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# ON UNIT-REGULAR RINGS SATISFYING S-COMPARABILITY

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## 1. Preliminaries and notations

In [2] and [3], we studied the properties of unit-regular rings satisfying the comparability axiom. In this paper, we shall investigate unit-regular rings satisfying $s$-comparability which is a generalized notion of the comparability axiom. In section 2, we shall show that these rings have the property (DF), that is, $P \oplus Q$ is directly finite for every two directly finite projective modules $P$ and $Q$. In section 3, we shall obtain a criterion of direct finiteness of projective modules over these rings (Proposition 4 and Theorem 7). Using this result, we can determine the types of directly finite projective modules and classify the family of all unit-regular rings satisfying $s$-comparability into three types; Types A, B and C (Theorem 12). In section 4, we shall give the ideal-theoretic characterization for Types A, B and C (Theorems 14, 15 and 16).

Throughout this paper, $R$ is a ring with identity and all modules are unital right $R$-modules.

NOTATION. If $M$ and $N$ are $R$-modules, then the notation $N \lesssim M$ (resp. $N \lesssim \oplus\ M$) means that $N$ is isomorphic to a submodule of $M$ (resp. $N$ is isomorphic to a direct summand of $M$). For a cardinal number $\alpha$ and an $R$-module $M$, $\alpha M$ denotes the direct sum of $\alpha$-copies of $M$. For a set $X$, we denote the cardinal number of $X$ by $|X|$. We denote by $N_0$ the set of all positive integers.

DEFINITION. A ring $R$ is *directly finite* if $xy=1$ implies $yx=1$ for all $x,y \in R$. An $R$-module $M$ is *directly finite* if $\mathrm{End}_R(M)$ is directly finite. A ring $R$ (a module $M$) is *directly infinite* if it is not directly finite. It is well-known that $M$ is directly finite if and only if $M$ is not isomorphic to a proper direct summand of $M$ itself. A ring $R$ is said to be a *unit-regular* ring if, for each $x \in R$, there exists a unit (i.e. an invertible element) $u$ of $R$ such that $xux=x$. Let $s$ be a positive integer. Then a regular ring $R$ is said to satisfy *s-comparability* provided that for any $x,y \in R$, either $xR \lesssim s(yR)$ or $yR \lesssim s(xR)$. Note that 1-comparability is called *the comparability axiom.*

Now we shall recall some elementary properties (see [2, Lemma 1]).

Let $R$ be a unit-regular ring. Then

(1) Every finitely generated projective $R$-module $P$ has the cancellation property, and so $P$ is directly finite.

(2) For any projective $R$-module $X$ and any finitely generated projective $R$-modules $Y_1, Y_2, \cdots$ such that $Y_1 \oplus \cdots \oplus Y_n \lesssim X$ for all positive integers $n$, we have that $\oplus_{n=1}^{\infty} Y_n \lesssim X$.

(3) Let $P$ and $Q$ be projective $R$-modules such that $Q$ is finitely generated. Then $P \oplus Q$ is directly finite if and only if so is $P$.

All basic results concerning regular rings can be found in a book by K. R. Goodearl [1].

## 2. The property (DF)

**Lemma 1.** *Let $R$ be a unit-regular ring, and $P$ be a projective $R$-module with a cyclic decomposition $P = \oplus_{i \in I} P_i$. Then the following conditions* (a)$\sim$(c) *are equivalent*:

(a) *$P$ is directly infinite.*

(b) *There exists a nonzero principal right ideal $X$ of $R$ such that $X \lesssim \oplus_{i \in I - \{i_1, \cdots, i_n\}} P_i$ for every finite subset $\{i_1, \cdots, i_n\}$ of $I$.*

(c) *There exists a nonzero principal right ideal $X$ of $R$ such that $\aleph_0 X \lesssim \oplus$ $P$.*

Proof. (b)$\Rightarrow$(c)$\Rightarrow$(a) are clear. We will show (a)$\Rightarrow$(b). Suppose $P$ is directly infinite. Then there exists a nonzero module $Y$ such that $P \simeq P \oplus Y$, and so we can take a nonzero principal right ideal $X$ of $R$ such that $X \lesssim Y$ and $X \lesssim P_{n(1)} \oplus \cdots \oplus P_{m(1)}$ for some finite subset $\{n(1), \cdots, m(1)\}$ of $I$. Put $I' = I - \{i_1, \cdots, i_n\}$. Using that $P \simeq P \oplus Y$, we have that

$$(P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus (P_{i_1} \oplus \cdots \oplus P_{i_n}) \oplus (\oplus_{i \in I'} P_i)$$

$$\simeq (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus (P_{i_1} \oplus \cdots \oplus P_{i_n}) \oplus (\oplus_{i \in I'} P_i) \oplus Y.$$

Noting that every finitely generated projective module has the cancellation property, we see that $\oplus_{i \in I'} P_i \simeq (\oplus_{i \in I'} P_i) \oplus Y$, and so $X \lesssim Y \lesssim \oplus_{i \in I'} P_i$ as desired.

**Proposition 2.** *Let $R$ be a unit-regular ring satisfying s-comparability, and $P$ be a projective $R$-module. Then $P$ is directly finite if and only if so is $nP$ for every positive integer $n$.*

Proof. "If part" is clear. We will show "Only if part". It is sufficient to prove that if $2P$ is directly infinite, then so is $P$. Let $P \simeq \oplus_{i \in I} P_i$ be a principal right ideal decomposition for $P$. Assume that $2P$ is directly infinite. From Lemma 1, there exists a nonzero principal right ideal $X$ of $R$ such that

$$X \lesssim (P_{n(1)} \oplus P_{n(1)+1}) \oplus \cdots \oplus (P_{m(1)} \oplus P_{m(1)+1}),$$
$$X \lesssim (P_{n(2)} \oplus P_{n(2)+1}) \oplus \cdots \oplus (P_{m(2)} \oplus P_{m(2)+1}),$$
$$\cdots\cdots$$

for some sequence $n(1)=n(1)+1<m(1)=m(1)+1<n(2)=n(2)+1<m(2)=m(2)+1<\cdots$ of $I$, and so $P_{n(i)}=P_{n(i)+1}$ and $P_{m(i)}=P_{m(i)+1}$ for every positive integer $i$. We shall argue in steps (I), (II) and (III).

