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On the Behaviour of Analytic Functions on Abstract Riemann Surfaces

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In this article we shall study mainly the structure of the covering surface, over the w-plane, of a function which is meromorphic on an abstract Riemann surface F. As a theorem of representation, we shall prove

Theorem 1. *Let F' be the remaining surface complementary to a compact subset F_0 of F. Then, if $F \notin O_G$*[1] *and $\in O_{HB}$ $(O_{HD} = O_{HBD})$, we have $F' \in O_{AB}(O_{AD})$.*

Proof of the former part. Since $F \notin O_G$, there exists a positive bounded harmonic function $\omega(p)$ on F' such that $\omega(p) = 0$ on ∂F_0[2]. Let F'^{∞} be the universal covering surface of F'. We map F'^{∞} conformally onto $|z| < 1$ by $z = \varphi(p)$. Assume $F' \notin O_{AB}$, then there exists a bounded analytic function $A(p)$ on F'. Consider $\omega(z) = \omega(\varphi^{-1}(z))$ on $|z| < 1$. Then there exists a set E of positive linear measure on $|z| = 1$ such that $\omega(z)$ has angular limits larger than $\delta(\delta > 0)$ on E. Let $\{\mathfrak{S}_n\}$ be a sequence of triangulation of the w-plane such that $\mathfrak{S}_{n+1}$ is a subdivision of $\mathfrak{S}_n$ and becomes as fine as we please when $n \to \infty$. Denote by $\{\Delta_n^i\}$[3] $(i = 1, 2, \cdots)$ the triangles of $\mathfrak{S}_n$. On account of Fatou's theorem $A(p(z))$ has angular limits almost everywhere on E. The subset of E, where $A(z)$ has angular limits contained in Δ_n^i will be denoted by E_n^i. Then every E_n^i is linearly measurable. There exists at least two E_n^i, $E_{n'}^{i'}$ such that $E_n^i \cap E_{n'}^{i'} = 0$ and both mes E_n^i and mes $E_{n'}^{i'}$ are positive. On the contrary, suppose for every n there exists $i(n)$ such that mes $E_n^i =$ mes E. $A(z)$ must be a constant contained in $\bigcap_{n=1}^{\infty} \Delta_n^i$[4]. Let $U(z)$ be a harmonic function in $|z| < 1$ such that $U(z) = 1$

1) O_G, O_{HP}, O_{HB}, O_{HD}, O_{AB} and O_{AD} are the classes of Riemann surfaces on which the Green's function, non-constant positive, bounded, Dirichlet-bounded, bounded analytic and Dirichlet-bounded analytic function does not exist respectively.

2) We denote by ∂S the relative boundary of S with respecto to F.

3) Δ_n^i are made half open so that they are mutually disjoint for fixed n.

4) M. Tsuji: Theory of meromorphic function in the neighbourhood of a closed set of capacity zero. Jap. Journ. 1944.

on E_n^i, $U(z)=0$ on the image of ∂F_0 on $|z|=1$ and $U(z)=-1$ on $E_{n'}^{i'}$. $U(z)$ can be considered on F', since it is automorphic. Let $\{F_n\}$[5] be an exhaustion of F and put $F^+=\varepsilon\left\{p: U(p)\geqq\frac{1}{2}\right\}$, $F^-=\varepsilon\left\{p: U(p)\leqq-\frac{1}{2}\right\}$. Then neither ∂F^+ nor ∂F^- intersects ∂F_0. Let $\{V_n(p)\}$ be a sequence of harmonic functions in F_n such that $V_n(p)=1$ on $\partial F_n \frown F^+$ and $V_n(p)=-1$ on $\partial F_n-(\partial F_n \frown F^+)$. Then we can[6] easily define a non-constant bounded harmonic function $V(p)$ from a sebsequence $\{V_{n'}(p(\}$ which converges uniformly in F. Hence $F\notin O_{HB}$.

Proof of the latter part. Assume, there exists a Dirichlet-bounded analytic function $A(p)$ on F'.

Case 1. The domain which is covered by $w=A(p)$ is dense in the w-plane. Since ∂F_0 is compact, $w=A(p)$ $(p\in\partial F_0)$ is bounded. Let M and N be the maximum and minimum of Re $A(p)$ $(p\in\partial F_0)$. Then there exist at least two components of F' on which Re $A(p)\geqq M'$ and Re $A(p)\leqq N'$ respectively, where $M'>M$, $N'>N$. We denote them by F^+ and F^-. Then neither ∂F^+ nor ∂F^- intersects ∂F_0. Consider Re $A(p)-M'$ and Re $A(p)-N'$ on F^+ and F^-. Then Re $A(p)-M'=0$ on ∂F^+, Re $A(p)-N'=0$ on ∂F^- and $D_{F^+}(\text{Re }A(p))<\infty$, $D_{F^-}(\text{Re }A(p))<\infty$. Let $\{F_n\}$ be an exhaustion of F and $\{V_n(p)\}$ be a sequence of harmonic function in F_n such that $V_n(p)=\text{Re }A(p)-M'$ on $\partial F_n\frown F^+$ and $=0$ on $\partial F_n-(\partial F_n\frown F^+)$. Then we can define[7] a non-constant Dirichlet-bounded function on F from uniformly convergent subsequence $\{V_{n'}(p)\}$.

Case 2. If $A(p)$ does not cover a domain, take a point w_0 in it. Then we see easily $D_{F'}\left(\frac{1}{A(p)-w_0}\right)<\infty$ and $\left|\frac{1}{A(p)-w_0}\right|<\infty$. We can suppose without loss of generality that $D_{F'}(A(p))<M_1<+\infty$ and $|A(p)|<M_2<+\infty$. We map the universal covering surface F'^∞ of F' conformally onto $|z|<1$ by $z=\varphi(p)$. Denote by E_0 the image of ∂F_0 on $|z|=1$ and by E_I the complementary set of E_0 on $|z|=1$. Since $F\notin O_G$, there exists a bounded harmonic function $\omega(p)$ on F' such that $\omega(p)=0$ on ∂F_0 and $\omega(\varphi^{-1}(z))=1$ almost everywhere on E_I. In the same manner as above, we can find triangles Δ_i and two subset E_i $(i=1, 2)$ of positive measure of E_I such that $A(p)=A(p(z))$ has angular limits, on E_i, contained in Δ_i. By a suitable choice of the coordinate axes we can suppose without loss of generality that $\delta_1\leqq\text{Re}(w)\leqq\delta_2$

5) In this article we assume $\{F_n\}$ has a compact relative boundary $\{\partial F_n\}$.

6) A. Mori: On the existance of harmonic functions on a Riemann surfaces, Journal of the Fac. Univ. Tokyo, 1951, 247–257.

M. Parrean: Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann, Annales de Fourier. 1952, 1–95.

7) See 6)

($w \in \Delta_1$), $\delta_3 \leqq \mathrm{Re}(w) \leqq \delta_4$ ($w \in \Delta_2$) ($\delta_2 < \delta_3$) and mes $E_1 > 0$, mes $E_2 > 0$. Let $U_1(p)$ be a harmonic function on F' such that $U_1(p) = \mathrm{Re}\, A(p)$ on ∂F_0 and $U_1(\varphi^{-1}(p)) = N_1$ on E_I, where $N_1 < N \leqq \mathrm{Re}\, A(p) \leqq M_2$. Then $D_{F'}(U_1(p)) < M_3 < \infty$. Put $U_2(p) = \mathrm{Re}\, A(p) - U_1(p)$ ($\not\equiv$ constant) and $F^+ = \mathcal{E}\left\{p\colon U_2(p) > \frac{\delta_2+\delta_3}{2} - N_1\right\}$ and $F^- = \mathcal{E}\left\{p\colon U(p) < \frac{\delta_2+\delta_3}{2} - N_1\right\}$. Then ∂F^+ does not intersect ∂F_0 and $D_{F^+}(U_2(p)) < M_4 < \infty$. Denote by C_ρ the ring: $\rho < |z| < 1$ ($\rho < 1$). Since $U_2(p)$ has angular limits between $\delta_3 - N_1$ and $\delta_4 - N_1$ almost everywhere on E_2, we can construct an angular domain D which contains an end part of every $A(\theta)$: $|\arg(1-e^{-i\theta}z)| < \frac{\pi}{4}$ ($e^{i\theta} \in E_2' \subset E_2$, mes $E_2' > 0$) and find ρ such that $\delta_3 - N_1 + \mathcal{E} \geqq U_2(z) \geqq \delta_4 - N_1 - \mathcal{E}$ in $D \cap C_\rho$, where $\mathcal{E} < \frac{\delta_3 - \delta_2}{4}$. Now $D \cap C_\rho$ consists of a finite number of domains, then there exists a domain D', such that D' has a subset $E_2'' \subset E_2'$ of linear measure positive on its boundary and that the boundary of D' is rectifiable. Hence there exists a non-constant bounded harmonic function $\omega'(z)$ in D' such that $\omega'(z) = 0$ on the boundary of D' except E_2'' and $0 \leqq \omega'(z) \leqq 1$. Let $\{F_n\}$ be an exhaustion of F and $\omega_n(p)$ be a sequence of harmonic function in $F_n \cap (F' - F^+)$ such that $\omega_n(p) = 0$ on $\partial F_0 + (\partial F^+ \cap F)$ and $= 1$ on $\partial F_n \cap (F' - F^+)$. Then for sufficiently large n the image of ∂F_n is contained in C_ρ and the image of $\partial F_0 + (\partial F^+ \cap F)$ does not fall in $D' (\subset C_\rho)$. Consider $\omega_n(p)$ in D'. Then we see $\omega_n(p) \geqq \omega'(z)$, whence $\omega(p) = \lim\limits_n \omega_n(p) \not\equiv 0$. Let $\{V_n(p)\}$ be a sequence of harmonic functions such that $V_n(p)$ is harmonic in F_n, $V_n(p) = U_2(p)$ on $\partial F_n \cap F^+$ and $V_n(p) = \frac{\delta_2+\delta_3}{2} - N_1$ on $\partial F_n - (\partial F_n \cap F^+)$. Then we can define a non-constant Dirichlet-bounded harmonic function[8] as above. Hence $F \notin O_{HD} = O_{HBD}$.

