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SELF HOMOTOPY EQUIVALENCES OF STIEFEL MANIFOLDS $W_{n,2}$ AND $V_{n,2}$

Dedicated to Professor Y. Matsushima on his 60th birthday

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1. Introduction

Let $\mathcal{E}(X)$ denote the group of homotopy classes of self homotopy equivalences of a space X , whose group structure is induced by map-composition. Very little is known about this group in case X is a simply-connected CW complex with three cells which is not an H-space. In this article we shall calculate $\mathcal{E}(X)$ for the real and complex Stiefel manifolds of orthonormal 2-frames in n -space, $V_{n,2}=O(n)/O(n-2)$ and $W_{n,2}=U(n)/U(n-2)$.

2. Statement of the results

As is well known, $W_{n,2}$ and $V_{n,2}$ are sphere-bundles over spheres:

$$S^{2n-3} \xrightarrow{l} W_{n,2} \xrightarrow{\pi} S^{2n-1}, \quad S^{n-2} \xrightarrow{l} V_{n,2} \xrightarrow{\pi} S^{n-1}$$

and have the following cell-structures (see James-Whitehead [9]);

$$W_{n,2} = (S^{2n-3} \bigcup_{\theta} e^{2n-1}) \bigcup_{\rho} e^{4n-4}, \quad V_{n,2} = (S^{n-2} \bigcup_{\theta} e^{n-1}) \bigcup_{\rho} e^{2n-3}$$

where θ in $W_{n,2}$ is the non-zero element $\eta_{2n-3} \in \pi_{2n-2}(S^{2n-3})$ for odd n and 0 for even n , and θ in $V_{n,2}$ is $2 \iota_{n-2}$ for odd n and 0 for even n . The characteristic element χ of the bundle, $\chi \in \pi_{2n-2}(O(2n-2))$ for $W_{n,2}$ and $\chi \in \pi_{n-2}(O(n-1))$ for $V_{n,2}$, is reduced to ξ , $\xi \in \pi_{2n-2}(O(2n-3))$ for $W_{n,2}$ and $\xi \in \pi_{n-2}(O(n-2))$ for $V_{n,2}$, if n is even.

We shall prove

Theorem 2.1. *Let n be odd, $n \geq 5$. Then there exists a split exact sequence*

$$1 \rightarrow \pi_{4n-4}(W_{n,2})/l_*(\text{Ker } S) \rightarrow \mathcal{E}(W_{n,2}) \rightarrow Z_2 \rightarrow 1,$$

where S is the suspension homomorphism $S: \pi_{4n-4}(S^{2n-3}) \rightarrow \pi_{4n-3}(S^{2n-2})$.

Theorem 2.2. *Let n be even, $n \geq 6$. Then there exists a split exact sequence*

$$1 \rightarrow \pi_{4n-4}(S^{2n-1}) + \pi_{4n-4}(S^{2n-3})/\text{Ker } S \rightarrow \mathcal{E}(W_{n,2}) \rightarrow Z_2 \rightarrow 1,$$

where S is the same as in Theorem 2.1. The action of $-1 \in Z_2$ is given by

$$(a, b) \rightarrow (-a, -(-\iota_{2n-3})b) \text{ for } a \in \pi_{4n-4}(S^{2n-1}), b \in \pi_{4n-4}(S^{2n-3})/\text{Ker } S.$$

Theorem 2.3. Let n be odd, $n \neq 3, 5, 9$ and let $\text{Tor } G$ denote the finite part of an abelian group G . Then $\mathcal{E}(V_{n,2})$ is isomorphic to $\text{Tor } \pi_{2n-3}(V_{n,2})$ for $n \equiv 3 \pmod{4}$ and, for $n \equiv 1 \pmod{4}$ there is an exact sequence

$$1 \rightarrow \text{Tor } \pi_{2n-3}(V_{n,2}) \rightarrow \mathcal{E}(V_{n,2}) \rightarrow Z_2 \rightarrow 1.$$

Theorem 2.4. Let n be even, $n \geq 6$ and $n \neq 8$. Then there exists a split exact sequence

$$1 \rightarrow \pi_{2n-3}(S^{n-1}) + \pi_{2n-3}(S^{n-2})/H \rightarrow \mathcal{E}(V_{n,2}) \rightarrow Z_2 \times Z_2 \rightarrow 1,$$

where H is the subgroup generated by $J(\xi\eta_{n-2})$ and the Whitehead product $[\gamma_{n-2}^2, \iota_{n-2}]$ (which is trivial for $n \equiv 0 \pmod{4}$). The action of $(-1, 1), (1, -1) \in Z_2 \times Z_2$ is given by

$$(-1, 1) \cdot (a, b) = (-(-\iota_{n-1})a, -b), \quad (1, -1) \cdot (a, b) = (-a, -(-\iota_{n-2})b)$$

for $a \in \pi_{2n-3}(S^{n-1}), b \in \pi_{2n-3}(S^{n-2})/H$.

REMARK. We can show that there exist exact sequences

$$1 \rightarrow Z_2 \rightarrow \mathcal{E}(V_{5,2}) \rightarrow Z_2 \rightarrow 1, \quad 1 \rightarrow (Z_2)^3 \rightarrow \mathcal{E}(V_{9,2}) \rightarrow Z_2 \rightarrow 1,$$

$$1 \rightarrow (Z_2)^2 \rightarrow \mathcal{E}(V_{4,2}) \rightarrow D(Z) \times Z_2 \rightarrow 1,$$

$$1 \rightarrow Z_2 + Z_{60} \rightarrow \mathcal{E}(V_{8,2}) \rightarrow (Z_2)^3 \rightarrow 1,$$

where $D(Z)$ denotes the generalized dihedral group.

3. Twisted homotopy operations and isotropy groups

Throughout this note we work in the category of based 1-connected CW complexes. Consider a situation shown by the following commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{\theta} & A & \xrightarrow{u} & X \\ & & \downarrow i & \nearrow v & \\ C & \xrightarrow{\rho} & E = C_\theta & \xrightarrow{p} & SB \\ & & \downarrow j & & \\ & & T = C_\rho & \xrightarrow{q} & SC \end{array}$$

where C_θ is the cofibre of θ and B, A and C are co H -groups.

Let $n: C \rightarrow C \vee C$ denote the comultiplication. The principal structure map

$\mu: E \rightarrow SB \vee E$ induces $\mu': T^A E \rightarrow S^2 B \vee E$ and n induces a homotopy equivalence $n': TC \rightarrow SC \vee C$, where TC is the reduced torus over C , $C \times S^1/* \times S^1$, and $T^A E$ the space obtained from TE by shrinking $i(a) \times S^1$ to a point for each $a \in A$. The coaction of SC on T , $T \rightarrow SC \vee T$, induces the action $[SC, X] \times [T, X] \rightarrow [T, X]$ which we denote by the dot.

