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BOUNDARY SLOPES FOR KNOTS

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Let T be a torus. By the *slope* of an essential simple closed curve on T we mean its isotopy class. The *distance* $\Delta(r_1, r_2)$ between two slopes r_1 and r_2 is defined to be $|\gamma_1 \cdot \gamma_2|$, where γ_1 and γ_2 are curves with slopes r_1 and r_2 and $\cdot$ denotes homological intersection number. (Note that this is independent of all orientations. Note also that Δ is not a metric on the set of slopes; the triangle inequality does not hold.)

Now let M be an irreducible, orientable 3-manifold and T a torus component of ∂M. Let $(F, \partial F) \subset (M, T)$ be an incompressible, boundary incompressible, orientable, genus g surface. Then the components of ∂F all have the same slope on T, and we call this the *boundary slope* of F. Let $S(M)_g$ denote the set of boundary slopes of such genus g surfaces. When M is an exterior $E(K)$ of a knot K, we write $S(E(K))_g$ as $S(K)_g$.

Gordon and Luecke gave estimations of ∂-slopes in $S(M)_0$ and $S(M)_1$, and showed that their estimations are the best possible (see [1], [3], [4]). So far, however, there is no estimation of ∂-slopes in $S(M)_g$ for $g \geq 2$.

In this paper, we give some estimation of ∂-slopes in $S(M)_g$ for arbitrary g when M has a certain geometric restriction, and we give an example which estimates the strength of the theorem.

Our main results are then the following.

Theorem 1. *If M has no essential annulus, then for any $g_1, g_2 \geq 1$, $r_1 \in S(M)_{g_1}$, $r_2 \in S(M)_{g_2}$, we have $\Delta(r_1, r_2) < 36(2g_1-1)(2g_2-1)$.*

Theorem 2. *Suppose a knot K has an m-string ∂-irreducible tangle decomposition.*

(i) *Let a/b $(\neq 0/1)$ be an element of $S(K)_g$, where a and b are coprime integers. Then $|b| \leq g/m$.*

(ii) *$g(K) \geq (m+1)/2$, where $g(K)$ is the genus of K.*

Theorem 3. *For any n non-trivial knots $K_1, \cdots, K_n$ and $a/b \in S(K_1 \# \cdots \#$*

$K_n)_g$, we have $|b|\leq g/(n-1)$.

The organization is as follows. In sections 1 and 2 we prove above theorems. In section 3 we give an example which concerns Theorem 2 and construct ∂-irreducible tangles systematically.

1. Proof of Theorem 1

Let $N(\cdot\cdot)$ denote a tublar neighbourhood. Let G be a finite graph in a closed surface S. We take edges and faces of G to be open edges and faces, i.e., components of $G-\{vertices\}$ and $S-G$, respectively. Then an edge e belongs to a face f if $e\subset cl(f)$, where $cl(f)$ denotes the closure of f in S. A face is 1-*sided* if it has only one edge (and one vertex).

To prove Theorem 1 we need the following lemma ([2, Lemma 6.2]).

Lemma 1.1. *Let Γ be a finite graph in a closed surface S, with V vertices and no 1-sided faces which are open discs. Suppose that, for some integer $n\geq 2$, every vertex of Γ has order greater than $(max\{[6(1-\chi(S)/V)], 1\})(n-1)$. Then Γ has n mutually parallel edges.*