Step (I). Noting that $X \lesssim (P_{n(1)} \oplus P_{n(1)+1}) \oplus \cdots \oplus (P_{m(1)} \oplus P_{m(1)+1})$, we have a decomposition $X=\oplus_{i_1} x_{i_1} R$ such that $x_{i_1} R \lesssim P_{i_1}$ for each $i_1=n(1), n(1)+1, \cdots, m(1), m(1)+1$ by [1, Lemma 2.7]. Using that $X \lesssim (P_{n(2)} \oplus P_{n(2)+1}) \oplus \cdots \oplus (P_{m(2)} \oplus P_{m(2)+1})$, we have a decomposition

$$X \simeq (x_{n(2)} R \oplus x_{n(2)+1} R) \oplus \cdots \oplus (x_{m(2)} R \oplus x_{m(2)+1} R)$$

for some $x_{i_2} R < \oplus\ P_{i_2}$, where $i_2=n(2), n(2)+1, \cdots, m(2),\ m(2)+1$ and

$$x_{i_1} R \simeq \oplus_{i_2} x_{i_2, i_1} R$$

for some $x_{i_2, i_1} R \leqq x_{i_2} R < \oplus\ P_{i_2}$ by [1, Corollary 2.9]. Therefore there exists a decomposition

$$X = \oplus_{i_1, i_2} x_{i_1 i_2} R$$

such that

$$x_{i_2, i_1} R \simeq x_{i_1 i_2} R \leqq x_{i_1} R \quad \text{and}$$
$$2(x_{i_1 i_2} R) \simeq x_{i_1 i_2} R \oplus x_{i_2, i_1} R \lesssim P_{i_1} \oplus P_{i_2}$$
$$\leqq (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus (P_{n(2)} \oplus \cdots \oplus P_{m(2)})\ (\leqq P).$$

Next, noting that $X \lesssim (P_{n(3)} \oplus P_{n(3)+1}) \oplus \cdots \oplus (P_{m(3)} \oplus P_{m(3)+1})$, we have decompositions

$$X = \oplus_{i_1, i_2} x_{i_1 i_2} R$$
$$\simeq (x_{n(3)} R \oplus x_{n(3)+1} R) \oplus \cdots \oplus (x_{m(3)} R \oplus x_{m(3)+1} R)$$

for some $x_{i_3} R < \oplus\ P_{i_3}$, where $i_3=n(3), n(3)+1, \cdots, m(3),\ m(3)+1$ and

$$X_{i_1 i_2} R \simeq \oplus_{i_3} x_{i_3, i_1 i_2} R$$

for some $x_{i_3,i_1i_2}R \leqq x_{i_3}R < \oplus\ P_{i_3}$. Therefore there exists a decomposition

$$x_{i_1}x_{i_2}R = \oplus_{i_3} x_{i_1i_2i_3}R$$

such that

$$x_{i_1i_2i_3}R \simeq x_{i_3,i_1i_2}R$$

and hence

$$X = \oplus_{i_1,i_2} x_{i_1i_2}R = \oplus_{i_1,i_2,i_3} x_{i_1i_2i_3}R$$

such that

$$\begin{aligned} &3(x_{i_1i_2i_3}R) \leqq 2(x_{i_1i_2i_3}R) \oplus x_{i_1i_2i_3}R \lesssim 2(x_{i_1i_2}R) \oplus x_{i_3,i_1i_2}R \\ &\leqq P_{i_1} \oplus P_{i_2} \oplus P_{i_3} \\ &\leqq (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus (P_{n(2)} \oplus \cdots \oplus P_{m(2)}) \oplus (P_{n(3)} \oplus \cdots \oplus P_{m(3)})\ (\leqq P). \end{aligned}$$

Continuing this procedure, we have a decomposition

$$X = \oplus_{i_1,i_2,\cdots,i_s} x_{i_1i_2\cdots i_s}R$$

such that

$$\begin{aligned} &s(x_{i_1i_2\cdots i_s}R) \\ &\lesssim (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus \cdots \oplus (P_{n(s)} \oplus \cdots \oplus P_{m(s)})\ (\leqq P) \end{aligned}$$

for each $i_1, i_2, \cdots, i_s$.

Step (II). Noting that $X \lesssim (P_{n(s+1)} \oplus P_{n(s+1)+1}) \oplus \cdots \oplus (P_{m(s+1)} \oplus P_{m(s+1)+1})$, we may assume with no loss of generality that $X \lesssim P_{n(s+1)} \oplus P_{n(s+1)+1}$, and so $X \simeq x_{n(s+1)}R \oplus x_{n(s+1)+1}R$ for some $x_{n(s+1)}R < \oplus\ P_{n(s+1)}$ and $x_{n(s+1)+1}R < \oplus\ P_{n(s+1)+1}$. We put $w_{n(s+1)}R = x_{n(s+1)}R \cap x_{n(s+1)+1}R$ in $R$, and so there exist principal right ideals $y_{n(s+1)}R$ and $y_{n(s+1)+1}R$ of $R$ such that

$$\begin{aligned} &x_{n(s+1)}R = w_{n(s+1)}R \oplus y_{n(s+1)}R, \\ &x_{n(s+1)+1}R = x_{n(s+1)}R \oplus y_{n(s+1)+1}R \quad \text{and} \\ &w_{n(s+1)}R \oplus y_{n(s+1)}R \oplus y_{n(s+1)+1}R \lesssim P_{n(s+1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} X &\simeq x_{n(s+1)}R \oplus x_{n(s+1)+1}R \\ &\simeq (w_{n(s+1)}R \oplus y_{n(s+1)}R \oplus y_{n(s+1)+1}R) \oplus w_{n(s+1)}R \\ &\lesssim P_{n(s+1)} \oplus w_{n(s+1)}R. \end{aligned}$$

