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SELF HOMOTOPY EQUIVALENCES OF STIEFEL MANIFOLDS $W_{n,2}$ AND $V_{n,2}$

Dedicated to Professor Y. Matsushima on his 60th birthday

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1. Introduction

Let $\mathcal{E}(X)$ denote the group of homotopy classes of self homotopy equivalences of a space X, whose group structure is induced by map-composition. Very little is known about this group in case X is a simply-connected CW complex with three cells which is not an H-space. In this article we shall calculate $\mathcal{E}(X)$ for the real and complex Stiefel manifolds of orthonormal 2-frames in n-space, $V_{n,2}=O(n)/O(n-2)$ and $W_{n,2}=U(n)/U(n-2)$.

2. Statement of the results

As is well known, $W_{n,2}$ and $V_{n,2}$ are sphere-bundles over spheres:

$$S^{2n-3} \xrightarrow{l} W_{n,2} \xrightarrow{\pi} S^{2n-1}, \quad S^{n-2} \xrightarrow{l} V_{n,2} \xrightarrow{\pi} S^{n-1}$$

and have the following cell-structures (see James-Whitehead [9]);

$$W_{n,2}=(S^{2n-3}\bigcup_{\theta} e^{2n-1})\bigcup_{\rho} e^{4n-4}, \quad V_{n,2}=(S^{n-2}\bigcup_{\theta} e^{n-1})\bigcup_{\rho} e^{2n-3}$$

where θ in $W_{n,2}$ is the non-zero element $\eta_{2n-3}\in\pi_{2n-2}(S^{2n-3})$ for odd n and 0 for even n, and θ in $V_{n,2}$ is $2\iota_{n-2}$ for odd n and 0 for even n. The characteristic element χ of the bundle, $\chi\in\pi_{2n-2}(O(2n-2))$ for $W_{n,2}$ and $\chi\in\pi_{n-2}(O(n-1))$ for $V_{n,2}$, is reduced to ξ, $\xi\in\pi_{2n-2}(O(2n-3))$ for $W_{n,2}$ and $\xi\in\pi_{n-2}(O(n-2))$ for $V_{n,2}$, if n is even.

We shall prove

Theorem 2.1. *Let n be odd, $n\geqq 5$. Then there exists a split exact sequence*

$$1\rightarrow\pi_{4n-4}(W_{n,2})/l_*(\mathrm{Ker}\ S)\rightarrow\mathcal{E}(W_{n,2})\rightarrow Z_2\rightarrow 1\,,$$

where S is the suspension homomorphism S: $\pi_{4n-4}(S^{2n-3})\rightarrow\pi_{4n-3}(S^{2n-2})$.

Theorem 2.2. *Let n be even, $n\geqq 6$. Then there exists a split exact sqeuence*

$$1 \to \pi_{4n-4}(S^{2n-1}) + \pi_{4n-4}(S^{2n-3})/\mathrm{Ker}\, S \to \mathcal{E}(W_{n,2}) \to Z_2 \to 1\,,$$

where S is the same as in Theorem 2.1. The action of $-1 \in Z_2$ is given by

$$(a, b) \to (-a, -(-\iota_{2n-3})b) \quad \textit{for } a \in \pi_{4n-4}(S^{2n-1}),\ b \in \pi_{4n-4}(S^{2n-3})/\mathrm{Ker}\, S\,.$$

Theorem 2.3. *Let n be odd, $n \neq 3, 5, 9$ and let* Tor G *denote the finite part of an abelian group G. Then $\mathcal{E}(V_{n,2})$ is isomorphic to* Tor $\pi_{2n-3}(V_{n,2})$ *for $n \equiv 3$* mod 4 *and, for $n \equiv 1$* mod 4 *there is an exact sequence*

$$1 \to \mathrm{Tor}\, \pi_{2n-3}(V_{n,2}) \to \mathcal{E}(V_{n,2}) \to Z_2 \to 1\,.$$

Theorem 2.4. *Let n be even, $n \geqq 6$ and $n \neq 8$. Then there exists a split exact sequence*

$$1 \to \pi_{2n-3}(S^{n-1}) + \pi_{2n-3}(S^{n-2})/H \to \mathcal{E}(V_{n,2}) \to Z_2 \times Z_2 \to 1\,,$$

where H is the subgroup generated by $J(\xi\eta_{n-2})$ and the Whitehead product $[\eta^2_{n-2}, \iota_{n-2}]$ (which is trivial for $n \equiv 0$ mod 4*). The action of $(-1, 1)$, $(1, -1) \in Z_2 \times Z_2$ is given by*

$$(-1, 1)\cdot(a, b) = (-(-\iota_{n-1})a, -b), \quad (1, -1)\cdot(a, b) = (-a, -(-\iota_{n-2})b)$$

for $a \in \pi_{2n-3}(S^{n-1})$, $b \in \pi_{2n-3}(S^{n-2})/H$.

REMARK. We can show that there exist exact sequences

$$1 \to Z_2 \to \mathcal{E}(V_{5,2}) \to Z_2 \to 1\,, \quad 1 \to (Z_2)^3 \to \mathcal{E}(V_{9,2}) \to Z_2 \to 1\,,$$
$$1 \to (Z_2)^2 \to \mathcal{E}(V_{4,2}) \to D(Z) \times Z_2 \to 1\,,$$
$$1 \to Z_2 + Z_{60} \to \mathcal{E}(V_{8,2}) \to (Z_2)^3 \to 1\,,$$

where $D(Z)$ denotes the generalized dihedral group.

3. Twisted homotopy operations and isotropy groups

Throughout this note we work in the category of based 1-connected CW complexes. Consider a situation shown by the following commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\theta"] & A \arrow[r, "u"] \arrow[d, "i"'] & X \\
C \arrow[r, "\rho"] & E=C_\theta \arrow[r, "p"] \arrow[d, "j"'] \arrow[ur, dashed, "v"'] & SB \\
& T=C_\rho \arrow[r, "q"] & SC
\end{tikzcd}$$

where C_θ is the cofibre of θ and B, A and C are co H-groups.

Let n: $C \to C \vee C$ denote the comultiplication. The principal structure map

$\mu: E \to SB \vee E$ induces $\mu': T^A E \to S^2 B \vee E$ and n induces a homotopy equivalence $n': TC \to SC \vee C$, where TC is the reduced torus over C, $C \times S^1/* \times S^1$, and $T^A E$ the space obtained from TE by shrinking $i(a) \times S^1$ to a point for each $a \in A$. The coaction of SC on T, $T \to SC \vee T$, induces the action $[SC, X] \times [T, X] \to [T, X]$ which we denote by the dot.

Given an extension $w: T \to X$ of v, let $I(w)$ denote the isotropy group of w under the above action, that is, $I(w) = \{\gamma \in [SC, X]: \gamma \cdot w \simeq w\}$. Further we consider another kind of isotropy group

$$I^A(w) = \{\gamma \in [SC, X]: \gamma \cdot w \simeq^A w\} ,$$

in which $\simeq^A$ indicates a homotopy under A. We blur the distinction between a map and the homotopy class it represents.