Suppose, for a contradiction, $\Delta=\Delta(r_1, r_2)\geq 36(2g_1-1)(2g_2-1)$. Let F_i be an incompressible, ∂-incompressible, orientable, connected, genus g_i surface with ∂-slope r_i ($i=1, 2$). After an isotopy of F_i we may assume that F_1 and F_2 intersect transversely, and each component of ∂F_1 [resp. ∂F_2] intersects that of ∂F_2 [resp. ∂F_2] exactly $\Delta(r_1, r_2)$ times. Then $F_1\cap F_2=A\amalg S$, where A is a disjoint union of properly embedded arcs and S is a disjoint union of simple closed curves. By a standard disc swapping argument, using the incompressibility of F_1 [resp. F_2], we may assume that no component of S bounds a disc on F_2 [resp. F_1]. As in [2], we form graphs G_{F_1}, G_{F_2} as follows. Let $\hat{F}_i$ be the closed surface obtained by capping off the boundary components of F_i by disc ($i=1, 2$). We obtain a graph G_{F_1} in $\hat{F}_1$ by taking as the "fat" vertices of G_{F_1} the discs attached as above, and as the edges of G_{F_1}, the arcs in A. Similarly we obtain the graph G_{F_2} in $\hat{F}_2$. Since F_1 [resp. F_2] is ∂-incompressible, we may assume (again by a standard disc swapping argument) that G_{F_2} [resp. G_{F_1}] has no 1-sided faces. Let n_i denote the number of boundary components of G_{F_i} ($i=1, 2$). Then G_{F_i} has n_i vertices, each of order Δn_j ($i\neq j$). By a homological argument, we may assume $n_2\geq 2$. (If $n_1=1$, then ∂F_i is null-homologus in $H_1(E(K))$. Hence ∂-slope of F_i is 0/1.) Then by the assumption, $\Delta n_2\geq 36(2g_1-1)(2g_2-1)n_2>6(2g_1-1)\{6(2g_2-1)n_2-1\}\geq[6(1-(2-2g_1)/n_1)]\{6(2g_2-1)n_2-1\}$. Hence by Lemma 1.1, G_{F_1} has $6(2g_2-1)n_2$ mutually parallel edges. Let Γ be the subgraph of G_{F_2} arising from these edges. Then the order of each vertex of Γ is $6(2g_2-1)$. Since $6(2g_2-1)>[6(1-(2-2g_2)/n_2)]$, by Lemma 1.1 again, Γ has parallel edges. Let e_1 and e_2 be edges of Γ which are parallel and adjacent in F_2, and let B [resp. E] be the disc in F_1 [resp. F_2] cut off by e_1 and e_2. Put $A=B\cup E$, then A is either an annulus or a Möbius band properly embedded in M.

Case 1. $A=B\cup E$ is an annulus: Then ∂A is a union of two essential simple loops on T.

CLAIM. *A is not boundary parallel.*

Proof. If A is boundary parallel, then e_1 partially bounds a boundary compression disc of F_1, a contradiction.

By Claim and the assumption of the theorem, A is compressible, and hence T is compressible. Then it follows that M is a solid torus by the irreducibility of M. This is a contradiction, since a solid torus has only one ∂-slope.

Case 2. $A=B\cup E$ is a Möbius band: If ∂A is an inessential loop on T, then the disc on T bounded by ∂A and A make P^2 in M ; therefore $M=M'\#P^3$, for some 3-manifold M', a contradiction. Hence, ∂A is an essential loop on T. Since M is orientable, $N(A)$ is a twisted I-bundle over A. Therefore $\tilde{A}=FrN(A)$ is an annulus properly embedded in M, where $F_rN(A)$ is the frontier of $N(A)$ in M. By the assumption, $\tilde{A}$ is compressible or boundary compressible. If $\tilde{A}$ is compressible, then by the argument in Case 1, we have a contradiction. Hence $\tilde{A}$ is boundary compressible. Thus we see $\tilde{A}$ is boundary parallel by using the irreducibility of M and the fact that $\tilde{A}=FrN(A)$ is separating. Therefore M is a union of $N(A)$ and $\tilde{A}\times I$ along $\tilde{A}\times 0$; so $M\cong N(A)\cong S^1\times D^2$, a contradiction. This completes the proof of Theorem 1.

2. Proof of Theorem 2 and Theorem 3

Let K be a knot in S^3. The exterior of K is $E(K)=S^3-intN(K)$. A *tangle* (B, t) is a pair that consists of a 3-ball B and a 1-dimensional manifold t properly embedded in B. A tangle (B, t) is an *m-string tangle* if t consists of m number of arcs. A tangle (B, t) is called *∂-irreducible* if $\partial(cl(B-N(t)))$ is incompressible in $cl(B-N(t))$. We say that K has an m-string ∂-irreducible tangle decomposition if it can be expressed as a sum of two m-string ∂-irreducible tangles, i.e., there is a sphere S meeting K transversely in $2m$ points, such that each of the balls bounded by S determines, with its intersection with K, an m-string ∂-irreducible tangle.

