M_1 < \frac{\beta}{l} < M_2$ ) so that  $\lambda$  may be always smaller than  $\lambda_0$  ( $\lambda_0 < 1$ ).

A) In the  $z$ -plane denote by  $J_{n\nu}$  ( $n$  is fixed;  $\nu = 1, 2, \dots, 2^{\mu_n}$ ) a circular rectangle, containing  $I_{n\nu}$ , with sides  $\partial J_{n\nu}$  as follows:

$$\begin{aligned} |z| = r_n - 3s_n, \frac{2\pi\nu}{2^{\mu_n}} - \frac{\pi}{2^2 \cdot 2^{\mu_n}} \leq \arg z \leq \frac{2\pi\nu}{2^{\mu_n}} + \frac{\pi}{2^2 \cdot 2^{\mu_n}}, \\ r_n - 3s_n \leq |z| \leq r_n + 3s_n, \arg z = \frac{2\pi\nu}{2^{\mu_n}} - \frac{\pi}{2^2 \cdot 2^{\mu_n}}, \\ |z| = r_n + 3s_n, \frac{2\pi\nu}{2^{\mu_n}} - \frac{\pi}{2^2 \cdot 2^{\mu_n}} \leq \arg z \leq \frac{2\pi\nu}{2^{\mu_n}} + \frac{\pi}{2^2 \cdot 2^{\mu_n}}, \\ r_n - 3s_n \leq |z| \leq r_n + 3s_n, \arg z = \frac{2\pi\nu}{2^{\mu_n}} + \frac{\pi}{2^2 \cdot 2^{\mu_n}}. \end{aligned}$$

Let  $L_{n\nu}, L'_{n\nu}$  be half straight lines such that

$$L_{n\nu}: 0 \leq |z| < \infty, \arg z = \frac{2\pi\nu}{2^{\mu_n}} + \pi - \frac{\pi}{2 \cdot 2^{\mu_n}},$$

$$L'_{n\nu}: 0 \leq |z| < \infty, \arg z = \frac{2\pi\nu}{2} + \pi + \frac{\pi}{2 \cdot 2^{\mu_n}} \quad (\text{Fig. 2}).$$

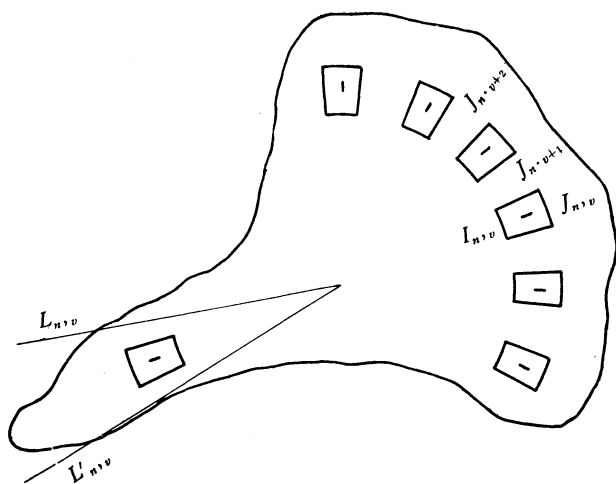


Fig. 2

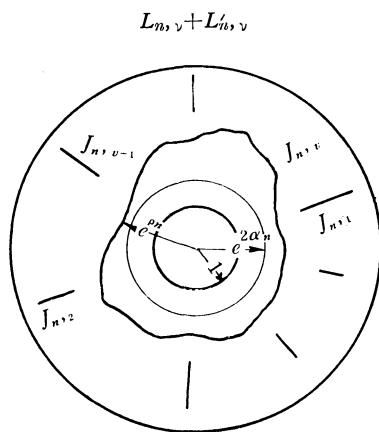


Fig. 3

We denote by  $G_{n\nu}$  the domain with boundaries  $L_{n\nu}, L'_{n\nu}, \sum_{i \pm \nu, \nu'} \partial J_{ni}$  ( $\nu' = \nu + \frac{\mu_n}{2}$ ) and  $I_{n\nu}$ , and map it conformally by  $w = \varphi(z)$  onto the ring  $1 \leq |w| \leq e^{\mathfrak{M}_n^{(1)}}$  so that  $L_{n\nu} + L'_{n\nu}, I_{n\nu}, \sum_{i \pm \nu'} \partial J_{ni}$  may be transformed onto  $|w| = e^{\mathfrak{M}_n^{(1)}}$ ,  $|w| = 1$  and radial slits  $\sum_{i \pm \nu'} \partial J_{ni(w)}$  respectively. In this mapping any measurable set of positive angular measure  $\leq \frac{2}{k_n}$  on  $|w| = 1$  mapped onto a measurable set of positive linear measure  $\leq$  length of  $I_{n\nu}/N_n$  on  $I_{n\nu}$ , where  $\kappa_n$  depends only on the mapping function  $w = \varphi(z)$  and does not depend on the situation of the set on  $|w| = 1$ . And the doubly connected domain bounded by  $\partial J_{n\nu}$  and  $I_{n\nu}$  of module  $\mathfrak{M}_n^{(2)}$  is mapped onto a domain bounded by  $|w| = 1$  and the image  $\partial J_{n\nu(w)}$  of  $J_{n\nu}$ .