Given an extension $w: T \rightarrow X$ of v , let $I(w)$ denote the isotropy group of w under the above action, that is, $I(w) = \{\gamma \in [SC, X]: \gamma \cdot w \simeq w\}$. Further we consider another kind of isotropy group

$$I^A(w) = \{\gamma \in [SC, X]: \gamma \cdot w \simeq^A w\},$$

in which \simeq^A indicates a homotopy under A . We blur the distinction between a map and the homotopy class it represents.

Barcus-Barratt [2] and Rutter [23] have defined the homomorphisms

$$\nabla(u, \theta): [SA, X] \rightarrow [SB, X]$$

and

$$\nabla(v, \rho): [SE, X] \rightarrow [SC, X] \quad \text{if } \theta \text{ is a suspension,}$$

such that $\text{Im } \nabla(u, \theta) = I(v)$ and $\text{Im } \nabla(v, \rho) = I(w)$. Similarly we may define

$$\nabla^i(v, \rho): [S^2 B, X] \rightarrow [SC, X]$$

by setting

$$(T\rho)^* \mu'^* \{\beta, v\} = n'^* \{\nabla^i(v, \rho)\beta, \rho^* v\} \quad \text{for } \beta \in [S^2 B, X],$$

where $T\rho: TC \rightarrow TE$ is the induced map. Note that, if $A = *$ then $\nabla^i(v, \rho) = \nabla(v, \rho)$.

Lemma 3.1. *If w is an extension of v to T , then $\text{Im } \nabla^i(v, \rho) = I^A(w)$.*

Lemma 3.2 (Functoriality). *Suppose f is induced by the top square in the commutative diagram*

$$\begin{array}{ccccc} B' & \xrightarrow{g} & B & & \\ \theta' \downarrow & & \downarrow \theta & & \\ A' & \xrightarrow{\bar{g}} & A & \xrightarrow{u} & X \\ \downarrow i' & & \downarrow i & \nearrow v & \\ C & \xrightarrow{\rho'} & C_{\theta'} & \xrightarrow{f} & C_{\theta} \end{array}$$

Then we have $\nabla^i(v, f\rho')\beta = \nabla^i(vf, \rho')(S^2 g)^\beta$.*

As a dual counter-part of the operation in [16], we may define a secondary homotopy operation

$$\Psi = \Psi^\theta(v, \rho): \text{Ker } \nabla(u, \theta) \rightarrow \text{Cok } \nabla^i(v, \rho)$$

having the following property (the detail is worked out in [19]).

Theorem 3.3. *The image of Ψ coincides with $I(w)/I^A(w)$, where w is an extension of v .*

Corollary 3.4. *If $\nabla(u, \theta)$ is monic or $\nabla^i(v, \rho)$ is epic, then $I(w) = \text{Im } \nabla^i(v, \rho)$.*

We say that the iterated cofibration ji is *stable* if there exists $c: C \rightarrow SB \vee A$ such that the composite $C \xrightarrow{c} SB \vee A \rightarrow A$ is null-homotopic and $\mu\rho \simeq (1 \vee i)c + i_2\rho$, where $i_2: E \rightarrow SB \vee E$ is the injection. Let $c': SC \rightarrow S^2B \vee A$ be the map induced by c . The following theorem is dual to Theorem (4.2) of James-Thomas [11].

Theorem 3.5. $\nabla^i(v, \rho)\beta = c'^* \{\beta, vi\}$.

4. Sphere-bundles over spheres

Let $S^m \xrightarrow{l} T \xrightarrow{\pi} S^n$ be a S^m -bundle over S^n , $n > 1$, and let $\chi(T) \in \pi_{n-1}(O(m+1))$ denote the characteristic element of this bundle. Let $\theta \in \pi_{n-1}(S^m)$ be the image of $\chi(T)$ under $\pi_{n-1}(O(m+1)) \rightarrow \pi_{n-1}(S^m)$. James-Whitehead [10] have shown that T has a cell-structure shown in the following diagram

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{\theta} & S^m & & \\ & & \downarrow i & & \\ S^{m+n-1} & \xrightarrow{\rho} & C_\theta = E & \xrightarrow{p} & S^n \\ & & \downarrow j & & \\ & & C_p = T & \xrightarrow{q} & S^{m+n} \end{array}$$

Lemma 4.1. *Under the above notation we have*

- 1) $l \simeq ji$, $\pi j \simeq p$; hence, $p\rho \simeq 0$.
- 2) *If π admits a cross-sections, then there is $\xi \in \pi_{n-1}(O(m))$ such that ξ goes to $\chi(T)$ under $\pi_{n-1}(O(m)) \rightarrow \pi_{n-1}(O(m+1))$, and*

$$\rho = i_2 J(\xi) + [i_1 \iota_n, i_2 \iota_m] \quad \text{and} \quad [s, l] = l_* J(\xi),$$

where $S^n \xrightarrow{i_1} C_\theta = S^n \vee S^m \xleftarrow{i_2} S^m$ denote the injections.

- 3) (G. Whitehead [27; p. 289]) *Let H be the Hopf invariant and let J be the Hopf-Whitehead J homomorphism. Then*

$$HJ((\chi T)) = \pm S^{m+1}\theta.$$

- 4) (I. M. James [8]) *We have $S\rho \simeq (Si)J(\chi(T))$.*
- 5) (I. M. James [6]) *ji is stable with $[i_1 \iota_n, i_2 \iota_m]$ as c , where $2 \leq m \leq n-1$.*

REMARK. James proved 5) for $m < n-1$. The assertion for $m = n-1$ and $\theta = 2\iota_m$ can be seen by inspection of cohomology with coefficients in Z_2 .

5. Proofs of Theorems 2.1 and 2.2

In this section $\chi(W_{n,2})$ is abbreviated as χ . The self homeomorphism $g: W_{n,2} \rightarrow W_{n,2}$ given by

$$g(z_1, \dots, z_n; w_1, \dots, w_n) = (\bar{z}_1, \dots, \bar{z}_n; \bar{w}_1, \dots, \bar{w}_n),$$

where z_k and w_k are complex numbers such that $\sum_k |z_k|^2 = 1 = \sum_k |w_k|^2$, induces maps of degree $(-1)^n$ and $(-1)^{n-1}$ on cells e^{2n-1} and S^{2n-3} . We say that a self homotopy equivalence of $W_{n,2}$ is of type (e_1, e_2) if it induces maps of degree e_1 and e_2 on cells e^{2n-1} and S^{2n-3} respectively.

Lemma 5.1. *Let $\chi': S^{2n-2} \times S^{2n-3} \rightarrow S^{2n-3}$ be the adjoint of χ . Then, for odd $n \geq 3$, $\chi'(\iota_{2n-2} \times (-\iota_{2n-3}))$ is not homotopic to $(-\iota_{2n-3})\chi'$.*

Proof. It is obvious that the map obtained from $\chi'(\iota_{2n-2} \times (-\iota_{2n-3}))$ by the Hopf construction represents $-J(\chi)$. But, we see from Lemma 4.1, 3) that $HJ(\chi) = \eta_{4n-5}$. Since $[\iota_{2n-2}, \iota_{2n-2}]\eta_{4n-5} = [\eta_{2n-2}, \iota_{2n-2}] \neq 0$ by Hilton [3], it follows that

$$(-\iota_{2n-2})J(\chi) = -J(\chi) + [\iota_{2n-2}, \iota_{2n-2}]HJ(\chi) \neq -J(\chi),$$

thereby our assertion.

