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BOUNDARY SLOPES FOR KNOTS

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Let $T$ be a torus. By the slope of an essential simple closed curve on $T$ we mean its isotopy class. The distance $\Delta(r_1, r_2)$ between two slopes $r_1$ and $r_2$ is defined to be $|\gamma_1 \cdot \gamma_2|$, where $\gamma_1$ and $\gamma_2$ are curves with slopes $r_1$ and $r_2$ and $\cdot$ denotes homological intersection number. (Note that this is independent of all orientations. Note also that $\Delta$ is not a metric on the set of slopes; the triangle inequality does not hold.)

Now let $M$ be an irreducible, orientable 3-manifold and $T$ a torus component of $\partial M$. Let $(F, \partial F) \subset (M, T)$ be an incompressible, boundary incompressible, orientable, genus $g$ surface. Then the components of $\partial F$ all have the same slope on $T$, and we call this the boundary slope of $F$. Let $S(M)_g$ denote the set of boundary slopes of such genus $g$ surfaces. When $M$ is an exterior $E(K)$ of a knot $K$, we write $S(E(K))_g$ as $S(K)_g$.

Gordon and Luecke gave estimations of $\partial$-slopes in $S(M)_0$ and $S(M)_1$, and showed that their estimations are the best possible (see [1], [3], [4]). So far, however, there is no estimation of $\partial$-slopes in $S(M)_g$ for $g \geq 2$.

In this paper, we give some estimation of $\partial$-slopes in $S(M)_g$ for arbitrary $g$ when $M$ has a certain geometric restriction, and we give an example which estimates the strength of the theorem.

Our main results are then the following.

**Theorem 1.** If $M$ has no essential annulus, then for any $g_1, g_2 \geq 1$, $r_1 \in S(M)_{g_1}$, $r_2 \in S(M)_{g_2}$, we have $\Delta(r_1, r_2) < 36(2g_1 - 1)(2g_2 - 1)$.

**Theorem 2.** Suppose a knot $K$ has an $m$-string $\partial$-irreducible tangle decomposition.

(i) Let $a/b$ ($\neq 0/1$) be an element of $S(K)_g$, where $a$ and $b$ are coprime integers. Then $|b| \leq g/m$.

(ii) $g(K) \geq (m + 1)/2$, where $g(K)$ is the genus of $K$.

**Theorem 3.** For any $n$ non-trivial knots $K_1, \ldots, K_n$ and $a/b \in S(K_1 \# \cdots \# K_n)$
\(K_n\), we have \(|b| \leq g/(n-1)\).

The organization is as follows. In sections 1 and 2 we prove above theorems. In section 3 we give an example which concerns Theorem 2 and construct \(\partial\)-irreducible tangles systematically.

1. Proof of Theorem 1

Let \(N(\cdot)\) denote a tublar neighbourhood. Let \(G\) be a finite graph in a closed surface \(S\). We take edges and faces of \(G\) to be open edges and faces, i.e., components of \(G - \{\text{vertices}\}\) and \(S - G\), respectively. Then an edge \(e\) belongs to a face \(f\) if \(e \subseteq \operatorname{cl}(f)\), where \(\operatorname{cl}(f)\) denotes the closure of \(f\) in \(S\). A face is \(1\)-sided if it has only one edge (and one vertex).

To prove Theorem 1 we need the following lemma (\cite[Lemma 6.2]{2}).

**Lemma 1.1.** Let \(\Gamma\) be a finite graph in a closed surface \(S\), with \(V\) vertices and no \(1\)-sided faces which are open discs. Suppose that, for some integer \(n \geq 2\), every vertex of \(\Gamma\) has order greater than \((\max\{6(1 - x(S)/V), 1\})(n-1)\). Then \(\Gamma\) has \(n\) mutually parallel edges.