Let  $e^{\rho_n}$  be the distance between  $\partial J_{n\nu(w)}$  and the point  $w=0$ .

Put

$$\rho'_n = \frac{\mathfrak{M}_n^{(2)}}{2\pi\kappa_n(n+1)^2 P_{n+1}^{2+2\delta} \cdot 2^{\mu_n}} \quad (1)$$

We choose  $\alpha_n$  so that  $e^{\alpha_n} \leq e^{\rho'_n}$  and  $e^{2\alpha_n} \leq e^{\rho_n}$  (Fig. 3).

Let  $s_n$  and  $q_n$  ( $n=1, 2, \dots$ ) integers such that

$$\frac{s_n}{2\pi(n+1)^2\kappa_n(\mathfrak{M}_n^{(1)}-\alpha_n)} \geq \frac{P_{n+1}^{2+2\delta} 2^{\mu_n}}{\mathfrak{M}_n^{(2)}}, \quad (2)$$

$$2P_{n+1}^{1+\delta} \lambda_0^{q_n-s_n} \leq \frac{1}{n+1}. \quad (3)$$

In the ring:  $1 \leq |w| \leq e^{2\alpha_n}$ , we define, two systems of rings  $\{{}^\alpha C_{n\nu}^{ij}\}$ , ( $\alpha=1, 2; n, \nu$  are fixed;  $i \geq j, i, j, =1, 2, \dots, q_n$ ) and a circle  $H_{n\nu}$  as follows:

$$\begin{aligned} {}^1C_{n\nu}^{ij} &: 2\alpha_n - \gamma(i(i-1) + 2j - 1) \geq \log |w| \geq 2\alpha_n - \gamma(i(i-1) + 2j), \\ {}^2C_{n\nu}^{ij} &: \gamma(i(i-1) + 2j - 1) \leq \log |w| \leq \gamma(i(i-1) + 2j), \\ H_{n\nu} &: \log |w| = \alpha_n, \end{aligned}$$

where  $\gamma = \frac{\alpha_n}{(q_n^2 + q_n + 1)}$  and  $M_1 \leq \frac{\gamma}{l_n} \leq M_2$  (see lemma 3).

Let  $\{S_{ijk}^{n\nu}, \{S_{ijk}^{n\nu}, \{\tilde{S}_{ijk}^{n\nu}, \{S_{ijk}^{n\nu}\}$  ( $n, \nu$  are fixed,  $i \geq j, i, j=1, 2, \dots, q_n; k=1, 2, \dots, l_n$ ) be slits such that  $\{S_{ijk}^{n\nu}, \{S_{ijk}^{n\nu}, \{\tilde{S}_{ijk}^{n\nu}, \{S_{ijk}^{n\nu}\}$  are contained in  $\{C_{n\nu}^{ij}\}$  and  $\{\tilde{C}_{n\nu}^{ij}\}$  respectively, (Fig. 4).

$$\begin{aligned} {}^1S_{ijk}^{n\nu} &: 2\alpha_n - \gamma\left(i(i-1) + 2j - 1 + \frac{2}{6}\right) \leq \log |w| \leq 2\alpha_n - \gamma\left(i(i-1) + 2j - 1 + \frac{1}{6}\right), \\ \arg w &= \frac{2\pi k}{l_n} \end{aligned}$$

$$\begin{aligned} {}^2S_{ijk}^{n\nu} &: 2\alpha_n - \gamma\left(i(i-1) + 2j - 1 + \frac{5}{6}\right) \leq \log |w| \leq 2\alpha_n - \gamma\left(i(i-1) + 2j - 1 + \frac{4}{6}\right), \\ \arg w &= \frac{2\pi k}{l_n} \end{aligned}$$

$$\begin{aligned} \tilde{S}_{ijk}^{n\nu} &: 2\gamma\left(i(i-1) + 2j - 1 + \frac{4}{6}\right) \leq \log |w| \leq \gamma\left(i(i-1) + 2j - 1 + \frac{5}{6}\right), \\ \arg w &= \frac{2\pi k}{l_n} \end{aligned}$$

$$\begin{aligned} {}^1\tilde{S}_{ijk}^{n\nu} &: \gamma\left(i(i-1) + 2j - 1 + \frac{1}{6}\right) \leq \log |w| \leq \gamma\left(i(i-1) + 2j - 1 + \frac{2}{6}\right), \\ \arg w &= \frac{2\pi k}{l_n} \end{aligned}$$