We can take a direct summand $z_1R$ of $X$ such that $z_1R \simeq w_{n(s+1)}R$. Next, noting that $w_{n(s+1)}R \lesssim X \lesssim (P_{n(s+2)} \oplus P_{n(s+2)+1}) \oplus \cdots \oplus (P_{m(s+2)} \oplus P_{m(s+2)+1})$, we have that $w_{n(s+1)}R \simeq (x_{n(s+2)}R \oplus x_{n(s+2)+1}R) \oplus \cdots \oplus (x_{m(s+2)}R \oplus x_{m(s+2)+1}R)$ for some $x_{n(s+2)}R <\oplus\ P_{n(s+2)}$, $x_{n(s+2)+1}R <\oplus\ P_{n(s+2)+1}, \cdots, x_{m(s+2)}R <\oplus\ P_{m(s+2)}$ and $x_{m(s+2)+1}R <\oplus\ P_{m(s+2)+1}$. We put $w_iR = x_iR \cap x_{i+1}R$ in $R$ for each $i = n(s+2), \cdots, m(s+2)$, from which we have decompositions

$$x_iR = w_iR \oplus y_iR,$$

$$x_{i+1}R = w_iR \oplus y_{i+1}R \quad \text{for some} \quad y_iR <\oplus\ x_iR \quad \text{and} \quad y_{i+1}R <\oplus\ x_{i+1}R$$

and

$$x_iR \oplus x_{i+1}R \simeq (w_iR \oplus y_iR \oplus y_{i+1}R) \oplus w_iR \lesssim P_i \oplus w_iR$$

for each $i = n(s+2), \cdots, m(s+2)$. Therefore $w_{n(s+1)}R \lesssim (P_{n(s+2)} \oplus \cdots \oplus P_{m(s+2)}) \oplus (w_{n(s+2)}R \oplus \cdots \oplus w_{m(s+2)}R)$ and $2(w_{n(s+2)}R \oplus \cdots \oplus w_{m(s+2)}R) \lesssim w_{n(s+1)}R$. We can take a direct summand $z_2R$ of $w_{n(s+1)}R$ such that $z_2R \simeq w_{n(s+2)}R \oplus \cdots \oplus w_{m(s+2)}R$, and hence

$$X \lesssim (P_{n(s+1)} \oplus \cdots \oplus P_{m(s+1)}) \oplus (P_{n(s+2)} \oplus \cdots \oplus P_{m(s+2)}) \oplus z_2R$$

and

$$2(z_2R) \lesssim w_{n(s+1)}R \simeq z_1R.$$

Continuing this procedure, we can take a family $\{z_kR\}_{k=1}^{\infty}$ of principal right ideals of $R$ such that

$$X \lesssim (P_{n(s+1)} \oplus \cdots \oplus P_{m(s+1)}) \oplus \cdots \oplus (P_{n(s+k)} \oplus \cdots \oplus P_{m(s+k)}) \oplus z_kR,$$

$$X \geqq z_1R \gtrsim z_2R \gtrsim \cdots, \quad \text{and that}$$

$2(z_{k+1}R) \lesssim z_kR$ for each positive integer $k$.

Step (III). We claim that $z_kR \lesssim (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus \cdots \oplus (P_{n(s)} \oplus \cdots \oplus P_{m(s)})$ for some positive integer $k$. We assume that $z_kR \not\lesssim s(x_{i_1i_2\cdots i_s}R)$ for all $i_1, i_2, \cdots, i_s$ and $k$, and so $z_kR \neq 0$. Using that $R$ satisfies $s$-comparability, $x_{i_1i_2\cdots i_s}R \lesssim s(z_kR)$, and so we have that $X = x_{i_1,i_2,\cdots i_s}x_{i_1i_2\cdots i_s}R \lesssim s^l(z_{k'}R) \lneqq (s^l+1)(z_{k'}R) \lesssim X$ for some positive integer $l$ and $k'$ by step (II), which contradicts the direct finiteness of $X$. Hence there exist positive integers $i_1, \cdots, i_s$ and $k$ such that $z_kR \lesssim s(x_{i_1i_2\cdots i_s}R) \lesssim (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus \cdots \oplus (P_{n(s)} \oplus \cdots \oplus P_{m(s)})$ by step (I).

Combining steps (II) and (III), we see that $X \lesssim (P_{n(s+1)} \oplus \cdots \oplus P_{m(s+1)}) \oplus \cdots \oplus (P_{n(s+k)} \oplus \cdots \oplus P_{m(s+k)}) \oplus z_kR \lesssim (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus \cdots \oplus (P_{n(s+k)} \oplus \cdots \oplus P_{m(s+k)})$ ($\leqq P$). Similarly, we apply the above discussion for $I - \{n(1), n(1)+1, \cdots, m(s+k), m(s+k)+1\}$. Continuing this procedure, we have that $\aleph_0 X \lesssim \oplus\ P$, from which $P$ is directly infinite. Therefore the proof of the proposition is complete.

