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IDEAL TETRAHEDRAL DECOMPOSITIONS OF HYPERBOLIC 3-MANIFOLDS

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1. Introduction

Since W. Thurston investigated cusped hyperbolic 3-manifolds by decomposing them into ideal tetrahedra [2], the method has become an indispensable tool for the reseachers of hyperbolic 3-manifolds. Although it is not known whether a noncompact hyperbolic 3-manifold of finite volume always admits a decomposition into ideal tetrahedra, most of us believe the following:

Conjecture. *Every noncompact hyperbolic 3-manifold of finite volume admits a decomposition into convex ideal tetrahedra.*

In [1], Epstein and Penner have shown that every noncompact hyperbolic 3-manifold of finite volume has a canonical decomposition into convex ideal polyhedra. Therefore, in order to prove the above conjecture it suffices to show that every hyperbolic 3-manifold obtained by glueing ideal polyhedra admits a decomposition into ideal tetrahedra. Wada recently proved that if a noncompact hyperbolic 3-manifold M consists of one convex ideal polyhedron then M can be decomposed into ideal tetrahedra [3].

In this paper, we show the following theorem.

Main Theorem. *Suppose that a noncompact hyperbolic 3-manifold M is obtained by glueing two convex ideal polyhedra P_1 and P_2 in such a way that every face of P_1 is pasted with a face of P_2. Then M can be decomposed into ideal tetrahedra.*

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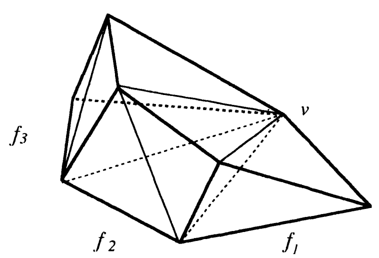
2. Cone Decomposition

Let P be a polyhedron and v a vertex of P. Let $f_1, \cdots, f_r$ denote the faces of P not incident with the vertex v. Now take a triangulation τ of $\bigcup_{i=1}^{r} f_i$ without adding a new vertex. Then a triangulation $K(\tau)$ of P can be defined as follows. The sets $K^i(\tau)$ of i-simplices of $K(\tau)$ are given by

$$K^3(\tau)=\{\langle v, \sigma_2\rangle | \sigma_2 \text{ is a 2-simplex of } \tau\},$$
$$K^2(\tau)=\{\langle v, \sigma_1\rangle | \sigma_1 \text{ is a 1-simplex of } \tau\} \cup \{\text{2-simplices of } \tau\},$$
$$K^1(\tau)=\{\langle v, \sigma_0\rangle | \sigma_0 \text{ is a 0-simplex of } \tau\} \cup \{\text{1-simplices of } \tau\},$$
$$K^0(\tau)=\{\text{vertices of } P\}.$$



This polyhedron is divided into 5 tetrahedra.

figure 1

We call such a triangulation $K(\tau)$ a *cone decomposition* of P from the vertex v.

We say that M admits a decomposition into ideal polyhedra (tetrahedra) if M is obtained by glueing together the faces of ideal polyhedra (tetrahedra). Note that taking cone decompositions of P_1 and P_2 from arbitrary vertices dose not necessary give a decomposition of M, since the triangulations induced by those of P_1 and P_2 on identified faces might not agree.

3. Good Vertex

By V_i, E_i, and F_i, we denote the sets of vertices, edges and faces of P_i respectively. We decompose $F_i=\bigcup_{n\geq 3} F_{n,i}$ where $F_{n,i}$ is the set of n-gons of $F_i (i=1, 2)$. Furthermore we decompose $F_{n,2}=F'_{n,2} \cup F''_{n,2}$ where $F'_{n,2}$ is the set of n-gons glued to faces of P_1 and $F''_{n,2}$ is the set of n-gons glued to faces of P_2.