B) Let  $F_n(1), F_n(2), \dots, F_n(2^{q_n-1}), \hat{F}_n(1), \hat{F}_n(2), \dots, \hat{F}_n(2^{q_n-1})$  be  $2^{q_n}$  leaves of rings with slits  $\{S_{ijk}^{n\nu}, \{S_{ijk}^{n\nu}, \{\tilde{S}_{ijk}^{n\nu}, \{S_{ijk}^{n\nu}\}$  ( $n, \nu$  are fixed;  $i \geq j, i, j, =1, 2, \dots, q_n; k=1, 2, \dots, l_n$ ). Connect  $F_n(1)$  and  $\hat{F}_n(1)$  ( $i=1, 2, \dots, 2^{q_n-1}$ ) crosswise on  $\{S_{i1k}^{n\nu}, \{S_{i1k}^{n\nu}, \{\tilde{S}_{i1k}^{n\nu}, \{S_{i1k}^{n\nu}\}$  ( $n, \nu$  are fixed  $i=1, 2, \dots, q_n; k=1, \dots, l_n$ ) and call this connection 1-connecting.

2-Connecting is as follows:

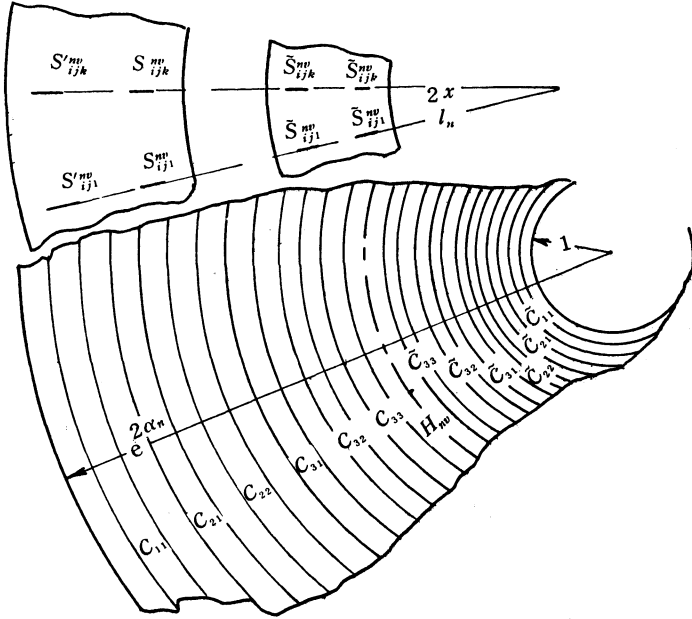
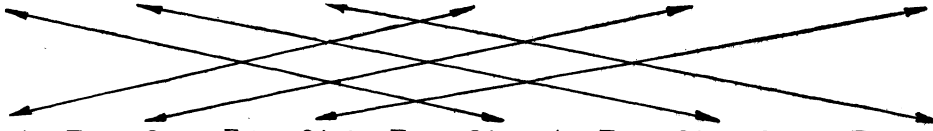


Fig. 4

$$\begin{array}{ccccccc}
 F_n(1) & F_n(2) & F_n(3) & F_n(4) & \dots & F_n(2^{q_n-1}-1) & F_n(2^{q_n-1}) \\
 \swarrow & \searrow & \swarrow & \searrow & & \swarrow & \searrow \\
 \hat{F}_n(1) & \hat{F}_n(2) & \hat{F}_n(3) & \hat{F}_n(4) & \dots & \hat{F}_n(2^{q_n-1}-1) & \hat{F}_n(2^{q_n-1}) \\
 \text{on } \{S_{i2k}^{n\nu}, S_{i2k}^{n\nu}, \tilde{S}_{ijk}^{n\nu}, \tilde{S}_{ijk}^{n\nu}\}.
 \end{array}$$

In general  $j_0$ -connecting is  $(i_0=1, \dots, q_n)$

$$F_n(a+1), F_n(a+2), \dots, F_n(a+2^{j_0-2}), F_n(a+2^{j_0-2}+1), F_n(a+2^{j_0-2}+2), \dots, F_n(a+2^{j_0-1}),$$



$$F_n(a+1), F_n(a+2), \dots, F_n(a+2^{j_0-2}), F_n(a+2^{j_0-2}+1), F_n(a+2^{j_0-2}+2), \dots, F_n(a+2^{j_0-1}),$$

on  $\{S_{i j_0 k}^{n\nu}, S_{i j_0 k}^{n\nu}, \tilde{S}_{i j_0 k}^{n\nu}, \tilde{S}_{i j_0 k}^{n\nu}\}$  ( $n, \nu$  are fixed;  $i=j_0, j_0+1, \dots, q_n$ ;  $k=1, 2, \dots, l_n$ ),

where  $a=0, 2^{j_0-1}, 2 \cdot 2^{j_0-1}, \dots, (2^{q_n-1}-1)2^{j_0-1}$ .