Proof of Theorem 2. Suppose K is expressed as the sum of two m-string ∂-irreducible tangles (B_1, t_1) and (B_2, t_2). Let P denote $\partial B_1\cap E(K)$ $(=\partial B_2\cap E(K))$, then P is incompressible and ∂-incompressible by the definition of a ∂-irreducible tangle and an argument in [6, Lemma 1.10]. Let F be an incompressible, ∂-incompressible, orientable, connected, genus g surface with ∂-slope a/b. As in the proof of Theorem 1, we may assume that each component of ∂F [resp. ∂P] intersects that of ∂P [resp. ∂F] exactly $|b|$ times, and we define $\hat{F}$, $\hat{P}$, G_F, and G_P. Again we may assume that G_F and G_P have no 1-sided face.

Lemma 2.1. *There is no disc face in* G_F.

Proof. Suppose there is a disc face D in G_F, and D is contained in B_1. Then $cl(D)\cap\partial(cl(B_1-N(t_1)))$ is a simple loop in $\partial(cl(B_1-N(t_1)))$. By the definition of a ∂-irreducible tangle, $cl(D)\cap\partial(cl(B_1-N(t_1)))$ bounds a disc D' in $\partial(cl(B_1-N(t_1)))$. Let α be a component of $\partial D'\cap\partial P$ which is outermost disc in D', and let d be the (outermost) disc in D' cut off by α. Then d is contained in P and it produces a 1-sided face in G_P, a contradiction.

Let V and E, respectively, be the numbers of the vertices and the edges of G_F. Note that $E=m|b|V$.

Lemma 2.2. $g\geq V(m|b|-1)/2+1$

Proof. Note that $2-2g=\chi(\hat{F})=V-E+\sum_i\chi(F_i)$, where F_i runs over all faces of G_F. Since $\chi(F_i)\leq 0$ by Lemma 2.1, we have $2-2g\leq V-E=(1-m|b|)V$.

If F is a Seifert surface, then $a/b=0/1$ and $V=1$. Therefore, by Lemma 2.2, $g\geq(m+1)/2$.

If F is not a Seifert surface, then $V\geq 2$. Therefore, again by Lemma 2.2, $g\geq m|b|$.

This completes the proof of Theorem 2.

Proof of Theorem 3. Let $A_1, \cdots, A_{n-1}$ denote the annuli in $E(K)$ defining the connected sum as illustrated in Figure 2.1, and put $P=\bigcup_{i=1}^{n-1}A_i$.

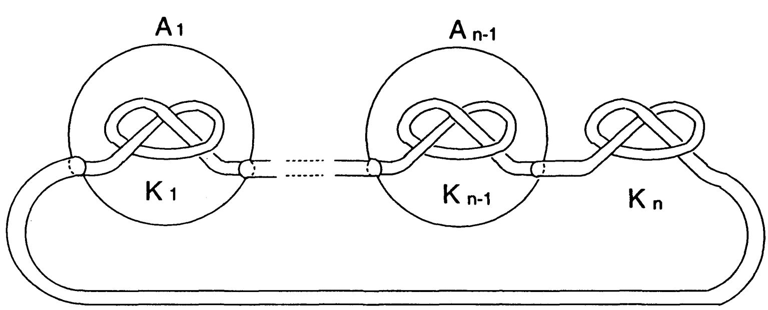

Figure 2.1

We define $\hat{F}$, $\hat{P}$, G_F, G_P, E, and V as in the proof of Theorem 2. Then G_F does not have a disc face. To show this note that P cuts $E(K)$ into the disjoint union $\coprod_{i=1}^{n} E(K_i)$. Suppose G_F has a disc face D. Then D is a properly embedded disc in some $E(K_i)$, and we can see that ∂D is essential in $\partial E(K_i)$. This implies that K_i is a trivial knot, a contradiction.