Barcus-Barratt [2] and Rutter [23] have defined the homomorphisms

$$\nabla(u, \theta): [SA, X] \to [SB, X]$$

and

$$\nabla(v, \rho): [SE, X] \to [SC, X] \quad \text{if } \theta \text{ is a suspension,}$$

such that Im $\nabla(u, \theta) = I(v)$ and Im $\nabla(v, \rho) = I(w)$. Similarly we may define

$$\nabla^i(v, \rho): [S^2 B, X] \to [SC, X]$$

by setting

$$(T\rho)^* \mu'^* \{\beta, v\} = n'^* \{\nabla^i(v, \rho)\beta, \rho^* v\} \quad \text{for } \beta \in [S^2 B, X] ,$$

where $T\rho: TC \to TE$ is the induced map. Note that, if $A = *$ then $\nabla^i(v, \rho) = \nabla(v, \rho)$.

Lemma 3.1. *If w is an extension of v to T, then* Im $\nabla^i(v, \rho) = I^A(w)$.

Lemma 3.2 (Functoriality). *Suppose f is induced by the top square in the commutative diagram*

$$\begin{tikzcd}
 & B' \arrow[r, "g"] \arrow[d, "\theta'"'] & B \arrow[d, "\theta"] & \\
 & A' \arrow[r, "g"] \arrow[d, "i'"'] & A \arrow[r, "u"] \arrow[d, "i"] & X \\
C \arrow[r, "\rho'"] & C_{\theta'} \arrow[r, "f"] & C_\theta \arrow[ur, "v"'] &
\end{tikzcd}$$

Then we have $\nabla^i(v, f\rho')\beta = \nabla^{i'}(vf, \rho')\,(S^2 g)^*\beta$.

As a dual counter-part of the operation in [16], we may define a secondary homotopy operation

$$\Psi = \Psi^{\theta}(v, \rho)\colon \operatorname{Ker} \nabla(u, \theta) \to \operatorname{Cok} \nabla^{i}(v, \rho)$$

having the following property (the detail is worked out in [19]).

Theorem 3.3. *The image of Ψ coincides with $I(w)/I^{A}(w)$, where w is an extension of v.*

Corollary 3.4. *If $\nabla(u, \theta)$ is monic or $\nabla^{i}(v, \rho)$ is epic, then $I(w) = \operatorname{Im} \nabla^{i}(v, \rho)$.*

We say that the iterated cofibration ji is *stable* if there exists $c\colon C \to SB \vee A$ such that the composite $C \xrightarrow{c} SB \vee A \to A$ is null-homotopic and $\mu\rho \simeq (1 \vee i)c + i_2\rho$, where $i_2\colon E \to SB \vee E$ is the injection. Let $c'\colon SC \to S^2B \vee A$ be the map induced by c. The following theorem is dual to Theorem (4.2) of James-Thomas [11].

Theorem 3.5. $\nabla^{i}(v, \rho)\beta = c'^{*}\{\beta, vi\}$.

4. Sphere-bundles over spheres

Let $S^m \xrightarrow{l} T \xrightarrow{\pi} S^n$ be a S^m-bundle over S^n, $n > 1$, and let $\chi(T) \in \pi_{n-1}(O(m+1))$ denote the characteristic element of this bundle. Let $\theta \in \pi_{n-1}(S^m)$ be the image of $\chi(T)$ under $\pi_{n-1}(O(m+1)) \to \pi_{n-1}(S^m)$. James-Whitehead [10] have shown that T has a cell-structure shown in the following diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\theta"] & S^m \arrow[d, "i"] & \\
S^{m+n-1} \arrow[r, "\rho"] & C_\theta = E \arrow[r, "p"] \arrow[d, "j"] & S^n \\
 & C_\rho = T \arrow[r, "q"] & S^{m+n}
\end{tikzcd}$$

Lemma 4.1. *Under the above notation we have*

1) $l \simeq ji$, $\pi j \simeq p$; *hence*, $p\rho \simeq 0$.

2) *If π admits a cross-sections, then there is $\xi \in \pi_{n-1}(O(m))$ such that ξ goes to $\chi(T)$ under $\pi_{n-1}(O(m)) \to \pi_{n-1}\,(O(m+1))$, and*

$$\rho = i_2 J(\xi) + [i_1\iota_n, i_2\iota_m] \quad \textit{and} \quad [s, l] = l_* J(\xi)\,,$$

where $S^n \xrightarrow{i_1} C_\theta = S^n \vee S^m \xleftarrow{i_2} S^m$ denote the injections.

3) (G. Whitehead [27; p. 289]) *Let H be the Hopf invariant and let J be the Hopf-Whitehead J homomorphism. Then*

$$HJ((\chi T)) = \pm S^{m+1}\theta\,.$$

4) (I. M. James [8]) *We have $S\rho \simeq (Si)J(\chi(T))$.*

5) (I. M. James [6]) *ji is stable with $[i_1\iota_n, i_2\iota_m]$ as c, where $2 \leqq m \leqq n-1$.*

REMARK. James proved 5) for $m<n-1$. The assertion for $m=n-1$ and $\theta=2\iota_m$ can be seen by inspection of cohomology with coefficients in Z_2.

5. Proofs of Theorems 2.1 and 2.2

In this section $\chi(W_{n,2})$ is abbreviated as χ. The self homeomorphism g: $W_{n,2}\to W_{n,2}$ given by

$$g(z_1, \cdots, z_n; w_1, \cdots, w_n) = (\bar{z}_1, \cdots, \bar{z}_n; \bar{w}_1, \cdots, \bar{w}_n),$$

where z_k and w_k are complex numbers such that $\sum_k |z_k|^2=1=\sum_k |w_k|^2$, induces maps of degree $(-1)^n$ and $(-1)^{n-1}$ on cells e^{2n-1} and S^{2n-3}. We say that a self homotopy equivalence of $W_{n,2}$ is of type (e_1, e_2) if it induces maps of degree e_1 and e_2 on cells e^{2n-1} and S^{2n-3} respectively.

Lemma 5.1. *Let χ': $S^{2n-2}\times S^{2n-3}\to S^{2n-3}$ be the adjoint of χ. Then, for odd $n\geqq 3$, $\chi'(\iota_{2n-2}\times(-\iota_{2n-3}))$ is not homotopic to $(-\iota_{2n-3})\chi'$.*

Proof. It is obvious that the map obtained from $\chi'(\iota_{2n-2}\times(-\iota_{2n-3}))$ by the Hopf construction represents $-J(\chi)$. But, we see from Lemma 4.1, 3) that $HJ(\chi)=\eta_{4n-5}$. Since $[\iota_{2n-2}, \iota_{2n-2}]\eta_{4n-5}=[\eta_{2n-2}, \iota_{2n-2}]\neq 0$ by Hilton [3], it follows that

$$(-\iota_{2n-2})J(\chi) = -J(\chi)+[\iota_{2n-2}, \iota_{2n-2}]HJ(\chi)\neq -J(\chi),$$

thereby our assertion.

