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THE EXPLOSION PROBLEM FOR BRANCHING MARKOV PROCESS

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0. Introduction

Consider a single-type branching process. Then a well-known result of Dynkin is the following: explosion happens (i.e., the number of particles will be infinite in a finite time with positive probability) iff $\int_{1-\varepsilon}^1 \frac{du}{u-h(u)}$ converges for every $\varepsilon > 0$, where h is the generating function of new-born particles (see, e.g., [3, p. 106]). N. Ikeda [4] has also given an interesting proof of this fact using probabilistic techniques. Indeed he shows that the convergence of $\int_{1-\varepsilon}^1 \frac{du}{u-h(u)}$ is equivalent to the finiteness of the expected value of e_Δ , the time of explosion (i.e., the first time when the number of particles is infinite).

The purpose of this paper is to investigate the explosion problem for a more general class of branching processes: branching Markov process¹ (see Ikeda, Nagasawa and Watanabe [5]). For a large class of bmp. we are able to show that a sufficient condition for explosion (non-explosion) is the convergence (divergence) of a particular integral. In many cases of interest, this condition is also necessary and sufficient.

In §1 we introduce the necessary terminology and notation; in §2 we generalize the methods of Ikeda and thus treat the problem from a probabilistic viewpoint; in §3, we consider the explosion problem from the analytical viewpoint. These results are of a more local character than those of §2 and hence give stronger results in some sense. Section 4 is devoted to applications. In particular, we consider branching diffusion processes with absorbing boundary. Another interesting application is that of branching Brownian motion whose splits occur only on a "fat" Cantor set.

It should be remarked that the explosion problem is intimately related to the uniqueness (or non-uniqueness) of solution of certain semi-linear parabolic equations. Such questions have been considered by Fujita and Watanabe [2].

1. We usually abbreviate this as bmp.

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1. Definitions and statement of problem

Let S be a locally compact, second-countable, Hausdorff topological space. Form the n -fold direct-product topological space $S^{(n)}$. Let $S^n = S^{(n)}/\sim$ be the quotient topological space induced by the equivalence relation \sim of permutation: $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$ iff there exists a permutation π on $\{1, \dots, n\}$ such that $x_i = y_{\pi i}$, all $i=1, \dots, n$. The topological sum $\bigcup_{n=0}^{\infty} S^n$ is denoted by S , where $S^0 = \{\partial\}$, ∂ being an isolated point. Since S is locally compact (but not compact) we let $\hat{S} = S \cup \{\Delta\}$ be its one-point compactification.

In order to define a branching Markov process, it is convenient to introduce the mapping $\wedge : B_1(S) \rightarrow B(\hat{S})^2$ defined by

$$\hat{f}(x) = \begin{cases} 1 & \text{if } x = \partial, \\ \prod_{i=1}^n f(x_i) & \text{if } x = [x_1, \dots, x_n] \in S^n, \\ 0 & \text{if } x = \Delta. \end{cases}$$

Another mapping that we shall have occasion to use is the following: given $f, g \in B_1(S)$, we define the $B(\hat{S})$ -measurable function $\langle f|g \rangle$ by

$$\langle f|g \rangle(x) = \begin{cases} \sum_{i=1}^n g(x_i) \prod_{\substack{j=1 \\ j \neq i}}^n f(x_j) & \text{if } x = [x_1, \dots, x_n] \in S^n, \\ 0 & \text{if } x = \partial \text{ or } \Delta. \end{cases}$$

Now let $X = (\Omega, \mathcal{B}_t, P_x, X_t, \theta_t)$ be a Markov process on S^3 , and let T_t be the semi-group on $B(\hat{S})$ induced by X ; i.e., $T_t f(x) = E_x[f(X_t)]$. Following Ikeda, Nagasawa, and Watanabe, we say that X is a branching Markov process⁴ (on S) if

$$T_t \hat{f}(x) = (\widehat{T_t f})|_S(x)^5$$

2. For any topological space E , $\mathcal{B}(E)$ is the Borel sets, $B(E)$ the space of all (real-valued) bounded Borel-measurable functions, and $B_1(E) = \{f \in B(E) : \|f\| = \sup_{x \in E} |f(x)| \leq 1\}$.

3. We refer the reader to Dynkin [1] for the relevant definitions and properties concerning Markov processes.

4. For a clear and detailed exposition of such processes, see Ikeda, Nagasawa, and Watanabe [5].

5. For $f \in B(S)$, $f|_S$ means the restriction of f to S .

for all $t \geq 0, x \in \hat{S}$, and $f \in B_1(S)$. We shall always assume that X is right-continuous, strong Markov, and $\bar{\mathcal{B}}_t = \mathcal{B}_t$, $\mathcal{B}_{t+} = \mathcal{B}_t$, all $t \geq 0$.

One easily sees that Δ is a trap,⁶ and if e_Δ is the first hitting time of Δ , then $P_x(e_\Delta > t) = T_t \hat{1}(x)$. This representation will play an important role in §2. We shall call e_Δ the explosion time. Furthermore, letting $e_t(x) = P_x(e_\Delta > t)$ it follows that $e_t \downarrow e$ as $t \rightarrow \infty$, where $e(x) = P_x(e_\Delta = \infty)$.