Suppose, for a contradiction, \(\Delta = \Delta(r_1, r_2) \geq 36(2g_1 - 1)(2g_2 - 1)\). Let \(F_i\) be an incompressible, \(\partial\)-incompressible, orientable, connected, genus \(g_i\) surface with \(\partial\)-slope \(r_i\) \((i = 1, 2)\). After an isotopy of \(F_i\) we may assume that \(F_1\) and \(F_2\) intersect transversely, and each component of \(\partial F_1\) [resp. \(\partial F_2\)] intersects that of \(\partial F_2\) [resp. \(\partial F_2\)] exactly \(\Delta(r_1, r_2)\) times. Then \(F_1 \cap F_2 = A \cup S\), where \(A\) is a disjoint union of properly embedded arcs and \(S\) is a disjoint union of simple closed curves. By a standard disc swapping argument, using the incompressibility of \(F_1\) [resp. \(F_2\)], we may assume that no component of \(S\) bounds a disc on \(F_2\) [resp. \(F_1\)]. As in \cite{2}, we form graphs \(G_{F_1}, G_{F_2}\) as follows. Let \(\bar{F}_i\) be the closed surface obtained by capping off the boundary components of \(F_i\) by disc \((i = 1, 2)\). We obtain a graph \(G_{F_i}\) in \(\bar{F}_i\) by taking as the “fat” vertices of \(G_{F_i}\), the discs attached as above, and as the edges of \(G_{F_i}\), the arcs in \(A\). Similarly we obtain the graph \(G_{F_2}\) in \(\bar{F}_2\). Since \(F_1\) [resp. \(F_2\)] is \(\partial\)-incompressible, we may assume (again by a standard disc swapping argument) that \(G_{F_1}\) [resp. \(G_{F_2}\)] has no \(1\)-sided faces. Let \(n_i\) denote the number of boundary components of \(G_{F_i}\) \((i = 1, 2)\). Then \(G_{F_i}\) has \(n_i\) vertices, each of order \(\Delta n_i\) \((i \neq j)\). By a homological argument, we may assume \(n_2 \geq 2\). If \(n_1 = 1\), then \(\partial F_i\) is null-homologous in \(H_1(E(K))\). Hence \(\partial\)-slope of \(F_i\) is \(0/1\). Then by the assumption, \(\Delta n_2 \geq 36(2g_1 - 1)(2g_2 - 1)n_2 > 6(2g_1 - 1)(6(2g_2 - 1)n_2 - 1) \geq [6(1 - (2 - 2g_1)/n_1)][6(2g_2 - 1)n_2 - 1]. Hence by Lemma 1.1, \(G_{F_1}\) has \(6(2g_2 - 1)n_2\) mutually parallel edges. Let \(\Gamma\) be the subgraph of \(G_{F_2}\) arising from these edges. Then the order of each vertex of \(\Gamma\) is \(6(2g_2 - 1)\). Since \(6(2g_2 - 1) > [6(1 - (2 - 2g_2)/n_2)]\), by Lemma 1.1 again, \(\Gamma\) has parallel edges. Let \(e_1\) and \(e_2\) be edges of \(\Gamma\) which are
parallel and adjacent in $F_2$, and let $B$ [resp. $E$] be the disc in $F_1$ [resp. $F_2$] cut off by $e_1$ and $e_2$. Put $A = B \cup E$, then $A$ is either an annulus or a M"obius band properly embedded in $M$.

Case 1. $A = B \cup E$ is an annulus: Then $\partial A$ is a union of two essential simple loops on $T$.

**Claim.** $A$ is not boundary parallel.

**Proof.** If $A$ is boundary parallel, then $e_1$ partially bounds a boundary compression disc of $F_1$, a contradiction.

By Claim and the assumption of the theorem, $A$ is compressible, and hence $T$ is compressible. Then it follows that $M$ is a solid torus by the irreducibility of $M$. This is a contradiction, since a solid torus has only one $\partial$-slope.

Case 2. $A = B \cup E$ is a M"obius band: If $\partial A$ is an inessential loop on $T$, then the disc on $T$ bounded by $\partial A$ and $A$ make $P$ in $M$; therefore $M = M' \# P^2$, for some 3-manifold $M'$, a contradiction. Hence, $\partial A$ is an essential loop on $T$. Since $M$ is orientable, $N(A)$ is a twisted $I$-bundle over $A$. Therefore $\tilde{A} = \text{Fr} N(A)$ is an annulus properly embedded in $M$, where $\text{Fr} N(A)$ is the frontier of $N(A)$ in $M$. By the assumption, $\tilde{A}$ is compressible or boundary compressible. If $\tilde{A}$ is compressible, then by the argument in Case 1, we have a contradiction. Hence $\tilde{A}$ is boundary compressible. Thus we see $\tilde{A}$ is boundary parallel by using the irreducibility of $M$ and the fact that $\tilde{A} = \text{Fr} N(A)$ is separating. Therefore $M$ is a union of $N(A)$ and $\tilde{A} \times I$ along $\tilde{A} \times 0$; so $M \cong N(A) \cong S^1 \times D^2$, a contradiction. This completes the proof of Theorem 1.