Then we have  $2^{q_n}$ -sheeted covering surface  $R_{n\nu}$ . Let  $U(p)$  be a positive harmonic function on  $R_{n\nu}$  such that  $U(p) \leq P_{n+1}^{1+\delta}$  and  $T_{j_0}(P)$  be the conformal mapping  $p \leftrightarrow \tilde{p}$ , where  $p$  and  $\tilde{p}$  are points such that  $p$  and  $\tilde{p}$  have the same projections and are lying on the leaves respectively

which are connected by arrows in the above schema. Consider  $V(p) = U(p) - U(T_{j_0}(p))$  on  $F(1) + \hat{F}(j_0)$  of  $R_{n\nu}$ . Then  $V(p)$  is harmonic and vanishes at endpoints of  $\{S_{i j_0 k}^{n\nu}, S_{i j_0 k}^{n\nu}, \tilde{S}_{i j_0 k}^{n\nu}, \tilde{S}_{i j_0 k}^{n\nu}\}$   $n, \nu$  fixed;  $i = j_0, j_0 + 1, \dots, q_n, k = 1, 2, \dots, l_n$  and further harmonic on the remaining  $\{S_{i j k}^{n\nu}, S_{i j k}^{n\nu}, \tilde{S}_{i j k}^{n\nu}, \tilde{S}_{i j k}^{n\nu}\}$ . We have by the lemma and maximum principle

$$\begin{aligned} |V(p)| &\leq P_{n+1}^{1+\delta} \lambda_0 \text{ on } 2q_n \lambda \leq \log |w| \leq 2\alpha_n - 2q_n \gamma, \\ |V(p)| &\leq P_{n+1}^{1+\delta} \lambda_0^2 \text{ on } (4q_n - 2)\gamma \leq \log |w| \leq 2\alpha_n - (4q_n - 2)\gamma, \\ &\dots\dots\dots \\ |V(p)| &\leq P_{n+1}^{1+\delta} \lambda_0^{q_n - j_0 + 1} \text{ on } \log |w| = \alpha_n. \end{aligned}$$

If we denote by  $|F_n(1) - \hat{F}_n(j)|$  the maximum of  $|U(p_1) - U(p_2)|$ , where  $p_1$  and  $p_2$  have the same projections and lie on  $H_{n\nu}$  of  $F(1)$  and  $\hat{F}(j)$  respectively, we have  $|F(1) - \hat{F}(2^{j-1})| < P_{n+1}^{1+\delta} \lambda_0^{q_n - j + 1}$ . Taking account of the property  $s_n$  and  $q_n$ , we see that there exist at least  $s_n$  leaves such that  $|F(1) - \hat{F}(i)| < \frac{1}{n+1}$  ( $i = i_1, \dots, i_{tn}$ ).

**Construction of the Surface.**

We mapped  $G_{n\nu}$  by  $w = \varphi(z)$  onto the ring  $1 \leq |w| \leq e^{\mathfrak{M}_n^{(1)}}$  and defined slits  $\{S_{i j k}^{n\nu}, S_{i j k}^{n\nu}, \tilde{S}_{i j k}^{n\nu}, \tilde{S}_{i j k}^{n\nu}\}$  and  $H_{n\nu}$ . Consider the inverse image in the  $z$ -plane of them and denote them by the same letters. Then  $\{H_{n\nu}\}$  ( $n$  is fixed,  $\nu = 1, 2, \dots, 2^{\mu_n}$ ) are approximate ellipses enclosing  $\{I_{n\nu}\}$ , and  $\{\tilde{S}_{i j k}^{n\nu}, \tilde{S}_{i j k}^{n\nu}\}$  ( $i \geq j, i, j = 1, 2, \dots, q_n; k = 1, \dots, l_n$ ) lie approximately radially in the approximate ring bounded by  $I_{n\nu}$  and  $H_{n\nu}$  and  $\{S_{i j k}^{n\nu}, S_{i j k}^{n\nu}\}$  lie approximately radially outside of  $H_{n\nu}$ . Denote by  $F_m(1)$  the  $z$ -plane with the slits  $I_{m\nu}$  and  $\{S_{i j k}^{m\nu}, S_{i j k}^{m\nu}, \tilde{S}_{i j k}^{m\nu}, \tilde{S}_{i j k}^{m\nu}\}$  and by  $F_m(2) \dots F_m(2^{q_m - 1}), \hat{F}_m(1), \hat{F}_m(2) \dots \hat{F}_m(2^{q_m - 1})$  the  $z$ -plane with slits  $\{S_{i j k}^{m\nu}, S_{i j k}^{m\nu}, \tilde{S}_{i j k}^{m\nu}, \tilde{S}_{i j k}^{m\nu}\}$ , where  $m$  is fixed;  $\nu = 1, 2, \dots, 2^{\mu_m}; i \geq j, i, j = 1, 2, 3, \dots, q_m; k = 1, 2, \dots, l_m$ . Let  $F_{0n}$  be the part of  $F_0$  such that  $0 \leq |z| \leq r_n + 3s_n$ , where  $F_0$  is the unit circle with  $\{I_{n\nu}\}$  ( $n = 1, 2, \dots$ ). Connect  $F_{0n}$  with  $\{F_m(1)\}$  on  $\{I_{m\nu}\}$  ( $\nu = 1, 2, \dots, 2^{\mu_m}$ ) crosswise and connect every  $F_m(1)$  with  $F_m(l)$  and  $\hat{F}_m(l')$  ( $l = 2, 3, \dots, 2^{q_m - 1}, l' = 1, 2, \dots, 2^{q_m - 1}$ ) on  $\{S_{i j k}^{m\nu}, S_{i j k}^{m\nu}, \tilde{S}_{i j k}^{m\nu}, \tilde{S}_{i j k}^{m\nu}\}$  in the manner mentioned in (B) ( $m = 1, 2, \dots, n$ ). Then we have a Riemann surface denoted by  $\mathfrak{F}_n$ .  $\mathfrak{F}_n$  covers the part  $|z| \leq r_n + 3s_n$   $1 + 2^{q_1}; \dots + 2^{q_n}$  times and covers the part  $|z| \geq r_n + 3s_n$   $2^{q_1} + \dots + 2^{q_n}$  times. Put  $\bigcup_n \mathfrak{F}_n = F$ . Then  $F$  is the required Riemann surface, (Fig. 5).

