| Title | Ideal tetrahedral decompositions of hyperbolic <br> 3-manifolds |
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| Author(s) | Yoshida, Han |
| Citation | 0saka Journal of Mathematics. 1996, 33(1), p. <br> $37-46$ |
| Version Type | VoR |
| URL | https://doi.org/10.18910/10590 |
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# IDEAL TETRAHEDRAL DECOMPOSITIONS OF HYPERBOLIC 3-MANIFOLDS 

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## 1. Introduction

Since W. Thurston investigated cusped hyperbolic 3-manifolds by decomposing them into ideal tetrahedra [2], the method has become an indispensable tool for the reseachers of hyperbolic 3-manifolds. Although it is not known whether a noncompact hyperbolic 3-manifold of finite volume always admits a decomposition into ideal tetrahedra, most of us believe the following :

Conjecture. Every noncompact hyperbolic 3-manifold of finite volume admits a decomposition into convex ideal tetrahedra.

In [1], Epstein and Penner have shown that every noncompact hyperbolic 3-manifold of finite volume has a canonical decomposition into convex ideal polyhedra. Therefore, in order to prove the above conjecture it suffices to show that every hyperbolic 3 -manifold obtained by glueing ideal polyhedra admits a decomposition into ideal tetrahedra. Wada recently proved that if a noncompact hyperbolic 3-manifold $M$ consists of one convex ideal polyhedron then $M$ can be decomposed into ideal tetrahedra [3].

In this paper, we show the following theorem.
Main Theorem. Suppose that a noncompact hyperbolic 3-manifold $M$ is obtained by glueing two convex ideal polyhedra $P_{1}$ and $P_{2}$ in such a way that every face of $P_{1}$ is pasted with a face of $P_{2}$. Then $M$ can be decomposed into ideal tetrahedra.

The author would like to express her thanks to Professors Masaaki Wada and Yasushi Yamashita. Her interest in the problem in this paper comes from the discussion with them.

## 2. Cone Decomposition

Let $P$ be a polyhedron and $v$ a vertex of $P$. Let $f_{1}, \cdots, f_{r}$ denote the faces of $P$ not incident with the vertex $v$. Now take a triangulation $\tau$ of $\bigcup_{i=1}^{r} f_{i}$ without adding a new vertex. Then a triangulation $K(\tau)$ of $P$ can be defined as follows. The sets $K^{i}(\tau)$ of $i$-simplices of $K(\tau)$ are given by

$$
\begin{gathered}
K^{3}(\tau)=\left\{\left\langle v, \sigma_{2}\right\rangle \mid \sigma_{2} \text { is a } 2 \text {-simplex of } \tau\right\}, \\
K^{2}(\tau)=\left\{\left\langle v, \sigma_{1}\right\rangle \mid \sigma_{1} \text { is a } 1 \text {-simplex of } \tau\right\} \cup\{2 \text {-simplices of } \tau\}, \\
K^{1}(\tau)=\left\{\left\langle v, \sigma_{0}\right\rangle \mid \sigma_{0} \text { is a } 0 \text {-simplex of } \tau\right\} \cup\{1 \text {-simplices of } \tau\}, \\
K^{0}(\tau)=\{\text { vertices of } P\} .
\end{gathered}
$$



This polyhedron is divided into 5 tetrahedra.
figure 1

We call such a triangulation $K(\tau)$ a cone decomposition of $P$ from the vertex $v$.

We say that $M$ admits a decomposition into ideal polyhedra (tetrahedra) if $M$ is obtained by glueing together the faces of ideal polyhedra (tetrahedra). Note that taking cone decompositions of $P_{1}$ and $P_{2}$ from arbitrary vertices dose not necessary give a decomposition of $M$, since the triangulations induced by those of $P_{1}$ and $P_{2}$ on identified faces might not agree.

## 3. Good Vertex

By $V_{i}, E_{i}$, and $F_{i}$, we denote the sets of vertices, edges and faces of $P_{i}$ respectively. We decompose $F_{i}=\bigcup_{n 23} F_{n, i}$ where $F_{n, i}$ is the set of $n$-gons of $F_{i}(i=1,2)$. Furthermore we decompose $F_{n, 2}=F_{n, 2}^{\prime} \cup F_{n, 2}^{\prime \prime}$ where $F_{n, 2}^{\prime}$ is the set of $n$-gons glued to faces of $P_{1}$ and $F_{n, 2}^{\prime \prime}$ is the set of $n$-gons glued to faces of $P_{2}$.

