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IDEAL TETRAHEDRAL DECOMPOSITIONS OF HYPERBOLIC 3-MANIFOLDS

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1. Introduction

Since W. Thurston investigated cusped hyperbolic 3-manifolds by decomposing them into ideal tetrahedra [2], the method has become an indispensable tool for the reseachers of hyperbolic 3-manifolds. Although it is not known whether a noncompact hyperbolic 3-manifold of finite volume always admits a decomposition into ideal tetrahedra, most of us believe the following:

Conjecture. *Every noncompact hyperbolic* 3-*manifold of finite volume admits a decomposition into convex ideal tetrahedra.*

In [1], Epstein and Penner have shown that every noncompact hyperbolic 3-manifold of finite volume has a canonical decomposition into convex ideal polyhedra. Therefore, in order to prove the above conjecture it suffices to show that every hyperbolic 3-manifold obtained by glueing ideal polyhedra admits a decomposition into ideal tetrahedra. Wada recently proved that if a noncompact hyperbolic 3-manifold M consists of one convex ideal polyhedron then M can be decomposed into ideal tetrahedra [3].

In this paper, we show the following theorem.

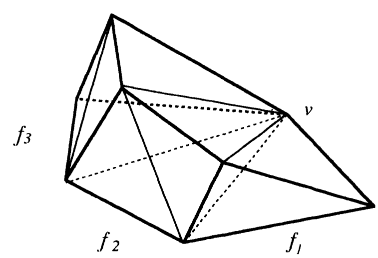
Main Theorem. *Suppose that a noncompact hyperbolic* 3-*manifold M is obtained by glueing two convex ideal polyhedra P_1 and P_2 in such a way that every face of P_1 is pasted with a face of P_2. Then M can be decomposed into ideal tetrahedra.*

The author would like to express her thanks to Professors Masaaki Wada and Yasushi Yamashita. Her interest in the problem in this paper comes from the discussion with them.

2. Cone Decomposition

Let P be a polyhedron and v a vertex of P. Let $f_1, \cdots, f_r$ denote the faces of P not incident with the vertex v. Now take a triangulation τ of $\bigcup_{i=1}^{r} f_i$ without adding a new vertex. Then a triangulation $K(\tau)$ of P can be defined as follows. The sets $K^i(\tau)$ of i-simplices of $K(\tau)$ are given by

$$K^3(\tau)=\{\langle v, \sigma_2\rangle \mid \sigma_2 \text{ is a 2-simplex of } \tau\},$$
$$K^2(\tau)=\{\langle v, \sigma_1\rangle \mid \sigma_1 \text{ is a 1-simplex of } \tau\} \cup \{\text{2-simplices of } \tau\},$$
$$K^1(\tau)=\{\langle v, \sigma_0\rangle \mid \sigma_0 \text{ is a 0-simplex of } \tau\} \cup \{\text{1-simplices of } \tau\},$$
$$K^0(\tau)=\{\text{vertices of } P\}.$$



This polyhedron is divided into 5 tetrahedra.

figure 1

We call such a triangulation $K(\tau)$ a *cone decomposition* of P from the vertex v.

We say that M admits a decomposition into ideal polyhedra (tetrahedra) if M is obtained by glueing together the faces of ideal polyhedra (tetrahedra). Note that taking cone decompositions of P_1 and P_2 from arbitrary vertices dose not necessary give a decomposition of M, since the triangulations induced by those of P_1 and P_2 on identified faces might not agree.

3. Good Vertex

By V_i, E_i, and F_i, we denote the sets of vertices, edges and faces of P_i respectively. We decompose $F_i = \bigcup_{n\geq 3} F_{n,i}$ where $F_{n,i}$ is the set of n-gons of $F_i (i=1, 2)$. Furthermore we decompose $F_{n,2} = F'_{n,2} \cup F''_{n,2}$ where $F'_{n,2}$ is the set of n-gons glued to faces of P_1 and $F''_{n,2}$ is the set of n-gons glued to faces of P_2.