Lemma 5.2. *For odd $n\geqq 5$, there is no homotopy equivalence $W_{n,2}\to W_{n,2}$ of type* $(1, -1)$.

Proof. We show that, if a homotopy equivalence f: $W_{n,2}\to W_{n,2}$ is of type $(1, \varepsilon)$, $\varepsilon=\pm 1$, then there exists a homotopy equivalence f': $W_{n,2}\to W_{n,2}$ of type $(1, \varepsilon)$ such that $\pi f'=\pi$. Assuming this, we infer from naturality of the clutching function χ' that $f'\chi'(\pi(z), z)=\chi'(\pi(z), f'(z))$ and hence $(\varepsilon\iota_{2n-3})\chi'\simeq\chi'(\iota_{2n-2}\times(\varepsilon\iota_{2n-3}))$. Thus, by Lemma 5.1, $\varepsilon\neq -1$.

Now let f be of type $(1, \varepsilon)$. Since the assertion is trivial if $\varepsilon=1$, we may assume $\varepsilon=-1$. Then $fj\simeq j(f|E)$, $p(f|E)\simeq p$ and $fl\simeq l(-\iota_{2n-3})$, which implies $\pi fj\simeq\pi j$ by $p\simeq\pi j$. Thus there is α: $S^{3n-4}\to S^{2n-1}$ with $\pi f\simeq\alpha\cdot\pi$, where the dot denotes the coaction. We shall show that $\pi_*\alpha'=\alpha$ for some $\alpha'\in\pi_{4n-4}(W_{n,2})$; then $f'=(-\alpha')\cdot f$ is what we wanted by naturality of the coaction.

Let η denote η_{2n-3}. Since $\alpha=S\alpha''$ for some $\alpha''\in\pi_{4n-5}(S^{2n-2})$, it suffices to prove that $\eta\alpha''=0$. $(S\eta)\pi j\simeq 0$ yields a $\beta\in\pi_{4n-4}(S^{2n-2})$ with $(S\eta)\pi\simeq\beta q$. Since $qf\simeq(-\iota_{4n-4})q\simeq qg$ and $\pi g\simeq(-\iota_{2n-1})\pi$, we have

$$\begin{aligned}(S\eta)\pi&\simeq(S\eta)(-\iota_{2n-1})\pi\simeq(S\eta)\pi g\simeq\beta qg\simeq\beta(-\iota_{4n-4})q\\&\simeq\beta qf\simeq(S\eta)\pi f\simeq S(\eta\alpha'')\cdot[(S\eta)\pi],\end{aligned}$$

which means that $S(\eta\alpha'')\in I((S\eta)\pi)$.

We now see that $(Si)^*\colon [SC_\eta, S^{2n-2}]\to[S^{2n-2}, S^{2n-2}]$ is monic with image generated by $2\iota_{2n-2}$. It follows from Lemma 4.1, 4) and 3.3.1 of Rutter [23] that the image of

$$\nabla((S\eta)p, \rho)=\nabla(*, \rho)=(S\rho)^*=(J\chi)^*(Si)^*\colon [SC_\eta, S^{2n-2}]\to[S^{4n-4}, S^{2n-2}]$$

is generated by

$$(J\chi)^*(2\iota_{2n-2})=2J(\chi)+[\iota_{2n-2}, \iota_{2n-2}]HJ(\chi)=[\eta_{2n-2}, \iota_{2n-2}],$$

since $\pi_{2n-2}(O(2n-2))=(Z_2)^2$ or $(Z_2)^3$ by Kervaire [12]. Thus, by the relation $[\eta_{4k}, \iota_{4k}]\in\eta_{4k*}\pi_{8k}(S^{4k+1})$, $k>1$, proved in [17], we have $S(\eta\alpha'')=0$ in view of $I((S\eta)\pi)=\mathrm{Im}\ \nabla((S\eta)p, \rho)$. This implies $\eta\alpha''=0$ by $[\eta^2_{2n-3}, \iota_{2n-3}]=0$ (see Hilton [3]).

We now proceed to prove Theorem 2.1. It is known (see e.g. [21]) that $\mathcal{E}(C_\eta)\cong Z_2\times Z_2$ is generated by $g|C_\eta$ and g', where $g'|e^{2n-2}$ and $g'|S^{2n-3}$ are of degree 1 and -1 respectively. Since $\pi_{4n-4}(W_{n,2})$ is finite by p. 494 of Serre [25], we may infer from the exact sequence

$$\pi_k(S^{4n-5})\xrightarrow{\rho_*}\pi_k(C_\eta)\xrightarrow{j_*}\pi_k(W_{n,2})\xrightarrow{q_*}\pi_k(S^{4n-4})\qquad(k\leqq 6n-10)$$

that $j_*\colon \pi_{4n-4}(C_\eta)\to\pi_{4n-4}(W_{n,2})$ is epic. Thus, since ρ is of infinite order, we obtain an exact sequence

$$1\to I(1_{W_{n,2}})\to\pi_{4n-4}(W_{n,2})\to\mathcal{E}(W_{n,2})\to Z_2\to 1$$

by Lemma 5.2 and Theorem (6.1) of Barcus-Barratt [2] (cf. [21),] [24]), where g gives a splitting. Now, since $\pi_{2n-2}(W_{n,2})=0$, we see from Corollary 3.4 that $I(1_{W_{n,2}})$ coincides with the image of

$$\nabla^i(j, \rho)\colon \pi_{2n}(W_{n,2})\to\pi_{4n-4}(W_{n,2}).$$

Observe that $l_*\colon \pi_{2n}(S^{2n-3})\to\pi_{2n}(W_{n,2})$ is epic. Hence, by Theorem 3.5 and Lemma 4.1, 5), we have

$$\begin{aligned}\nabla^i(j, \rho)1_*\pi_{2n}(S^{2n-3})&=[l_*\pi_{2n}(S^{2n-3}), ji]\\&=l_*[\pi_{2n}(S^{2n-3}), \iota_{2n-3}]=l_*\mathrm{Ker}\ S,\end{aligned}$$

which completes the proof of Theorem 2.1.

Note that the action of $-1\in Z_2$ is given by $\alpha\mapsto -g_*\alpha$ for $\alpha\in\pi_{4n-4}(W_{n,2})/l_*\mathrm{Ker}\ S$.