Iversen's Property. Let F be an abstract Riemann surface, $\{F_n\}$ be its exhaustion and $\omega_n(p)$ be the harmonic measure of ∂F_n with respect to $F_n - F_0$, i.e., $\omega_n(p) = 0$ on ∂F_0 and $\omega_n(p) = 1$ on ∂F_n. Denote by ${}^\rho C_n$ the niveau curve of $\omega_n(p)$ with height ρ. ${}^\rho C_n$ consists of a finite number of analytic curves ${}^\rho l_n^1, {}^\rho l_n^2, \cdots, {}^\rho l_n^{K_n}$. Put ${}^\rho L_n^i = \int_{{}^\rho l_n^i} \frac{\partial \omega_n}{\partial n} ds$ and $\Lambda_n(\rho) = \max\limits_i {}^\rho L_n^i$.

Theorem 2. *If*

$$\lim_{n=\infty} \int_0^1 e^{4\int_0^{\rho_n} \frac{d\rho}{\Lambda_n(\rho)}} d\rho_n = \infty ,$$

8) A. Mori: A remark on the class of O_{HD} of Riemann surfaces, Kōdai Math. Semi. Report, No. 2 June 1952, 57–58.

then every connected piece of F over $|w-w_0|<S$ covers every point except possibly a null set of E_{AB}[9].

We can prove the theorem similarly as in the previous[10].

REMARK. Pfluger[11] proved, if $\lim_{n=\infty}\int_0^1 e^{\int_0^{\rho_n}\frac{d\rho}{\Lambda_n(\rho)}}d\rho_n=\infty$, $F\in O_{AB}$.

Let F be a Riemann surface of finite genus. Then F can be mapped conformally onto a subsurface F of another closed Riemann surface F^* of the same genus. Suppose $F\in O_{AB}$ is represented as a covering surface F_w over the w-plane by a mapping function $w=f(p)$.

Let $V_\rho(w_0)$ be a connected piece of F_w over the circle $|w-w_0|<\rho$. Then $V_\rho(w)$ has a finite or enumerably infinite number of analytic curves α_n lying on $|w-w_0|=\rho$ as its relative boundary. Let $f^{-1}(\alpha_n)$ be the image of α_n on F^*. Then $\sum f^{-1}(\alpha_n)$ and a subset of (F^*-F) will be a finite or infinite number of continua which are denoted by $b_i (i=1, 2, \cdots)$. b_i and their limit points enclose a domain V such that $V\supset f^{-1}(V_\rho(w_0))$.

Theorem 3. *Let F be a Riemann surface of finite genus and F^*-F $(F\in O_{AB})$ be a set of linear measure zero. If the number of continuum boundary components of V is finite, the connected piece $V_\rho(w_0)$ covers $|w-w_0|<\rho$ except possibly a null set of E_{AB}.*

Proof. We see that every b_i consists of finite or an enumerably infinite number of $f^{-1}(\alpha_n)$ and a subset of F^*-F. Let β be a subarc of b_i and let us draw a rectifiable curve γ connecting two endpoints of β such that β and γ encloses a simply connected subdomain N of V. Let G be a simply connected domain such that $G\supset N$ and the distance between ∂N and ∂G is positive. We map conformally G onto $|z|<1$, and N onto $|\xi|<1$ by $z=\varphi(p)$ and $\xi=\psi(p)$ respectively. Since the composed function $z=\varphi(\psi^{-1}(p))=z(\xi)$ is bounded in $|\xi|<1$, $z(\xi)$ has angular limits and angular derivatives (containing) infinity. Denote by E the set on $|\xi|=1$, where $\frac{dz(\xi)}{d\xi}=\infty$ and the angular domain: $|\arg(1-e^{-i\theta}\xi)|<\frac{\pi}{4}$ at $e^{i\theta}$ by $A(e)$. If E is a set of positive linear measure, we can find a closed subset $E'(\subset E)$ of positive measure such that $\frac{dz(\xi)}{d\xi}$ tends to the angular limit ∞ uniformly as $\xi\to e^{i\theta}\in E'$ from the

9) E_{AB} is the boundary of a domain $\in O_{AB}$ on the w-plane.

10) Z. Kuramochi: On covering property of abstract Riemann surfaces, Osaka Math. Journ., 6 (1954).

11) A. Pfluger: Über das Anwachsen eindeutiger analytischen Funktionen auf offene Riemannschen Flächen, Annales Acad. Fenn., 1948.

inside of $A(\theta)$. In usual manner we get a domain $D(\subset|\xi|<1)$, which contains an end part of every $A(\theta)$ for $e^{i\theta}\in E'$, and is bounded by a rectifiable curve C consisting of E' and segments lying on the boundary of $A(\theta)$ ($e^{i\theta}\in E'$) and further an analytic curve. Denote by ξ_i points in $|\xi|<1$ where $\frac{dz}{d\xi}=\infty$. We can suppose $\left|\frac{dz}{d\xi}\right|>1$ ($\xi\in D$), therefore the characteristic function $T\left(\frac{dz}{d\xi}\right)$ of $\frac{dz}{d\xi}$ is bounded, which implies that $\sum_i G(\xi, \xi_i)<\infty$, where $G(\xi, \xi_i)$ is the Green's function of D with its pole at ξ_i. We map conformally D onto $|\zeta|<1$ by $\xi=\xi(\zeta)$. Then E' is transformed onto a set E_ζ' of positive linear measure on $|\zeta|=1$. Since D has a rectifiable boundary, we can construct a domain D' containing an end part $|\arg(1-e^{-i\theta}\zeta)|<\frac{\pi}{4}$ for $e^{i\theta}\in E_\zeta'$, where E_ζ' is a set of positive linear measure on which $\log\left|\frac{dz}{d\xi}\right|-\sum_i G(\zeta, \zeta_i)=U(\zeta)$ tends uniformly to ∞, when ζ tends to E' inside D'. It follows that $U(\zeta)\equiv\infty$. Hence E is a set of linear measure zero.

If $\beta\frown(F^*-F)$ is mapped onto a set E_1 of a positive measure on $|\xi|=1$, we can find a set $E_1'(\subset E_1)$ of positive measure and we construct an angular domain D_1 containg the end part $A(\theta)$ for $e^{i\theta}\in E_1'$ and having a rectifiable boundary C such that $\left|\frac{dz}{d\xi}\right|<M<\infty$ on the boundary of D_1. We see at once $\varphi^{-1}(C+M_1')$ is rectifiable. On account of Riesz's theorem E_1' corresponds to a set of linear measure positive of β. This contradicts our assumption. Since b_i is covered by a finite number of subarcs, we observe that $\sum b_i\frown(F^*-F)$ is a set of harmonic measure zero with respect to V.

If $V_\rho(w_0)$ the connected piece, on $|w-w_0|<\rho$, does not covers a set larger than E_{AB}, there would exist a non constant bounded analytic function $A(w)=U(w)+iV(w)$ such that $U(w)=0$ on $|w-w_0|=\rho$. If $A(p)=A(f^{-1}(p))$ regular throught V, $A(p)$ must be a constant, because $U(p)\equiv 0$, whence there exists a closed set E^* where $A(p)$ is not regular. Since $E^*(\subset(F^*-F))$ is totally disconnected, we can find a domain $G(\subset V)$ containing $E^{**}(\subset E^*)$ such that ∂G has a positive distance from $\sum b_i+E^{**}$. We can find a non constant harmonic function $\tilde{U}(p)$ by Neumann's method such that $\tilde{U}(p)$ is bounded in F^*-G, $U(p)-\tilde{U}(p)$ is bounded and the conjugate function of $\tilde{U}(p)$ is one valued and bounded in $G-E^{**}$. Since the genus of F^* is finite, we can construct a non constant bounded analytic function on F from $\tilde{U}(p)$ with a linear form of Abel's first kind integrals. This contradicts the fact $F\in O_{AB}$. Hence $V_\rho(w_0)$ covers $|w-w_0|<\rho$ except possibly E_{AB}.

Remark. We constructed a covering surface F[12] over the z–plane such that F satisfies the conditions of theorem 3, i. e., 1°) harmonic measure of (F^*-F) is zero with respect to V (above defined) of every connected piece. 2°) F is mapped conformally onto a domain on the w–plane, whose boundary is a set of linear measure zero on the real axis, and we proved that F has not Gross's property. Hence the conditions of the theorem are sufficient to have Iversen's property but not Gross's property for F.