Lemma 5.2. *For odd $n \geq 5$, there is no homotopy equivalence $W_{n,2} \rightarrow W_{n,2}$ of type $(1, -1)$.*

Proof. We show that, if a homotopy equivalence $f: W_{n,2} \rightarrow W_{n,2}$ is of type $(1, \varepsilon)$, $\varepsilon = \pm 1$, then there exists a homotopy equivalence $f': W_{n,2} \rightarrow W_{n,2}$ of type $(1, \varepsilon)$ such that $\pi f' = \pi$. Assuming this, we infer from naturality of the clutching function χ' that $f'\chi'(\pi(z), z) = \chi'(\pi(z), f'(z))$ and hence $(\varepsilon\iota_{2n-3})\chi' \simeq \chi'(\iota_{2n-2} \times (\varepsilon\iota_{2n-3}))$. Thus, by Lemma 5.1, $\varepsilon \neq -1$.

Now let f be of type $(1, \varepsilon)$. Since the assertion is trivial if $\varepsilon = 1$, we may assume $\varepsilon = -1$. Then $fj \simeq j(f|E)$, $p(f|E) \simeq p$ and $fl \simeq l(-\iota_{2n-3})$, which implies $\pi fj \simeq \pi j$ by $p \simeq \pi j$. Thus there is $\alpha: S^{3n-4} \rightarrow S^{2n-1}$ with $\pi f \simeq \alpha \cdot \pi$, where the dot denotes the coaction. We shall show that $\pi_* \alpha' = \alpha$ for some $\alpha' \in \pi_{4n-4}(W_{n,2})$; then $f' = (-\alpha') \cdot f$ is what we wanted by naturality of the coaction.

Let η denote η_{2n-3} . Since $\alpha = S\alpha''$ for some $\alpha'' \in \pi_{4n-5}(S^{2n-2})$, it suffices to prove that $\eta\alpha'' = 0$. $(S\eta)\pi j \simeq 0$ yields a $\beta \in \pi_{4n-4}(S^{2n-2})$ with $(S\eta)\pi \simeq \beta q$. Since $qf \simeq (-\iota_{4n-4})q \simeq qg$ and $\pi g \simeq (-\iota_{2n-1})\pi$, we have

$$\begin{aligned} (S\eta)\pi &\simeq (S\eta)(-\iota_{2n-1})\pi \simeq (S\eta)\pi g \simeq \beta qg \simeq \beta(-\iota_{4n-4})q \\ &\simeq \beta qf \simeq (S\eta)\pi f \simeq S(\eta\alpha'') \cdot [(S\eta)\pi], \end{aligned}$$

which means that $S(\eta\alpha'') \in I((S\eta)\pi)$.

We now see that $(Si)^*: [SC_\eta, S^{2n-2}] \rightarrow [S^{2n-2}, S^{2n-2}]$ is monic with image generated by $2\iota_{2n-2}$. It follows from Lemma 4.1, 4) and 3.3.1 of Rutter [23] that the image of

$$\nabla((S\eta)p, \rho) = \nabla(*, \rho) = (S\rho)^* = (J\chi)^*(Si)^*: [SC_\eta, S^{2n-2}] \rightarrow [S^{4n-4}, S^{2n-2}]$$

is generated by

$$(J\chi)^*(2\iota_{2n-2}) = 2J(\chi) + [\iota_{2n-2}, \iota_{2n-2}]HJ(\chi) = [\eta_{2n-2}, \iota_{2n-2}],$$

since $\pi_{2n-2}(O(2n-2)) = (Z_2)^2$ or $(Z_2)^3$ by Kervaire [12]. Thus, by the relation $[\eta_{4k}, \iota_{4k}] \in \eta_{4k} \pi_{8k}(S^{4k+1})$, $k > 1$, proved in [17], we have $S(\eta\alpha'') = 0$ in view of $I((S\eta)\pi) = \text{Im } \nabla((S\eta)p, \rho)$. This implies $\eta\alpha'' = 0$ by $[\eta_{2n-3}^2, \iota_{2n-3}] = 0$ (see Hilton [3]).

We now proceed to prove Theorem 2.1. It is known (see e.g. [21]) that $\mathcal{E}(C_\eta) \cong Z_2 \times Z_2$ is generated by $g|C_\eta$ and g' , where $g'|e^{2n-2}$ and $g'|S^{2n-3}$ are of degree 1 and -1 respectively. Since $\pi_{4n-4}(W_{n,2})$ is finite by p. 494 of Serre [25], we may infer from the exact sequence

$$\pi_k(S^{4n-5}) \xrightarrow{\rho_*} \pi_k(C_\eta) \xrightarrow{j_*} \pi_k(W_{n,2}) \xrightarrow{q_*} \pi_k(S^{4n-4}) \quad (k \leq 6n-10)$$

that $j_*: \pi_{4n-4}(C_\eta) \rightarrow \pi_{4n-4}(W_{n,2})$ is epic. Thus, since ρ is of infinite order, we obtain an exact sequence

$$1 \rightarrow I(1_{W_{n,2}}) \rightarrow \pi_{4n-4}(W_{n,2}) \rightarrow \mathcal{E}(W_{n,2}) \rightarrow Z_2 \rightarrow 1$$

by Lemma 5.2 and Theorem (6.1) of Barcus-Barratt [2] (cf. [21], [24]), where g gives a splitting. Now, since $\pi_{2n-2}(W_{n,2}) = 0$, we see from Corollary 3.4 that $I(1_{W_{n,2}})$ coincides with the image of

$$\nabla^i(j, \rho): \pi_{2n}(W_{n,2}) \rightarrow \pi_{4n-4}(W_{n,2}).$$

Observe that $l_*: \pi_{2n}(S^{2n-3}) \rightarrow \pi_{2n}(W_{n,2})$ is epic. Hence, by Theorem 3.5 and Lemma 4.1, 5), we have

$$\begin{aligned} \nabla^i(j, \rho) l_* \pi_{2n}(S^{2n-3}) &= [l_* \pi_{2n}(S^{2n-3}), j i] \\ &= l_* [\pi_{2n}(S^{2n-3}), \iota_{2n-3}] = l_* \text{Ker } S, \end{aligned}$$

which completes the proof of Theorem 2.1.

Note that the action of $-1 \in Z_2$ is given by $\alpha \mapsto -g_* \alpha$ for $\alpha \in \pi_{4n-4}(W_{n,2})/l_* \text{Ker } S$.