**Theorem 3.** *Let $R$ be a unit-regular ring satisfying s-comparability. Then $R$ has the property* (DF), *that is, $P\oplus Q$ is directly finite for every two directly finite projective $R$-modules $P$ and $Q$.*

Proof. Let $P=\oplus_{i\in I}P_i$ and $Q=\oplus_{i\in I'}Q_i$ be cyclic decompositions of $P$ and $Q$. Assume that $P\oplus Q$ is directly infinite. We may assume, without loss of generality, that $I=I'$ and $|I|=\infty$ by the elementary properties (1) and (3). From Lemma 1, there exist a nonzero principal right ideal $X$ of $R$ and a sequence $n(1)<\cdots<m(1)<n(2)<\cdots<m(2)<\cdots$ of $I$ such that

$$X\lesssim(P_{n(1)}\oplus Q_{n(1)})\oplus\cdots\oplus(P_{m(1)}\oplus Q_{m(1)}),$$
$$X\lesssim(P_{n(2)}\oplus Q_{n(2)})\oplus\cdots\oplus(P_{m(2)}\oplus Q_{m(2)}),$$
$$\cdots\cdots\cdots,$$

from which we have a decomposition $X=p_iR\oplus q_iR$ for each positive integer $i$ such that $p_iR\lesssim P_{n(i)}\oplus\cdots\oplus P_{m(i)}$ and $q_iR\lesssim Q_{n(i)}\oplus\cdots\oplus Q_{m(i)}$. Set $J=\{i\in N_0\mid p_iR\lesssim s(q_iR)\}$. If $|J|=\infty$, then $X=p_iR\oplus q_iR\lesssim(s+1)q_iR$ for all $i\in J$, and so $\aleph_0X\simeq|J|X\lesssim\oplus\ (s+1)(\oplus_{i\in J}q_iR)\lesssim\oplus\ (s+1)Q$. Therefore $(s+1)Q$ is directly infinite, and so is $Q$ by Proposition 2. Otherwise $|J|<\infty$. We see that $|J'|=\infty$, where $J'=N_0-J$. Then $q_iR\lesssim s(p_iR)$ for all $i\in J'$, and so $X=p_iR\oplus q_iR\lesssim(s+1)(p_iR)$, hence $\aleph_0X\lesssim\oplus\ \oplus_{i\in J'}\ (s+1)(p_iR)\lesssim\oplus\ (s+1)P$. Therefore $(s+1)P$ is directly infinite, ans so is $P$. Thus the theorem is proved.

## 3. Directly finite projective modules

In this section, we shall determine the types of directly finite projective modules.

**Proposition 4.** *Let $R$ be a unit-regular ring satisfying s-comparability, and $P$ be a non-finitely and non-countably generated projective $R$-module with a cyclic decomposition $P=\oplus_{i\in I}P_i$, where $|I|>\aleph_0$. Then $P$ is directly infinite.*

Proof. For each $i\in I$, put $I_i=\{j\in I\mid P_i\lesssim sP_j\}$. If $|I_i|\geqq\aleph_0$ for some $i\in I$, then we see that $\aleph_0P_i\lesssim\oplus\ (\oplus_{i\in I_i}sP_j)<\oplus\ sP$, from which $sP$ is directly infinite, hence so is $P$ from Proposition 2. Thus we may assume that $|I_i|$ is finite for all $i\in I$. We can take $i_1\in I$, $i_2\in I-I_{i_1}$, $i_3\in I-(I_{i_1}\cup I_{i_2})$ and so on. By the calculation of cardinal numbers, we see that $I-(I_{i_1}\cup I_{i_2}\cup\cdots)$ is a nonempty set, and so there exists $i_0\in I-(I_{i_1}\cup I_{i_2}\cup\cdots)$. Nothing that $i_0\notin I_{i_1}\cup I_{i_2}\cup\cdots$ and that $R$ satisfies s-comparability, we see that $P_{i_0}\lesssim sP_i$ for each $i=i_1,i_2,\cdots$. Hence $\aleph_0P_{i_0}\lesssim\oplus\ s(P_{i_1}\oplus P_{i_2}\oplus\cdots)<\oplus\ sP$. Thus $sP$ is directly infinite, hence so is $P$ as desired.

NOTE. Let $R$ be a unit-regular ring satisfying s-comparability. Then we see that every directly finite projective $R$-module is finitely generated or countably generated from the above proposition.

**Lemma 5.** *Let R be a unit-regular ring satisfying s-comparability, and let* $P_1,P_2,\cdots,P_n$ *be cyclic projective R-modules, where* $n\geqq 2$. *Then there is a set* $\{i_1,i_2,\cdots,i_n\}=\{1,2,\cdots,n\}$ *such that* $sP_{i_1}\gtrsim P_{i_2}$, $s^2P_{i_1}\gtrsim P_{i_3},\cdots$ *and* $s^{n-1}P_{i_1}\gtrsim P_{i_n}$.

Proof. We shall prove this lemma by the induction on $n$ ($\geqq 2$). For cyclic projective modules $P_1$ and $P_2$, we have that $sP_1\gtrsim P_2$ or $sP_2\gtrsim P_1$, and so the lemma holds when $n=2$. Assume that $sP_{i_1}\gtrsim P_{i_2},\cdots$ and $s^{n-1}P_{i_1}\gtrsim P_{i_n}$. For $P_{i_1}$ and $P_{n+1}$, we have that $sP_{i_1}\gtrsim P_{n+1}$ or $sP_{n+1}\gtrsim P_{i_1}$. In the first case, $s^nP_{i_1}\geqq sP_{i_1}\gtrsim P_{n+1}$, hence we can take $i_{n+1}=n+1$. In the second case, we see that

$$\begin{aligned} &sP_{n+1}\gtrsim P_{i_1},\\ &s^2P_{n+1}\gtrsim sP_{i_1}\gtrsim P_{i_2},\\ &\qquad\cdots\\ &s^nP_{n+1}\gtrsim s^{n-1}P_{i_1}\gtrsim P_{i_n}, \end{aligned}$$

from which we can take $j_1=n+1$, $j_2=i_1,\cdots$ and $j_{n+1}=i_n$, and so $\{j_1,\cdots,j_{n+1}\}=\{1,\cdots,n+1\}$. Therefore the induction argument works.