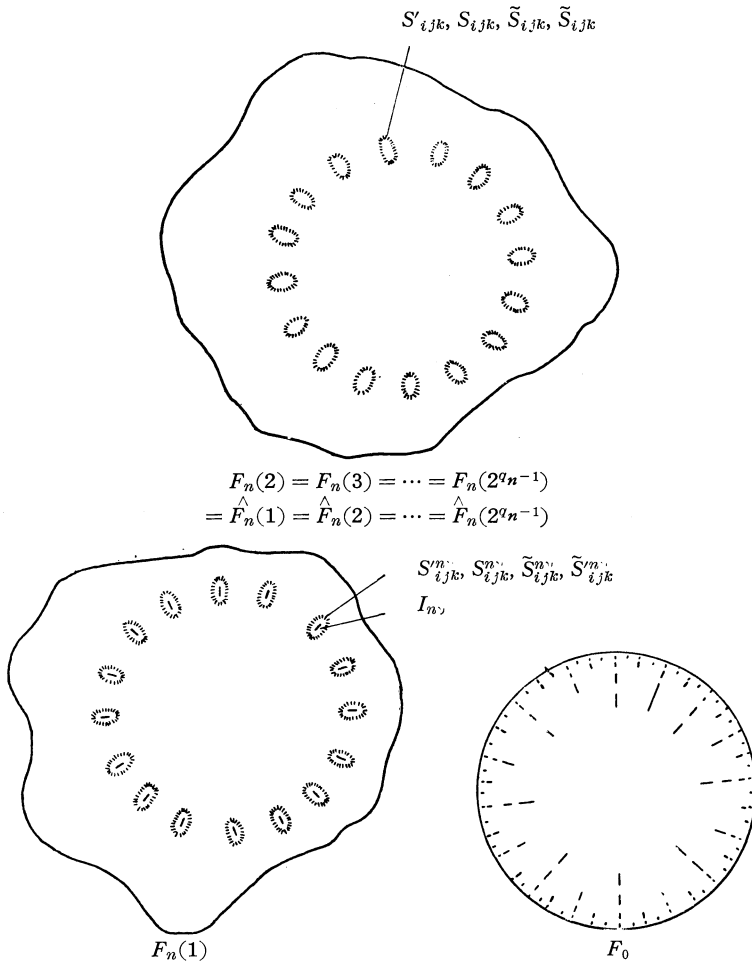


Fig. 5

**Proof of the theorem.**

Let  $|z| = r_{n+1} - \frac{6}{5} s_{n+1}$  be a circle on  $F_0$ , which is a dividing cut of  $F$ . If we denote by  $\mathfrak{F}'_n$  the compact surface which is one of the two divided by the above cut, then  $F = \bigcup_n \mathfrak{F}'_n$ . Denote by  $L_n$  the maximum of  $U(p)$  on  $|z| = r_n + \frac{6}{5} s_n$  on  $F_0$ . Let  $U(p) = L_n$  on  $p_0$  on  $|z| = r_n + \frac{6}{5} s_n$  of  $F_0$ , we see, by maximum principle, that there exists a Jordan curve  $C$  joining two boundary components of the ring  $r_n + \frac{6}{5} s_n \leq |z| \leq r_{n+1} - \frac{6}{5} s_{n+1}$  such that  $U(p) \geq L_n$  on  $C$ . Then by notation 1, there exists a dividing cut  $l$  in the ring such that  $U(p) \geq \frac{L_n}{P_n}$  on  $l$ . If  $\overline{\lim}_{n=\infty} L_n \geq P_n^{1+\delta}$ ,