To prove the main theorem, it suffices to show that we can so triangulate polyhedra $P_{1}$ and $P_{2}$ without adding a new vertex that triangulations induced on each pair of faces identified under the glueing map agree. We say that $\left(v_{1}, v_{2}\right) \in$
$V_{1} \times V_{2}$ is a good pair if some cone decomposition of $P_{1}$ from $v_{1}$ and some cone decomposition of $P_{2}$ from $v_{2}$ agree on each pair of faces identified under the glueing map. We call $v_{1} \in V_{1}$ a good vertex if there exists a vertex $v_{2} \in V_{2}$ such that ( $v_{1}, v_{2}$ ) is a good pair. Otherwise, we say that $v_{1}$ is a bad vertex. Therefore it suffices to show the existence of a good vertex.

In the following we will derive a sufficient condition for the existence of a good vertex.

Fix a vertex $v \in V_{1}$. Let $A(v)$ be the set of vertices $v_{2} \in V_{2}$ such that for any cone decomposition of $P_{1}$ from $v$ and any cone decomposition of $P_{2}$ from $v_{2}$, the triangulations induced by those of $P_{1}$ and $P_{2}$ do not agree on some faces $f \in F_{1}$ and $f^{\prime} \in F_{2}$ when glueing $P_{1}$ and $P_{2}$. Then for $w \in A(v),(v, w)$ is not a good pair. Let $B$ be the set of vertices $v_{2} \in V_{2}$ such that for any cone decomposition of $P_{2}$ from $v_{2}$, the triangulations induced by that of $P_{2}$ do not agree on some faces $f_{2}, f_{2}^{\prime} \in F_{2}^{\prime \prime}$ when pasted. Thus if $v_{2} \in B,\left(v, v_{2}\right)$ is not a good pair for every $v \in V_{1}$.

Let $v$ be a bad vertex. By the definition, $\left(v, v_{2}\right) \in V_{1} \times V_{2}$ is not a good pair for all $v_{2} \in V_{2}$. Then we have

$$
\begin{equation*}
|A(v)|+|B| \geq\left|V_{2}\right| . \tag{3.1}
\end{equation*}
$$

First we consider about $A(v)$. Suppose $w \in A(v)$ and take cone decompositions of $P_{1}$ and $P_{2}$ from the vertices $v$ and $w$ respectively. Then there exist faces $f_{1} \in F_{1}$ and $f_{2} \in F_{2}$ on which the triangulations induced by those of $P_{1}$ and $P_{2}$ do not agree. If a face $f_{1}$ (resp. $f_{2}$ ) is not incident with a vertex $v$ (resp. $w$ ), we can replace the triangulation of $f_{1}$ (resp. $f_{2}$ ) so that it matches that of $f_{2}$ (resp. $f_{1}$ ) under the glueing map. The number of vertices of a face $f$ is called degree of $f$, and denoted by $\operatorname{deg} f$. We may assume that $\operatorname{deg} f_{1}=\operatorname{deg} f_{2} \geq 4$ and that the faces $f_{1}$ and $f_{2}$ are incident with the vertices $v$ and $w$ respectively. Since the triangulations do not agree on $f_{1}$ and $f_{2}$, if $\operatorname{deg} f_{1}=\operatorname{deg} f_{2} \geq 5, w$ dose not correspond to $v$ when glueing $f_{1}$ and $f_{2}$. Suppose that the faces $f_{1}$ and $f_{2}$ are quadrilaterals. Let $u$ be the vertex of $f_{1}$ which is not adjacent to $v$. The vertex $w$ does not correspond to neither $v$ nor $u$ when glueing $f_{1}$ and $f_{2}$. (See figure 2)
Then we have

$$
\begin{equation*}
|A(v)| \leq \sum_{v \in f} d(f) \tag{3.2}
\end{equation*}
$$

where

$$
d(f)= \begin{cases}\operatorname{deg} f-1 & \text { if } \operatorname{deg} f \geq 5 \\ 2 & \text { if } \operatorname{deg} f=4 \\ 0 & \text { if } \operatorname{deg} f=3\end{cases}
$$

Next we consider about $B$. Suppose that we take a cone decomposition of $P_{2}$

figure 2
from a vertex $w \in B$. Then there exist a pair of faces $f_{2}, f_{2}^{\prime} \in F_{2}^{\prime \prime}$ on which the triangulations induced by that of $P_{2}$ do not agree. We can arbitrarily choose the triangulations on the faces which are not incident with $w$. Hence we may assume that a pair of faces $f_{2}, f_{2}^{\prime} \in F_{2}^{\prime \prime}$ satisfy the following conditions ;
(1) $w \in f_{2} \cap f_{2}^{\prime}$,
(2) $f_{2}$ is glued to $f_{2}^{\prime}$,
(3) $\operatorname{deg} f_{2}=\operatorname{deg} f_{2}^{\prime} \geq 4$.