Next we remark $E=(n-1)|b|V$. Then as in the proof of Lemma 2.2, we see $2-2g \leq V-E=V-(n-1)|b|V$. Hence, $g \geq V\{(n-1)|b|-1\}/2+1$.

If $V \geq 2$, then we obtain $g \geq (n-1)|b|$. If $V=1$, then F is a Seifert surface, and hence $g(K_1 \# \cdots \# K_n) \geq n > n-1$.

This completes the proof of Theorem 3.

3. Constructing ∂-irreducible Tangles

In this section, we give a systematic construction of ∂-irreducible tangles. And combining the results of [5] with this construction, we present examples of knots which estimate the strength of Theorem 2.

A *Montesinos tangle* $T(r_1, \cdots, r_n)$ ($r_i \in \boldsymbol{Q} \cup \{1/0\}$) is a tangle illustrated in Figure 3.1.

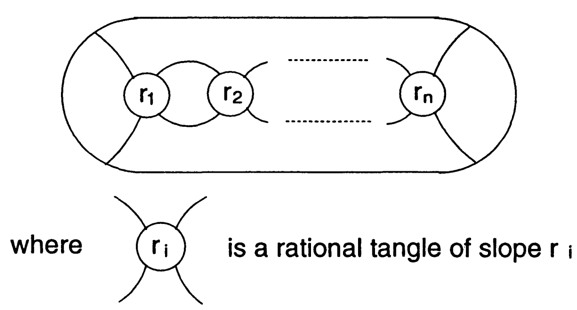

Figure 3.1

First we study which Montesions tangle is ∂-irreducible.

Theorem 4. *Suppose $n \geq 2$, $r_i \notin \boldsymbol{Z} \cup \{1/0\}$ $(1 \leq i \leq n)$, and r_1, $r_n \notin \{q/2 | q \in \boldsymbol{Z}\}$. Then $T(r_1, \cdots, r_n)$ is a ∂-irreducible tangle.*

REMARK.

(1) *$(B, t)=T(1/2, p/q)$ is not a ∂-irreducible tangle, indeed, $cl(B-N(t))$ is a genus 2 handlebody.*

(2) *After having done this work, the author learned that Wu [7] had proved that, except for trivial cases, a Montesinos tangle which is not ∂-irreducible*

is $T(1/2, p/q)$.

Proof of Theorem 4. Put $T(r_1, \cdots, r_n)=(B, t)$, $E(t)=cl(B-N(t))$, and let $A_1, \cdots, A_{n-1}$ be the surfaces in $E(t)$ as illustrated in Figure 3.2.

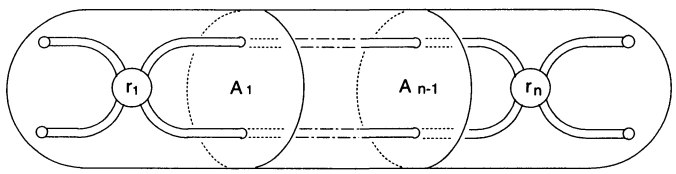

Figure 3.2

Then $\bigcup_{i=1}^{n-1} A_i$ decomposes $E(t)$ into $\coprod_{i=1}^{n} E(t_i)$, where $E(t_i)$ is the exterior $cl(B_i - N(t_i))$ of a rational tangle (B_i, t_i) of slope r_i $(1 \leq i < n)$.

Suppose $E(t)$ has a compressing disc D. Then we may assume D intersects $\cup A_i$ transversely. By using the assumption that $r_i \neq 1/0$ $(1 \leq i \leq n)$, we can isotope D so that $D \cap (\cup A_i)$ consists of only arcs. In the following, we assume that $|D \cap (\cup A_i)|$ is minimized; we see this number is not zero by using the same assumption. Let α be a component of $D \cap (\cup A_i)$ which is outermost in D, and let E be the disc in D cut off by α, such that $(\text{int } E) \cap (\cup A_i) = \emptyset$. Then α lies in some A_i, and (A_i, α) is of one of the six types illustrated in Figure 3.3.