GROSS'S PROPERTY. It is well known

Theorem 4.[13] *Let $z=z(p)$ be a meromorphic function on an abstract Riemann surface $\in O_G$. If we denote by $p=p(z)$ its inverse function and if $p(z)$ is regular at z_0, we can continue $z(p)$ analytically on half lines: $z=z_0+re^{i\theta}(0\leqq r<\infty)$ except for θ of angular measure zero.*

We have, however,

Theorem 5. *A Riemann surface of O_{HP} has not necessarily Gross's property.*

In order to prove the theorem, it is sufficient to construct a Riemann surface which has not Gross's property but on which no non-constant single-valued positive harmonic function exists. As preparations, we shall prove some lemmas and define notations as follows.

Lemma 1. *Let G be a curvilinear rectangle on the z--$(=x+iy)$ plane whose sides are C_1: $-a\leqq x\leqq a$, $y=0$, C_2: $x-a=\varphi(y)$, $(0\leqq y\leqq b)$, $y=y$, C_3: $\varphi(b)+a\geqq x\geqq \varphi(b)-a$, $y=b$ and C_4: $x+a=\varphi(y)$, $y=y(b\geqq y\geqq 0)$, where $\varphi(y)$ is a continuous function such that $\varphi(0)=0$. Suppose a positive harmonic function $U(z)$ on G such that $U(z)\geqq M$ on C_2+C_4 and $U(z)\geqq 0$ on C_1+C_3, then there exists a curve connecting C_2 with C_4 on which $U(z)$ is larger than $M\omega(a, b)$, where $\omega(a, b)$ is the value at $w=\frac{ib}{2}$ of the harmonic measure of sides $(\overleftrightarrow{p_1\ p_4})+(\overleftrightarrow{p_3\ p_2})$ with respect to the rectangle, on the w–plane with vertices such that p_1: $w=-a$, p_2: $w=a$, p_3: $w=a+ib$ and p_4: $w=-a+ib$.*

Proof. We map G conformally by the function $z=f(w)$ onto a rectangle with vertices q_1: $w=-a$, q_2: $w=a$, q_3: $w=a+ib'$ and q_4: $w=-a+ib'$. Then $2a\leqq\int_{-a+\varphi}^{a+\varphi}\left|\frac{\partial f}{\partial u}\right|du$ $(w=u+iv)$, and by Schwarz's inequality we have

12) See 10).

13) R. Nevanlinna: Eindeutige Analytische Funktionen, 1936.

Z. Yujobo: On the Riemann surfaces, no Green's functions of which exists, Mathematica Japonica, No. 2 (1951), 61–68,

$$4a^2b \leq 2a\int_0^{b'}\int_{-a+\varphi}^{a+\varphi}\left|\frac{\partial f}{\partial u}\right|^2 \; dudv = 4a^2b' .$$

It follows $b \leq b'$. Consider $U(z)$ in the w-plane, then we see that $U(f(w)) \geq M\omega(a, b') \geq M\omega(a, b)$ on the segment $w = u + \frac{ib'}{2}$ $(-a \leq u \leq a)$. If we denote by l in the z-plane the image of the segment, l is the required curve.

NOTATION 1. The number $P_{n-1}(n=2, 3, \cdots ; \lim P_{n-1} = \infty)$.

Put $r_n = \frac{2^{1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}}}{4}$, $s_n = \frac{r_{n+1}-r_n}{6}$ and let R_n be a ring such that $r_{n-1} + \frac{6}{5}s_{n-1} \leq |z| \leq r_n - \frac{6}{5}s_n$ and M_n be the module of R_n: $M_n = \log \left(\frac{11-6\cdot 2^{\frac{1}{2^n}}}{6-2^{\frac{1}{2^{n-1}}}}\right)$. The transformations: $R_n \rightarrow$ the rectangle with vertices $(-\pi, 0)$, $(\pi, 0)$, (π, iM_n), $(-\pi, iM_n)$ in the ξ-plane $\rightarrow$ the upper η-half plane $\left(A = -\frac{1}{K}, B = -1, D = 1, E = \frac{1}{K}\right) \rightarrow$ the unit circle of ζ-plane are carried by $\xi = \log z$,

$$\xi = \frac{1}{h}\int_0^{\eta}\frac{d\eta}{\sqrt{(1-\eta^2)(1-K^2\eta^2)}}, \quad \zeta = \frac{(1+i)\eta + \sqrt{\frac{1}{K}}(1-i)}{(1-i)\eta + \sqrt{\frac{1}{K}}(1+i)}$$

respectively. We have $\omega(\pi, M_n) \doteq e^{-\pi^2\left(\frac{1}{M_n}\right)}/32$ by some calculations. Put $P_{n-1} = \frac{1}{\omega(\pi, M_n)}$.

NOTATION 2. $\mu_n'(\mu_n'$ is an integer) $(n=1, 2, \cdots ; \lim_n \mu_n' = \infty)$.

Let $\{I_{n\nu}\}$ $(n=1, 2, \cdots ; \nu = 1, 2, 3, \cdots 2^{\mu_n})$ be slits such that $r_n - s_n \leq |z| \leq r_n + s_n$, $\arg z = \frac{2\pi\nu}{2^{\mu_n}}$ and denote by R_n' the ring such that $r_{n-1} - \frac{11}{10}s_{n-1} \leq |z| \leq r_n + \frac{11}{10}s_n$ and $U(z)$ be a harmonic function in R_n' such that $0 \leq U(z) \leq P_{n+1}^{1+\delta}(\delta > 0)$ and $U(z) = 0$ on $\sum_{\nu=1}^{2^{\mu_n}} I_{n\nu} + \sum_{\nu=1}^{2^{\mu_{n-1}}} I_{n-1}^{\nu}$. Then there exist $\{\mu_n'\}$ such that maximum of $U(z)$ in R_n is smaller than $\frac{1}{n}$ for $\mu_n \geq \mu_n'$, $\mu_{n-1} \geq \mu_{n-1}'$, (Fig. 1).

NOTATION 3. μ_{n-1}'' and μ_n (μ_n'' and μ_n are integers) $(n=2, 3, \cdots;$

$\lim_n \mu''_{n-1}=\infty$).

Let $U(z)$ be a harmonic function as above. Then $U(z)=\frac{1}{2\pi}\int_{\partial R_n} U(\zeta)\frac{\partial g(z,\zeta)}{\partial n(\zeta)}ds(\zeta)$, where $g(z,\zeta)$ is the Green's function of R_n with pole at z. Since $\frac{\partial g(z,\zeta)}{\partial n(\zeta)}$ is continuous function of z for fixed ζ, there exist $\{\mu''_{n-1}\}$ such that $|U(z_1)-U(z_2)|<\frac{1}{n}$ for every pair z_1 and z_2 lying on the circle $|z|=\frac{r_n+r_{n-1}}{2}$ such that $|\arg z_1-\arg z_2|<\frac{2\pi}{2^{\mu_{n-1}}}$ for $\mu_{n-1}\geqq\mu''_{n-1}$. Put $\mu_n=\text{Max}\,(\mu_n', \mu_n'')$.

NOTATION 4. The number N_n (N_n is an integer) $(n=1, 2, 3, \cdots)$.

Let $U(z)$ be a harmonic function on R_n' such that $0\leqq U(z)\leqq P_{n+1}^{1+\delta}$ on R_n', $U(z)\leqq\frac{3}{n}$ on $\sum_\nu I_{n-1,\nu}$, and $U(z)\leqq\frac{3}{n+1}$ on $\sum_\nu I_{n\nu}$ except possibly a set of measure smaller than length of $I_{n-1\nu}/N'$, length of $I_{n\nu}/N''$ respectively. Since, if N' and $N''=\infty$, we have $U(z)\leqq\frac{1}{n}$ on R_n by Notation 1, there exist $\{N_n\}$ such that $U(z)\leqq\frac{3}{n}$ on R_n for $N'\geqq N_{n-1}$, and $N''\geqq N_n$, (Fig. 1).

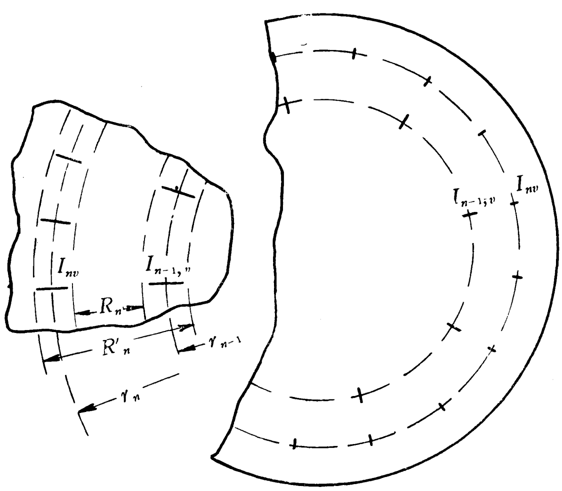

Fig. 1

Lemma 2. *Let G be a domain in the z-plane with boundaries consisting of analytic curves $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}, \gamma_n$. Map G conformally onto a ring R_ζ in the ζ-plane such that $1\leqq|\zeta|\leqq e^{\mathfrak{M}}$ so that $\gamma_1, \gamma_n, \gamma_2, \cdots, \gamma_{n-1}$ may correspond to $|\zeta|=1$, $|\zeta|=e^{\mathfrak{M}}$ and radial slits respectively in this ring. Let $U(z)$ be harmonic in G with boundary value $\varphi_1(z)$ on γ_1 and $\varphi_2(z)$ on γ_n respectively, where $\varphi_i(z)$ $(i=1, 2)$ is a continuous function of z. Then*

$$D_G(U(z))=D_{R_\zeta}(U(\zeta))\geqq\frac{1}{2\pi\mathfrak{M}}\int_0^{2\pi}|\varphi_1(\theta)-\varphi_2(\theta)|^2d\theta\,,$$

where $U(\zeta)$ and $\varphi_i(\theta)$ $(i=1, 2)$ are transformed functions from $U(z)$ and $\varphi_i(z)$.