**Lemma 6.** *Let R be a unit-regular ring satisfying s-comparability. Let* $P_1,P_2,\cdots,P_n$ *and X be cyclic projective R-modules such that* $X\not\lesssim s(P_1\oplus\cdots\oplus P_n)$. *Then* $P_1\oplus\cdots\oplus P_n\lesssim \bar{s}X$, *where* $\bar{s}=s^0+s^1+\cdots+s^{s-1}$.

Proof. Assume that $X\not\lesssim s(P_1\oplus\cdots\oplus P_n)$. Then we see that $P_1\lesssim sX$ by the $s$-comparability and $X\not\lesssim sP_1$. Then we have decompositions

$$X=X_{11}\oplus X_{11}^*=X_{12}\oplus X_{12}^*=\cdots=X_{1s}\oplus X_{1s}^*$$

such that

$$X_{11}\oplus\cdots\oplus X_{1s}\simeq P_1.$$

From Lemma 5, we may assume that $sX_{11}\gtrsim X_{12}$, $s^2X_{11}\gtrsim X_{13},\cdots$ and $s^{s-1}X_{11}\gtrsim X_{1s}$, from which $P_1\lesssim\bar{s}X_{11}$. Note that $X_{11}^*\not\lesssim sP_2$. If not, we see that $X=X_{11}\oplus X_{11}^*\lesssim P_1\oplus sP_2\lesssim s(P_1\oplus P_2)$, which contradicts the assumption. Hence we have that $P_2\lesssim sX_{11}^*$, and that

$$P_1\oplus P_2\lesssim\bar{s}X_{11}\oplus sX_{11}^*\leqq\bar{s}(X_{11}\oplus X_{11}^*)\leqq\bar{s}X.$$

Noting that $P_2\lesssim\bar{s}X_{11}^*$, we have decompositions

$$X_{11}^*=X_{21}\oplus X_{21}^*=X_{22}\oplus X_{22}^*=\cdots=X_{2s}\oplus X_{2s}^*$$

such that

$$X_{21}\oplus\cdots\oplus X_{2s}\simeq P_2.$$

From Lemma 5, we may assume that $sX_{21}\gtrsim X_{22}$, $s^2X_{21}\gtrsim X_{23},\cdots$, and $s^{s-1}X_{21}\gtrsim X_{2s}$. Hence $P_2\lesssim\bar{s}X_{21}$. Note that $X_{21}^*\not\lesssim sP_3$. If not,

$$X=X_{11}\oplus X_{11}^*=X_{11}\oplus(X_{21}\oplus X_{21}^*)\lesssim P_1\oplus P_2\oplus sP_3\leqq s(P_1\oplus P_2\oplus P_3),$$

which contradicts the assumption. Then $P_3\lesssim\bar{s}X_{21}^*$, and so $P_1\oplus P_2\oplus P_3\lesssim\bar{s}X_{11}\oplus(\bar{s}X_{21}\oplus\bar{s}X_{21}^*)\leqq\bar{s}(X_{11}\oplus X_{11}^*)\leqq\bar{s}X$. Continuing this procedure, we see that $P_1\oplus\cdots\oplus P_n\simeq\bar{s}X$.

Henceforth we put $\bar{s}=s^0+s^1+\cdots+s^{s-1}$ for $s$.

**Theorem 7.** *Let $R$ be a unit-regular ring satisfying s-comparability, and let $P$ be a countably generated projective $R$-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$. Then $P$ is directly finite if and only if, for each nonzero cyclic projective $R$-module $X$ there exist positive integers $n$ and $t$ such that $\oplus_{i=n}^{\infty}P_i\lesssim tX$.*

Proof. "Only if part". Assume that $P$ is directly finite, hence so is $sP$. From Lemma 1, we see that, for each nonzero cyclic projective module $X$ there exists a positive integer $n$ such that $X\not\lesssim s(P_n\oplus P_{n+1}\oplus\cdots)$. We see that $P_n\oplus P_{n+1}\oplus\cdots\lesssim\bar{s}X$ from Lemma 6 and the elementary property (2).

"If part". Assume that for each nonzero cyclic projective module $X$ there exist positive integers $n$ and $t$ such that $\oplus_{i=n}^{\infty}P_i\lesssim tX$, and that $P$ is directly infinite. There exists a nonzero principal right ideal $Y$ of $R$ such that $Y\lesssim\oplus_{I-\{i_1,\cdots,i_n\}}P_i$ for every finite subset $\{i_1,\cdots,i_n\}$ of $I$ from Lemma 1, and we can take positive integers $n$ and $t$ such that $\oplus_{i=n}^{\infty}P_i\lesssim tY$. Then we see that

$$tY\lesssim\aleph_0Y\lesssim(\oplus_{i=n}^{\infty}P_i)\lesssim tY,$$

which contradicts the direct finiteness of $tY$.

**Corollary 8.** *Let $R$ be a simple unit-regular ring satisfying s-comparability, and let $P$ be a countably generated projective $R$-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$. Then the following conditions* (a)$\sim$(c) *are equivalent:*

(a) *$P$ is directly finite.*

(b) *There exist positive integers $n$ and $t$ such that $\oplus_{i=n}^{\infty}P_i\lesssim tR$.*

(c) *$P\lesssim t'R$ for some positive integer $t'$.*

Proof. Note that $R$ is simple. Then for each nonzero principal right ideal $X$ of $R$, there exist positive integers $t_1$ and $t_2$ such that $X\lesssim t_1R$ and $R\lesssim t_2X$ by

[1, Corollary 2.23]. Combining this result with Theorem 7, we see that this corollary holds.

NOTE. It is known from [4] that simple directly finite regular rings satisfying $s$-comparability are unit-regular.