REMARK. Using the fact $[SC_\eta, W_{n,2}]=l_*\pi_{2n}(S^{2n-3})(Sp)\cong Z_6$, we may infer by the same argument as in the proof of Lemma 6.4 invoking Lemma 3.2 that $\nabla(j, \rho)l_*\pi_{2n}(S^{2n-3})(Sp)=\nabla^i(j, \rho)l_*\pi_{2n}(S^{2n-3})$.

$$\begin{aligned}\nabla(j,\ [i_1\iota_{2n-1},\ i_2\iota_{2n-3}])\,(\alpha_1,\ \alpha_2) &= \nabla(\{ji_1,\ ji_2\},\ [i_1\iota_{2n-1},\ i_2\iota_{2n-3}])\,(\alpha_1,\ \alpha_2)\\ &= -[\alpha_1,\ ji_2]+[ji_1,\ \alpha_2]\,,\end{aligned}$$

where n: $S^{4n-5}\to S^{4n-5}\vee S^{4n-5}$ is the comultiplication. Therefore,

$$\begin{aligned}&\nabla(j,\rho)\,(s_*\eta_{2n-1},0) = 0-[s_*\theta_{2n-1},ji_2] = [s,l]\eta_{4n-5} = l_*J(\xi)\eta_{4n-5} \text{ by Lemma 4.1}\\ &\nabla(j,\rho)\,(l_*\pi_{2n}(S^{2n-3}),0) = 0-l_*[\pi_{2n}(S^{2n-3}),\ \iota_{2n-3}] = l_*\mathrm{Ker}\ S\,,\\ &\nabla(j,\rho)\,(0,l_*\eta_{2n-3}) = l_*\eta_{2n-3}SJ(\xi)+[ji_1,\ l_*\eta_{2n-3}]\\ &\qquad = l_*\eta_{2n-3}SJ(\xi)+[s,\ l]\eta_{4n-5}\\ &\qquad = l_*\eta_{2n-3}SJ(\xi)+l_*J(\xi)\eta_{4u-5}\,.\end{aligned}$$

But

$$\begin{aligned}S(\eta_{2n-3}SJ(\xi)) &= \eta_{2n-2}[\iota_{2n-1},\ \iota_{2n-1}] = [\eta^2_{2n-2},\ \iota_{2n-2}] = 0\,,\\ S(J(\xi)\eta_{4n-5}) &= SJ(\xi\eta_{2n-2}) = -Jh(\xi\eta_{2n-2}) = 0\,,\end{aligned}$$

since $\pi_{2n-1}(O(2n-2))\cong Z$ by Kervaire [12], where h: $\pi_*(O(2n-3))\to\pi_*(O(2n-2))$. This shows that $\mathrm{Im}\,\nabla(j,\rho)=1_*\mathrm{Ker}\ S$. As in the previous case g gives a splitting.

REMARK. We may show, using $\pi_6(O(5))=0$ and $[\eta^2_5,\ \iota_5]=0$, that there exists an exact sequence $1\to Z_{30}\to\mathcal{E}(W_{4,2})\to(Z_2)^3\to 1$.

6. Proofs of Theorems 2.3 and 2.4

In this section we take $B=A=S^{n-2}$ and $C=S^{2n-4}$. We denote a Z_2-Moore space $K'(Z_2,\ r)$ by K_r. There is the Puppe sequence

$$S^r \xrightarrow{2\iota} S^r \xrightarrow{i_r} K_r \xrightarrow{p_r} S^{r+1} \xrightarrow{2\iota} S^{r+1} \to \cdots\,.$$

Lemma 6.1. *For n odd, $n\geqq 5$, $[SK_{n-2},\ V_{n,2}]\cong Z_2+Z_2$ are generated by $l(S\bar{\eta})$ and $j\tilde{\eta}(Sp)$, where $\bar{\eta}$: $K_{n-2}\to S^{n-3}$ and $\tilde{\eta}$: $S^n\to K_{n-2}$ are, respectively, an extension of η_{n-3} and a coextension of η_{n-2} with respect to 2ι: $S^{n-2}\to S^{n-2}$.*

This follows from Theorem 4.1 of Araki-Toda [1] and the isomorphism j_*: $[SK_{n-2},\ K_{n-2}]\cong[SK_{n-2},\ V_{n,2}]$.

Lemma 6.2 (cf. 4.15 of Araki-Toda [1]). *$\pi_{r+s}(K_r\wedge K_s)\cong Z_2$ is generated by $i_r\wedge i_s$ and $\pi_{r+s+1}(K_r\wedge K_s)\cong Z_4$ is generated by* $\mathrm{Coext}(i_r\wedge 1)$ *(or* $\mathrm{Coext}(1\wedge i_s)$*) with* $2\,\mathrm{Coext}(i_r\wedge 1)=(i_r\wedge i_s)\eta_{r+s}$, *where the coextension is taken with respect to 2: $K_{r+s}=K_r\wedge S^s\to K_{r+s}$.*

Proof. The first half follows from the Künneth and Hurewicz theorems and, for the second half it suffices to use (4.2) of Araki-Toda [1] in the Puppe sequence of $1_{K_r}\wedge 2\iota_s$ and to observe that $\{1\wedge 2\iota_s,\ i_r\wedge 1,\ 2\iota_{r+s}\}\equiv(i_r\wedge 1)\eta_{r+s}$.

Lemma 6.3. *For $n\equiv 3$ mod 4, $n\geqq 11$, there exists $\tau\in\pi_{2n-4}(S^{n-3})$ such that*

Lemma 5.3. *For even n, $n \geqq 6$, ξ is a generator of $\pi_{2n-2}(O(2n-3)) \cong Z_8$ and $4J(\xi)=[\eta^2_{2n-3}, \iota_{2n-3}] \neq 0$.*

Proof. James-Whitehead [10] have shown that ξ goes to $[\iota_{2n-1}, \iota_{2n-1}]$ of order 2 via the composite (see Kervaire [12])

$$\pi_{2n-2}(O(2n-3)) \xrightarrow{h} \pi_{2n-2}(O(2n-2)) = Z_4 \to \pi_{2n-2}(O(2n-1)) = Z_2 \xrightarrow{J} \pi_{4n-3}(S^{2n-1}) .$$

It follows that ξ is a generator and that Ker h is generated by 4ξ which is the image of η^2_{2n-3} under $\partial\colon \pi_{2n-1}(S^{2n-3}) \to \pi_{2n-2}(O(2n-3))$. Thus the assertion follows from Lemma (5.1) of Hsiang-Levine-Szczarba [4].