$U(p)$  must be constant infinity by minimum principle. Hence, without loss of generality, we can suppose

$$\overline{\lim}_{n \rightarrow \infty} L_n < P_n^{1+\delta}.$$

$(\mathfrak{F}_n - \mathfrak{F}_{n-1})$  is the Riemann surface which consists of the ring, of  $F_0$  such that  $r_{n-1} + 3s_{n-1} \leq |z| \leq r_n + 3s_n$  and leaves  $\sum_{m=1}^{2^{q_n-1}} ((F_n(m) + \hat{F}_n(m)))$ .  $(\mathfrak{F}_n - \mathfrak{F}_{n-1})$  has two circles  $|z| = r_n + 3s_n$  and  $|z| = r_{n-1} + 3s_{n-1}$  as its boundary components  $\gamma_n$  and  $\gamma'_n$  respectively.

Let  $T_\lambda(p)$  be a conformal mapping in  $(\mathfrak{F}_n - \mathfrak{F}_{n-1})$   $p \leftrightarrow \tilde{p}$ , where  $\tilde{p}$  is the symmetric point of  $p$  with respect to the straight line  $\overset{*}{L}_{n\lambda} + \overset{*}{L}'_{n\lambda}$  such that  $\overset{*}{L}_{n\lambda} : 0 \leq |z| < \infty, \arg z = \frac{2\pi\lambda}{2^{\mu_n}} + \frac{\pi}{2}$ .  $0 \leq |z| < \infty, \overset{*}{L}'_{n\lambda} : \arg z = \frac{2\pi\lambda}{2^{\mu_n}} - \frac{\pi}{2} (\lambda = 1, 2, \dots, \frac{2^{\mu_n}}{2})$ . Put  $U_\lambda(p) = U(p) - U(T_\lambda(p))$ . Then  $|U_\lambda(p)|$  is subharmonic on  $(\mathfrak{F}_n - \mathfrak{F}_{n-1})$ ,  $\leq P_{n+1}^{1+\delta}$  on  $\gamma_n + \gamma'_n$  and vanishes at every point whose projection is on  $\overset{*}{L}_{n\lambda} + \overset{*}{L}'_{n\lambda}$ . Let  $(\mathfrak{F}_n - \mathfrak{F}_{n-1})^*$  be the surface consisting of  $\sum_{\nu=1}^{2^{\mu_n}} J_{n\nu}$  of  $F_0$  and  $\sum_{m=1}^{2^{q_n-1}} (F_n(m) + \hat{F}_n(m))$ , and let  $(\mathfrak{F}_n - \mathfrak{F}_{n-1})'_k$  be the part of  $(\mathfrak{F}_n - \mathfrak{F}_{n-1})'$  which is over the part of the  $z$ -plane such that

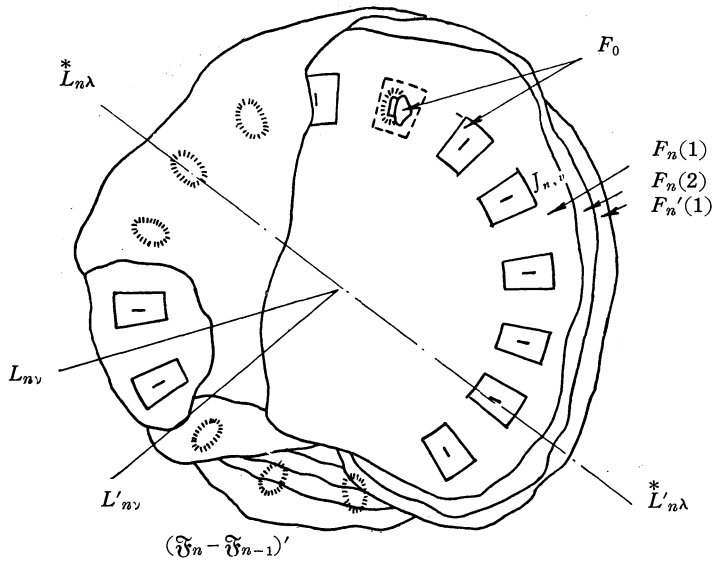


Fig. 6

$$0 \leq |z| < \infty, \frac{2\pi k}{2^{\mu_n}} + \pi - \frac{\pi}{2 \cdot 2^{\mu_n}} \leq \arg z \leq \frac{2\pi k}{2^{\mu_n}} + \pi + \frac{\pi}{2 \cdot 2^{\mu_n}}$$

$$\left( \text{containing } \frac{2\pi k}{2^{\mu_n}} \right), \text{ if } 1 \leq k \leq \frac{2^{\mu_n}}{2}.$$

$$0 \leq |z| < \infty, \frac{2\pi k}{2^{\mu_n}} - \pi - \frac{\pi}{2 \cdot 2^{\mu_n}} \leq \arg z \leq \frac{2\pi k}{2^{\mu_n}} - \pi + \frac{\pi}{2 \cdot 2^{\mu_n}}$$

$$\left( \text{containing } \frac{2\pi k}{2^{\mu_n}} \right), \text{ if } \frac{2^{\mu_n}}{2} \leq k \leq 2^{\mu_n}. \quad (\text{Fig. 6}).$$