Proof. Let $\tau(\zeta)$ be a harmonic function on R_ζ such that $\tau(\zeta)=\varphi_1(\theta)$

on $|\zeta|=1$, $\tau(\zeta)=\varphi_2(\theta)$ on $|\zeta|=e^{\mathfrak{M}}$ and $\frac{\partial\tau}{\partial n}=0$ on radial slits. We can prove easily $D(U(\zeta))\geqq D(\tau(\zeta))$.

Now we divide the ring into sufficiently narrow circular regtangles A_j: $1\leqq|\zeta|\leqq e^{\mathfrak{M}}$, $\theta_j\leqq\arg\zeta<\theta_{j+1}$ $(j=1, 2, \cdots, m)$ such that Max $\varphi_1^j(\theta)-$ Min $\varphi_1^j(\theta)\leqq\frac{1}{n}$ and Max $\varphi_2^j(\theta)-$Min $\varphi_2^j(\theta)\leqq\frac{1}{n}$, where Max $\varphi_i^j(\theta)$, Min $\varphi_i^j(\theta)$ is the maximunm and minimum of $\varphi_i(\theta)$ $(i=1, 2, \theta_j\leqq\theta\leqq\theta_{j+1})$ respectively.

Let $\{A_j'\}$ and $\{A_j''\}$ be A_j such that Max $\varphi_1^j(\theta)\leqq$Min $\varphi_2^j(\theta)$ and Min$_1$ $\varphi^j(\theta)\geqq$Max $\varphi_1^j(\theta)$ respectively and $\{A_j'''\}$ be rectangles contained neither in $\{A_j'\}$ nor in $\{A_j''\}$. If $A_j\in\{A_j'\}$, let $\tilde{U}_j(\zeta)$ be a harmonic function in A_j such that $\tilde{U}_j(\zeta)=\varphi_1(\theta)$ on γ_1, $\tilde{U}_j(\zeta)=\varphi_2(\theta)$ on γ_n and $\frac{\partial\tilde{U}_j(\zeta)}{\partial n}=0$ on two segments: $1\leqq|\zeta|\leqq e^{\mathfrak{M}}$, arg $\zeta=\theta_j$, $1\leqq|\zeta|\leqq e^{\mathfrak{M}}$, arg $\zeta=\theta_{j+1}$ and on radial slits in A_j. Let $U_j^*(\zeta)$ be a harmonic function in A_j such that $U_j^*(\zeta)=$Max $\varphi_1^j(\theta)$ on γ_1, $U_j^*(\zeta)=$Min $\varphi_2^j(\theta)$ on γ_n and $\frac{\partial U_j^*(\zeta)}{\partial n}=0$ on boundary segments and radial slits. Since $\tilde{U}_j(\zeta)-U_j^*(\zeta)\leqq0$ and $\frac{\partial U_j^*(\zeta)}{\partial n}\leqq0$ on γ_1 and $\tilde{U}_j(\zeta)-U_j^*(\zeta)\geqq0$ and $\frac{\partial U_j^*(\zeta)}{\partial n}\geqq0$ on γ_n, we have $D(U_j^*(\zeta), \tilde{U}_j(\zeta)-U_j^*(\zeta))\leqq0$. Clearly $D_{A_j}(U_j^*(\zeta))=\frac{(\theta_{j+1}-\theta_j)}{2\pi\mathfrak{M}}\,|$ Max $\varphi_2^j(\theta)-$ Min $\varphi_1^j(\theta)|^2$ and $D_{A_j}(U(\zeta))\geqq D_{A_j}(\tilde{U}_j(\zeta))\geqq D_{A_j}(U_j^*(\zeta))+D_{A_j}(\tilde{U}_j(\zeta)-U_j^*(\zeta))+2D_{A_j}(U_j^*(\zeta), (\tilde{U}_j(\zeta)-U_j^*(\zeta))\geqq D_{A_j}(U_j^*(\zeta))\geqq\frac{1}{2\pi\mathfrak{M}}\int_{\theta_j\leqq\theta<\theta_{j+1}}|\varphi_2(\theta)-\varphi_1(\theta)|^2d\theta-\frac{4(\theta_{j+1}-\theta_j)}{2\pi n^2}$. If $A_j\in\{A_j'')$, we can prove similarly the above inequality. If $A_j\in\{A_j'''\}$, let $U_j^*(\zeta)\equiv0$. Then

$$D_{A_j}(U(\zeta))\geqq 0\geqq\frac{1}{2\pi\mathfrak{M}}\int_{\theta_j\leqq\theta<\theta_{j+1}}|\varphi_2(\theta)-\varphi_1(\theta)|^2d\theta-\frac{4(\theta_{j+1}-\theta_j)}{2\pi n^2}.$$

Hence

$$D_{R_\zeta}(U(\zeta))\geqq\sum_{j=1}^{m}\left(\frac{1}{2\pi\mathfrak{M}}\int_{\theta_j\leqq\theta<\theta_{j+1}}|\varphi_2(\theta)-\varphi_1(\theta|^2d\theta\right)-\frac{4}{n^2}.$$

Let $n\to\infty$. We have the lemma.

Lemma 3. *Let R_1 and R_2 be rings $1\leqq|\zeta|\leqq e^{\beta}$ with the same slits $\{{}^iS^k\}$ $(i=1, 2,)$ such that*

$${}^1S^k:\ e^{\frac{\beta}{6}}\leqq|\zeta|\leqq e^{\frac{2\beta}{6}},\ \arg\zeta=\frac{2k\pi}{l}\ (k=1, 2, \cdots, l)$$

$${}^2S^k:\ e^{\frac{5\beta}{6}}\leqq|\zeta|\leqq e^{\frac{5\beta}{6}},\ \arg\zeta=\frac{2k\pi}{l}\ (\qquad ,, \qquad).$$

Connect R_1 with R_2 on $\{{}^iS^k\}$, $(i=1, 2;\ k=1, 2, \cdots, l)$ crosswise. Then we have two-sheeted Riemann surface R. Denote by $\tilde{\zeta}$ a point such that $\tilde{\zeta}$ has the same projection as ζ and let $U(\zeta)$ be a single-valued positive harmonic function on R such that $U(\zeta)\leqq P$ on R. Then

$$|V(\zeta)|=|U(\zeta)-U(\tilde{\zeta})|<\lambda P$$

at every point whose projection lies on $|\zeta|=e^{\frac{\beta}{2}}$, where λ (<1) depends continuously on the ratio $\frac{\beta}{l}$ only.

Proof. Map R conformally onto a strip R_η by $\eta=\log\zeta$ and consider $V(\zeta)=V(\log\eta)$ on this strip. Then $V(\eta)$ is harmonic and vanishes at every end points of the image $\{{}^i_\eta S^k\}$ of $\{{}^iS^k\}$, and has the same absolute value with opposite sign on two sides of every $\{{}^i_\eta S^k\}$. Now we fix η. Let $q, \hat{q}$ be two points facing each other on two sides of $\{{}^i_\eta S^k\}$ and let $\{{}^i_\eta \hat{S}^k\}$ be a set of point $\hat{q}$ on $\{{}^i_\eta S^k\}$ (one of q and $\hat{q}$) such that $\frac{\partial g(\hat{q}, \eta)}{\partial n(q)}\geqq\frac{\partial g(q, \eta)}{\partial n(q)}\geqq 0$.

Let $V^*(\eta)$ be a harmonic function on R_η such that $V^*(\eta)=P$ on $\mathrm{Re}\,\eta=1$, $\mathrm{Re}\,\eta=\beta$ and $\{{}^i_\eta \hat{S}^k\}$ and that $V^*(\zeta)=0$ on the remaining boundary of R_η. Since $\int_{\partial R_\eta}\frac{\partial g(q, \eta)}{\partial n}ds(q)=2\pi$, we have $V^*(\zeta)=\frac{1}{2\pi}\int_{\partial R_\eta-\{\hat{S}\}} P\frac{\partial g(q, \zeta)}{\partial n}ds<\lambda(\eta)P$, $(\lambda(\eta)<1)$. On the other hand, $V^*(\eta)=V^*\left(\eta+\frac{2\pi}{l}\right)$ and $\lambda(\eta)$ is continuous with respect to η. Then we have $V^*(\eta)\leqq\lambda P(\lambda)$ on $\mathrm{Re}\,\eta=\frac{\beta}{2}$. By maximum and minimum principle, we have $|V(\eta)|\leqq|V^*(\eta)|<\lambda P$, where λ depends continuously on the ratio $\frac{\beta}{l}$. In the following we fix two bounds M_1 and M_2 $(M_1<\frac{\beta}{l}<M_2)$ so that λ may be always smaller than λ_0 $(\lambda_0<1)$.