To prove the main theorem, it suffices to show that we can so triangulate polyhedra P_1 and P_2 without adding a new vertex that triangulations induced on each pair of faces identified under the glueing map agree. We say that $(v_1, v_2) \in$

$V_1 \times V_2$ is a ***good pair*** if some cone decomposition of P_1 from v_1 and some cone decomposition of P_2 from v_2 agree on each pair of faces identified under the glueing map. We call $v_1 \in V_1$ a ***good vertex*** if there exists a vertex $v_2 \in V_2$ such that (v_1, v_2) is a good pair. Otherwise, we say that v_1 is a ***bad vertex***. Therefore it suffices to show the existence of a good vertex.

In the following we will derive a sufficient condition for the existence of a good vertex.

Fix a vertex $v \in V_1$. Let $A(v)$ be the set of vertices $v_2 \in V_2$ such that for any cone decomposition of P_1 from v and any cone decomposition of P_2 from v_2, the triangulations induced by those of P_1 and P_2 do not agree on some faces $f \in F_1$ and $f' \in F_2$ when glueing P_1 and P_2. Then for $w \in A(v)$, (v, w) is not a good pair. Let B be the set of vertices $v_2 \in V_2$ such that for any cone decomposition of P_2 from v_2, the triangulations induced by that of P_2 do not agree on some faces $f_2, f_2' \in F_2''$ when pasted. Thus if $v_2 \in B$, (v, v_2) is not a good pair for every $v \in V_1$.

Let v be a bad vertex. By the definition, $(v, v_2) \in V_1 \times V_2$ is not a good pair for all $v_2 \in V_2$. Then we have

$$|A(v)| + |B| \geq |V_2|. \tag{3.1}$$

First we consider about $A(v)$. Suppose $w \in A(v)$ and take cone decompositions of P_1 and P_2 from the vertices v and w respectively. Then there exist faces $f_1 \in F_1$ and $f_2 \in F_2$ on which the triangulations induced by those of P_1 and P_2 do not agree. If a face f_1 (resp. f_2) is not incident with a vertex v (resp. w), we can replace the triangulation of f_1 (resp. f_2) so that it matches that of f_2 (resp. f_1) under the glueing map. The number of vertices of a face f is called *degree of f*, and denoted by $\deg f$. We may assume that $\deg f_1 = \deg f_2 \geq 4$ and that the faces f_1 and f_2 are incident with the vertices v and w respectively. Since the triangulations do not agree on f_1 and f_2, if $\deg f_1 = \deg f_2 \geq 5$, w dose not correspond to v when glueing f_1 and f_2. Suppose that the faces f_1 and f_2 are quadrilaterals. Let u be the vertex of f_1 which is not adjacent to v. The vertex w does not correspond to neither v nor u when glueing f_1 and f_2. (See figure 2)
Then we have

$$|A(v)| \leq \sum_{v \in f} d(f) \tag{3.2}$$

where

$$d(f) = \begin{cases} \deg f - 1 & \text{if } \deg f \geq 5, \\ 2 & \text{if } \deg f = 4, \\ 0 & \text{if } \deg f = 3. \end{cases}$$

Next we consider about B. Suppose that we take a cone decomposition of P_2

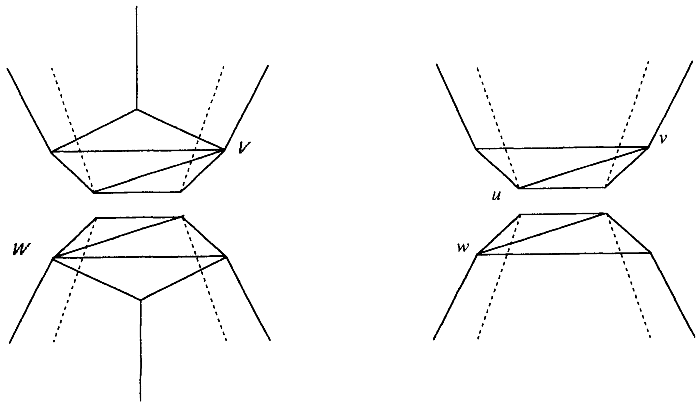


figure 2

from a vertex $w \in B$. Then there exist a pair of faces f_2, $f_2' \in F_2''$ on which the triangulations induced by that of P_2 do not agree. We can arbitrarily choose the triangulations on the faces which are not incident with w. Hence we may assume that a pair of faces f_2, $f_2' \in F_2''$ satisfy the following conditions;

(1) $w \in f_2 \cap f_2'$,
(2) f_2 is glued to f_2',
(3) $\deg f_2 = \deg f_2' \geq 4$.