Lemma 5.4. *For even n, $n \geqq 6$, the image of the canonical homomorphism $\mathcal{E}(W_{n,2}) \to \mathcal{E}(S^{2n-1} \vee S^{2n-3})$ is generated by $\iota_{2n-1} \vee (-\iota_{2n-3})$.*

Proof. Since $[\eta^2_{2n-4}, \iota_{2n-4}] \neq 0$ by Hilton [3], we see from Lemma 4.1, 3) and from the exact sequence

$$\begin{tikzcd} \pi_{2n-2}(O(2n-3)) \arrow[r] & \pi_{2n-2}(S^{2n-4}) \arrow[r, "\partial"] \arrow[dr, "P"'] & \pi_{2n-3}(O(2n-4)) \arrow[d, "J"] \\ & & \pi_{4n-7}(S^{2n-4}) \end{tikzcd}$$

that $HJ(\xi)=0$ and hence $(-\iota_{2n-3})J(\xi) = -J(\xi)$. By Cor. 1.14 of [21], $\mathcal{E}(S^{2n-1} \vee S^{2n-3})$ is isomorphic to $(Z_2)^3$ with generators $\iota_{2n-1} \vee (-\iota_{2n-3})$, $(-\iota_{2n-1}) \vee \iota_{2n-3}$ and $\{i_2\eta^2_{2n-3} + i_1\iota_{2n-1}, i_2\iota_{2n-3}\}$.

By Lemma 4.1, 2) we have $\rho = i_2J(\xi) + [i_1\iota_{2n-1}, i_2\iota_{2n-3}]$. Thus, using Lemma 5.3, we can show that $\iota_{2n-1} \vee (-\iota_{2n-3})$ is the only element k of $\mathcal{E}(S^{2n-1} \vee S^{2n-3})$ that satisfies $k\rho \simeq \pm\rho$.

Let n be even and let us prove Theorem 2.2. Since ρ is of infinite order and $j_*\pi_{4n-4}(S^{2n-1} \vee S^{2n-3}) = s_*\pi_{4n-4}(S^{2n-1}) + l_*\pi_{4n-4}(S^{2n-3}) = \pi_{4n-4}(W_{n,2})$, it follows from Lemma 5.4 and Theorem (6.1) of [2] that there is an exact sequence

$$1 \to \pi_{4n-4}(W_{n,2})/\mathrm{Im}\,\nabla(j, \rho) \to \mathcal{E}(W_{n,2}) \to Z_2 \to 1 .$$

Now we shall compute $\nabla(j, \rho)\colon [S^{2n} \vee S^{2n-2}, W_{n,2}] \to \pi_{4n-4}(W_{n,2})$. It is readily seen that $[S^{2n} \vee S^{2n-2}, W_{n,2}]$ is generated by $s_*\eta_{2n-1}$, $l_*\pi_{2n}(S^{2n-3})$ and $l_*\eta_{2n-3}$. Note that $\nabla(j, \rho) = \nabla(j, i_2J(\xi)) + \nabla(j, [i_1\iota_{2n-1}, i_2\iota_{2n-3}])$. Using properties described in 3.3 and 3.4 of Rutter [23] we have, for $\alpha_1 \in \pi_{2n}(W_{n,2})$ and $\alpha_2 \in \pi_{2n-2}(W_{n,2})$,

$$\begin{aligned} \nabla(j, i_2J(\xi))(\alpha_1, \alpha_2) &= \nabla(\{ji_1, ji_2\}, (* \vee J(\xi))n)(\alpha_1, \alpha_2) \\ &= \nabla(\{*, ji_2J(\xi)\}, n)\nabla(\{ji_1, ji_2\}, * \vee J(\xi))(\alpha_1, \alpha_2) \\ &= (Sn)^*(\nabla(ji_1, *)\alpha_1, \nabla(ji_2, J(\xi))\alpha_2) \\ &= SJ(\xi)^*\alpha_2 , \end{aligned}$$

$$[\iota_n, \iota_n] = S^3\tau,\ [\eta_{n-1}, \iota_{n-1}] = 2S^2\tau,\ [\eta^2_{n-2}, \iota_{n-2}] = 4S\tau \neq 0\,.$$

Proof. Since we have $HJ(\xi)=0$ in the proof of Lemma 5.4, it suffices to take for τ a desuspension of $J(\xi)$. We note that $[\eta_5^2, \iota_5]=0$ by (5.13) of Toda [26].

Lemma 6.4. *Let* $[\eta_{n-1}]$ *denote a generator of* $\pi_n(V_{n,2}) \cong Z_4$ *with* $\pi_*[\eta_{n-1}] = \eta_{n-1}$, *where n is odd,* $n \geqq 5$. *Then* $[[\eta_{n-1}], l]=0$ *and* Im $\nabla(j, \rho)$ *is trivial.*

Proof. First we show that Im $\nabla(j, \rho)$ is generated by $[[\eta_{n-1}], l]$. In view of Lemma 6.1 we have only to compute $\nabla(j, \rho)j\tilde{\eta}(Sp)$ and $\nabla(j, \rho)l(S\bar{\eta})$. Applying Lemma 3.2 to the diagram

$$\begin{tikzcd}
& K_{n-3} \arrow[r, "p_{n-3}"] \arrow[d] & S^{n-2} \arrow[d, "2\iota"] & \\
& * \arrow[r] \arrow[d] & S^{n-2} \arrow[r, "l"] \arrow[d, "i"] & V_{n,2} \\
S^{2n-4} \arrow[r, "\rho"] & K_{n-2} \arrow[r, "1"] & K_{n-2} \arrow[ur, dashed, "j"'] &
\end{tikzcd}$$

we have

$$\begin{aligned}
\nabla(j, \rho)j\tilde{\eta}(Sp) &= \nabla(j, \rho)\,(S^2p_{n-3})^*j\tilde{\eta} \\
&= \nabla^i(j, 1\circ\rho)j\tilde{\eta} = [j\tilde{\eta}, ji] \quad \text{by Theorem 3.5 and Lemma 4.1} \\
&= [[\eta_{n-1}], l] \qquad\qquad \text{by } \pi j\tilde{\eta} = \eta_{n-1}
\end{aligned}$$

Now observe that the generalized Hopf invariant $H(\rho)$ of ρ lies in $\pi_{2n-4}(S(K_{n-3}\wedge K_{n-3})) = \pi_{2n-4}(K_{n-2}\wedge K_{n-3})$. It follows from Theorem 3.4.3 of Rutter [23], Lemma 6.2 and the relation $J(\chi) = -[\iota_{n-1}, \iota_{n-1}]$ (see James-Whitehead [10]) that

$$\begin{aligned}
\nabla(j, \rho)l(S\bar{\eta}) &= l(S\bar{\eta})\,(S\rho) + [l(S\bar{\eta}), j]SH(\rho) \\
&= l(S\bar{\eta})\,(Si)\,[\iota_{n-1}, \iota_{n-1}] + [l(S\bar{\eta}), j]SH(\rho) \\
&= l_*[\eta^2_{n-2}, \iota_{n-2}] + [l(S\bar{\eta}), j]SH(\rho)\,,
\end{aligned}$$

$$[l(S\bar{\eta}), j]S\ \mathrm{Coext}(1\wedge i_{n-3}) = [l, j]S(\bar{\eta}\wedge 1)S\ \mathrm{Coext}(1\wedge i_{n-3})\,.$$