REMARK. Using the fact $[SC_\eta, W_{n,2}] = l_* \pi_{2n}(S^{2n-3})(Sp) \cong Z_6$, we may infer by the same argument as in the proof of Lemma 6.4 invoking Lemma 3.2 that $\nabla(j, \rho) l_* \pi_{2n}(S^{2n-3})(Sp) = \nabla^i(j, \rho) l_* \pi_{2n}(S^{2n-3})$.

$$\begin{aligned}\nabla(j, [i_1\iota_{2n-1}, i_2\iota_{2n-3}])(\alpha_1, \alpha_2) &= \nabla(\{ji_1, ji_2\}, [i_1\iota_{2n-1}, i_2\iota_{2n-3}])(\alpha_1, \alpha_2) \\ &= -[\alpha_1, ji_2] + [ji_1, \alpha_2],\end{aligned}$$

where $n: S^{4n-5} \rightarrow S^{4n-5} \vee S^{4n-5}$ is the comultiplication. Therefore,

$$\begin{aligned}\nabla(j, \rho)(s_*\eta_{2n-1}, 0) &= 0 - [s_*\theta_{2n-1}, ji_2] = [s, l]\eta_{4n-5} = l_*J(\xi)\eta_{4n-5} \text{ by Lemma 4.1} \\ \nabla(j, \rho)(l_*\pi_{2n}(S^{2n-3}), 0) &= 0 - l_*[\pi_{2n}(S^{2n-3}), \iota_{2n-3}] = l_*\text{Ker } S, \\ \nabla(j, \rho)(0, l_*\eta_{2n-3}) &= l_*\eta_{2n-3}SJ(\xi) + [ji_1, l_*\eta_{2n-3}] \\ &= l_*\eta_{2n-3}SJ(\xi) + [s, l]\eta_{4n-5} \\ &= l_*\eta_{2n-3}SJ(\xi) + l_*J(\xi)\eta_{4n-5}.\end{aligned}$$

But

$$\begin{aligned}S(\eta_{2n-3}SJ(\xi)) &= \eta_{2n-2}[\iota_{2n-1}, \iota_{2n-1}] = [\eta_{2n-2}^2, \iota_{2n-2}] = 0, \\ S(J(\xi)\eta_{4n-5}) &= SJ(\xi\eta_{2n-2}) = -Jh(\xi\eta_{2n-2}) = 0,\end{aligned}$$

since $\pi_{2n-1}(O(2n-2)) \cong Z$ by Kervaire [12], where $h: \pi_*(O(2n-3)) \rightarrow \pi_*(O(2n-2))$. This shows that $\text{Im } \nabla(j, \rho) = 1_*\text{Ker } S$. As in the previous case g gives a splitting.

REMARK. We may show, using $\pi_6(O(5))=0$ and $[\eta_5^2, \iota_5]=0$, that there exists an exact sequence $1 \rightarrow Z_{30} \rightarrow \mathcal{C}(W_{4,2}) \rightarrow (Z_2)^3 \rightarrow 1$.

6. Proofs of Theorems 2.3 and 2.4

In this section we take $B=A=S^{n-2}$ and $C=S^{2n-4}$. We denote a Z_2 -Moore space $K'(Z_2, r)$ by K_r . There is the Puppe sequence

$$S^r \xrightarrow{2\iota} S^r \xrightarrow{i_r} K_r \xrightarrow{p_r} S^{r+1} \xrightarrow{2\iota} S^{r+1} \rightarrow \dots$$

Lemma 6.1. *For n odd, $n \geq 5$, $[SK_{n-2}, V_{n,2}] \cong Z_2 + Z_2$ are generated by $l(S\bar{\eta})$ and $j\bar{\eta}(Sp)$, where $\bar{\eta}: K_{n-2} \rightarrow S^{n-3}$ and $\bar{\eta}: S^n \rightarrow K_{n-2}$ are, respectively, an extension of η_{n-3} and a coextension of η_{n-2} with respect to $2\iota: S^{n-2} \rightarrow S^{n-2}$.*

This follows from Theorem 4.1 of Araki-Toda [1] and the isomorphism $j_*: [SK_{n-2}, K_{n-2}] \cong [SK_{n-2}, V_{n,2}]$.

Lemma 6.2 (cf. 4.15 of Araki-Toda [1]). *$\pi_{r+s}(K_r \wedge K_s) \cong Z_2$ is generated by $i_r \wedge i_s$ and $\pi_{r+s+1}(K_r \wedge K_s) \cong Z_4$ is generated by $\text{Coext}(i_r \wedge 1)$ (or $\text{Coext}(1 \wedge i_s)$) with $2 \text{Coext}(i_r \wedge 1) = (i_r \wedge i_s)_{\eta_{r+s}}$, where the coextension is taken with respect to $2: K_{r+s} \rightarrow K_r \wedge S^s \rightarrow K_{r+s}$.*

Proof. The first half follows from the Künneth and Hurewicz theorems and, for the second half it suffices to use (4.2) of Araki-Toda [1] in the Puppe sequence of $1_{K_r} \wedge 2\iota_s$ and to observe that $\{1 \wedge 2\iota_s, i_r \wedge 1, 2\iota_{r+s}\} \equiv (i_r \wedge 1)_{\eta_{r+s}}$.

Lemma 6.3. *For $n \equiv 3 \pmod{4}$, $n \geq 11$, there exists $\tau \in \pi_{2n-4}(S^{n-3})$ such that*

Lemma 5.3. *For even n , $n \geq 6$, ξ is a generator of $\pi_{2n-2}(O(2n-3)) \cong Z_8$ and $4J(\xi) = [\eta_{2n-3}^2, \iota_{2n-3}] \neq 0$.*

Proof. James-Whitehead [10] have shown that ξ goes to $[\iota_{2n-1}, \iota_{2n-1}]$ of order 2 via the composite (see Kervaire [12])

$$\pi_{2n-2}(O(2n-3)) \xrightarrow{h} \pi_{2n-2}(O(2n-2)) = Z_4 \rightarrow \pi_{2n-2}(O(2n-1)) = Z_2 \xrightarrow{J} \pi_{4n-3}(S^{2n-1}).$$

It follows that ξ is a generator and that $\text{Ker } h$ is generated by 4ξ which is the image of η_{2n-3}^2 under $\partial: \pi_{2n-1}(S^{2n-3}) \rightarrow \pi_{2n-2}(O(2n-3))$. Thus the assertion follows from Lemma (5.1) of Hsiang-Levine-Szczarba [4].