A) In the z-plane denote by $J_{n\nu}$ (n is fixed; $\nu=1, 2, \cdots, 2^{\mu_n}$) a circular rectangle, containing $I_{n\nu}$, with sides $\partial J_{n\nu}$ as follows:

$$|z|=r_n-3s_n,\ \frac{2\pi\nu}{2^{\mu_n}}-\frac{\pi}{2^2\cdot 2^{\mu_n}}\leqq\arg z\leqq\frac{2\pi\nu}{2^{\mu_n}}+\frac{\pi}{2^2\cdot 2^{\mu_n}},$$

$$r_n-3s_n\leqq|z|\leqq r_n+3s_n,\ \arg z=\frac{2\pi\nu}{2^{\mu_n}}-\frac{\pi}{2^2\cdot 2^{\mu_n}},$$

$$|z|=r_n+3s_n,\ \frac{2\pi\nu}{2^{\mu_n}}-\frac{\pi}{2^2\cdot 2^{\mu_n}}\leqq\arg z\leqq\frac{2\pi\nu}{2^{\mu_n}}+\frac{\pi}{2^2\cdot 2^{\mu_n}},$$

$$r_n-3s_n\leqq|z|\leqq r_n+3s_n,\ \arg z=\frac{2\pi\nu}{2^{\mu_n}}+\frac{\pi}{2^2\cdot 2^{\mu_n}}.$$

Let $L_{n\nu}$, $L'_{n\nu}$ be half straight lines such that

$$L_{n\nu}:\quad 0 \leqq |z| < \infty,\ \arg z = \frac{2\pi\nu}{2^{\mu_n}} + \pi - \frac{\pi}{2\cdot 2^{\mu_n}},$$

$$L'_{n\nu}:\quad 0 \leqq |z| < \infty,\ \arg z = \frac{2\pi\nu}{2} + \pi + \frac{\pi}{2\cdot 2^{\mu_n}} \quad \text{(Fig. 2)}.$$

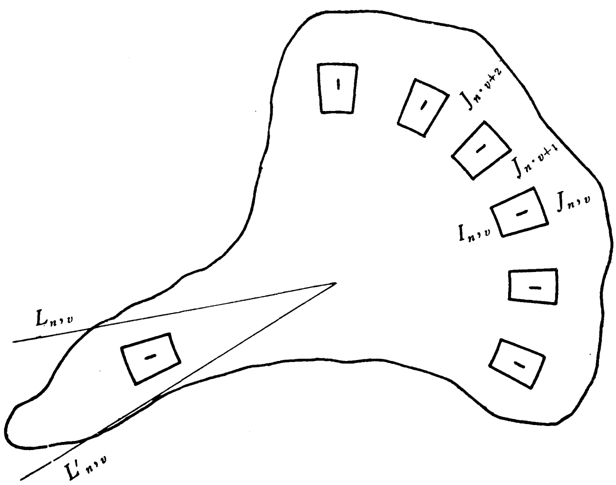

Fig. 2

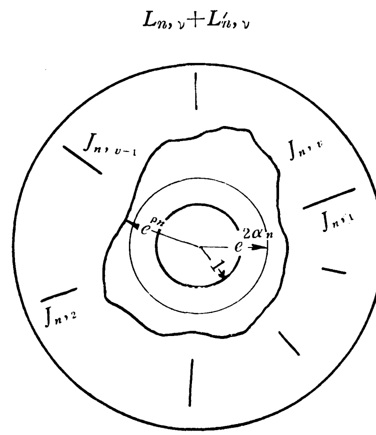

Fig. 3

We denote by $G_{n\nu}$ the domain with boundaries $L_{n\nu}$, $L'_{n\nu}$, $\sum_{i\neq\nu,\nu'} \partial J_{ni}$ $\left(\nu' = \nu + \frac{\mu^n}{2}\right)$ and $I_{n\nu}$, and map it conformally by $w = \varphi(z)$ onto the ring $1 \leqq |w| \leqq e^{\mathfrak{M}_n^{(1)}}$ so that $L_{n\nu} + L'_{n\nu}$, $I_{n\nu}$ $\sum_{i\neq\nu'} \partial J_{ni}$ may be transformed onto $|w| = e^{\mathfrak{M}_n^{(1)}}$, $|w| = 1$ and radial slits $\sum_{i\neq\nu'} \partial J_{ni(w)}$ respectively. In this mapping any measurable set of positive angular measure $\leqq \frac{2}{k_n}$ on $|w| = 1$ mapped onto a measurable set of positive linear measure $\leqq$ length of $I_{n\nu}/N_n$ on $I_{n\nu}$, where κ_n depends only on the mapping function $w = \varphi(z)$ and does not depend on the situation of the set on $|w| = 1$. And the doubly connected domain bounded by $\partial J_{n\nu}$ and $I_{n\nu}$ of module $\mathfrak{M}_n^{(2)}$ is mapped onto a domain bounded by $|w| = 1$ and the image $\partial J_{n\nu(w)}$ of $J_{n\nu}$.

Let e^{ρ_n} be the distance between $\partial J_{n\nu(w)}$ and the point $w = 0$.

Put

$$\rho_n' = \frac{\mathfrak{M}_n^{(2)}}{2\pi\kappa_n(n+1)^2 P_{n+1}^{2+2\delta}\cdot 2^{\mu_n}}. \tag{1}$$

We choose α_n so that $e^{\alpha_n} \leqq e^{\rho'_n}$ and $e^{2\alpha_n} \leqq e^{\rho_n}$ (Fig. 3).

Let s_n and q_n $(n=1, 2, \cdots)$ integers such that

$$\frac{s_n}{2\pi(n+1)^2\kappa_n(\mathfrak{M}_n^{(1)}-\alpha_n)} \geqq \frac{P_{n+1}^{2+2\delta}\,2^{\mu_n}}{\mathfrak{M}_n^{(2)}}, \tag{2}$$

$$2P_{n+1}^{1+\delta}\,\lambda_0^{q_n-s_n} \leqq \frac{1}{n+1}. \tag{3}$$

In the ring: $1\leqq|w|\leqq e^{2\alpha_n}$, we define, two systems of rings $\{{}^\alpha C_{n\nu}^{ij}\}$, $(\alpha=1, 2;\ n, \nu$ are fixed; $i\geqq j, i, j,=1, 2, \cdots, q_n)$ and a circle $H_{n\nu}$ as follows:

$${}^1C_{n\nu}^{ij}:\quad 2\alpha_n-\gamma(i(i-1)+2j-1)\geqq \log|w|\geqq 2\alpha_n-\gamma(i(i-1)+2j),$$

$${}^2C_{n\nu}^{ij}:\quad \gamma(i(i-1)+2j-1)\leqq \log|w|\leqq \gamma(i(i-1)+2j),$$

$$H_{n\nu}:\quad \log|w|=\alpha_n,$$

where $\gamma=\dfrac{\alpha_n}{(q_n^2+q_n+1)}$ and $M_1\leqq\dfrac{\gamma}{l_n}\leqq M_2$ (see lemma 3).

Let $\{'S_{ijk}^{n\nu}\}$, $\{S_{ijk}^{n\nu}\}$, $\{\tilde{S}_{ijk}^{n\nu}\}$, $\{'\tilde{S}_{ijk}^{n\nu}\}$ $(n, \nu$ are fixed, $i\geqq j, i, j=1, 2, \cdots, q_n;\ k=1, 2, \cdots, l_n)$ be slits such that $\{'S_{ijk}^{n\nu}\}$, $\{S_{ijk}^{n\nu}\}$, $\{\tilde{S}_{ijk}^{n\nu}\}$, $\{'\tilde{S}_{ijk}^{n\nu}\}$ are contained in $\{C_{n\nu}^{ij}\}$ and $\{\tilde{C}_{n\nu}^{ij}\}$ respectively, (Fig. 4).

$${}'S_{ijk}^{n\nu}:\ 2\alpha_n-\gamma\left(i(i-1)+2j-1+\frac{2}{6}\right)\leqq \log|w|\leqq 2\alpha_n-\gamma\left(i(i-1)+2j-1+\frac{1}{6}\right),\quad \arg w=\frac{2\pi k}{l_n}$$

$$S_{ijk}^{n\nu}:\ 2\alpha_n-\gamma\left(i(i-1)+2j-1+\frac{5}{6}\right)\leqq \log|w|\leqq 2\alpha_n-\gamma\left(i(i-1)+2j-1+\frac{4}{6}\right),\quad \arg w=\frac{2\pi k}{l_n}$$

$$\tilde{S}_{ijk}^{n\nu}:\ 2\gamma\left(i(i-1)+2j-1+\frac{4}{6}\right)\leqq \log|w|\leqq \gamma\left(i(i-1)+2j-1+\frac{5}{6}\right),\quad \arg w=\frac{2\pi k}{l_n}$$

$${}'\tilde{S}_{ijk}^{n\nu}:\ \gamma\left(i(i-1)+2j-1+\frac{1}{6}\right)\leqq \log|w|\leqq \gamma\left(i(i-1)+2j-1+\frac{2}{6}\right),\quad \arg w=\frac{2\pi k}{l_n}$$

B) Let $F_n(1), F_n(2), \cdots, F_n(2^{q_n-1})$, $\hat{F}_n(1), \hat{F}_n(2), \cdots, \hat{F}_n(2^{q_n-1})$ be 2^{q_n} leaves of rings with slits $\{\{'S_{ijk}^{n\nu}\}, \{S_{ijk}^{n\nu}\}, \{\tilde{S}_{ijk}^{n\nu}\}, \{'\tilde{S}_{ijk}^{n\nu}\}$ $(n, \nu$ are fixed; $i\geqq j, i, j, 1, 2, \cdots, q_n;\ k=1, 2, \cdots, l_n)$. Connect $F_n(1)$ and $\hat{F}_n(1)$ $(i=1, 2, \cdots, 2^{q_n-1})$ crosswise on $\{'S_{i1k}^{n\nu}\}$, $\{S_{i1k}^{n\nu}\}$, $\{\tilde{S}_{i1k}^{n\nu}\}$, $\{'\tilde{S}_{i1k}^{n\nu}\}$ $(n, \nu$ are fixed $i=1, 2, \cdots, q_n;\ k=1, \cdots, l_n)$ and call this connection 1-connecting.