To prove the main theorem, it suffices to show that we can so triangulate polyhedra P_1 and P_2 without adding a new vertex that triangulations induced on each pair of faces identified under the glueing map agree. We say that $(v_1, v_2) \in$

$V_1 \times V_2$ is a ***good pair*** if some cone decomposition of P_1 from v_1 and some cone decomposition of P_2 from v_2 agree on each pair of faces identified under the glueing map. We call $v_1 \in V_1$ a ***good vertex*** if there exists a vertex $v_2 \in V_2$ such that (v_1, v_2) is a good pair. Otherwise, we say that v_1 is a *bad vertex*. Therefore it suffices to show the existence of a good vertex.

In the following we will derive a sufficient condition for the existence of a good vertex.

Fix a vertex $v \in V_1$. Let $A(v)$ be the set of vertices $v_2 \in V_2$ such that for any cone decomposition of P_1 from v and any cone decomposition of P_2 from v_2, the triangulations induced by those of P_1 and P_2 do not agree on some faces $f \in F_1$ and $f' \in F_2$ when glueing P_1 and P_2. Then for $w \in A(v)$, (v, w) is not a good pair. Let B be the set of vertices $v_2 \in V_2$ such that for any cone decomposition of P_2 from v_2, the triangulations induced by that of P_2 do not agree on some faces $f_2, f_2' \in F_2''$ when pasted. Thus if $v_2 \in B$, (v, v_2) is not a good pair for every $v \in V_1$.

Let v be a bad vertex. By the definition, $(v, v_2) \in V_1 \times V_2$ is not a good pair for all $v_2 \in V_2$. Then we have

$$|A(v)| + |B| \geq |V_2|. \tag{3.1}$$

First we consider about $A(v)$. Suppose $w \in A(v)$ and take cone decompositions of P_1 and P_2 from the vertices v and w respectively. Then there exist faces $f_1 \in F_1$ and $f_2 \in F_2$ on which the triangulations induced by those of P_1 and P_2 do not agree. If a face f_1 (resp. f_2) is not incident with a vertex v (resp. w), we can replace the triangulation of f_1 (resp. f_2) so that it matches that of f_2 (resp. f_1) under the glueing map. The number of vertices of a face f is called *degree of* f, and denoted by $\deg f$. We may assume that $\deg f_1 = \deg f_2 \geq 4$ and that the faces f_1 and f_2 are incident with the vertices v and w respectively. Since the triangulations do not agree on f_1 and f_2, if $\deg f_1 = \deg f_2 \geq 5$, w dose not correspond to v when glueing f_1 and f_2. Suppose that the faces f_1 and f_2 are quadrilaterals. Let u be the vertex of f_1 which is not adjacent to v. The vertex w does not correspond to neither v nor u when glueing f_1 and f_2. (See figure 2)
Then we have

$$|A(v)| \leq \sum_{v \in f} d(f) \tag{3.2}$$

where

$$d(f) = \begin{cases} \deg f - 1 & \text{if } \deg f \geq 5, \\ 2 & \text{if } \deg f = 4, \\ 0 & \text{if } \deg f = 3. \end{cases}$$

Next we consider about B. Suppose that we take a cone decomposition of P_2

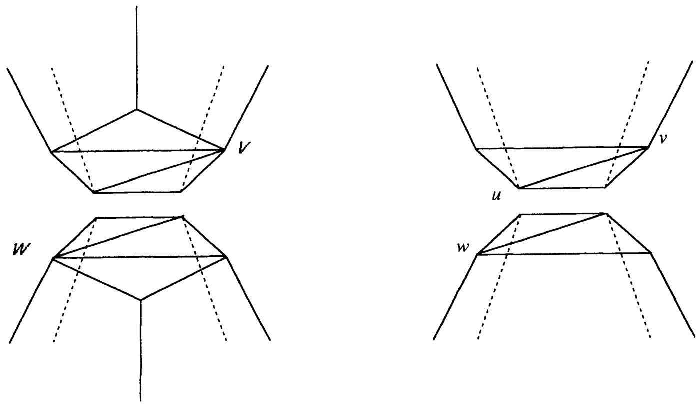


figure 2

from a vertex $w \in B$. Then there exist a pair of faces f_2, $f_2' \in F_2''$ on which the triangulations induced by that of P_2 do not agree. We can arbitrarily choose the triangulations on the faces which are not incident with w. Hence we may assume that a pair of faces f_2, $f_2' \in F_2''$ satisfy the following conditions;

(1) $w \in f_2 \cap f_2'$,
(2) f_2 is glued to f_2',
(3) $\deg f_2 = \deg f_2' \geq 4$.