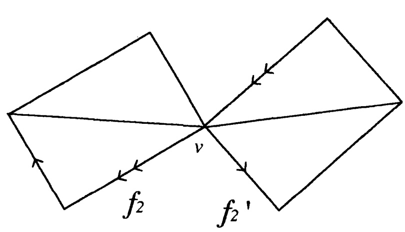


figure 3

Since the faces f_2 and f_2' share at most two vertices, we get

$$|B| \leq \sum_{n \geq 4} |F_{n,2}''|. \tag{3.3}$$

Therefore by (3.1), (3.2) and (3.3), we obtain

$$\sum_{v \in f} d(f) + \sum_{n \geq 4} |F_{n,2}''| \geq |V_2| \tag{3.4}$$

for every bad vertex v.

Using this result, we have the following lemma.

Lemma 1. *If v is a bad vertex, then*

$$\sum_{v\in f} d(f)\geq |V_1|.$$

Proof. We have the following formulas,

$$|V_2|-|E_2|+|F_2|=2,$$
$$|E_2|=\frac{1}{2}\sum_{n\geq 3} F_{n,2},$$
$$|F'_{n,2}|=|F_{n,1}|.$$

As we put $F_{n,2}=F'_{n,2}\cup F''_{n,2}$, we have

$$|F_{n,2}|=|F'_{n,2}|+|F''_{n,2}|.$$

From the above formulas, we get

$$|V_2|=2+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F_{n,1}|+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F''_{n,2}|. \tag{3.5}$$

Similarly we can get

$$|V_1|=2+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F_{n,1}|. \tag{3.6}$$

From (3.4) and (3.5), we obtain

$$\begin{aligned}\sum_{v\in f} d(f)&\geq 2+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F_{n,1}|+\sum_{n\geq 4}\left(\frac{n}{2}-2\right)|F''_{n,2}|+\frac{1}{2}|F''_{3,2}|\\ &\geq 2+\sum_{n\geq 3}\left(\frac{n}{2}-1\right)|F_{n,1}|\end{aligned} \tag{3.7}$$

for every bad vertex v. The inequality of Lemma 1 follows from (3.6) and (3.7).

The following lemma gives the sufficient condition for the existence of a good vertex, which we have been seeking.

Lemma 2. *If*

$$|V_1|^2-\sum_{f\in F_1} d(f)\deg f>0,$$

there exists a good vertex.

Proof. If no vertex of P_1 is a good vertex, i.e. if all vertices are bad vertices, then we obtain

$$|V_1|^2-\sum_{v\in V_1}\sum_{v\in f}d(f)\le 0$$

by summing the inequality of Lemma 1 over all vertices $v\in V_1$. Since

$$\sum_{v\in V_1}\sum_{v\in f}d(f)=\sum_{f\in F_1}d(f)\deg f,$$

we have Lemma 2.

We prepare the following lemma in order to prove the main theorem. Let $M=\max\{n\in\boldsymbol{N}\ ;\ |F_{n,1}|\neq 0\}$.

Lemma 3. *If one of the following conditions holds,*
(1) $M=4$ *and* $|V_1|\ge 6$,
(2) $M=5$ *and* $|V_1|\ge 12$,
(3) $M\ge 6$ *and* $|V_1|\ge 2M+1$,
then there exists a good vertex.