Lemma 5.4. *For even n , $n \geq 6$, the image of the canonical homomorphism $\mathcal{E}(W_{n,2}) \rightarrow \mathcal{E}(S^{2n-1} \vee S^{2n-3})$ is generated by $\iota_{2n-1} \vee (-\iota_{2n-3})$.*

Proof. Since $[\eta_{2n-4}^2, \iota_{2n-4}] \neq 0$ by Hilton [3], we see from Lemma 4.1, 3) and from the exact sequence

$$\begin{array}{ccc} \pi_{2n-2}(O(2n-3)) & \rightarrow & \pi_{2n-2}(S^{2n-4}) \xrightarrow{\partial} \pi_{2n-3}(O(2n-4)) \\ & \searrow P & \downarrow J \\ & & \pi_{4n-7}(S^{2n-4}) \end{array}$$

that $HJ(\xi) = 0$ and hence $(-\iota_{2n-3})J(\xi) = -J(\xi)$. By Cor. 1.14 of [21], $\mathcal{E}(S^{2n-1} \vee S^{2n-3})$ is isomorphic to $(Z_2)^3$ with generators $\iota_{2n-1} \vee (-\iota_{2n-3})$, $(-\iota_{2n-1}) \vee \iota_{2n-3}$ and $\{i_2\eta_{2n-3}^2 + i_1\iota_{2n-1}, i_2\iota_{2n-3}\}$.

By Lemma 4.1, 2) we have $\rho = i_2J(\xi) + [i_1\iota_{2n-1}, i_2\iota_{2n-3}]$. Thus, using Lemma 5.3, we can show that $\iota_{2n-1} \vee (-\iota_{2n-3})$ is the only element k of $\mathcal{E}(S^{2n-1} \vee S^{2n-3})$ that satisfies $k\rho \simeq \pm\rho$.

Let n be even and let us prove Theorem 2.2. Since ρ is of infinite order and $j_*\pi_{4n-4}(S^{2n-1} \vee S^{2n-3}) = s_*\pi_{4n-4}(S^{2n-1}) + l_*\pi_{4n-4}(S^{2n-3}) = \pi_{4n-4}(W_{n,2})$, it follows from Lemma 5.4 and Theorem (6.1) of [2] that there is an exact sequence

$$1 \rightarrow \pi_{4n-4}(W_{n,2}) / \text{Im } \nabla(j, \rho) \rightarrow \mathcal{E}(W_{n,2}) \rightarrow Z_2 \rightarrow 1.$$

Now we shall compute $\nabla(j, \rho): [S^{2n} \vee S^{2n-2}, W_{n,2}] \rightarrow \pi_{4n-4}(W_{n,2})$. It is readily seen that $[S^{2n} \vee S^{2n-2}, W_{n,2}]$ is generated by $s_*\eta_{2n-1}$, $l_*\pi_{2n}(S^{2n-3})$ and $l_*\eta_{2n-3}$. Note that $\nabla(j, \rho) = \nabla(j, i_2J(\xi)) + \nabla(j, [i_1\iota_{2n-1}, i_2\iota_{2n-3}])$. Using properties described in 3.3 and 3.4 of Rutter [23] we have, for $\alpha_1 \in \pi_{2n}(W_{n,2})$ and $\alpha_2 \in \pi_{2n-2}(W_{n,2})$,

$$\begin{aligned} \nabla(j, i_2J(\xi))(\alpha_1, \alpha_2) &= \nabla(\{ji_1, ji_2\}, (* \vee J(\xi))n)(\alpha_1, \alpha_2) \\ &= \nabla(\{*, ji_2J(\xi)\}, n)\nabla(\{ji_1, ji_2\}, * \vee J(\xi))(\alpha_1, \alpha_2) \\ &= (Sn)^*(\nabla(ji_1, *)\alpha_1, \nabla(ji_2, J(\xi))\alpha_2) \\ &= SJ(\xi)^*\alpha_2, \end{aligned}$$

$$[\iota_n, \iota_n] = S^3\tau, [\eta_{n-1}, \iota_{n-1}] = 2S^2\tau, [\eta_{n-2}^2, \iota_{n-2}] = 4S\tau \neq 0.$$

Proof. Since we have $HJ(\xi)=0$ in the proof of Lemma 5.4, it suffices to take for τ a desuspension of $J(\xi)$. We note that $[\eta_5^2, \iota_5]=0$ by (5.13) of Toda [26].

Lemma 6.4. *Let $[\eta_{n-1}]$ denote a generator of $\pi_n(V_{n,2}) \cong Z_4$ with $\pi_*[\eta_{n-1}] = \eta_{n-1}$, where n is odd, $n \geq 5$. Then $[[\eta_{n-1}], l] = 0$ and $\text{Im } \nabla(j, \rho)$ is trivial.*

Proof. First we show that $\text{Im } \nabla(j, \rho)$ is generated by $[[\eta_{n-1}], l]$. In view of Lemma 6.1 we have only to compute $\nabla(j, \rho)j\tilde{\eta}(Sp)$ and $\nabla(j, \rho)l(S\bar{\eta})$. Applying Lemma 3.2 to the diagram

$$\begin{array}{ccccc} K_{n-3} & \xrightarrow{p_{n-3}} & S^{n-2} & & \\ \downarrow & & \downarrow 2\iota & \searrow l & \\ * & \longrightarrow & S^{n-2} & \xrightarrow{l} & V_{n,2} \\ \downarrow & & \downarrow i & \nearrow j & \\ S^{2n-4} & \xrightarrow{\rho} & K_{n-2} & \xrightarrow{1} & K_{n-2} \end{array}$$

we have

$$\begin{aligned} \nabla(j, \rho)j\tilde{\eta}(Sp) &= \nabla(j, \rho)(S^2p_{n-3})^*j\tilde{\eta} \\ &= \nabla^i(j, 1 \circ \rho)j\tilde{\eta} = [j\tilde{\eta}, j\iota] \text{ by Theorem 3.5 and Lemma 4.1} \\ &= [[\eta_{n-1}], l] \quad \text{by } \pi j\tilde{\eta} = \eta_{n-1} \end{aligned}$$

Now observe that the generalized Hopf invariant $H(\rho)$ of ρ lies in $\pi_{2n-4}(S(K_{n-3} \wedge K_{n-3})) = \pi_{2n-4}(K_{n-2} \wedge K_{n-3})$. It follows from Theorem 3.4.3 of Rutter [23], Lemma 6.2 and the relation $J(X) = -[\iota_{n-1}, \iota_{n-1}]$ (see James-Whitehead [10]) that

$$\begin{aligned} \nabla(j, \rho)l(S\bar{\eta}) &= l(S\bar{\eta})(S\rho) + [l(S\bar{\eta}), j]SH(\rho) \\ &= l(S\bar{\eta})(Si)[\iota_{n-1}, \iota_{n-1}] + [l(S\bar{\eta}), j]SH(\rho) \\ &= l_*[\eta_{n-2}^2, \iota_{n-2}] + [l(S\bar{\eta}), j]SH(\rho), \\ [l(S\bar{\eta}), j]S \text{Coext}(1 \wedge i_{n-3}) &= [l, j]S(\bar{\eta} \wedge 1)S \text{Coext}(1 \wedge i_{n-3}). \end{aligned}$$