2. **Proof of Theorem 2 and Theorem 3**

Let $K$ be a knot in $S^3$. The exterior of $K$ is $E(K) = S^3 - \text{int} N(K)$. A **tangle** $(B, t)$ is a pair that consists of a 3-ball $B$ and a 1-dimensional manifold $t$ properly embedded in $B$. A tangle $(B, t)$ is an $m$-**string tangle** if $t$ consists of $m$ number of arcs. A tangle $(B, t)$ is called $\partial$-**irreducible** if $\partial(\text{cl}(B - N(t)))$ is incompressible in $\text{cl}(B - N(t))$. We say that $K$ has an $m$-string $\partial$-irreducible tangle decomposition if it can be expressed as a sum of two $m$-string $\partial$-irreducible tangles, i.e., there is a sphere $S$ meeting $K$ transversely in $2m$ points, such that each of the balls bounded by $S$ determines, with its intersection with $K$, an $m$-string $\partial$-irreducible tangle.

**Proof of Theorem 2.** Suppose $K$ is expressed as the sum of two $m$-string $\partial$-irreducible tangles $(B_1, t_1)$ and $(B_2, t_2)$. Let $P$ denote $\partial B_1 \cap E(K) (= \partial B_2 \cap E(K))$, then $P$ is incompressible and $\partial$-incompressible by the definition of a $\partial$-irreducible tangle and an argument in [6, Lemma 1.10]. Let $F$ be an incompressible, $\partial$-incompressible, orientable, connected, genus $g$ surface with $\partial$-slope $a/b$. As
in the proof of Theorem 1, we may assume that each component of \( \partial F \) [resp. \( \partial P \)] intersects that of \( \partial P \) [resp. \( \partial F \)] exactly \(|b|\) times, and we define \( \tilde{F}, \tilde{P}, G_F, \) and \( G_P \). Again we may assume that \( G_F \) and \( G_P \) have no 1-sided face.

**Lemma 2.1.** There is no disc face in \( G_F \).

Proof. Suppose there is a disc face \( D \) in \( G_F \), and \( D \) is contained in \( B_1 \). Then \( \text{cl}(D) \cap \partial(\text{cl}(B_1 - N(t_1))) \) is a simple loop in \( \partial(\text{cl}(B_1 - N(t_1))) \). By the definition of a \( \partial \)-irreducible tangle, \( \text{cl}(D) \cap \partial(\text{cl}(B_1 - N(t_1))) \) bounds a disc \( D' \) in \( \partial(\text{cl}(B_1 - N(t_1))) \). Let \( a \) be a component of \( \partial D' \cap \partial P \) which is outermost disc in \( D' \), and let \( d \) be the (outermost) disc in \( D' \) cut off by \( a \). Then \( d \) is contained in \( P \) and it produces a 1-sided face in \( G_P \), a contradiction.

Let \( V \) and \( E \), respectively, be the numbers of the vertices and the edges of \( G_F \). Note that \( E = m|b|V \).

**Lemma 2.2.** \( g \geq V(m|b| - 1)/2 + 1 \)

Proof. Note that \( 2 - 2g = \chi(F) = V - E + \sum \chi(F_i) \), where \( F_i \) runs over all faces of \( G_F \). Since \( \chi(F_i) \leq 0 \) by Lemma 2.1, we have \( 2 - 2g \leq V - E = (1 - m|b|) V \).

If \( F \) is a Seifert surface, then \( a/b = 0/1 \) and \( V = 1 \). Therefore, by Lemma 2.2, \( g \geq (m+1)/2 \).

If \( F \) is not a Seifert surface, then \( V \geq 2 \). Therefore, again by Lemma 2.2, \( g \geq m|b| \).

This completes the proof of Theorem 2.

**Proof of Theorem 3.** Let \( A_1, \ldots, A_{n-1} \) denote the annuli in \( E(K) \) defining the connected sum as illustrated in Figure 2.1, and put \( P = \bigcup_{i=1}^{n-1} A_i \).

*Figure 2.1*
We define \( \bar{F}, \bar{P}, G_F, G_P, E, \) and \( V \) as in the proof of Theorem 2. Then \( G_F \) does not have a disc face. To show this note that \( P \) cuts \( E(K) \) into the disjoint union \( \prod_{i=1}^k E(K_i) \). Suppose \( G_F \) has a disc face \( D \). Then \( D \) is a properly embedded disc in some \( E(K_i) \), and we can see that \( \partial D \) is essential in \( \partial E(K_i) \). This implies that \( K_i \) is a trivial knot, a contradiction.

Next we remark \( E = (n-1)|b|V \). Then as in the proof of Lemma 2.2, we see \( 2-2g \leq V - E = V - (n-1)|b|V \). Hence, \( g \geq \frac{V((n-1)|b|-1)}{2} + 1 \).