Proof. Using the formula (3.6), we have

$$\begin{aligned}|V_1|^2-\sum_{f\in F_1}d(f)\deg f&=|V_1|\Big(2+\sum_{n\ge 3}\Big(\frac{n}{2}-1\Big)|F_{n,1}|\Big)-\sum_{f\in F_1}d(f)\deg f\\&=2|V_1|+\sum|V_1|\Big(\frac{n}{2}-1\Big)|F_{n,1}|\\&\quad-\big(\sum_{n\ge 5}n(n-1)|F_{n,1}|+8|F_{4,1}|\big)\\&=2\Big(2+\sum_{n\ge 3}\Big(\frac{n}{2}-1\Big)|F_{n,1}|\Big)+\sum_{n\ge 3}|V_1|\Big(\frac{n}{2}-1\Big)|F_{n,1}|\\&\quad-\sum_{n\ge 5}(n^2-n)|F_{n,1}|-8|F_{4,1}|\\&=4+\Big(1+\frac{1}{2}|V_1|\Big)|F_{3,1}|+(|V_1|-6)|F_{4,1}|\\&\quad+\sum_{n\ge 5}\Big(|V_1|\Big(\frac{n}{2}-1\Big)-n^2+2n-2\Big)|F_{n,1}|.\end{aligned}$$

By the definition of M, we have $|F_{n,1}|=0$ for $n>M$. It is not hard to see that the coefficient of $|F_{n,1}|$ in the above formula is nonnegative for $n\le M$ in each case of the assumption. Hence we obtain

$$|V_1|^2-\sum_{f\in F_1}d(f)\deg(f)>0.$$

Therefore there exists a good vertex by Lemma 2.

4. Proof of Main Theorem

We divide the proof into several cases.
Case 1. Suppose that $M=3$. Then all vertices of P_1 are good vertices.
Case 2. Suppose that $M=4$. Clearly $|V_1|\ge 5$. If $|V_1|\ge 6$, there exists a good vertex

by Lemma 3. If $|V_1|=5$, then the faces of P_1 consist of one quadrilateral and four triangles, and the inequality of Lemma 2 holds. Therefore there exists a good vertex.

Case 3. Suppose that $M=5$. Assume that all vertices of P_1 were bad vertices. By Lemma 2, we have

$$20|F_{5,1}|+|F_{4,1}|\geq|V_1|^2. \tag{4.1}$$

We have the following conditions about P_1. By the formula (3.6),

$$|V_1|=\sum_{n=3}^{5}\left(\frac{n}{2}-1\right)|F_{n,1}|+2. \tag{4.2}$$

Since a pentagon is adjacent to at least five faces,

$$\sum_{n=3}^{5}|F_{n,1}|\geq 6. \tag{4.3}$$

Because every vertex is incident with at least three faces,

$$\sum_{n=3}^{5}n|F_{n,1}|\geq 3|V_1|. \tag{4.4}$$

If $|V_1|\geq 12$, there exists a good vertex by Lemma 3. Therefore we may assume $6\leq |V_1|\leq 11$. Furthermore we may assume that $|F_{4,1}|$ and $|F_{3,1}|$ are nonnegative integers and that $|F_{5,1}|$ is a positive integer. However there exists no possible combination $(|V_1|, |F_{3,1}|, |F_{4,1}|, |F_{5,1}|)$ which satisfies (4.1) (4.2) (4.3) and (4.4). Therefore P_1 must have a good vertex.

Case 4. Suppose that $M\geq 6$. Let f_0 be one of the M-gonal faces of P_1. Assuming that all vertices of f_0 are bad vertices, we will show

$$|V_1|\geq 2M+1$$

step by step. Then it follows that there exists a good vertex somewhere else by Lemma 3.

Note that, we have

$$|V_1|\geq 2+\left(\frac{\deg f_0}{2}-1\right)+\sum_{f\text{ is adjacent to }f_0}\left(\frac{\deg f}{2}-1\right) \tag{4.5}$$

by (3.6). We denote the last term of right side of (4.5) by $V(f_0)$.

Step 1. In this step, we show $|V_1|\geq 5M/4+1$.

Let v be a vertex of f_0. By the assumption, v is a bad vertex and $|V_1|\geq M+1$. Thus by Lemma 1, we have

$$\sum_{v\in f}d(f)\geq M+1.$$

We denote by $I(v)$ the set of faces other than f_0 which are incident with v. Since $\sum_{v\in f} d(f)=\sum_{f\in I(v)} d(f)+M-1$, it follows

$$\sum_{f\in I(v)} d(f)\geq 2.$$

This implies that $I(v)$ contains a face of degree n (≥ 4). Since every vertex is incident with at least three faces, $V(f_0)$ is minimal if a vertex v is incident with a triangle, a quadrilateral and f_0 for all $v\in f_0$.