But the commutative diagram

$$\begin{array}{ccccccc} S^{n-2} \wedge S^{n-3} & \xrightarrow{1 \wedge i_{n-3}} & S^{n-2} \wedge K_{n-3} & \xrightarrow{1 \wedge p_{n-3}} & S^{n-2} \wedge S^{n-2} & \xrightarrow{1 \wedge 2\iota} & S^{n-2} \wedge S^{n-2} \\ & & \parallel & & \downarrow 2\iota \wedge 1 & & \downarrow \text{Coext}(1 \wedge i_{n-3}) \\ & & S^{n-2} \wedge K_{n-3} & \xrightarrow{2\iota \wedge 1} & S^{n-2} \wedge K_{n-3} & \xrightarrow{i \wedge 1} & K_{n-2} \wedge K_{n-3} \\ & & & & \parallel & & \downarrow \bar{\eta} \wedge 1 \\ & & & & S^{n-2} \wedge K_{n-3} & \xrightarrow{\eta \wedge 1} & S^{n-3} \wedge K_{n-3} \end{array}$$

together with the relation $2\iota \wedge 1 = 2 \cdot 1_{K_{2n-5}} = i_{2n-5} \eta_{2n-5} p_{2n-5}$, reveals that $\overline{2\iota \wedge 1} = i_{2n-5} \eta_{2n-5}$ and hence $2(\bar{\eta} \wedge 1) \text{Coext}(1 \wedge i_{n-3}) = i_{2n-6} \eta_{2n-6}^2$. Thus we see from (4.2') of Araki-Toda [1] that

$$(\bar{\eta} \wedge 1) \text{Coext}(1 \wedge i_{n-3}) = \pm \tilde{\eta}_{2n-5}.$$

This implies that

$$\begin{aligned} [l(S\bar{\eta}), j]S \text{Coext}(1 \wedge i_{n-3}) &= \pm [l, j]S \tilde{\eta}_{2n-5} = \pm [l, j]S(1 \wedge \tilde{\eta}_{n-3}) \\ &= [l, j \tilde{\eta}_{n-2}] = [l, [\eta_{n-1}]]. \end{aligned}$$

We see from Lemma 6.3 that, for the transgression $\partial: \pi_*(S^{n-1}) \rightarrow \pi_{*-1}(S^{n-2})$ of the fibration π ,

$$\partial[\eta_{n-1}, \iota_{n-1}] = 2\iota_{n-2} \circ 2S\tau = 4S\tau = [\eta_{n-2}^2, \iota_{n-2}] \quad \text{for } n \equiv 3 \pmod{4}$$

which implies $l_*[\eta_{n-2}^2, \iota_{n-2}] = 0$. It follows that $\nabla(j, \rho)l(S\bar{\eta})$ lies in the subgroup generated by $[[\eta_{n-1}], l]$.

The fact that $[[\eta_{n-1}], l] = 0$ can be deduced from the following proposition, setting $\gamma = P_2 s_2 \iota_n$ and noting $2\pi_{n-1}(O(n-1)) = 0$ (see Kervaire [12]) for $n \equiv 1 \pmod{4}$, and taking $\beta = s_3 \iota_n$, $s = k = 1$ for $n \equiv 3 \pmod{4}$ where $H_1 P_3 s_3 \iota_n = \partial_3 s_3 \iota_n = \partial_3 p_3 s_4 \iota_n = 0$, in which $s_k: \pi_*(S^n) \rightarrow \pi_*(V_{n+1, k})$ denotes the homomorphism induced by a section.

Proposition. *Let r be odd and let $q \leq 2r - 3$.*

- 1) *Suppose that $\gamma \in \pi_{q+r}(S^{r+1})$ is of order 2 and that $S: \pi_{q+r-1}(S^r) \rightarrow \pi_{q+r}(S^{r+1})$ is monic. Then, for $\alpha \in \pi_q(V_{r+2, 2})$ with $p_1 \alpha = EH_1 \gamma$, we have $[\alpha, l_1 \iota_r] = 0$.*
- 2) *Suppose that $2l_s \beta = l_{k+s} \alpha$ for $\alpha \in \pi_q(V_{r+m, m})$ and $\beta \in \pi_q(V_{r+m+k, m+k})$ and that $[SH_1 P_{m+k} \beta, \iota_r] = 0$. Then we have $[\alpha, l_{m-1} \iota_r] = 0$.*

Here we use the homomorphisms in the homotopy exact sequence

$$\cdots \rightarrow \pi_*(V_{u-k, v-k}) \xrightarrow{l_k} \pi_*(V_{u, v}) \xrightarrow{p_k} \pi_*(V_{u, k}) \xrightarrow{\partial_k} \pi_{*-1}(V_{u-k, v-k}) \rightarrow \cdots$$

Proof. We need the formula

$$[\alpha, l_{m-1} \iota_r] = l_{m-1} P_m \alpha$$

due to I.M. James [9], where $P_m: \pi_q(V_{r+m, m}) \rightarrow \pi_{q+r-1}(S^r)$ is the composite

$$\pi_q(O(r+m)/O(r)) \xrightarrow{\partial} \pi_{q-1}(O(r)) \xrightarrow{J} \pi_{q+r-1}(S^r).$$

Note that $S^{m-1} P_m \alpha = (-1)^{m-1} P_1 p_1 \alpha = (-1)^m [p_1 \alpha, \iota_{r+m-1}]$. We see from [18] that

$$\begin{aligned} S\partial_1 \gamma &= 2\gamma + [SH_1 \gamma, \iota_{r+1}] = [SH_1 \gamma, \iota_{r+1}] \\ &= [p_1 \alpha, \iota_{r+1}] = SP_2 \alpha. \end{aligned}$$

Thus our assumption implies that $\partial_1 \gamma = P_2 \alpha$, hence $l_1 P_2 \alpha = 0$. This proves 1).

To prove 2) we introduce the diagram

$$\begin{array}{ccccc}
\pi_q(V_{r+m,m}) & \xrightarrow{P_m} & \pi_{q+r-1}(S^r) & \xleftarrow{\partial_1} & \pi_{q+r}(S^{r+1}) \\
\downarrow l_k & \nearrow P_{m+k} & \downarrow H_1 & & \\
\pi_q(V_{r+m-k,m+k}) & \xrightarrow{\partial_{m+k}} & \pi_{q-1}(S^{r-1}) & \xrightarrow{l_{m+k}} & \pi_{q-1}(V_{r+m+k,m+k+1}) \\
\downarrow l_s & \nearrow \partial_{m+k+s} & & & \\
\pi_q(V_{r+m+k+s,m+k+s}) & & & &
\end{array}$$

which is commutative by a result of James [5]. Since the characteristic element of the fibration $V_{r+2,2} \rightarrow S^{r+1}$ is $2\iota_r$, it follows from the assumption that

$$\begin{aligned}
\partial_1 S P_{m+k} \beta &= 2\iota_r \circ P_{m+k} \beta = 2\iota_r \circ P_{m+k+s} l_s \beta \\
&= 2P_{m+k+s} l_s \beta + [\iota_r, \iota_r] S^r H_1 P_{m+k+s} l_s \beta \\
&= P_{m+k+s} l_{k+s} \alpha + [S H_1 P_{m+k} \beta, \iota_r] = P_m \alpha,
\end{aligned}$$

which yields that $l_1 P_m \alpha = 0$, hence $l_{m-1} P_m \alpha = 0$.