figure 3

Since the faces $f_{2}$ and $f_{2}^{\prime}$ share at most two vertices, we get

$$
\begin{equation*}
|B| \leq \sum_{n \geq 4}\left|F_{n, 2}^{\prime \prime}\right| . \tag{3.3}
\end{equation*}
$$

Therefore by (3.1), (3.2) and (3.3), we obtain

$$
\begin{equation*}
\sum_{v \in f} d(f)+\sum_{n \geq 4}\left|F_{n, 2}^{\prime \prime}\right| \geq\left|V_{2}\right| \tag{3.4}
\end{equation*}
$$

for every bad vertex $v$.

Using this result, we have the following lemma.

Lemma 1. If $v$ is a bad vertex, then

$$
\sum_{v \in f} d(f) \geq\left|V_{1}\right| .
$$

Proof. We have the following formulas,

$$
\begin{gathered}
\left|V_{2}\right|-\left|E_{2}\right|+\left|F_{2}\right|=2, \\
\left|E_{2}\right|=\frac{1}{2} \sum_{n \geq 3} F_{n, 2}, \\
\left|F_{n, 2}^{\prime}\right|=\left|F_{n, 1}\right| .
\end{gathered}
$$

As we put $F_{n, 2}=F_{n, 2}^{\prime} \cup F_{n, 2}^{\prime \prime}$, we have

$$
\left|F_{n, 2}\right|=\left|F_{n, 2}^{\prime}\right|+\left|F_{n, 2}^{\prime \prime}\right| .
$$

From the above formulas, we get

$$
\begin{equation*}
\left|V_{2}\right|=2+\sum_{n \geq 3}\left(\frac{n}{2}-1\right)\left|F_{n, 1}\right|+\sum_{n \geq 3}\left(\frac{n}{2}-1\right)\left|F_{n, 2}^{\prime \prime}\right| . \tag{3.5}
\end{equation*}
$$

Similarly we can get

$$
\begin{equation*}
\left|V_{1}\right|=2+\sum_{n \geq 3}\left(\frac{n}{2}-1\right)\left|F_{n, 1}\right| . \tag{3.6}
\end{equation*}
$$

From (3.4) and (3.5), we obtain

$$
\begin{align*}
\sum_{v \in f} d(f) & \geq 2+\sum_{n \geq 3}\left(\frac{n}{2}-1\right)\left|F_{n, 1}\right|+\sum_{n \geq 4}\left(\frac{n}{2}-2\right)\left|F_{n, 2}^{\prime \prime}\right|+\frac{1}{2}\left|F_{3,2}^{\prime \prime}\right| \\
& \geq 2+\sum_{n \geq 3}\left(\frac{n}{2}-1\right)\left|F_{n, 1}\right| \tag{3.7}
\end{align*}
$$

for every bad vertex $v$. The inequality of Lemma 1 follows from (3.6) and (3.7).
The following lemma gives the sufficient condition for the existence of a good vertex, which we have been seeking.

Lemma 2. If

$$
\left|V_{1}\right|^{2}-\sum_{f \in F_{1}} d(f) \operatorname{deg} f>0,
$$

there exists a good vertex.
Proof. If no vertex of $P_{1}$ is a good vertex, i.e. if all vertices are bad vertices, then we obtain

$$
\left|V_{1}\right|^{2}-\sum_{v \in V_{1}} \sum_{v \in f} d(f) \leq 0
$$

by summing the inequality of Lemma 1 over all vertices $v \in V_{1}$. Since

$$
\sum_{v \in V_{1}} \sum_{v \in f} d(f)=\sum_{f \in F_{1}} d(f) \operatorname{deg} f
$$

we have Lemma 2.
We prepare the following lemma in order to prove the main theorem. Let $M$ $=\max \left\{n \in \boldsymbol{N} ;\left|F_{n, 1}\right| \neq 0\right\}$.