If \( V \geq 2 \), then we obtain \( g \geq (n-1)|b| \). If \( V = 1 \), then \( F \) is a Seifert surface, and hence \( g(K_1\#\cdots\#K_n) \geq n > n-1 \). This completes the proof of Theorem 3.

3. Constructing \( \partial \)-irreducible Tangles

In this section, we give a systematic construction of \( \partial \)-irreducible tangles. And combining the results of [5] with this construction, we present examples of knots which estimate the strength of Theorem 2.

A Montesinos tangle \( T(r_1, \cdots, r_n) (r_i \in \mathbb{Q} \cup \{1/0\}) \) is a tangle illustrated in Figure 3.1.

First we study which Montesinos tangle is \( \partial \)-irreducible.

**Theorem 4.** Suppose \( n \geq 2 \), \( r_i \in \mathbb{Z} \cup \{1/0\} \) (\( 1 \leq i \leq n \)), and \( r_i, r_n \in \{q/2|q \in \mathbb{Z}\} \). Then \( T(r_1, \cdots, r_n) \) is a \( \partial \)-irreducible tangle.

**Remark.**

1. \( (B, t) = T(1/2, p/q) \) is not a \( \partial \)-irreducible tangle, indeed, \( \text{cl}(B - N(t)) \) is a genus 2 handlebody.

2. After having done this work, the author learned that Wu [7] had proved that, except for trivial cases, a Montesinos tangle which is not \( \partial \)-irreducible
is $T(1/2, p/q)$.

Proof of Theorem 4. Put $T(r_1, \cdots, r_n) = (B, t), E(t) = cl(B - N(t))$, and let $A_1, \cdots, A_{n-1}$ be the surfaces in $E(t)$ as illustrated in Figure 3.2.

Then $\bigcup_{i=1}^n A_i$ decomposes $E(t)$ into $\prod_{i=1}^n E(t_i)$, where $E(t_i)$ is the exterior $cl(B_i - N(t_i))$ of a rational tangle $(B_i, t_i)$ of slope $r_i$ ($1 \leq i < n$).

Suppose $E(t)$ has a compressing disc $D$. Then we may assume $D$ intersects $\bigcup A_i$ transversely. By using the assumption that $r_i \neq 1/0$ ($1 \leq i \leq n$), we can isotope $D$ so that $D \cap (\bigcup A_i)$ consists of only arcs. In the following, we assume that $|D \cap (\bigcup A_i)|$ is minimized; we see this number is not zero by using the same assumption. Let $a$ be a component of $D \cap (\bigcup A_i)$ which is outermost in $D$, and let $E$ be the disc in $D$ cut off by $a$, such that $(int E) \cap (\bigcup A_i) = \emptyset$. Then $a$ lies in some $A_i$, and $(A_i, a)$ is of one of the six types illustrated in Figure 3.3.
Then one of the following three cases occurs.

1. $E$ is contained in $E(t_i)$ (or $E(t_n)$) and $E(t_i)$ (or $E(t_n)$) is of $X$-type.
2. $E$ is contained in $E(t_i)$ (or $E(t_n)$) and $E(t_i)$ (or $E(t_n)$) is of $Y$-type.
3. $E$ is contained in $E(t_j)$, where $2 \leq j \leq n - 1$.

Here, we say that $E(t_i)$ [resp. $E(t_n)$] is of $X$-type if each component of $FrN(t_i)$ [resp. $FrN(t_n)$] has one boundary component in $A_i$ [resp. $A_{n-1}$], $Y$-type otherwise. We show that we can find a contradiction in any case. We consider only Case (2), because the arguments for Cases (1) and (3) are similar to that for Case (2). It should be noted that $(A_i, a)$ is not of type (vi) since $E(t_i)$ is of $Y$-type.

Without loss of generality we assume $E$ is contained in $E(t_i)$. Put $\beta = \text{cl}(\partial E - a)$, and $T_1$ and $T_2$ the components of $FrN(t_i)$; we assume that $T_1 \cap a \neq \emptyset$ in case $a$ is of type (ii), (iii), (iv), or (vi). By elementary but careful arguments, we may assume $(E(t_i), A_i, E)$ is as illustrated in Figure 3.4 (i)-(v) according as the type of $(A_i, a)$. Here, in case $(A_i, a)$ is of type (v), Figure 3.4 (v) illustrates $(E(t_i), A_i, E)$ only modulo integral twists of $E(t_i)$.

![Figure 3.4](image)

These figures imply that (1) if $(A_i, a)$ is of type (i), (ii), (iii), or (iv), then $r_1 = 1/0$ and (2) if $(A_i, a)$ is of type (v), then $r_1 = q/2$ with $q$ an odd integer. This is a contradiction.