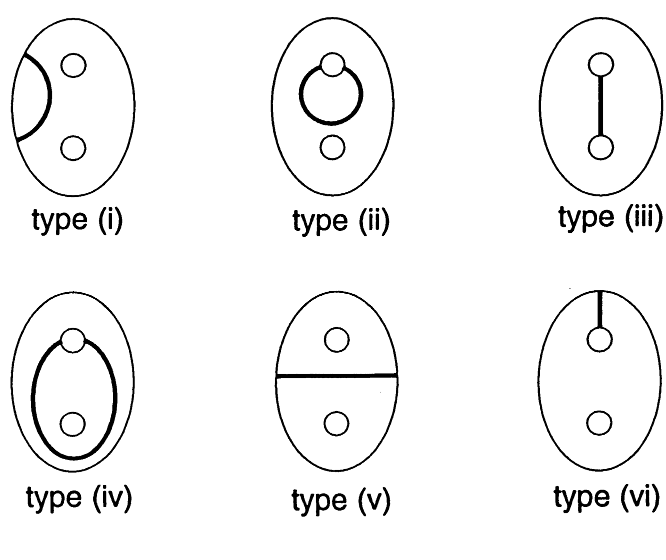

Figure 3.3

Then one of the following three cases occurs.

(1) E is contained in $E(t_1)$ (or $E(t_n)$) and $E(t_1)$ (or $E(t_n)$) is of X-type.

(2) E is contained in $E(t_1)$ (or $E(t_n)$) and $E(t_1)$ (or $E(t_n)$) is of Y-type.

(3) E is contained in $E(t_j)$, where $2 \leq j \leq n-1$.

Here, we say that $E(t_1)$ [resp. $E(t_n)$] is of X-type if each component of $FrN(t_1)$ [resp. $FrN(t_n)$] has one boundary component in A_1 [resp. A_{n-1}], Y-type otherwise. We show that we can find a contradiction in any case. We consider only Case (2), because the arguments for Cases (1) and (3) are similar to that for Case (2). It should be noted that (A_1, α) is not of type (vi) since $E(t_1)$ is of Y-type.

Without loss of generality we assume E is contained in $E(t_1)$. Put $\beta = cl(\partial E - \alpha)$, and T_1 and T_2 the components of $FrN(t_1)$; we assume that $T_1 \cap \alpha \neq \emptyset$ in case α is of type (ii), (iii), (iv), or (vi). By elementary but careful arguments, we may assume $(E(t_1), A_1, E)$ is as illustrated in Figure 3.4 (i)-(v) according as the type of (A_1, α). Here, in case (A_1, α) is of type (v), Figure 3.4 (v) illustrates $(E(t_1), A_1, E)$ only modulo integral twists of $E(t_1)$.

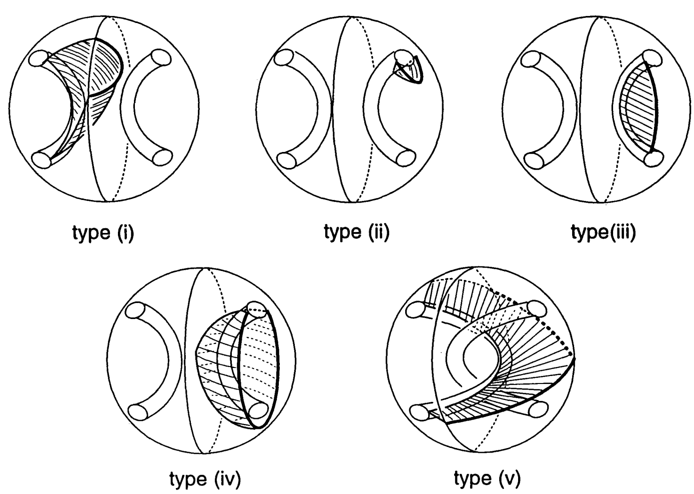

Figure 3.4

These figures imply that (1) if (A_1, α) is of type (i), (ii), (iii), or (iv), then $r_1 = 1/0$ and (2) if (A_1, α) is of type (v), then $r_1 = q/2$ with q an odd integer. This is a contradiction.