2-Connecting is as follows:

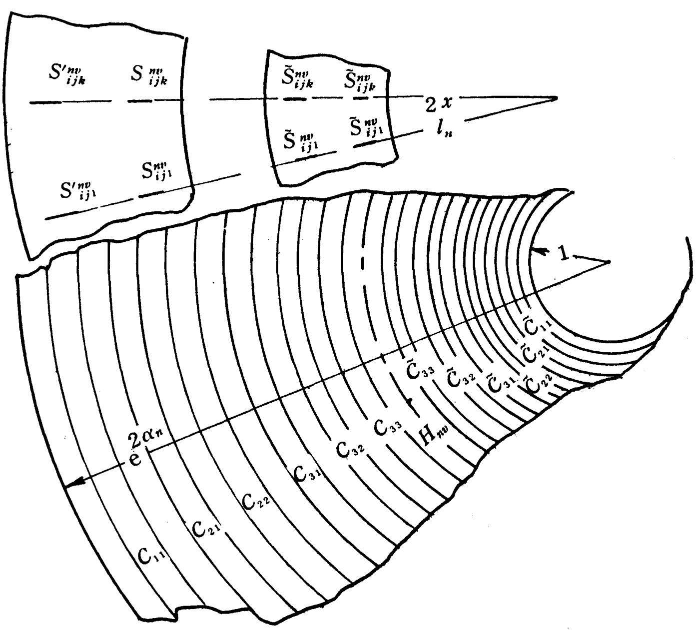

Fig. 4

$$
\begin{array}{cccccc}
F_n(1) & F_n(2) & F_n(3) & F_n(4) \cdots F_n 2^{q_{n-1}}-1) & F_n(2^{q_{n-1}}) \\
\hat{F}_n(1) & \hat{F}_n(2) & \hat{F}_n(3) & \hat{F}_n(4) \cdots \hat{F}_n(^{q_{n-1}}-1) & \hat{F}_n(2^{q_{n-1}})
\end{array}
$$

on $\{'S^{n\nu}_{i2k}, S^{n\nu}_{i2k}, \tilde{S}^{n\nu}_{ijk}, '\tilde{S}^{n\nu}_{ijk}\}$.

In general j_0-connecting is ($j_0=1, \cdots, q_n$)

$F_n(a+1)$, $F_n(a+2)$, $\cdots F_n(2+2^{j_0-2})$, $F_n(a+2^{j_0-2}+1)$, $F_n(a+2^{j_0-2}+2)$, $\cdots$, $F_n(a+2^{j_0-1})$,

$F_n(a+1)$, $F_n(a+2)$, $\cdots F_n(a+2^{j_0-2})$, $F_n(a+2^{j_0-2}+1)$, $F_n(a+2^{j_0-2}+2)$, $\cdots$, $F_n(a+2^{j_0-1})$,

on $\{'S^{n\nu}_{ij_0k}, S^{n\nu}_{ij_0k}, \tilde{S}^{n\nu}_{ij_0k}, '\tilde{S}^{n\nu}_{ij_0k}\}$ (n, ν are fixed; $i=j_0, j_0+1, \cdots, q_n$; $k=1, 2, \cdots, l_n$),

where $$a=0,\ 2^{j_0-1},\ 2\cdot 2^{j_0-1}, \cdots , (2^{q_{n-1}}-1)2^{j_0-1}.$$

Then we have 2^{q_n}-sheeted covering surface $R_{n\nu}$. Let $U(p)$ be a positive harmonic function on $R_{n\nu}$ such that $U(p)\leqq P^{1+\delta}_{n+1}$ and $T_{j_0}(P)$ be the conformal mapping $p\leftrightarrow\tilde{p}$, where p and $\tilde{p}$ are points such that p and $\tilde{p}$ have the same projections and are lying on the leaves respectively which are connected by arrows in the above schema. Consider $V(p)=U(p)-U(T_{j_0}(p))$ on $F(1)+\hat{F}(j_0)$ of $R_{n\nu}$. Then $V(p)$ is harmonic and vanishes at endpoints of $\{'S^{n\nu}_{ij_0k}, S^{n\nu}_{ij_0k}, \tilde{S}^{n\nu}_{ij_0k}, '\tilde{S}^{n\nu}_{ij_0k}\}$ n, ν fixed; $i=j_0, j_0+1, \cdots, q_n, k=1, 2, \cdots, l_n)$ and further harmonic on the remaining $\{'S^{n\nu}_{ijk}, S^{n\nu}_{ijk}, \tilde{S}^{n\nu}_{ijk}, 'S^{n\nu}_{ijk}\}$. We have by the lemma and maximum principle

$$|V(p)| \leqq P^{1+\delta}_{n+1}\lambda_0 \text{ on } 2q_n\lambda \leqq \log|w| \leqq 2\alpha_n - 2q_n\gamma,$$
$$|V(p)| \leqq P^{1+\delta}_{n+1}\lambda_0^2 \text{ on } (4q_n-2)\gamma \leqq \log|w| \leqq 2\alpha_n-(4q_n-2)\gamma,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$|V(p)| \leqq P^{1+\delta}_{n+1}\lambda_0^{q_n-j_0+1} \text{ on } \log|w|=\alpha_n.$$

If we denote by $|F_n(1)-\hat{F}_n(j)|$ the maximum of $|U(p_1)-U(p_2)|$, where p_1 and p_2 have the same projections and lie on $H_{n\nu}$ of $F(1)$ and $\hat{F}(j)$ respectively, we have $|F(1)-\hat{F}(2^{j-1})|<P^{1+\delta}_{n+1}\lambda_0^{q_n-j+1}$. Taking account of the property s_n and q_n, we see that there exist at least s_n leaves such that $|F(1)-\hat{F}(i)|<\dfrac{1}{n+1}$ $(i=i_1, \cdots, i_{tn})$.

Construction of the Surface.

We mapped $G_{n\nu}$ by $w=\varphi(z)$ onto the ring $1\leqq|w|\leqq e^{\mathfrak{M}^{(1)}_n}$ and defined slits $\{'S^{n\nu}_{ijk}, S^{n\nu}_{ijk}, \tilde{S}^{n\nu}_{ijk}, '\tilde{S}^{n\nu}_{ijk}\}$ and $H_{n\nu}$. Consider the inverse image in the z-plane of them and denote them by the same letters. Then $\{H_{n\nu}\}$ (n is fixed, $\nu=1, 2, \cdots, 2^{\mu_n}$) are approximate ellipses enclosing $\{I_{n\nu}\}$, and $\{'\tilde{S}^{n\nu}_{ijk}, \tilde{S}^{n\nu}_{ijk}\}$ $(i\geqq j, i, j=1, 2, \cdots, q_n; k=1, \cdots, l_n)$ lie approximately radially in the approximate ring bounded by $I_{n\nu}$ and $H_{n\nu}$ and $\{'S^{n\nu}_{ijk}, S^{n\nu}_{ijk}\}$ lie approximately radially outside of $H_{n\nu}$. Denote by $F_m(1)$ the z-plane with the slits $I_{m\nu}$ and $\{'S^{m\nu}_{ijk}, S^{m\nu}_{ijk}, \tilde{S}^{m\nu}_{ijk}, '\tilde{S}^{m\nu}_{ijk}\}$ and by $F_m(2)\cdots F_m(2^{q_m-1})$, $\hat{F}_m(1)$, $\hat{F}_m(2)\cdots\hat{F}_m(2^{q_m-1})$ the z-plane with slits $\{'S^{m\nu}_{ijk}, S^{m\nu}_{ijk}, \tilde{S}^{m\nu}_{ijk}, '\tilde{S}^{m\nu}_{ijk}\}$, where m is fixed; $\nu=1, 2, \cdots, 2^{\mu_m}$; $i\geqq j, i, j=1, 2, 3, \cdots, q_m$; $k=1, 2, \cdots, l_m$. Let F_{0n} be the part of F_0 such that $0\leqq|z|\leqq r_n+3s_n$, where F_0 is the unit circle with $\{I_{n\nu}\}$ $(n=1, 2, \cdots)$. Connect F_{0n} with $\{F_m(1)\}$ on $\{I_{m\nu}\}$ $(\nu=1, 2, \cdots, 2^{\mu_m})$ crosswise and connect every $F_m(1)$ with $F_m(l)$ and $\hat{F}_m(l')$ $(l=2, 3, \cdots, 2^{q_m-1}, l'=1, 2, \cdots, 2^{q_m-1})$ on $\{'S^{m\nu}_{ijk}, S^{m\nu}_{ijk}, \tilde{S}^{m\nu}_{ijk}, '\tilde{S}^{m\nu}_{ijk}\}$ in the manner mentioned in (B) $(m=1, 2, \cdots, n)$. Then we have a Riemann surface denoted by $\mathfrak{F}_n$. $\mathfrak{F}_n$ covers the part $|z|\leqq r_n+3s_n$ $1+2^{q_1}; \cdots +2^{q_n}$ times and covers the part $|z|\geqq r^n+3s_n$ $2^{q_1}+\cdots+2^{q_n}$ times. Put $\bigcup_n \mathfrak{F}_n=F$. Then F is the required Riemann surface, (Fig. 5).