But the commutative diagram

$$\begin{tikzcd}
S^{n-2}\wedge S^{n-3} \arrow[r, "1\wedge i_{n-3}"] & S^{n-2}\wedge K_{n-3} \arrow[r, "1\wedge p_{n-3}"] \arrow[d, equal] & S^{n-2}\wedge S^{n-2} \arrow[r, "1\wedge 2\iota"] \arrow[d, "\overline{2\iota\wedge 1}"] & S^{n-2}\wedge S^{n-2} \arrow[d, "\mathrm{Coext}(1\wedge i_{n-3})"] \\
& S^{n-2}\wedge K_{n-3} \arrow[r, "2\iota\wedge 1"] & S^{n-2}\wedge K_{n-3} \arrow[r, "i\wedge 1"] \arrow[d, equal] & K_{n-2}\wedge K_{n-3} \arrow[d, "\bar{\eta}\wedge 1"] \\
& & S^{n-2}\wedge K_{n-3} \arrow[r, "\eta\wedge 1"] & S^{n-3}\wedge K_{n-3}
\end{tikzcd}$$

together with the relation $2\iota\wedge 1=2\cdot 1_{K_{2n-5}}=i_{2n-5}\eta_{2n-5}p_{2n-5}$, reveals that $\overline{2\iota\wedge 1}=i_{2n-5}\eta_{2n-5}$ and hence $2(\bar{\eta}\wedge 1)$ Coext $(1\wedge i_{n-3})=i_{2n-6}\eta^2_{2n-6}$. Thus we see from (4.2′) of Araki-Toda [1] that

$$(\bar{\eta}\wedge 1)\ \mathrm{Coext}\ (1\wedge i_{n-3})=\pm\tilde{\eta}_{2n-5}\,.$$

This implies that

$$\begin{aligned}[l(S\bar{\eta}), j]S\ \mathrm{Coext}\ (1\wedge i_{n-3}) &= \pm[l, j]S\tilde{\eta}_{2n-5}=\pm[l, j]S(1\wedge\tilde{\eta}_{n-3})\\ &= [l, j\tilde{\eta}_{n-2}]=[l, [\eta_{n-1}]]\,.\end{aligned}$$

We see from Lemma 6.3 that, for the transgression $\partial\colon \pi_*(S^{n-1})\rightarrow\pi_{*-1}(S^{n-2})$ of the fibration π,

$$\partial[\eta_{n-1}, \iota_{n-1}]=2\iota_{n-2}\circ 2S\tau=4S\tau=[\eta^2_{n-2}, \iota_{n-2}]\qquad \text{for } n\equiv 3 \bmod 4$$

which implies $l_*[\eta^2_{n-2}, \iota_{n-2}]=0$. It follows that $\nabla(j, \rho)l(S\bar{\eta})$ lies in the subgroup generated by $[[\eta_{n-1}), l]$.

The fact that $[[\eta_{n-1}], l]=0$ can be deduced from the following proposition, setting $\gamma=P_2s_2\iota_n$ and noting $2\pi_{n-1}(O(n-1))=0$ (see Kervaire [12]) for $n\equiv 1 \bmod 4$, and taking $\beta=s_3\iota_n$, $s=k=1$ for $n\equiv 3 \bmod 4$ where $H_1P_3s_3\iota_n=\partial_3s_3\iota_n=\partial_3p_3s_4\iota_n=0$, in which $s_k\colon\pi_*(S^n)\rightarrow\pi_*(V_{n+1,k})$ denotes the homomorphism induced by a section.

Proposition. *Let r be odd and let $q\leqq 2r-3$.*

1) *Suppose that $\gamma\in\pi_{q+r}(S^{r+1})$ is of order 2 and that $S\colon\pi_{q+r-1}(S^r)\rightarrow\pi_{q+r}(S^{r+1})$ is monic. Then, for $\alpha\in\pi_q(V_{r+2,2})$ with $p_1\alpha=EH_1\gamma$, we have $[\alpha, l_1\iota_r]=0$.*

2) *Suppose that $2l_s\beta=l_{k+s}\alpha$ for $\alpha\in\pi_q(V_{r+m,m})$ and $\beta\in\pi_q(V_{r+m+k,m+k})$ and that $[SH_1P_{m+k}\beta, \iota_r]=0$. Then we have $[\alpha, l_{m-1}\iota_r]=0$.*

Here we use the homomorphisms in the homotopy exact sequence

$$\cdots\rightarrow\pi_*(V_{u-k,v-k})\xrightarrow{l_k}\pi_*(V_{u,v})\xrightarrow{p_k}\pi_*(V_{u,k})\xrightarrow{\partial_k}\pi_{*-1}(V_{u-k,v-k})\rightarrow\cdots$$

Proof. We need the formula

$$[\alpha, l_{m-1}\iota_r]=l_{m-1}P_m\alpha$$

due to I.M. James [9], where $P_m\colon\pi_q(V_{r+m,m})\rightarrow\pi_{q+r-1}(S^r)$ is the composite

$$\pi_q(O(r+m)/O(r))\xrightarrow{\partial}\pi_{q-1}(O(r))\xrightarrow{J}\pi_{q+r-1}(S^r)\,.$$

Note that $S^{m-1}P_m\alpha=(-1)^{m-1}P_1p_1\alpha=(-1)^m[p_1\alpha, \iota_{r+m-1}]$. We see from [18] that

$$\begin{aligned}S\partial_1\gamma &= 2\gamma+[SH_1\gamma, \iota_{r+1}]=[SH_1\gamma, \iota_{r+1}]\\ &= [p_1\alpha, \iota_{r+1}]=SP_2\alpha\,.\end{aligned}$$

Thus our assumption implies that $\partial_1\gamma=P_2\alpha$, hence $l_1P_2\alpha=0$. This proves 1).

To prove 2) we introduce the diagram

$$\begin{tikzcd}
\pi_q(V_{r+m,m}) \arrow[r, "P_m"] \arrow[d, "l_k"] & \pi_{q+r-1}(S^r) \arrow[d, "H_1"] & \pi_{q+r}(S^{r+1}) \arrow[l, "\partial_1"'] \\
\pi_q(V_{r+m+k,m+k}) \arrow[r, "P_{m+k}"] \arrow[d, "l_s"] \arrow[ur] & \pi_{q-1}(S^{r-1}) \arrow[r, "l_{m+k}"] \arrow[l, "\partial_{m+k}"] & \pi_{q-1}(V_{r+m+k,m+k+1}) \\
\pi_q(V_{r+m+k+s,m+k+s}) \arrow[ur, "\partial_{m+k+s}"'] & &
\end{tikzcd}$$

which is commutative by a result of James [5]. Since the characteristic element of the fibration $V_{r+2,2}\to S^{r+1}$ is $2\iota_r$, it follows from the assumption that

$$\begin{aligned}
\partial_1 SP_{m+k}\beta &= 2\iota_r\circ P_{m+k}\beta = 2\iota_r\circ P_{m+k+s}l_s\beta \\
&= 2P_{m+k+s}l_s\beta+[\iota_r,\ \iota_r]S^rH_1P_{m+k+s}l_s\beta \\
&= P_{m+k+s}l_{k+s}\alpha+[SH_1P_{m+k}\beta,\ \iota_r] = P_m\alpha\ ,
\end{aligned}$$

which yields that $l_1P_m\alpha=0$, hence $l_{m-1}P_m\alpha=0$.