Lemma 3. If one of the following conditions holds,
(1) $M=4$ and $\left|V_{1}\right| \geq 6$,
(2) $M=5$ and $\left|V_{1}\right| \geq 12$,
(3) $M \geq 6$ and $\left|V_{1}\right| \geq 2 M+1$,
then there exists a good vertex.
Proof. Using the formula (3.6), we have

$$
\begin{aligned}
\left|V_{1}\right|^{2}-\sum_{f \in F_{1}} d(f) \operatorname{deg} f= & \left|V_{1}\right|\left(2+\sum_{n \geq 3}\left(\frac{n}{2}-1\right)\left|F_{n, 1}\right|\right)-\sum_{f \in F_{1}} d(f) \operatorname{deg} f \\
= & 2\left|V_{1}\right|+\sum\left|V_{1}\right|\left(\frac{n}{2}-1\right)\left|F_{n, 1}\right| \\
& -\left(\sum_{n \geq 5} n(n-1)\left|F_{n, 1}\right|+8\left|F_{4,1}\right|\right) \\
= & 2\left(2+\sum_{n \geq 3}\left(\frac{n}{2}-1\right)\left|F_{n, 1}\right|\right)+\sum_{n \geq 3}\left|V_{1}\right|\left(\frac{n}{2}-1\right)\left|F_{n, 1}\right| \\
& -\sum_{n \geq 2}\left(n^{2}-n\right)\left|F_{n, 1}\right|-8\left|F_{4,1}\right| \\
= & 4+\left(1+\frac{1}{2}\left|V_{1}\right|\right)\left|F_{3,1}\right|+\left(\left|V_{1}\right|-6\right)\left|F_{4,1}\right| \\
& +\sum_{n \geq 5}\left(\left|V_{1}\right|\left(\frac{n}{2}-1\right)-n^{2}+2 n-2\right)\left|F_{n, 1}\right| .
\end{aligned}
$$

By the definition of $M$, we have $\left|F_{n, 1}\right|=0$ for $n>M$. It is not hard to see that the coefficient of $\left|F_{n, 1}\right|$ in the above formula is nonnegative for $n \leq M$ in each case of the assumption. Hence we obtain

$$
\left|V_{1}\right|^{2}-\sum_{f \in F_{1}} d(f) \operatorname{deg}(f)>0
$$

Therefore there exists a good vertex by Lemma 2.

## 4. Proof of Main Theorem

We divide the proof into several cases.
Case 1. Suppose that $M=3$. Then all vertices of $P_{1}$ are good vertices.
Case 2. Suppose that $M=4$. Clearly $\left|V_{1}\right| \geq 5$. If $\left|V_{1}\right| \geq 6$, there exists a good vertex
by Lemma 3. If $\left|V_{1}\right|=5$, then the faces of $P_{1}$ consist of one quadrilateral and four triangles, and the inequality of Lemma 2 holds. Therefore there exists a good vertex.
Case 3. Suppose that $M=5$. Assume that all vertices of $P_{1}$ were bad vertices. By Lemma 2, we have

$$
\begin{equation*}
20\left|F_{5,1}\right|+\left|F_{4,1}\right| \geq\left|V_{1}\right|^{2} \tag{4.1}
\end{equation*}
$$

We have the following conditions about $P_{1}$. By the formula (3.6),

$$
\begin{equation*}
\left|V_{1}\right|=\sum_{n=3}^{5}\left(\frac{n}{2}-1\right)\left|F_{n, 1}\right|+2 \tag{4.2}
\end{equation*}
$$

Since a pentagon is adjacent to at least five faces,

$$
\begin{equation*}
\sum_{n=3}^{5}\left|F_{n, 1}\right| \geq 6 \tag{4.3}
\end{equation*}
$$

Because every vertex is incident with at least three faces,

$$
\begin{equation*}
\sum_{n=3}^{5} n\left|F_{n, 1}\right| \geq 3\left|V_{1}\right| . \tag{4.4}
\end{equation*}
$$

If $\left|V_{1}\right| \geq 12$, there exists a good vertex by Lemma 3. Therefore we may assume $6 \leq$ $\left|V_{1}\right| \leq 11$. Furthermore we may assume that $\left|F_{4,1}\right|$ and $\left|F_{3,1}\right|$ are nonnegative integers and that $\left|F_{5,1}\right|$ is a positive integer. However there exists no possible combination ( $\left|V_{1}\right|,\left|F_{3,1}\right|,\left|F_{4,1}\right|,\left|F_{5,1}\right|$ ) which satisfies (4.1) (4.2) (4.3) and (4.4). Therefore $P_{1}$ must have a good vertex.

Case 4. Suppose that $M \geq 6$. Let $f_{0}$ be one of the $M$-gonal faces of $P_{1}$. Assuming that all vertices of $f_{0}$ are bad vertices, we will show

$$
\left|V_{1}\right| \geq 2 M+1
$$

step by step. Then it follows that there exists a good vertex somewhere else by Lemma 3.