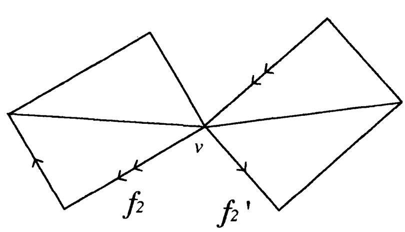


figure 3

Since the faces f_2 and f_2' share at most two vertices, we get

$$|B| \leq \sum_{n \geq 4} |F''_{n,2}|. \tag{3.3}$$

Therefore by (3.1), (3.2) and (3.3), we obtain

$$\sum_{v \in f} d(f) + \sum_{n \geq 4} |F''_{n,2}| \geq |V_2| \tag{3.4}$$

for every bad vertex v.

Using this result, we have the following lemma.

Lemma 1. *If v is a bad vertex, then*

$$\sum_{v\in f} d(f)\geq |V_1|.$$

Proof. We have the following formulas,

$$|V_2|-|E_2|+|F_2|=2,$$
$$|E_2|=\frac{1}{2}\sum_{n\geq 3} F_{n,2},$$
$$|F'_{n,2}|=|F_{n,1}|.$$

As we put $F_{n,2}=F'_{n,2}\cup F''_{n,2}$, we have

$$|F_{n,2}|=|F'_{n,2}|+|F''_{n,2}|.$$

From the above formulas, we get

$$|V_2|=2+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F_{n,1}|+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F''_{n,2}|. \tag{3.5}$$

Similarly we can get

$$|V_1|=2+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F_{n,1}|. \tag{3.6}$$

From (3.4) and (3.5), we obtain

$$\begin{aligned}\sum_{v\in f} d(f)&\geq 2+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F_{n,1}|+\sum_{n\geq 4}\left(\frac{n}{2}-2\right)|F''_{n,2}|+\frac{1}{2}|F''_{3,2}|\\ &\geq 2+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F_{n,1}|\end{aligned} \tag{3.7}$$

for every bad vertex v. The inequality of Lemma 1 follows from (3.6) and (3.7).

The following lemma gives the sufficient condition for the existence of a good vertex, which we have been seeking.

Lemma 2. *If*

$$|V_1|^2-\sum_{f\in F_1} d(f)\deg f>0,$$

there exists a good vertex.

Proof. If no vertex of P_1 is a good vertex, i.e. if all vertices are bad vertices, then we obtain

$$|V_1|^2-\sum_{v\in V_1}\sum_{v\in f}d(f)\leq 0$$

by summing the inequality of Lemma 1 over all vertices $v\in V_1$. Since

$$\sum_{v\in V_1}\sum_{v\in f}d(f)=\sum_{f\in F_1}d(f)\deg f,$$

we have Lemma 2.

We prepare the following lemma in order to prove the main theorem. Let $M=\max\{n\in \boldsymbol{N}\,;\ |F_{n,1}|\neq 0\}$.

Lemma 3. *If one of the following conditions holds*,
(1) $M=4$ *and* $|V_1|\geq 6$,
(2) $M=5$ *and* $|V_1|\geq 12$,
(3) $M\geq 6$ *and* $|V_1|\geq 2M+1$,
then there exists a good vertex.