This completes the proof of Theorem 4.\(\square\)

From now, a thorough understanding of [5] is assumed, and we investigate the genus of the surface realizing a $\partial$-slope in the following proposition.
Proposition 1 ([5, Proposition 2.2]). For each \( p/q \in \mathbb{Q} \), there exists an incompressible, \( \delta \)-incompressible, orientable surface in the complement of some Montesinos knot, with \( \delta \)-slope \( p/q \).

Concerning Theorem 2, we look for surfaces whose \( \delta \)-slopes have denominators \( q \). Then, following the first half of the proof of the above proposition, we obtain the following theorem.

Theorem 5. For a natural number \( q \), let \( K_q \) be the Montesinos knot \( M(2/7, 1/(8q+13), -1/3, 1/(8q+13), -1/3) \) or \( M(2/7, 1/(8q+13), -1/3, 5/18, 1/(8q+13), -1/3) \) according as \( q \) is odd or even. Then \( E(K_q) \) contains an incompressible, \( \delta \)-incompressible, orientable surface \( S_q \), such that the denominator of the \( \delta \)-slope of \( S_q \) is \( q \), and the genus of \( S_q \) is at most \( cq \) where \( c \) is a constant independent of \( q \).

Remark. By Theorem 4, \( K_q \) has a 2-string \( \delta \)-irreducible tangle decomposition.

We give the proof only for the case where \( q \) is odd, because the proof for the case where \( q \) is even is similar.

Proof of Theorem 5. We shall use notations of [5]. First we recall the construction of the \( K_q \) (see [5, pp. 455-456]). Viewing \( S^3 \) as the join of two circles \( A \) and \( B \), let the circle \( B \) be subdivided as a six-sided polygon. Then the join of \( A \) with the \( i \)th edge of \( B \) is a ball \( B_i \). Put \( H_i := B_i \cap B_{i+1} \cap \text{int} B_{i+1} \), then \( \partial B_i = H_{i-1} \cup H_i \). The 6 balls \( B_i (1 \leq i \leq 6) \) cover \( S^3 \). Recall that \( (S^3, K_q) \) is constructed as the union \( (B_1, t_1) \cup \cdots \cup (B_6, t_6) \), where \((B_1, t_1), \cdots, (B_6, t_6)\) are rational tangles of slopes \( 2/7, \cdots, -1/3 \) respectively. In the proof of Proposition 1, an incompressible, \( \delta \)-incompressible, orientable, candidate surface \( S_q \) is constructed as follows. For each rational tangle \((B_i, t_i)\), choose an edgepath \( \gamma_i \) as follows.

- \( \gamma_1 \) goes linearly from \((1, 6, 2)\) to the point \( A = (4q-3)(1, 2, 1) + (1, 6, 2) \).
- \( \gamma_2 \) and \( \gamma_5 \) go linearly from \((1, 8q+12, 1)\) to the point \( B = (4q-1)(1, 0, 0) + (1, 8q+12, 1) \).
- \( \gamma_3 \) and \( \gamma_6 \) are constant, at the point \( C = (4q, 8q+12, -4-4q) \).
- \( \gamma_4 \) first goes linearly from \((1, 17, 5)\) to \((1, 6, 2)\) and second goes linearly \((1, 6, 2)\) to the point \( A \).

To each \( \gamma_i \), a surface \( S_i \) in \((B_i, t_i)\) is associated so that \( S_i \cap H_i = S_{i+1} \cap H_i \). Then \( S_q = \bigcup_{i=1}^6 S_i \). The \( \delta \)-slope of \( S_q \) is given by \( \tau(S_q) - \tau(S_0) \), where \( \tau(S_q) = 2/q - 2 \) and \( \tau(S_0) \) is a certain integer associated with a Seifert surface \( S_0 \). Hence the denominator of the \( \delta \)-slope of \( S_q \) is \( q \).

Finally we roughly estimate the genus of \( S_q \).

1. Through all \( S_i \) the number of saddles is less than \( c_1 q \), where \( c_1 \) is a
constant independent of \( q \) \((1 \leq i \leq 6)\).

(2) The number of arcs in \( H_i \cap S_i \) is less than \( c_2q \), where \( c_2 \) is a constant independent of \( q \) \((1 \leq i \leq 6)\).

Hence we can see that the genus of \( S_q \) is at most \( cq \), where \( c \) is a constant independent of \( q \).

This completes the proof of Theorem 5.

**Remark.** In Theorem 5 we can take \( c = 100 \).

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**References**


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