REMARK. From the comparison of the above computation of $\nabla(j, \rho) j \tilde{\eta}(S p)$ with the one using Theorem 3.4.3 of Rutter [23] we may infer that

$$H(\rho) \equiv \text{Coext}(1 \wedge i_{n-3}) \bmod (i_{n-2} \wedge i_{n-3}) \eta_{2n-5}.$$

Lemma 6.5. *Let n be odd, $n \neq 5, 9$. Then*

- 1) *the free part of $\pi_{2n-3}(V_{n,2})$ is generated by $\pi_*^{-1}(d[\iota_{n-1}, \iota_{n-1}])$ where $d=1$ or 2 according as $n \equiv 1$ or $3 \bmod 4$, and the finite part coincides with $\text{Ker } q_*$,*
- 2) *(James [7]) the order of the attaching map ρ is $4d$, and*
- 3) *$i_*[\eta_{n-2}, \iota_{n-2}] = 4\rho$ for $n \equiv 3 \bmod 4$, $n \geq 7$.*

Proof. Using the EHP sequence we see that $\pi_{2n-3}(S^{n-1}) = Z + S^2 \pi_{2n-5}(S^{n-3})$, where Z is generated by $[\iota_{n-1}, \iota_{n-1}]$. Consider the boundary homomorphism $\partial: \pi_*(S^{n-1}) \rightarrow \pi_{*-1}(S^{n-2})$ for the fibration π . By a result of James [7] (see also [18]) we have

$$S\partial[\iota_{n-1}, \iota_{n-1}] = 2[\iota_{n-1}, \iota_{n-1}] - [2\iota_{n-1}, \iota_{n-1}] = 0.$$

Thus we have $\partial[\iota_{n-1}, \iota_{n-1}] = 0$ for $n \equiv 1 \bmod 4$ by $[\eta_{n-2}, \iota_{n-2}] = 0$ (see Hilton [3]). For $n \equiv 3 \bmod 4$, $n \geq 11$, we have $\partial[\eta_{n-1}, \iota_{n-1}] = [\eta_{n-2}^2, \iota_{n-2}] \neq 0$ by the argument as in the proof of Lemma 6.4, a fortiori $\partial[\iota_{n-1}, \iota_{n-1}] \neq 0$, so that $\partial[\iota_{n-1}, \iota_{n-1}] = [\eta_{n-2}, \iota_{n-2}]$ (this is valid for $n=7$, since $\pi_{10}(V_{7,2}) = 0$ by Paechter [22] and $[\eta_5, \iota_5] = \nu_5 \eta_5^2$ by Toda [26]). This proves the first half of 1).

Now introduce the homotopy-commutative diagram

$$\begin{array}{ccccccc}
S^{2n-4} & \longrightarrow & K_{n-2} & \xrightarrow{j} & V_{n,2} & \xrightarrow{q} & S^{2n-3} \xrightarrow{S\rho} K_{n-1} \\
& & \parallel & & \downarrow \pi & & \downarrow -\tilde{\rho} \\
& & K_{n-2} & \xrightarrow{p} & S^{n-1} & \xrightarrow{2\iota} & S^{n-1} \xrightarrow{Si} K_{n-1}
\end{array}$$

where $\bar{\rho}$ is a coextension of ρ . Since $S\rho \simeq (Si)J(\mathcal{X}) \simeq -(Si)[\iota_{n-1}, \iota_{n-1}]$ by Lemma 4.1, we may infer that

$$\bar{\rho} \equiv -[\iota_{n-1}, \iota_{n-1}] \bmod \text{Im } (2\iota)_* + \{2[\iota_{n-1}, \iota_{n-1}]\}.$$

We observe that $2\iota \circ [\iota_{n-1}, \iota_{n-1}] = 4[\iota_{n-1}, \iota_{n-1}]$ and that, in the exact sequence

$$\pi_r(K_{n-2}) \xrightarrow{j_*} \pi_r(V_{n,2}) \xrightarrow{q_*} \pi_r(S^{2n-3}) \quad (r \leq 3n-7)$$

with 2-primary $\pi_r(K_{n-2})$, q_* is monic on the free part of $\pi_r(V_{n,2})$. Hence we conclude that the second half of 1) holds and

$$q_*(\pi_*^{-1}(d[\iota_{n-1}, \iota_{n-1}])) = 4d\iota_{2n-3}.$$

Thus, inspection of the commutative diagram

$$\begin{array}{ccccccc}
 & & \pi_{2n-3}(S^{n-1}) & \xrightarrow{\quad \partial \quad} & & & \\
 & \nearrow \pi_* & \uparrow \cong & & \searrow & & \\
 & & \pi_{2n-3}(V_{n,2}, S^{n-2}) & \longrightarrow & \pi_{2n-4}(S^{n-2}) & & \\
 & & \downarrow & & \downarrow i_* & & \\
 \pi_{2n-3}(K_{n-2}) & \xrightarrow{j_*} & \pi_{2n-3}(V_{n,2}) & \longrightarrow & \pi_{2n-3}(V_{n,2}, K_{n-2}) & \longrightarrow & \pi_{2n-4}(K_{n-2}) \\
 & & \searrow q_* & & \downarrow \cong & & \\
 & & & & \pi_{2n-3}(S^{2n-3}) & &
 \end{array}$$

shows that ρ is of order $4d$ and that, for $n \equiv 3 \pmod 4$ where $d=2$, $[\iota_{n-1}, \iota_{n-1}]$ is related to $4\iota_{2n-3}$ via vertical homomorphisms, thereby obtaining 3).

We now prove Theorem 2.3. Since $[K_{n-2}, K_{n-2}] = Z_4$ is generated by the identity 1 of K_{n-2} with $2 \cdot 1 = i\eta p$ (see Theorem 4.1 of Araki-Toda [1]), we have that $\mathcal{E}(K_{n-2}) = \{1, 1+i\eta p\}$. Hence, using the remark after Lemma 6.4 and Lemma 6.5, 3), we may compute

$$\begin{aligned}
 (1+i\eta p)\rho &= \rho + i\eta p\rho + [1, i\eta p] \text{ Coext } (i_{n-3} \wedge 1) \\
 &= \rho + [1, i\eta] S(1 \wedge p_{n-3}) \text{ Coext } (i_{n-3} \wedge 1) \\
 &= \rho + [1, i\eta] (i_{n-2} \wedge 1) \\
 &= \rho + i_*[\iota_{n-2}, \eta_{n-2}] \\
 &= \begin{cases} \rho & \text{for } n \equiv 1 \pmod 4 \\ 5\rho & \text{for } n \equiv 3 \pmod 4. \end{cases}
 \end{aligned}$$

It follows that the canonical homomorphism $\mathcal{E}(V_{n,2}) \rightarrow \mathcal{E}(K_{n-2})$ is epic for $n \equiv 1 \pmod 4$ and trivial for $n \equiv 3 \pmod 4$. Thus, by Theorem (6.1) of Barcus-Barratt [2] and by Lemma 6.5, 2) we obtain an exact sequence

$$1 \rightarrow j_*\pi_{2n-3}(K_{n-2})/\text{Im } \nabla(j, \rho) \rightarrow \mathcal{E}(V_{n,2}) \rightarrow \mathcal{E}(K_{n-2}).$$

But, by Lemma 6.5, 1), $j_*\pi_{2n-3}(K_{n-2}) = \text{Ker } q_*$ is the finite part of $\pi_{2n-3}(V_{n,2})$. Hence Lemma 6.4 completes the proof of Theorem 2.3.