Proof. Using the formula (3.6), we have

$$\begin{aligned}|V_1|^2-\sum_{f\in F_1}d(f)\deg f&=|V_1|\Big(2+\sum_{n\geq 3}\Big(\frac{n}{2}-1\Big)|F_{n,1}|\Big)-\sum_{f\in F_1}d(f)\deg f\\&=2|V_1|+\sum|V_1|\Big(\frac{n}{2}-1\Big)|F_{n,1}|\\&\quad-\Big(\sum_{n\geq 5}n(n-1)|F_{n,1}|+8|F_{4,1}|\Big)\\&=2\Big(2+\sum_{n\geq 3}\Big(\frac{n}{2}-1\Big)|F_{n,1}|\Big)+\sum_{n\geq 3}|V_1|\Big(\frac{n}{2}-1\Big)|F_{n,1}|\\&\quad-\sum_{n\geq 5}(n^2-n)|F_{n,1}|-8|F_{4,1}|\\&=4+\Big(1+\frac{1}{2}|V_1|\Big)|F_{3,1}|+(|V_1|-6)|F_{4,1}|\\&\quad+\sum_{n\geq 5}\Big(|V_1|\Big(\frac{n}{2}-1\Big)-n^2+2n-2\Big)|F_{n,1}|.\end{aligned}$$

By the definition of M, we have $|F_{n,1}|=0$ for $n>M$. It is not hard to see that the coefficient of $|F_{n,1}|$ in the above formula is nonnegative for $n\leq M$ in each case of the assumption. Hence we obtain

$$|V_1|^2-\sum_{f\in F_1}d(f)\deg(f)>0.$$

Therefore there exists a good vertex by Lemma 2.

4. Proof of Main Theorem

We divide the proof into several cases.
Case 1. Suppose that $M=3$. Then all vertices of P_1 are good vertices.
Case 2. Suppose that $M=4$. Clearly $|V_1|\geq 5$. If $|V_1|\geq 6$, there exists a good vertex

by Lemma 3. If $|V_1|=5$, then the faces of P_1 consist of one quadrilateral and four triangles, and the inequality of Lemma 2 holds. Therefore there exists a good vertex.

Case 3. Suppose that $M=5$. Assume that all vertices of P_1 were bad vertices. By Lemma 2, we have

$$20|F_{5,1}|+|F_{4,1}|\geq|V_1|^2. \tag{4.1}$$

We have the following conditions about P_1. By the formula (3.6),

$$|V_1|=\sum_{n=3}^{5}\left(\frac{n}{2}-1\right)|F_{n,1}|+2. \tag{4.2}$$

Since a pentagon is adjacent to at least five faces,

$$\sum_{n=3}^{5}|F_{n,1}|\geq 6. \tag{4.3}$$

Because every vertex is incident with at least three faces,

$$\sum_{n=3}^{5}n|F_{n,1}|\geq 3|V_1|. \tag{4.4}$$

If $|V_1|\geq 12$, there exists a good vertex by Lemma 3. Therefore we may assume $6\leq |V_1|\leq 11$. Furthermore we may assume that $|F_{4,1}|$ and $|F_{3,1}|$ are nonnegative integers and that $|F_{5,1}|$ is a positive integer. However there exists no possible combination $(|V_1|, |F_{3,1}|, |F_{4,1}|, |F_{5,1}|)$ which satisfies (4.1) (4.2) (4.3) and (4.4). Therefore P_1 must have a good vertex.

Case 4. Suppose that $M\geq 6$. Let f_0 be one of the M-gonal faces of P_1. Assuming that all vertices of f_0 are bad vertices, we will show

$$|V_1|\geq 2M+1$$

step by step. Then it follows that there exists a good vertex somewhere else by Lemma 3.

Note that, we have

$$|V_1|\geq 2+\left(\frac{\deg f_0}{2}-1\right)+\sum_{f\text{ is adjacent to }f_0}\left(\frac{\deg f}{2}-1\right) \tag{4.5}$$

by (3.6). We denote the last term of right side of (4.5) by $V(f_0)$.

Step 1. In this step, we show $|V_1|\geq 5M/4+1$.