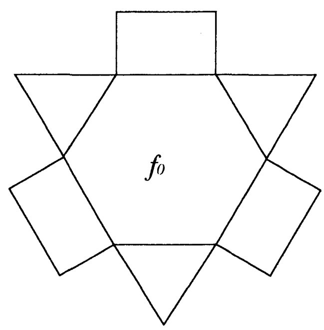


figure 4

Hence by (4.5), we get

$$|V_1|\geq\frac{5}{4}M+1.$$

Step 2. In this step, we show $|V_1|\geq 3M/2+1$.

By the assumption, v is a bad vertex, and $|V_1|\geq 5M/4+1$ from Step 1. By Lemma 1, we have

$$\sum_{v\in f} d(f)\geq\frac{5}{4}M+1.$$

Since we denote by $I(v)$ the set of faces other than f_0 which are incident with v, and $\sum_{v\in f} d(f)=\sum_{f\in I(v)} d(f)+M-1$, it follows

$$\sum_{f\in I(v)} d(f)\geq\frac{1}{4}M+2.$$

By the assumption, $M\geq 6$, then

$$\sum_{f\in I(v)} d(f)\geq\frac{5}{2}.$$

This implies that

(i) $I(v)$ contains a face of degree n (≥ 5), or

(ii) $I(v)$ contains two quadrilaterals,
for every vertex $v \in f_0$. We will evaluate the minimal value of $V(f_0)$. Then we may assume that every vertex of f_0 is incident with only three faces. Let $v_1, \cdots, v_k$ be the vertices of f_0 which satisfy (i) and $v_{k+1}, \cdots, v_M$ the vertices which satisfy (ii). Thus we may assume that, for $i=1, \cdots, k$, $I(v_i)$ consists of a triangle and a pentagon, or a quadrilateral and a pentagon and for $i=k+1, \cdots, M$, $I(v_i)$ consists of two quadrilaterals. If we replace the faces of $I(v_i)$ by quadrilaterals for $i=i, \cdots, k$, the value of $V(f_0)$ does not increase. Therefore we have minimal value of $V(f_0)$ when $I(v)$ consists of two quadrilaterals for every vertex $v \in f_0$.

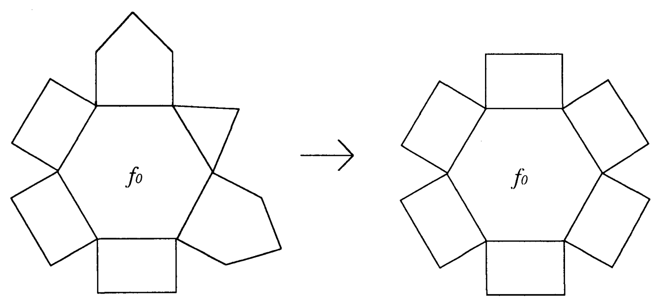


figure 5

Hence by (4.5) we get

$$|V_1| \geq \frac{3}{2}M+1.$$

Step 3. We repeat the argument, assuming the above inequality. For every vertex $v \in f_0$,

(i) $I(v)$ contains a face of degree n (≥ 6),
(ii) $I(v)$ contains a pentagon and a face of degree n $(=4, 5)$ or
(iii) $I(v)$ contains three quadrilaterals.

We have the minimal value of $V(f_0)$ when $I(v)$ consists of a triangle and a hexagon for every vertex $v \in f_0$. Then we have

$$|V_1| = \frac{7}{4}M+1.$$

Step 4. We repeat the argument once more, assuming the above inequality. We have the minimal value of $V(f_0)$ when $I(v)$ consists of two pentagons for every vertex $v \in f_0$. Then we obtain the desired inequality

$$|V_1| \geq 2M+1.$$

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