REMARK. From the comparison of the above computation of $\nabla(j,\ \rho)j\tilde{\eta}(Sp)$ with the one using Theorem 3.4.3 of Rutter [23] we may infer that

$$H(\rho)\equiv \text{Coext}\ (1\wedge i_{n-3})\ \text{mod}\ (i_{n-2}\wedge i_{n-3})\eta_{2n-5}\ .$$

Lemma 6.5. *Let n be odd, $n\neq 5,\ 9$. Then*

1) *the free part of $\pi_{2n-3}(V_{n,2})$ is generated by $\pi_*^{-1}(d[\iota_{n-1},\ \iota_{n-1}])$ where $d=1$ or 2 according as $n\equiv 1$ or 3 mod 4, and the finite part coincides with* Ker q_*,

2) (James [7]) *the order of the attaching map ρ is $4d$, and*

3) $i_*[\eta_{n-2},\ \iota_{n-2}]=4\rho$ *for* $n\equiv 3$ mod 4, $n\geqq 7$.

Proof. Using the EHP sequence we see that $\pi_{2n-3}(S^{n-1})=Z+S^2\pi_{2n-5}(S^{n-3})$, where Z is generated by $[\iota_{n-1},\ \iota_{n-1}]$. Consider the boundary homomorphism $\partial\colon \pi_*(S^{n-1})\to\pi_{*-1}(S^{n-2})$ for the fibration π. By a result of James [7] (see also [18]) we have

$$S\partial[\iota_{n-1},\ \iota_{n-1}] = 2[\iota_{n-1},\ \iota_{n-1}]-[2\iota_{n-1},\ \iota_{n-1}] = 0\ .$$

Thus we have $\partial[\iota_{n-1},\ \iota_{n-1}]=0$ for $n\equiv 1$ mod 4 by $[\eta_{n-2},\ \iota_{n-2}]=0$ (see Hilton [3]). For $n\equiv 3$ mod 4, $n\geqq 11$, we have $\partial[\eta_{n-1},\ \iota_{n-1}]=[\eta_{n-2}^2,\ \iota_{n-2}]\neq 0$ by the argument as in the proof of Lemma 6.4, a fortiori $\partial[\iota_{n-1},\ \iota_{n-1}]\neq 0$, so that $\partial[\iota_{n-1},\ \iota_{n-1}]=[\eta_{n-2},\ \iota_{n-2}]$ (this is valid for $n=7$, since $\pi_{10}(V_{7,2})=0$ by Paechter [22] and $[\eta_5,\ \iota_5]=\nu_5\eta_8^2$ by Toda [26]). This proves the first half of 1).

Now introduce the homotopy-commutative diagram

$$\begin{tikzcd}
S^{2n-4} \arrow[r] & K_{n-2} \arrow[r, "j"] \arrow[d, equal] & V_{n,2} \arrow[r, "q"] \arrow[d, "\pi"] & S^{2n-3} \arrow[r, "-S\rho"] \arrow[d, "-\tilde{\rho}"] & K_{n-1} \arrow[d, equal] \\
& K_{n-2} \arrow[r, "p"] & S^{n-1} \arrow[r, "2\iota"] & S^{n-1} \arrow[r, "Si"] & K_{n-1}
\end{tikzcd}$$

where $\tilde{\rho}$ is a coextension of ρ. Since $S\rho \simeq (Si)J(\chi) \simeq -(Si)[\iota_{n-1}, \iota_{n-1}]$ by Lemma 4.1, we may infer that

$$\tilde{\rho} \equiv -[\iota_{n-1}, \iota_{n-1}] \bmod \operatorname{Im}(2\iota)_* + \{2[\iota_{n-1}, \iota_{n-1}]\} .$$

We observe that $2\iota \circ [\iota_{n-1}, \iota_{n-1}] = 4[\iota_{n-1}, \iota_{n-1}]$ and that, in the exact sequence

$$\pi_r(K_{n-2}) \xrightarrow{j_*} \pi_r(V_{n,2}) \xrightarrow{q_*} \pi_r(S^{2n-3}) \quad (r \leqq 3n-7)$$

with 2-primary $\pi_r(K_{n-2})$, q_* is monic on the free part of $\pi_r(V_{n,2})$. Hence we conclude that the second half of 1) holds and

$$q_*(\pi_*^{-1}(d[\iota_{n-1}, \iota_{n-1}])) = 4d\iota_{2n-3} .$$

Thus, inspection of the commutative diagram

$$\begin{tikzcd}
 & & \pi_{2n-3}(S^{n-1}) \arrow[dr, "\partial"] & \\
 & & \pi_{2n-3}(V_{n,2}, S^{n-2}) \arrow[u, "\simeq"'] \arrow[r] \arrow[d] & \pi_{2n-4}(S^{n-2}) \arrow[d, "i_*"] \\
\pi_{2n-3}(K_{n-2}) \arrow[r, "j_*"] & \pi_{2n-3}(V_{n,2}) \arrow[uur, "\pi_*"] \arrow[r] \arrow[dr, "q_*"'] & \pi_{2n-3}(V_{n,2}, K_{n-2}) \arrow[r] \arrow[d, "\simeq"] & \pi_{2n-4}(K_{n-2}) \\
 & & \pi_{2n-3}(S^{2n-3}) &
\end{tikzcd}$$

shows that ρ is of order $4d$ and that, for $n \equiv 3 \bmod 4$ where $d=2$, $[\iota_{n-1}, \iota_{n-1}]$ is related to $4\iota_{2n-3}$ via vertical homomorphisms, thereby obtaining 3).

We now prove Theorem 2.3. Since $[K_{n-2}, K_{n-2}] = Z_4$ is generated by the identity 1 of K_{n-2} with $2 \cdot 1 = i\eta p$ (see Theorem 4.1 of Araki-Toda [1]), we have that $\mathcal{E}(K_{n-2}) = \{1, 1+i\eta p\}$. Hence, using the remark after Lemma 6.4 and Lemma 6.5, 3), we may compute

$$\begin{aligned}
(1+i\eta p)\rho &= \rho + i\eta p\rho + [1, i\eta p] \operatorname{Coext}(i_{n-3} \wedge 1) \\
&= \rho + [1, i\eta] S(1 \wedge p_{n-3}) \operatorname{Coext}(i_{n-3} \wedge 1) \\
&= \rho + [1, i\eta](i_{n-2} \wedge 1) \\
&= \rho + i_*[\iota_{n-2}, \eta_{n-2}] \\
&= \begin{cases} \rho & \text{for } n \equiv 1 \bmod 4 \\ 5\rho & \text{for } n \equiv 3 \bmod 4 . \end{cases}
\end{aligned}$$

It follows that the canonical homomorphism $\mathcal{E}(V_{n,2}) \to \mathcal{E}(K_{n-2})$ is epic for $n \equiv 1 \bmod 4$ and trivial for $n \equiv 3 \bmod 4$. Thus, by Theorem (6.1) of Barcus-Barratt [2] and by Lemma 6.5, 2) we obtain an exact sequence

$$1 \to j_*\pi_{2n-3}(K_{n-2})/\operatorname{Im} \nabla(j, \rho) \to \mathcal{E}(V_{n,2}) \to \mathcal{E}(K_{n-2}) .$$

But, by Lemma 6.5, 1), $j_*\pi_{2n-3}(K_{n-2})=\mathrm{Ker}\ q_*$ is the finite part of $\pi_{2n-3}(V_{n,2})$. Hence Lemma 6.4 completes the proof of Theorem 2.3.