On the other hand let  $V_k(p)$  be a harmonic function on  $(\mathfrak{F}_n - \mathfrak{F}_{n-1})_k'$  such that  $V_k(p) = P_{n+1}^{1+\delta}$  on the boundary of  $\sum_{\nu=k+\frac{2^{\mu_n}}{2}} \partial J_{n\nu}$  and vanishes at every point whose projection lies on the straight lines  $L_{nk}$  and  $L'_{nk}$ , where  $L_{nk}: 0 \leq |z| < \infty, \arg z = \frac{2k\pi}{2^{\mu_n}} + \pi + \frac{\pi}{2 \cdot 2^{\mu_n}}, L'_{nk}: 0 \leq |z| < \infty, \arg z = \frac{2k\pi}{2^{\mu_n}} + \pi - \frac{\pi}{2 \cdot 2^{\mu_n}}.$

Since  $|U_\lambda(p)|$  is subharmonic and the angular domain bounded by  $L_{nk} + L'_{nk}$  contains the half plane bounded by  $L_{n\lambda}^*$  and  $L'_{n\lambda}^*$ , we have  $V_k(p) \geq |U_\lambda(p)|$ , where

$$k \neq \lambda + \frac{1}{4} 2^{\mu_n}, \neq \lambda - \frac{1}{4} 2^{\mu_n} \text{ if } \frac{3}{4} 2^{\mu_n} \geq \lambda \geq \frac{1}{4} 2^{\mu_n},$$

$$k \neq \lambda + \frac{1}{4} 2^{\mu_n}, \neq \lambda + \frac{3}{4} 2^{\mu_n} \text{ if } \frac{1}{4} 2^{\mu_n} \geq \lambda \geq 0,$$

$$k \neq \lambda - \frac{3}{4} 2^{\mu_n}, \neq \lambda - \frac{1}{4} 2^{\mu_n} \text{ if } 2^{\mu_n} \geq \lambda \geq \frac{3}{4} 2^{\mu_n}.$$

In order to estimate the value of  $U_\lambda(p)$  on  $I_{ns}$  ( $s = \lambda, \lambda + 1, \dots, \lambda + \frac{1}{4} 2^{\mu_n} - 1, \lambda = \frac{3}{4} 2^{\mu_n} + 1, \dots, 2^{\mu_n}, 1, 2, \dots, \lambda - 1$ ). We consider  $V_k(p)$  on  $I_{nk}$ , (Fig. 6).

Let  $V(p)$  be a harmonic function on  $\sum_{\nu=1}^{2^{\mu_n}} (J_{n\nu} - I_{n\nu})$  such that  $V(p) = 0$  on  $\sum_{\nu=1}^{2^{\mu_n}} I_{n\nu}$  and  $V(p) = P_{n+1}^{1+\delta}$  on  $\sum_{\nu=1}^{2^{\mu_n}} \partial J_{n\nu}$ . Then we have by Dirichlet principle

$$\frac{2^{\mu_n} P_{n+1}^{2+2\delta}}{\mathfrak{M}_n^{(2)}} = D_{\mathfrak{Z}(J-I)}(V(p)) \geq D_{(\mathfrak{F}_n - \mathfrak{F}_{n-1})_k'}(V_k(p)) \quad (4)$$

Map conformally the domain of  $F_n(1)$  with boundary  $L_{nk} + L'_{nk} + \sum_{i \neq k, k'} \partial J_{ni} \left( k' = k + \frac{2^{\mu_n}}{2} \right) + I_{nk}$  by  $w = \varphi(z)$  onto the ring  $1 \leq |w| \leq e^{\mathfrak{M}_n^{(1)}}$  as defined in (A) and consider the composed function  $V_k(p) = V_k(\varphi(z))$ . Then  $V_k(p) = 0$  on  $|w| = e^{\mathfrak{M}_n^{(1)}}$ . If  $V_k(p) \geq \frac{2}{n+1}$  on a set of angular measure larger than  $\frac{1}{\kappa_n}$  on  $H_{nk}$ , there exists at least  $s_n$  leaves of  $\{F_n(i), \hat{F}_n(j)\}$  by (B), whose  $H_{nk}$  have the property such that  $V_k(p) \geq \frac{1}{n+1}$  on the set of angular measure larger than  $\frac{1}{\kappa_n}$  on  $H_{nk}$ . Let  $D(V_k(p))$  be the Dirichlet

integral over  $e^{\alpha_n} \leq |w| \leq e^{\mathfrak{M}_n^{(1)}}$  of these leaves. Then we have by lemma (3) and (2)

$$D(V_k(p)) \geq \frac{s_n}{2\pi(n+1)^2 \kappa_n (\mathfrak{M}_n^{(1)} - \alpha_n)} \geq \frac{P_{n+1}^{2+2\delta} 2^{\mu_n}}{\mathfrak{M}_n^{(2)}}. \tag{5}$$

(5) contradicts (4), whence  $V_k(p) \leq \frac{2}{n+1}$  on  $H_{nk}$  of  $F_n(1)$  except possibly a set of angular measure smaller than  $\frac{1}{\kappa_n}$ .