DEFINITION. Let $R$ be a unit-regular ring satisfying $s$-comparability. Let CP($R$) be the family of cyclic projective $R$-modules. For elements $A$ and $B$ in CP($R$), we define the relation "$\sim$" as follows: $A \sim B$ provided that $A \lesssim t_1 B$ and that $B \lesssim t_2 A$ for some positive integers $t_1$ and $t_2$. It is clear that the relation "$\sim$" is an equivalence relation. Put $[A] = \{B \in \mathrm{CP}(R)\} \mid A \sim B\}$ for each $A \in \mathrm{CP}(R)$. We see that $(\{[A] \mid B \in \mathrm{CP}(R)\}, \geqq)$ is a linearly ordered set, where $[A] \geqq [B]$ means that $B \lesssim tA$ for some positive integer $t$. Note that this definition is well-defined. We define $[A] \gneqq [B]$ if $[A] \geqq [B]$ and $[A] \neq [B]$, and this is equivalent to saying that $\aleph_0 B \lesssim \bar{s}A$ by the following Lemma 9.

**Lemma 9.** *Let $R$ be a unit-regular ring satisfying s-comparability, and let $A$ and $B$ be elements in* CP($R$). *Then $A \lesssim tB$ for some positive integer $t$ if and only if $\aleph_0 B \not\lesssim \bar{s}A$.*

Proof. "If part". Assume that $\aleph_0 B \not\lesssim \bar{s}A$. If $A \not\lesssim tB$ for all positive integers $t$, we see that $A \not\lesssim s(tB)$ and so $tB \lesssim \bar{s}A$ for all $t$ from Lemma 6, hence $\aleph_0 B \lesssim \bar{s}A$ which contradicts the assumption.

"Only if part". Assume that $A \lesssim tB$ for some positive integer $t$. If $\aleph_0 B \lesssim \bar{s}A$, we see that $\bar{s}tB \lesssim \aleph_0 B \lesssim \bar{s}A \lesssim \bar{s}tB$, which contradicts the direct finiteness of $\bar{s}tB$.

Let $R$ be a unit-regular ring satisfying $s$-comparability. For a countably generated projective $R$-module $P$ with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$, we consider the following three conditions (*), (A) and (B) in order to investigate the direct finiteness of P:

(*) For each nonzero cyclic projective $R$-module $X$, $\{i \in N_0 \mid X \lesssim \bar{s}P_i\}$ is a finite set.

(A) (i) $[P_n] = [P_{n+1}] = \cdots$ for some positive integer $n$.

(ii) $\oplus_{i=n+1}^{\infty} P_i \lesssim tP_n$ for some positive integer $t$.

(B) There exists a sequence $i_1 < i_2 < \cdots$ of positive integers such that $[P_{i_1}] = [P_{i_1+1}] = \cdots \gneqq [P_{i_2}] = [P_{i_2+1}] = \cdots \gneqq [P_{i_3}] = [P_{i_3+1}] = \cdots$.

NOTES 1. If (*) holds, then $\{i \in N_0 \mid X \lesssim sP_i\}$ is a finite set for each nonzero cyclic projective $R$-module $X$.

2. If $P$ is directly finite then $P$ has the property (*). Because if $P$ does not have (*), then $\aleph_0 X \lesssim \oplus\ \bar{s}P$ and so $\bar{s}P$ is directly infinite, hence so is $P$.

3. For a countably generated projective $R$-module $P$ with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $P$ has (*), we see that either condition (A)(i) or (B) holds, but not both, by a linearly orderedness of $(\{[A] \mid A\in \mathrm{CP}(R)\}, \geqq)$.

**Proposition 10.** *Let $R$ be a unit-regular ring satisfying s-comparability, and $P$ be a countably generated projective $R$-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $P$ has (*) and* (A)(i). *Then $P$ is directly finite if and only if* (A)(ii) *holds.*

Proof. "If part". Assume that (A)(ii) holds. For each nonzero cyclic projective module $X$, there exists a positive integer $m$ $(\geqq n+1)$ such that $P_m \lesssim sX$ from the condition (*) and Note 1 above. Thus, using the condition (A)(i), we have that $\oplus_{i=n+1}^{\infty}P_i \lesssim tP_n \lesssim t'P_m \lesssim st'X$ for some $t'$. By Theorem 7, $P$ is directly finite.

"Only if part". Assume that $P$ is directly finite. By Theorem 7, there exist positive integers $k$ and $t$ such that $\oplus_{i=k}^{\infty}P_i \lesssim tP_n$. We may assume $n+1<k$. Then $\oplus_{i=n+1}^{\infty}P_i=(P_{n+1}\oplus\cdots\oplus P_{k-1})\oplus(\oplus_{i=k}^{\infty}P_i)\lesssim(P_{n+1}\oplus\cdots\oplus P_{k-1})\oplus tP_n \lesssim t'P_n$ for some $t'$ usig the condition (A)(i). Therefore (A)(ii) holds.

**Proposition 11.** *Let $R$ be a unit-regular ring satisfying s-comparability, and $P$ be a countably generated projective $R$-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $P$ has (*) and* (B). *Then $P$ is directly finite.*

Proof. Assume that $P$ is directly infinite and $P=\oplus_{i=1}^{\infty}P_i$ has (*) and (B). Then there exists asequence $n(1)<m(1)<n(2)<m(2)<\cdots$ of positive integers such that

$$X \lesssim P_{n(1)}\oplus\cdots\oplus P_{m(1)},$$
$$X \lesssim P_{n(2)}\oplus\cdots\oplus P_{m(2)},$$
$$\cdots\cdots\cdots,$$

for some nonzero cyclic projective module $X$, and that $X \not\lesssim sP_{i_n}$ for some $i_n$, and that $[P_{i_n}] \gneqq [P_{i_{n+1}}] \gneqq \cdots$ by (*) and (B). We take a positive integer $n(t+1)$ with $i_n<i_{n+1}<n(t+1)$. Then $s\bar{s}X \lesssim (P_{n(t+1)}\oplus\cdots\oplus P_{m(t+1)})\oplus\cdots\oplus(P_{n(t+s\bar{s})}\oplus\cdots\oplus P_{m(t+s\bar{s})}) \lesssim \aleph_0 P_{i_{n+1}} \lesssim \bar{s}P_{i_n} \lesssim s\bar{s}X$, which contradicts the direct finiteness of $s\bar{s}X$.