From now on we assume n is even; thus, by Lemma 4.1, 2),

$$\rho = i_2 J(\xi) + [i_1 \iota_{n-1}, i_2 \iota_{n-2}], \xi \in \pi_{n-2}(O(n-2)).$$

Further, by a result of [21], every element of $\mathcal{E}(E) = \mathcal{E}(S^{n-1} \vee S^{n-2}) \cong (Z_2)^3$ can be expressed as

$$\{i_2 \varepsilon \eta_{n-2} + i_1 \iota_{n-1}, i_2 \iota_{n-2}\} ((-\iota_{n-1})^k \vee (-\iota_{n-2})^l),$$

where ε, k and l are equal to 0 or 1, and i_1, i_2 are the inclusions. This element will be abbreviated as $\varepsilon \eta(k, l)$.

Lemma 6.6. *We have that*

- 1) $HJ(\xi) = 0$ and $2J(\xi) = [\eta_{n-2}, \iota_{n-2}]$ for $n \equiv 0 \pmod{4}$, $n \geq 12$
- 2) $HJ(\xi) = \eta_{2n-5}$ and $J(\xi)$ is of order 2 for $n \equiv 2 \pmod{4}$.

Proof. This is readily proved with the aid of the results of Kervaire [12] and Hilton [3] and using the commutative diagram

$$\begin{array}{ccccc} \pi_{n-2}(O(n-3)) & \longrightarrow & \pi_{n-2}O((n-2)) & \longrightarrow & \pi_{n-2}(O(n-1)) \\ & \nearrow \partial & \downarrow J & & \downarrow J \\ \pi_{n-1}(S^{n-2}) & & \pi_{2n-4}(S^{n-2}) & \xrightarrow{-S} & \pi_{2n-3}(S^{n-1}) \\ & \searrow P & & & \end{array}$$

Using Lemma 6.6 one can solve the equation $\varepsilon \eta(k, l)\rho = \pm \rho$ and show that the image of the canonical homomorphism $\mathcal{E}(V_{n,2}) \rightarrow \mathcal{E}(E)$ is

$$\begin{aligned} \{(0,0), \eta(1,0), (0,1), \eta(1,1)\} & \quad \text{for } n \equiv 0 \pmod{4} \\ \{(0,0), (1,0), \eta(0,1), \eta(1,1)\} & \quad \text{for } n \equiv 2 \pmod{4}. \end{aligned}$$

Therefore we have an exact sequence, by Theorem (6.1) of Barcus-Barratt [2],

$$1 \rightarrow j_*\pi_{2n-3}(S^{n-1} \vee S^{n-2})/H \rightarrow \mathcal{E}(V_{n,2}) \rightarrow Z_2 \times Z_2 \rightarrow 1,$$

where H denotes the image of $\nabla(j, \rho): [S^n \vee S^{n-1}, V_{n,2}] \rightarrow \pi_{2n-3}(V_{n,2})$. We observe that the self-homeomorphisms of $V_{n,2}$,

$$\begin{aligned} (x_1, \dots, x_n; y_1, \dots, y_n) & \rightarrow (x_1, \dots, x_n; -y_1, \dots, -y_n), \\ (x_1, \dots, x_n; y_1, \dots, y_n) & \rightarrow (x_1, -x_2, \dots, -x_n; y_1, -y_2, \dots, -y_n), \end{aligned}$$

give a splitting.

By an argument similar to the proof of Theorem 2.2 we may compute

$$\begin{aligned} \nabla(j, \rho)(s_*\eta_{n-1}, 0) &= l_*J(\xi)\eta_{2n-4}, \\ \nabla(j, \rho)(l_*\eta_{n-2}^2, 0) &= l_*[\eta_{n-2}^2, \iota_{n-2}] \end{aligned}$$

$$\begin{aligned} \nabla(j, \rho)(0, s_*\iota_{n-1}) &= \begin{cases} 0 & \text{for } n \equiv 0 \pmod{4} \\ l_*J(\xi)_{\eta_{2n-4}} & \text{for } n \equiv 2 \pmod{4} \end{cases} \\ \nabla(j, \rho)(0, l_*\eta_{n-2}) &= \begin{cases} l_*[\eta_{n-2}^2, \iota_{n-2}] + l_*J(\xi)_{\eta_{2n-4}} & \text{for } n \equiv 0 \pmod{4} \\ l_*J(\xi)_{\eta_{2n-4}} & \text{for } n \equiv 2 \pmod{4} \end{cases} \end{aligned}$$

This shows that $\text{Im } \nabla(j, \rho)$ is generated by $l_*J(\xi)_{\eta_{n-2}}$ and $l_*[\eta_{n-2}^2, \iota_{n-2}]$, which completes the proof of Theorem 2.4.

The following corollary may be deduced from our theorems by applying the method of Mimura-Toda [15] and using the results of Toda [26], Mimura [13] and Mimura-Mori-Oda [14] (see also [20])

Corollary 6.7. *There exist split exact sequences*

$$\begin{aligned} 1 &\rightarrow Z_{240} \rightarrow \mathcal{E}(W_{5,2}) \rightarrow Z_2 \rightarrow 1, \\ 1 &\rightarrow Z_{504} + Z_3 \rightarrow \mathcal{E}(W_{7,2}) \rightarrow Z_2 \rightarrow 1, \\ 1 &\rightarrow Z_{32} + Z_{60} \rightarrow \mathcal{E}(W_{9,2}) \rightarrow Z_2 \rightarrow 1, \\ 1 &\rightarrow Z_{264} + (Z_2)^2 \rightarrow \mathcal{E}(W_{11,2}) \rightarrow Z_2 \rightarrow 1, \\ 1 &\rightarrow Z_{504} + (Z_2)^3 \rightarrow \mathcal{E}(W_{6,2}) \rightarrow Z_2 \rightarrow 1, \\ 1 &\rightarrow Z_{480} + Z_2 + Z_3 \rightarrow \mathcal{E}(W_{8,2}) \rightarrow Z_2 \rightarrow 1, \\ 1 &\rightarrow Z_{264} + (Z_2)^5 \rightarrow \mathcal{E}(W_{10,2}) \rightarrow Z_2 \rightarrow 1, \\ 1 &\rightarrow Z_{144} + Z_8 + (Z_2)^3 + Z_3 \rightarrow \mathcal{E}(W_{12,2}) \rightarrow Z_2 \rightarrow 1. \end{aligned}$$

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