This completes the proof of Theorem 4.□

From now, a thorough understanding of [5] is assumed, and we investigate the genus of the surface realizing a ∂-slope in the following proposition.

Proposition 1 ([5, Proposition 2.2]). *For each $p/q \in \boldsymbol{Q}$, there exists an incompressible, ∂-incompressible, orientable surface in the complement of some Montesinos knot, with ∂-slope p/q.*

Concerning Theorem 2, we look for surfaces whose ∂-slopes have denominators q. Then, following the first half of the proof of the above proposition, we obtain the following theorem.

Theorem 5. *For a natural number q, let K_q be the Montesinos knot $M(2/7, 1/(8q+13), -1/3, 5/18, 1/(8q+13), -1/3)$ or $M(2/7, 1/(8q+13), -1/3, 2/7, 1/(8q+13), -1/3, 5/18, 1/(8q+13), -1/3)$ according as q is odd or even. Then $E(K_q)$ contains an incompressible, ∂-incompressible, orientable surface S_q, such that the denominator of the ∂-slope of S_q is q, and the genus of S_q is at most cq where c is a constant independent of q.*

REMARK. *By* Theorem 4, *K_q has a 2-string ∂-irreducible tangle decomposition.*

We give the proof only for the case where q is odd, because the proof for the case where q is even is similar.

Proof of Theorem 5. We shall use notations of [5]. First we recall the construction of the K_q (see [5, pp. 455-456]). Viewing S^3 as the join of two circles A and B, let the circle B be subdivided as a six-sided polygon. Then the join of A with the ith edge of B is a ball B_i. Put $H_i = B_i \cap B_{i+1} = \partial B_i \cap \partial B_{i+1}$, then $\partial B_i = H_{i-1} \cup H_i$. The 6 balls B_i $(1 \leq i \leq 6)$ cover S^3. Recall that (S^3, K_q) is constructed as the union $(B_1, t_1) \cup \cdots \cup (B_6, t_6)$, where $(B_1, t_1), \cdots, (B_6, t_6)$ are rational tangles of slopes $2/7, \cdots, -1/3$ respectively. In the proof of Proposition 1, an incompressible, ∂-incompressible, orientable, *candidate* surface S_q is constructed as follows. For each rational tangle (B_i, t_i), choose an *edgepath* γ_i as follows.

- γ_1 goes linearly from $(1, 6, 2)$ to the point $A = (4q-3)(1, 2, 1) + 3(1, 6, 2)$.
- γ_2 and γ_5 go linearly from $(1, 8q+12, 1)$ to the point $B = (4q-1)(1, 0, 0) + (1, 8q+12, 1)$.
- γ_3 and γ_6 are constant, at the point $C = (4q, 8q+12, -4-4q)$.
- γ_4 first goes linearly from $(1, 17, 5)$ to $(1, 6, 2)$ and second goes linearly $(1, 6, 2)$ to the point A.

To each γ_i, a surface S_i in (B_i, t_i) is associated so that $S_i \cap H_i = S_{i+1} \cap H_i$. Then $S_q = \bigcup_{i=1}^{n} S_i$. The ∂-slope of S_q is given by $\tau(S_q) - \tau(S_0)$, where $\tau(S_q) = 2/q - 2$, and $\tau(S_0)$ is a certain integer associated with a Seifert surface S_0. Hence the denominator of the ∂-slope of S_q is q.

Finally we roughly estimate the genus of S_q.

(1) Through all S_i the number of saddles is less than $c_1 q$, where c_1 is a

constant independent of q $(1 \leq i \leq 6)$.

(2) The number of arcs in $H_i \cap S_i$ is less than $c_2 q$, where c_2 is a constant independent of q $(1 \leq i \leq 6)$.

Hence we can see that the genus of S_q is at most cq, where c is a constant independent of q.

This completes the proof of Theorem 5.

REMARK. *In* Theorem 5 *we can take* $c=100$.

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