Therefore we have the following theorem from Propositions 4, 10 and 11.

**Theorem 12.** *Unit-regular rings $R$ satisfying s-comparability are of the following three* types A, B *and* C, *and exclusively*:

Type A. *There exists a countably generated directly finite projective $R$-module $P$ with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $P$ has (*) and* (A), *and every countably generated directly finite projective $R$-module has (*) and* (A) *for some cyclic decomposition.*

Type B. *There exists a countably generated directly finite projective $R$-module $P$ with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $P$ has (*) and* (B), *and every* *countably generated directly finite projective R-module has* (∗) *and* (B) *for some cyclic decomposition.*

Type C. *All directly finite projectgive R-modules are finitely generated.*

Proof. It is sufficient to prove that Types A and B are independent. For a unit-regular ring $R$ satisfying $s$-comparability, there exist countably generated projective $R$-modules $P$ and $Q$ with cyclic decompositions $P=\oplus_{i=1}^{\infty}P_i$ and $Q=\oplus_{i=1}^{\infty}Q_i$ such that $P$ has (∗) and (A) and that $Q$ has (∗) and (B). Then $P_n \not\lesssim sQ_{i_k}$ for some $i_k$ by (∗), and so $Q_{i_k} \lesssim sP_n$. Similarly, $P_m \lesssim sQ_{i_{k+1}}$ for some $m$ ($\geqq n$). Then $[P_n]=[P_m]$ and $[P_n]\geqq[Q_{i_k}]\gneqq[Q_{i_{k+1}}]\geqq[P_m]$, which contradicts the property of the order "$\geqq$".

NOTE. It is clear from the condition (∗) that every unit-regular ring satisfying $s$-comparability with $\mathrm{Soc}(R)\neq 0$ is of Type C.

## 4. Types A, B and C

In this section, we shall give an ideal-theoretic characterization of Types A, B and C. Some results in this section are similar to the ones of [3]. But our proofs require something extra from [3].

Let $R$ be a unit-regular ring satisfying $s$-comparability. We denote the family of all ideals of $R$ by $L_2(R)$. Then $L_2(R)$ is a linearly ordered set under inclusion by the proof of [1, Proposition 8.5]. We put $I_0(R)=\cap\{I \mid 0\neq I\in L_2(R)\}$.

DEFINITION. A subfamily $\{I_i\}_{i=1}^{\infty}$ of $L_2(R)$ is said to be a cofinal subfamily of $L_2(R)$ if all $I_i$ are nonzero, $I_1\gneqq I_2\gneqq\cdots$, and if for each nonzero $X$ in $L_2(R)$ there exists a positive integer $n$ such that $X\geqq I_n$.

For an element $a$ of a ring $R$, we put $\Sigma_a=\Sigma\{xR \mid x\in R \text{ and } xR\lesssim aR\}$.

**Lemma 13.** *Let R be a unit-regular ring satisfying s-comparability.*

(a) *For each* $a\in R$, $\Sigma_a$ *is the smallest ideal of R containing a, and hence* $\Sigma_a=RaR$.

(b) *For each* $a,b\in R$, $\Sigma_a\leqq\Sigma_b$ *if and only if* $aR\lesssim t(bR)$ *for some positive integer t.*

(c) *For* $a,b\in R$, $\Sigma_a\lneqq\Sigma_b$ *if and only if* $\aleph_0(aR)\lesssim\bar{s}(bR)$. *Therefore we see that* $\Sigma_a=\Sigma_b$ *if and only if* $[aR]=[bR]$, *and* $\Sigma_a\lneqq\Sigma_b$ *if and only if* $[aR]\lneqq[bR]$.

Proof. See the proof of [3, Lemma 3.2] and Lemma 6.

**Theorem 14.** *Let R be a unit-regular ring satisfying s-comparability. Then the following conditions* (a)∼(c) *are equivalent:*

(a) *R is of* Type A.

(b) $\mathrm{Soc}(R)=0$ *and* $I_0(R)\neq 0$.

(c) *There exists a countably generated directly finite projective R-module P with a cyclic decomposition* $P=\oplus_{i=1}^{\infty}P_i$ *such that P satisfies the condition* (*), $[P_1]=[P_2]=\cdots$ *and that* $\oplus_{i=2}^{\infty}P_i\lesssim tP_1$ *for some positive integer t.*

Proof. (a)$\Rightarrow$(b). Let $R$ be of Type A. It is clear from the Note following Theorem 12 that $\mathrm{Soc}(R)=0$. Then there exists a countably generated directly finite projective $R$-module $P$ with the properties (*) and (A). Put $P_i\simeq x_iR$ for some $x_i\in R$. If $I_0(R)=0$, then there exists a nonzero ideal $X$ of $R$ such that $X\subsetneqq\Sigma_{x_n}=\Sigma_{x_{n+1}}=\cdots$, and so we can take a nonzero element $x$ in $X$, hence $xR\leqq\aleph_0(xR)\lesssim\bar{s}(x_mR)$ for each $m$ ($\geqq n$) from Lemma 13 and $\aleph_0(xR)<\oplus\ \bar{s}P$. Therefore $\bar{s}P$ is directly infinite, hence so is $P$ which contradicts the direct finiteness of $P$. Thus (b) holds.