From now on we assume n is even; thus, by Lemma 4.1, 2),

$$\rho = i_2J(\xi)+[i_1\iota_{n-1},\, i_2\iota_{n-2}],\ \xi\in\pi_{n-2}(O(n-2))\,.$$

Further, by a result of [21], every element of $\mathcal{E}(E)=\mathcal{E}(S^{n-1}\vee S^{n-2})\cong(Z_2)^3$ can be expressed as

$$\{i_2\varepsilon\eta_{n-2}+i_1\iota_{n-1},\, i_2\iota_{n-2}\}\,((-\iota_{n-1})^k\vee(-\iota_{n-2})^l)\,,$$

where ε, k and l are equal to 0 or 1, and i_1, i_2 are the inclusions. This element will be abbreviated as $\varepsilon\eta(k, l)$.

Lemma 6.6. *We have that*

1) $HJ(\xi)=0$ *and* $2J(\xi)=[\eta_{n-2},\, \iota_{n-2}]$ *for* $n\equiv 0 \bmod 4$, $n\geqq 12$
2) $HJ(\xi)=\eta_{2n-5}$ *and* $J(\xi)$ *is of order* 2 *for* $n\equiv 2 \bmod 4$.

Proof. This is readily proved with the aid of the results of Kervaire [12] and Hilton [3] and using the commutative diagram

$$\begin{tikzcd}
\pi_{n-2}(O(n-3)) \arrow[r] & \pi_{n-2}O((n-2)) \arrow[r] \arrow[d, "J"] & \pi_{n-2}(O(n-1)) \arrow[d, "J"] \\
\pi_{n-1}(S^{n-2}) \arrow[ur, "\partial"] \arrow[r, "P"'] & \pi_{2n-4}(S^{n-2}) \arrow[r, "-S"] & \pi_{2n-3}(S^{n-1})
\end{tikzcd}$$

Using Lemma 6.6 one can solve the equation $\varepsilon\eta(k, l)\rho=\pm\rho$ and show that the image of the canonical homomorphism $\mathcal{E}(V_{n,2})\to\mathcal{E}(E)$ is

$$\{(0,0),\ \eta(1,0),\ (0,1),\ \eta(1,1)\}\qquad \text{for } n\equiv 0 \bmod 4$$
$$\{(0,0),\ (1,0),\ \eta(0,1),\ \eta(1,1)\}\qquad \text{for } u\equiv 2 \bmod 4\,.$$

Therefore we have an exact sequence, by Theorem (6.1) of Barcus-Barratt [2],

$$1\to j_*\pi_{2n-3}(S^{n-1}\vee S^{n-2})/H\to\mathcal{E}(V_{n,2})\to Z_2\times Z_2\to 1\,,$$

where H denotes the image of $\nabla(j, \rho)\colon [S^n\vee S^{n-1}, V_{n,2}]\to\pi_{2n-3}(V_{n,2})$. We observe that the self-homeomorphisms of $V_{n,2}$,

$$(x_1, \cdots, x_n;\, y_1, \cdots, y_n)\to(x_1, \cdots, x_n;\ -y_1, \cdots, -y_n)\,,$$
$$(x_1, \cdots, x_n;\, y_1, \cdots, y_n)\to(x_1, -x_2, \cdots, -x_n;\, y_1, -y_2, \cdots, -y_n)\,,$$

give a splitting.

By an argument similar to the proof of Theorem 2.2 we may compute

$$\nabla(j, \rho)\,(s_*\eta_{n-1},\, 0) = l_*J(\xi)\eta_{2n-4}\,,$$
$$\nabla(j, \rho)\,(l_*\eta^2_{n-2},\, 0) = l_*[\eta^2_{n-2},\, \iota_{n-2}]$$

$$\nabla(j, \rho)(0, s_* \iota_{n-1}) = \begin{cases} 0 & \text{for } n \equiv 0 \bmod 4 \\ l_* J(\xi)\eta_{2n-4} & \text{for } n \equiv 2 \bmod 4 \end{cases}$$

$$\nabla(j, \rho)(0, l_* \eta_{n-2}) = \begin{cases} l_*[\eta_{n-2}^2, \iota_{n-2}] + l_* J(\xi)\eta_{2n-4} & \text{for } n \equiv 0 \bmod 4 \\ l_* J(\xi)\eta_{2n-4} & \text{for } n \equiv 2 \bmod 4 \end{cases}$$

This shows that Im $\nabla(j, \rho)$ is generated by $l_* J(\xi\eta_{n-2})$ and $l_*[\eta_{n-2}^2, \iota_{n-2}]$, which completes the proof of Theorem 2.4.

The following corollary may be deduced from our theorems by applying the method of Mimura-Toda [15] and using the results of Toda [26], Mimura [13] and Mimura-Mori-Oda [14] (see also [20])

Corollary 6.7. *There exist split exact sequences*

$$1 \to Z_{240} \to \mathcal{E}(W_{5,2}) \to Z_2 \to 1\,,$$
$$1 \to Z_{504} + Z_3 \to \mathcal{E}(W_{7,2}) \to Z_2 \to 1\,,$$
$$1 \to Z_{32} + Z_{60} \to \mathcal{E}(W_{9,2}) \to Z_2 \to 1\,,$$
$$1 \to Z_{264} + (Z_2)^2 \to \mathcal{E}(W_{11,2}) \to Z_2 \to 1\,,$$
$$1 \to Z_{504} + (Z_2)^3 \to \mathcal{E}(W_{6,2}) \to Z_2 \to 1\,,$$
$$1 \to Z_{480} + Z_2 + Z_3 \to \mathcal{E}(W_{8,2}) \to Z_2 \to 1\,,$$
$$1 \to Z_{264} + (Z_2)^5 \to \mathcal{E}(W_{10,2}) \to Z_2 \to 1\,,$$
$$1 \to Z_{144} + Z_8 + (Z_2)^3 + Z_3 \to \mathcal{E}(W_{12,2}) \to Z_2 \to 1\,.$$

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