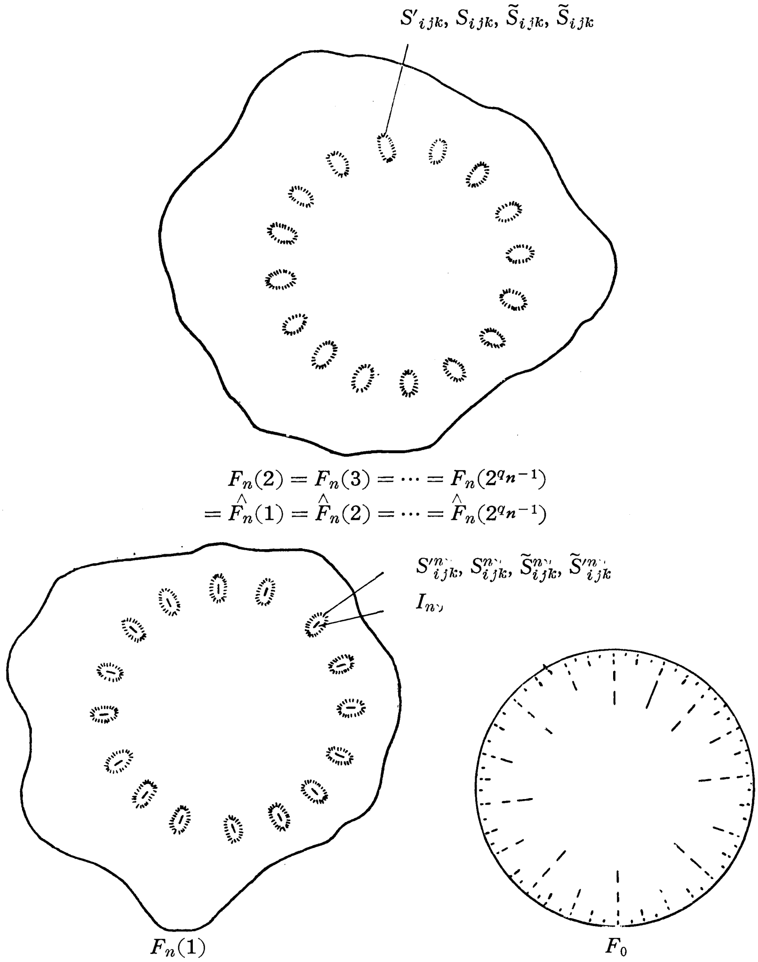

Fig. 5

Proof of the theorem.

Let $|z|=r_{n+1}-\frac{6}{5}s_{n+1}$ be a circle on F_0, which is a dividing cut of F. If we denote by $\mathfrak{F}_n'$ the compact surface which is one of the two divided by the above cut, then $F=\bigcup_n \mathfrak{F}_n'$. Denote by L_n the maximum of $U(p)$ on $|z|=r_n+\frac{6}{5}s_n$ on F_0. Let $U(p)=L_n$ on p_0 on $|z|=r_n+\frac{6}{5}s_n$ of F_0, we see, by maximum principle, that there exists a Jordan curve C joining two boundary components of the ring $r_n+\frac{6}{5}s_n\leqq|z|\leqq r_{n+1}-\frac{6}{5}s_{n+1}$ such that $U(p)\geqq L_n$ on C. Then by notation 1, there exists a divinding cut l in the ring such that $U(p)\geqq\frac{L_n}{P_n}$ on l. If $\overline{\lim_{n=\infty}}L_n\geqq P_n^{1+\delta}$,

$U(p)$ must be constant infinity by minimum principle. Hence, without loss of generality, we can suppose

$$\overline{\lim_{n=\infty}}\, L_n < P_n^{1+\delta}.$$

$(\mathfrak{F}_n - \mathfrak{F}_{n-1})$ is the Riemann surface which consists of the ring, of F_0 such that $r_{n-1} + 3s_{n-1} \leqq |z| \leqq r_n + 3s_n$ and leaves $\sum_{m=1}^{2^{q_n-1}} ((F_n(m) + \hat{F}_n(m))$. $(\mathfrak{F}_n - \mathfrak{F}_{n-1})$ has two circles $|z| = r_n + 3s_n$ and $|z| = r_{n-1} + 3s_{n-1}$ as its boundary components γ_n and γ_n' respectively.

Let $T_\lambda(p)$ be a conformal mapping in $(\mathfrak{F}_n - \mathfrak{F}_{n-1})$ $p \leftrightarrow \tilde{p}$, where $\tilde{p}$ is the symmetric point of p with respect to the straight line $\overset{*}{L}_{n\lambda} + \overset{*}{L}{}'_{n\lambda}$ such that $\overset{*}{L}_{n\lambda}$: $0 \leqq |z| < \infty$, $\arg z = \frac{2\pi\lambda}{2^{\mu_n}} + \frac{\pi}{2}$. $0 \leqq |z| < \infty$, $\overset{*}{L}{}'_{n\lambda}$: $\arg z = \frac{2\pi\lambda}{2^{\mu_n}} - \frac{\pi}{2}\left(\lambda = 1, 2, \cdots, \frac{2^{\mu_n}}{2}\right)$. Put $U_\lambda(p) = U(p) - U(T_\lambda(p))$. Then $|U_\lambda(p)|$ is subharmonic on $(\mathfrak{F}_n - \mathfrak{F}_{n-1})$, $\leqq P_{n+1}^{1+\delta}$ on $\gamma_n + \gamma_n'$ and vanishes at every point whose projection is on $\overset{*}{L}_{n\lambda} + \overset{*}{L}{}'_{n\lambda}$. Let $(\mathfrak{F}_n - \mathfrak{F}_{n-1})^*$ be the surface consisting of $\sum_{\nu=1}^{2^{\mu_n}} J_{n\nu}$ of F_0 and $\sum_{m=1}^{2^{q_n-1}} (F_n(m) + \hat{F}_n(m))$, and let $(\mathfrak{F}_n - \mathfrak{F}_{n-1})_k'$ be the part of $(\mathfrak{F}_n - \mathfrak{F}_{n-1})'$ which is over the part of the z-plane such that

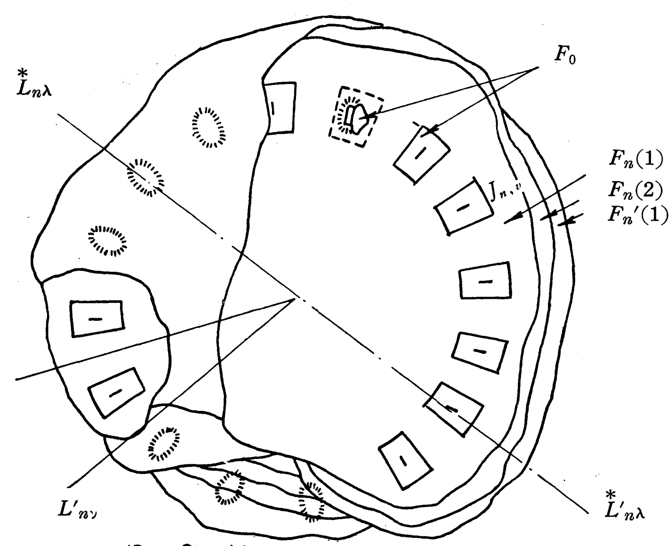

Fig. 6

$$0 \leqq |z| < \infty,\ \frac{2\pi k}{2^{\mu_n}} + \pi - \frac{\pi}{2\cdot 2^{\mu_n}} \leqq \arg z \leqq \frac{2\pi k}{2^{\mu_n}} + \pi + \frac{\pi}{2\cdot 2^{\mu_n}}$$

$$\left(\text{containing } \frac{2\pi k}{2^{\mu_n}}\right), \quad \text{if} \quad 1 \leqq k \leqq \frac{2^{\mu_n}}{2}.$$

$$0 \leqq |z| < \infty,\ \frac{2\pi k}{2^{\mu_n}} - \pi - \frac{\pi}{2\cdot 2^{\mu_n}} \leqq \arg z \leqq \frac{2\pi k}{2^{\mu_n}} - \pi + \frac{\pi}{2\cdot 2^{\mu_n}}$$

$$\left(\text{containing } \frac{2\pi k}{2^{\mu_n}}\right), \quad \text{if} \quad \frac{2^{\mu_n}}{2} \leqq k \leqq 2^{\mu_n}. \qquad \text{(Fig. 6).}$$

On the other hand let $V_k(p)$ be a harmonic function on $(\mathfrak{F}_n - \mathfrak{F}_{n-1})_k'$ such that $V_k(p) = P_{n+1}^{1+\delta}$ on the boundary of $\sum_{\nu \neq k + \frac{2^{\mu_n}}{2}} \partial J_{n\nu}$ and vanishes at every point whose projection lies on the straight lines L_{nk} and L'_{nk}, where L_{nk}: $0 \leqq |z| < \infty$, $\arg z = \frac{2k\pi}{2^{\mu_n}} + \pi + \frac{\pi}{2\cdot 2^{\mu_n}}$, L'_{nk}: $0 \leqq |z| < \infty$, $\arg z = \frac{2k\pi}{2^{\mu_n}} + \pi - \frac{\pi}{2\cdot 2^{\mu_n}}$.