Note that, we have

$$
\begin{equation*}
\left|V_{1}\right| \geq 2+\left(\frac{\operatorname{deg} f_{0}}{2}-1\right)+\sum_{f \text { is adjacent to } f_{0}}\left(\frac{\operatorname{deg} f}{2}-1\right) \tag{4.5}
\end{equation*}
$$

by (3.6). We denote the last term of right side of (4.5) by $V\left(f_{0}\right)$.
Step 1. In this step, we show $\left|V_{1}\right| \geq 5 M / 4+1$.
Let $v$ be a vertex of $f_{0}$. By the assumption, $v$ is a bad vertex and $\left|V_{1}\right| \geq M+1$. Thus by Lemma 1, we have

$$
\sum_{v \in f} d(f) \geq M+1
$$

We denote by $I(v)$ the set of faces other than $f_{0}$ which are incident with $v$. Since $\sum_{v \in f} d(f)=\sum_{f \in I(v)} d(f)+M-1$, it follows

$$
\sum_{f \in I(v)} d(f) \geq 2
$$

This implies that $I(v)$ contains a face of degree $n(\geq 4)$. Since every vertex is incident with at least three faces, $V\left(f_{0}\right)$ is minimal if a vertex $v$ is incident with a triangle, a quadrilateral and $f_{0}$ for all $v \in f_{0}$.

figure 4
Hence by (4.5), we get

$$
\left|V_{1}\right| \geq \frac{5}{4} M+1
$$

Step 2. In this step, we show $\left|V_{1}\right| \geq 3 M / 2+1$.
By the assumption, $v$ is a bad vertex, and $\left|V_{1}\right| \geq 5 M / 4+1$ from Step 1. By Lemma 1, we have

$$
\sum_{v \in f} d(f) \geq \frac{5}{4} M+1
$$

Since we denote by $I(v)$ the set of faces other than $f_{0}$ which are incident with $v$, and $\sum_{v \in f} d(f)=\sum_{f \in I(v)} d(f)+M-1$, it follows

$$
\sum_{f \in I(v)} d(f) \geq \frac{1}{4} M+2 .
$$

By the assumption, $M \geq 6$, then

$$
\sum_{f \in I(v)} d(f) \geq \frac{5}{2}
$$

This implies that
(i) $I(v)$ contains a face of degree $n(\geq 5)$, or
(ii) $I(v)$ contains two quadrilaterals,
for every vertex $v \in f_{0}$. We will evaluate the minimal value of $V\left(f_{0}\right)$. Then we may assume that every vertex of $f_{0}$ is incident with only three faces. Let $v_{1}, \cdots, v_{k}$ be the vertices of $f_{0}$ which satisfy (i) and $v_{k+1}, \cdots, v_{M}$ the vertices which satisfy (ii). Thus we may assume that, for $i=1, \cdots, k, I\left(v_{i}\right)$ consists of a triangle and a pentagon, or a quadrilateral and a pentagon and for $i=k+1, \cdots, M, I\left(v_{i}\right)$ consists of two quadrilaterals. If we replace the faces of $I\left(v_{i}\right)$ by quadrilaterals for $i=i, \cdots$, $k$, the value of $V\left(f_{0}\right)$ does not increase. Therefore we have minimal value of $V\left(f_{0}\right)$ when $I(v)$ consists of two quadrilaterals for every vertex $v \in f_{0}$.

figure 5

Hence by (4.5) we get

$$
\left|V_{1}\right| \geq \frac{3}{2} M+1
$$

Step 3. We repeat the argument, assuming the above inequality. For every vertex $v \in f_{0}$,
(i) $I(v)$ contains a face of degree $n(\geq 6)$,
(ii) $I(v)$ contains a pentagon and a face of degree $n(=4,5)$ or
(iii) $I(v)$ contains three quadrilaterals.

We have the minimal value of $V\left(f_{0}\right)$ when $I(v)$ consists of a triangle and a hexagon for every vertex $v \in f_{0}$. Then we have

$$
\left|V_{1}\right|=\frac{7}{4} M+1
$$

Step 4. We repeat the argument once more, assuming the above inequality. We have the minimal value of $V\left(f_{0}\right)$ when $I(v)$ consists of two pentagons for every vertex $v \in f_{0}$. Then we obtain the desired inequality

$$
\left|V_{1}\right| \geq 2 M+1
$$

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