Next consider  $V_k(p)$  in  $1 \leq |w| \leq e^{\alpha_n}$  (of the image of  $F_n(1)$ ). If  $|V_n(p_2) - V_k(p_1)| \geq \frac{1}{n+1}$  on a set of angular measure larger than  $\frac{1}{\kappa_n}$ , we have by lemma 3 and (1)

$$D(V_k(p)) \geq \frac{1}{2\pi(n+1)^2 \kappa_n \alpha_n} \geq \frac{P_{n+1}^{2+2\delta} 2^{\mu_n}}{\mathfrak{M}_n^{(2)}}, \tag{6}$$

where  $\arg p_1 = \arg p_2$ ,  $p_1$  and  $p_2$  lie on  $|w|=1$ , and  $|w|=e^{\alpha_n}$  respectively, and  $D(V_k(p))$  is the Dirichlet integral over  $1 \leq |w| \leq e^{\alpha_n}$  on  $F_n(1)$ . But (6) contradicts (4), whence  $V_k(p) \leq \frac{3}{n+1}$  on  $|w|=1$  except possibly a set of angular measure smaller than  $\frac{2}{\kappa_n}$ . If we consider the above results in  $F_0$ , we see that  $V_k(p) \leq \frac{3}{n+1}$  on  $I_{nk}$  except a set of measure smaller than length of  $I_{nk}/N_n$ .

We see at once  $U_\lambda(p) = 0$  on  $I_{n\beta}, I_{n\beta'}$  ( $\beta = \lambda + \frac{2^{\mu_n}}{4}, \beta' = \lambda - \frac{2^{\mu_n}}{4}$ ). On the other hand  $V_k(p) \geq |U_\lambda(p)|$  on  $I_{nk}$  ( $k \neq \beta, \beta'$ ). Thus  $|U_\lambda(p)| \leq \frac{3}{n+1}$  on every  $I_{n\nu}$  ( $\nu = 1, 2, \dots, 2^{\mu_n}$ ) except a set of linear measure smaller than the length  $I_{n\nu}/N_n$ . In the same manner we have in  $(\mathfrak{F}_{n-1} - \mathfrak{F}_{n-2})$ ,  $|U(p) - U(T_\lambda(p))| \leq \frac{3}{n}$  on every  $I_{n-1\nu}$  except a set of linear measure smaller than the length of  $I_{n-1\nu}/N_{n-1}$ .

Consider  $T_\nu(p)$  in  $(\mathfrak{F}_n - \mathfrak{F}_{n-2})$  ( $\nu = 1, 2, \dots, 2^{\mu_{n-1}}$ ). Then we see that  $|U_\nu(p)| \leq P_{n+1}^{1+\delta}$  on the ring  $R'_\nu$ , and that  $U(p)$  is symmetric with respect to  $2^{\mu_{n-1}}$  directions except at most  $\frac{6}{n}$  on  $R_n$ , (See Notation 4).

Then by (Notation 3)  $|U(p_1) - U(p_2)| \leq \frac{1}{n}$  on the circle  $C_n: |z_n| = \frac{r_n + r_{n-1}}{2}$  on  $F_0$  such that  $|\arg p_1 - \arg p_2| \leq \frac{1}{2^{\mu_{n-1}}}$ . If we denote by  $\text{Max } U(p)$ ,  $\text{Min } U(p)$  the maximum and minimum of  $U(p)$  on  $C_n$ , we have

$$|\text{Max } U(p) - \text{Min } U(p)| \leq \frac{7}{n}.$$

Denote by  $\mathfrak{F}_n''$  the surface such that  $\mathfrak{F}_n''$  consists of the part of  $F_0$  on  $|z| < \frac{r_n + r_{n-1}}{2}$  and  $\sum_{i=1}^n \sum_{m=1}^{2^{q_i}-1} (F_i(m) + \hat{F}_i(m))$ . Then  $F = \bigcup_n \mathfrak{F}_n''$ . Since every  $\mathfrak{F}_n''$  has only one boundary component lying on  $C_n$  on which the oscillation of  $U(p)$  is smaller than  $\frac{7}{n}$ . Let  $n \rightarrow \infty$ .  $U(p)$  must reduce to a constant on account of maximum and minimum principle. Hence  $F \in O_{HP}$ .

Since  $F_0$  is the unit circle, it is clear that  $F$  has not Gross's property and by theorem of Gross we see  $F \notin O_G$ .

From this example we see that the validity of Gross's property of Riemann surface does not depend upon the complexity of the boundary. It depends rather upon the "force" of the boundary, i. e., roughly speaking upon the size of the boundary.

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