Since $|U_\lambda(p)|$ is subharmonic and the angular domain bounded by $L_{nk} + L'_{nk}$ contains the half plane bounded by $\overset{*}{L}_{n\lambda}$ and $\overset{*}{L'}_{n\lambda}$, we have $V_k(p) \geqq |U_\lambda(p)|$, where

$$k \neq \lambda + \frac{1}{4}2^{\mu_n},\ \neq \lambda - \frac{1}{4}2^{\mu_n} \ \text{ if } \frac{3}{4}2^{\mu_n} \geqq \lambda \geqq \frac{1}{4}2^{\mu_n},$$

$$k \neq \lambda + \frac{1}{4}2^{\mu_n},\ \neq \lambda + \frac{3}{4}2^{\mu_n} \ \text{ if } \frac{1}{4}2^{\mu_n} \geqq \lambda \geqq 0,$$

$$k \neq \lambda - \frac{3}{4}2^{\mu_n},\ \neq \lambda - \frac{1}{4}2^{\mu_n} \ \text{ if } 2^{\mu_n} \geqq \lambda \geqq \frac{3}{4}2^{\mu_n}.$$

In order to estimate the value of $U_\lambda(p)$ on I_{ns} $(s = \lambda, \lambda+1, \cdots, \lambda + \frac{1}{4}2^{\mu_n} - 1,\ \lambda = \frac{3}{4}2^{\mu_n} + 1, \cdots, 2^{\mu_n}, 1, 2, \cdots, \lambda - 1)$. We consider $V_k(p)$ on I_{nk}, (Fig. 6).

Let $V(p)$ be a harmonic function on $\sum_{\nu=1}^{2^{\mu_n}} (J_{n\nu} - I_{n\nu})$ such that $V(p) = 0$ on $\sum_{\nu=1}^{2^{\mu_n}} I_{n\nu}$ and $V(p) = P_{n+1}^{1+\delta}$ on $\sum_{\nu=1}^{2^{\mu_n}} \partial J_{n\nu}$. Then we have by Dirichlet principle

$$\frac{2^{\mu_n} P_{n+1}^{2+2\delta}}{\mathfrak{M}_n^{(2)}} = D_{\Sigma(J-I)}(V(p)) \geqq D_{(\mathfrak{F}_n - \mathfrak{F}_{n-1})_{k'}}(V_k(p)) \tag{4}$$

Map conformally the domain of $F_n(1)$ with boundary $L_{nk} + L'_{nk} + \sum_{i \neq k, k'} \partial J_{ni} \left(k' = k + \frac{2^{\mu_n}}{2}\right) + I_{nk}$ by $w = \varphi(z)$ onto the ring $1 \leqq |w| \leqq e^{\mathfrak{M}_n^{(1)}}$ as defined in (A) and consider the composed function $V_k(p) = V_k(\varphi(z))$. Then $V_k(p) = 0$ on $|w| = e^{\mathfrak{M}_n^{(1)}}$. If $V_k(p) \geqq \frac{2}{n+1}$ on a set of angular measure larger than $\frac{1}{\kappa_n}$ on H_{nk}, there exists at least s_n leaves of $\{F_n(i), \hat{F}_n(j)\}$ by (B), whose H_{nk} have the property such that $V_k(p) \geqq \frac{1}{n+1}$ on the set of angular measure larger than $\frac{1}{\kappa_n}$ on H_{nk}. Let $D(V_k(p))$ be the Dirichlet integral over $e^{\alpha_n} \leqq |w| \leqq e^{\mathfrak{M}_n^{(1)}}$ of these leaves. Then we have by lemma (3) and (2)

$$D(V_k(p)) \geqq \frac{s_n}{2\pi(n+1)^2\kappa_n(\mathfrak{M}_n^{(1)}-\alpha_n)} \geqq \frac{P_{n+1}^{2+2\delta}2^{\mu_n}}{\mathfrak{M}_n^{(2)}}. \tag{5}$$

(5) contradicts (4), whence $V_k(p) \leqq \frac{2}{n+1}$ on H_{nk} of $F_n(1)$ except possibly a set of angular measure smaller than $\frac{1}{\kappa_n}$.

Next consider $V_k(p)$ in $1 \leqq |w| \leqq e^{\alpha_n}$ (of the image of $F_n(1)$). If $|V_n(p_2) - V_k(p_1)| \geqq \frac{1}{n+1}$ on a set of angular measure larger than $\frac{1}{\kappa_n}$, we have by lemma 3 and (1)

$$D(V_k(p)) \geqq \frac{1}{2\pi(n+1)^2\kappa_n\alpha_n} \geqq \frac{P_{n+1}^{2+2\delta}2^{\mu_n}}{\mathfrak{M}_n^{(2)}}, \tag{6}$$

where $\arg p_1 = \arg p_2$, p_1 and p_2 lie on $|w|=1$, and $|w|=e^{\alpha_n}$ respectively, and $D(V_k(p))$ is the Dirichlet integral over $1 \leqq |w| \leqq e^{\alpha_n}$ on $F_n(1)$. But (6) contradicts (4), whence $V_k(p) \leqq \frac{3}{n+1}$ on $|w|=1$ except possibly a set of angular measure smaller than $\frac{2}{\kappa_n}$. If we consider the above results in F_0, we see that $V_k(p) \leqq \frac{3}{n+1}$ on I_{nk} except a set of measure smaller than length of I_{nk}/N_n.

We see at once $U_\lambda(p)=0$ on $I_{n\beta}, I_{n\beta'}$ $\left(\beta = \lambda + \frac{2^{\mu_n}}{4}, \beta' = \lambda - \frac{2^{\mu_n}}{4}\right)$. On the other hand $V_k(p) \geqq |U_\lambda(p)|$ on I_{nk} $(k \neq \beta, \beta')$. Thus $|U_\lambda(p)| \leqq \frac{3}{n+1}$ on every $I_{n\nu}$ $(\nu = 1, 2, \cdots, 2^{\mu_n})$ except a set of linear measure smaller than the length $I_{n\nu}/N_n$. In the same manner we have in $(\mathfrak{F}_{n-1} - \mathfrak{F}_{n-2})$, $|U(p) - U(T_\lambda(p))| \leqq \frac{3}{n}$ on every $I_{n-1\nu}$ except a set of linear measure smaller than the length of $I_{n-1,\nu}/N_{n-1}$.

Consider $T_\nu(p)$ in $(\mathfrak{F}_n - \mathfrak{F}_{n-2})$ $(\nu = 1, 2, \cdots, 2^{\mu_{n-1}})$. Then we see that $|U_\nu(p)| \leqq P_{n+1}^{1+\delta}$ on the ring R_n', and that $U(p)$ is symmetric with respect to $2^{\mu_{n-1}}$ directions except at most $\frac{6}{n}$ on R_n, (See Notation 4).

Then by (Notation 3) $|U(p_1) - U(p_2)| \leqq \frac{1}{n}$ on the circle C_n: $|z_n| = \frac{r_n + r_{n-1}}{2}$ on F_0 such that $|\arg p_1 - \arg p_2| \leqq \frac{1}{2^{\mu_{n-1}}}$. If we denote by Max $U(p)$, Min $U(p)$ the maximum and minimum of $U(p)$ on C_n, we have

$$|\operatorname{Max} U(p) - \operatorname{Min} U(p)| \leqq \frac{7}{n}.$$

Denote by $\mathfrak{F}_n''$ the surface such that $\mathfrak{F}_n''$ consists of the part of F_0 on $|z| < \frac{r_n + r_{n-1}}{2}$ and $\sum_{l=1}^{n} \sum_{m=1}^{2^{q_l}-1} (F_l(m) + \hat{F}_l(m))$. Then $F = \bigcup_n \mathfrak{F}_n''$. Since every $\mathfrak{F}_n''$ has only one boundary component lying on C_n on which the oscilation of $U(p)$ is smaller than $\frac{7}{n}$. Let $n \to \infty$. $U(p)$ must reduce to a constant on account of maximum and minium principle. Hence $F \in O_{HP}$.

Since F_0 is the unit circle, it is clear that F has not Gross's property and by theorem of Gross we see $F \notin O_G$.

From this example we see that the validity of Gross's property of Riemann surface does not depend upon the complexity of the boundary. It depends rather upon the "force" of the boundary, i. e., roughly speaking upon the size of the boundary.

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