(b)$\Rightarrow$(c). Assume (b), and so there exists a nonzero element $x_1$ in $I_0(R)$ such that $\Sigma_{x_1}=I_0(R)$. Using that $\mathrm{Soc}(R)=0$, we can take nonzero elements $y_1$ and $z_1$ in $R$ such that $x_1R=y_1R\oplus z_1R$ and $y_1R\lesssim s(z_1R)$. From this, there exists a nonzero cyclic submodule $x_2R$ of $y_1R$ which is subisomorphic to $z_1R$ such that $2(x_2R)\lesssim x_1R$. Noting that $\mathrm{Soc}(R)=0$ again, we apply the above discussion to $x_2R$. Continuing this procedure, we obtain a nonzero submodule $x_{n+1}R$ of $y_nR$ such that $2(x_{n+1}R)\lesssim x_nR$ for $n=1,2,\cdots$. Put $P=\oplus_{i=1}^{\infty}x_iR$. We claim that $P$ is a desired one. Since $\Sigma_{x_1}$ is the smallest ideal of $R$, $\Sigma_{x_1}=\Sigma_{x_2}=\cdots$, and so $[x_1R]=[x_2R]=\cdots$. For each nonzero $y\in R$, assume that $yR\lesssim\bar{s}(x_iR)$ for each $i\in J$, where $J$ is an infinite set $\{j_1,j_2,\cdots\}$ of positive integers. Then $\Sigma_y\leqq\Sigma_{x_i}=\Sigma_{x_1}$ and so $\Sigma_y=\Sigma_{x_1}$. We can take positive integers $h$, $t$ and $m$ such that $t\bar{s}\leqq 2^h$ and $2^h(x_{j_m}R)\lesssim x_1R\lesssim t(yR)\lesssim t\bar{s}(x_{j_m}R)\leqq 2^h(x_{j_m}R)$, which is a contradiction. Thus (*) holds. It is clear that $\oplus_{i=2}^{\infty}x_iR\lesssim x_1R$. Therefore (c) holds. The implication (c)$\Rightarrow$(a) is clear by Theorem 12.

**Theorem 15.** *Let R be a unit-regular ring satisfying s-comparability. Then the following conditions* (a)$\sim$(c) *are equivalent*:

(a) *R is of* Type B.

(b) $\mathrm{Soc}(R)=0$, $I_0(R)=0$ *and* $\mathrm{L}_2(R)$ *has a cofinal subfamily.*

(c) *There exists a countably generated directly finite projective R-module with a cyclic decomposition* $P=\oplus_{i=1}^{\infty}P_i$ *such that P has* (*) *and* $[P_1]\gneqq[P_2]\gneqq\cdots$.

Proof. (a)$\Rightarrow$(b). If (a) holds, it is clear that $\mathrm{Soc}(R)=0$. Then there exists a countably generated directly finite projective module $P$ wihch has the properties (*) and (B). Put $P_i\simeq x_iR$ for some $x_i\in R$. Then $\cap\Sigma_{x_i}=0$. If not, there exists an nonzero element $x_0\in\cap\Sigma_{x_i}$, and so $\Sigma_{x_{i_1}}\gneqq\Sigma_{x_{i_2}}\gneqq\cdots\gneqq\Sigma_{x_0}$ by the condition (B) for $P$ and Lemma 13. Hence we see from Lemma 13(c) that $x_0R\lesssim\oplus\ \bar{s}(x_{i_j}R)$ for $j=1,2,\cdots$ and $\aleph_0(x_0R)\lesssim\oplus\ \bar{s}(x_{i_1}R\oplus x_{i_2}R\oplus\cdots)\lesssim\oplus\ \bar{s}P$. Therefore $\bar{s}P$ is directly infinite and $P$ is directly infinite which is a contradiction. Thus $I_0(R)=0$ and

$\{\Sigma_{x_i}\}$ is a cofinal subfamily of $L_2(R)$ from the linearly orderedness of $L_2(R)$.

(b) $\Rightarrow$ (c). If (b) holds, then there exists a cofinal subfamily $\{I_i\}_{i=1}^{\infty}$ of $L_2(R)$ such that $I_1 \gneqq I_2 \gneqq \cdots$. We can take an element $x_i \in I_i - I_{i+1}$, and so $I_i \geqq \Sigma_{x_i} \gneqq I_{i+1}$ by the linearly orderedness of $L_2(R)$. Then we see that $\Sigma_{x_1} \gneqq \Sigma_{x_2} \gneqq \cdots$ and $\{\Sigma_{x_i}\}$ is a cofinal subfamily of $L_2(R)$. Put $P = \oplus_{i=1}^{\infty} x_i R$. We claim that $P$ is a desired one. It is clear that $[x_1 R] \gneqq [x_2 R] \gneqq \cdots$. If $P$ does not satisfy (*), there exists a nonzero $x_0 \in R$ such that $x_0 R \lesssim_{\bar{s}} (x_i R)$ for $i \in J$, where $J$ is an infinite set of positive integers, and $\Sigma_{x_i} \geqq \Sigma_{x_0}$ for $i \in J$. We see taht $\Sigma_{x_0} = 0$ (i.e. $x_0 = 0$) by using that $\{\Sigma_{x_i}\}$ is a cofinal subfamily, which contradicts that $x_0 \neq 0$. Therefore (c) holds. The implication (c) $\Rightarrow$ (a) is clear.

As a direct result from the above Theorems 14 and 15, we have the following.

**Theorem 16.** *Let $R$ be a unit-regular ring satisfying s-comparability. Then the following conditions* (a) *and* (b) *are equivalent*:

(a) *$R$ is of* Type C.

(b) $\mathrm{Soc}(R) \neq 0$, *or* $I_0(R) = 0$ *and* $L_2(R)$ *has no cofinal subfamily of* $L_2(R)$.

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