Let v be a vertex of f_0. By the assumption, v is a bad vertex and $|V_1|\geq M+1$. Thus by Lemma 1, we have

$$\sum_{v\in f}d(f)\geq M+1.$$

We denote by $I(v)$ the set of faces other than f_0 which are incident with v. Since $\sum_{v\in f}d(f)=\sum_{f\in I(v)}d(f)+M-1$, it follows

$$\sum_{f\in I(v)} d(f)\geq 2.$$

This implies that $I(v)$ contains a face of degree n (≥ 4). Since every vertex is incident with at least three faces, $V(f_0)$ is minimal if a vertex v is incident with a triangle, a quadrilateral and f_0 for all $v\in f_0$.

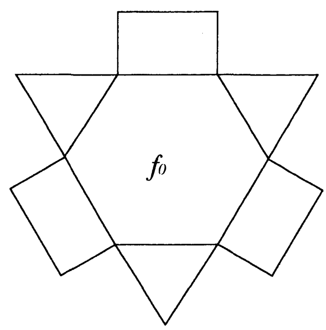


figure 4

Hence by (4.5), we get

$$|V_1|\geq\frac{5}{4}M+1.$$

Step 2. In this step, we show $|V_1|\geq 3M/2+1$.

By the assumption, v is a bad vertex, and $|V_1|\geq 5M/4+1$ from Step 1. By Lemma 1, we have

$$\sum_{v\in f} d(f)\geq\frac{5}{4}M+1.$$

Since we denote by $I(v)$ the set of faces other than f_0 which are incident with v, and $\sum_{v\in f}d(f)=\sum_{f\in I(v)}d(f)+M-1$, it follows

$$\sum_{f\in I(v)} d(f)\geq\frac{1}{4}M+2.$$

By the assumption, $M\geq 6$, then

$$\sum_{f\in I(v)} d(f)\geq\frac{5}{2}.$$

This implies that

(i) $I(v)$ contains a face of degree n (≥ 5), or

(ii) $I(v)$ contains two quadrilaterals,

for every vertex $v \in f_0$. We will evaluate the minimal value of $V(f_0)$. Then we may assume that every vertex of f_0 is incident with only three faces. Let $v_1, \cdots, v_k$ be the vertices of f_0 which satisfy (i) and $v_{k+1}, \cdots, v_M$ the vertices which satisfy (ii). Thus we may assume that, for $i=1, \cdots, k$, $I(v_i)$ consists of a triangle and a pentagon, or a quadrilateral and a pentagon and for $i=k+1, \cdots, M$, $I(v_i)$ consists of two quadrilaterals. If we replace the faces of $I(v_i)$ by quadrilaterals for $i=i, \cdots, k$, the value of $V(f_0)$ does not increase. Therefore we have minimal value of $V(f_0)$ when $I(v)$ consists of two quadrilaterals for every vertex $v \in f_0$.

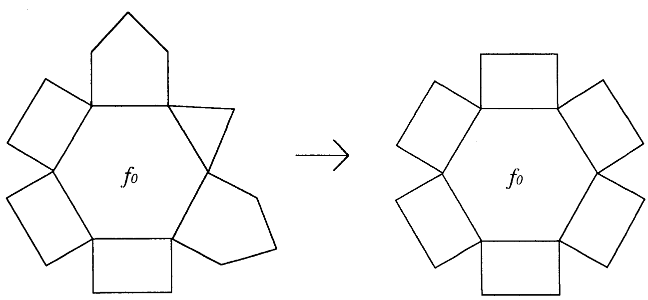


figure 5

Hence by (4.5) we get

$$|V_1| \geq \frac{3}{2}M+1.$$

Step 3. We repeat the argument, assuming the above inequality. For every vertex $v \in f_0$,

(i) $I(v)$ contains a face of degree n (≥ 6),
(ii) $I(v)$ contains a pentagon and a face of degree n $(=4, 5)$ or
(iii) $I(v)$ contains three quadrilaterals.

We have the minimal value of $V(f_0)$ when $I(v)$ consists of a triangle and a hexagon for every vertex $v \in f_0$. Then we have

$$|V_1| = \frac{7}{4}M+1.$$

Step 4. We repeat the argument once more, assuming the above inequality. We have the minimal value of $V(f_0)$ when $I(v)$ consists of two pentagons for every vertex $v \in f_0$. Then we obtain the desired inequality

$$|V_1| \geq 2M+1.$$

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