Let ξ_t be the number of particles at time t ; i.e., $\xi_t(\omega) = n$ if $X_t(\omega) \in S^n$, $n = 0, \dots, \infty$, where $S^\infty = \{\Delta\}$. Then the first splitting time τ is defined by

$$\tau(\omega) = \inf \{t: \xi_t(\omega) \neq \xi_0(\omega)\} \quad (\inf \phi = \infty).$$

The successive splitting times τ_n are defined inductively by $\tau_0 \equiv 0$ and $\tau_{n+1} = \tau_n + \tau \circ \theta_{\tau_n}$. Let $\tau_\infty = \lim_{n \rightarrow \infty} \tau_n$. We shall always assume that a bmp X satisfies the conditions

- (i) $P_x[\tau_\infty = e_\Delta; \tau_\infty < \infty] = P_x[\tau_\infty < \infty]$,
- (ii) $P_x[\tau = s] = 0$

for every $x \in S$ and $s \geq 0$.⁷

Given a bmp X , we call X^0 the non-branching part, where

$$X_t^0(\omega) = \begin{cases} X_t(\omega) & \text{if } t < \tau(\omega) \\ \Delta & \text{otherwise.} \end{cases}$$

We have the following important property for a bmp X . For every $f \in B_1(S)$, $u(t, x) = T_t \hat{f}(x)$ ($t \geq 0, x \in S$) is a solution of the S -equation with initial value f :

$$(1.1) \quad u(t, x) = T_t^0 f(x) + \int_0^t \int_S \Psi(x; ds d\mathbf{y}) \widehat{u(t-s, \cdot)}(\mathbf{y}),$$

where $T_t^0 f(x) = E_x[f(X_t); t < \tau]$ and $\Psi(x; ds d\mathbf{y}) = P_x[\tau \in ds, X_\tau \in d\mathbf{y}]$. Moreover, it is the minimal solution in the sense that when $0 \leq f \leq 1$ and if $0 \leq v \leq 1$ also satisfies (1.1), then $u \leq v$.

Two other properties enjoyed by a bmp which we shall have need of are

$$(1.2) \quad \begin{aligned} & \text{(i) } T_t^0 \hat{f}(x) = (\widehat{T_t^0 f})|_s(x) \\ & \text{(ii) if } x \in S^n, \\ & \int_0^t \int_{S^m} \Psi(x; ds d\mathbf{y}) \hat{f}(\mathbf{y}) = \begin{cases} \int_0^t \langle T_s^0 f | \int_{S^{m-n+1}} \Psi(\cdot; ds d\mathbf{y}) \hat{f}(\mathbf{y}) \rangle(x) \\ \quad \text{provided } m \neq n, m \geq n-1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for $f \in B_1(S)$.

6. $P_x[X_t = \Delta \Rightarrow X_s = \Delta, \forall s \geq t] = 1$, all $x \in \hat{S}$.

7. For most cases of interest, this constitutes no loss of generality. See [5] for more detail. There the conditions are labelled as (c. 1) and (c. 2) respectively.

8. When restricting our attention to $x \in S$, we often write x instead of \mathbf{x} .

A large class of bmp may be described in the following intuitive manner. Let $X^0 = (X_t^0, P_x^0)$ be a Markov process on $S \cup B \cup \{\nabla\}$, ∇ an isolated point (B may be empty). Let ζ be the first hitting time of the set $B \cup \{\nabla\}$. Then, a particle moves on S according to X^0 up to time ζ . If at time ζ , $X_{\zeta-}^0 \in B$, the particle is absorbed into ∂ ; otherwise, it splits into n -particles starting at $y \in S^n$ with probability $\pi(X_{\zeta-}^0, dy)$, where π is a given stochastic kernel on $S \times \mathcal{B}(\hat{S})$ ¹⁰ such that $\pi(x, S) = 0$, all $x \in S$. Each newborn particle then exhibits the same motion as the original independent of one another. The S -equation then becomes $u(t, x) = T_t^0 f(x) + h(t, x) + \int_0^t \int_S K(x; ds dy) F[y; u(t-s, \cdot)]$, where $T_t^0 f(x) = E_x^0[f(X_t^0); t < \zeta]$, $h(t, x) = P_x^0[\zeta \leq t, X_{\zeta-}^0 \in B]$, $K(x; ds dy) = P_x^0[\zeta \in ds, X_{\zeta-}^0 \in dy \cap S]$, and $F[y; g] = \int_{\hat{S}} \pi(y; dz) \hat{g}(z)$; furthermore, we have the relation $h(t, x) = 1 - T_t^0 1(x) - K(x; [0, t] \times S)$. In this case we say that X possesses the fundamental system (T_t^0, K, π) . In particular, if X^0 is obtained from a conservative Markov process $X = (X_t, P_x)$ by first absorbing it into δ (an isolated point) when it hits B and then killing this process with a non-negative measurable function k ($k = 0$ on δ), we say that the fundamental system (T_t^0, K, π) is determined by $[X, k, \pi]$, or briefly, that X possesses the regular fundamental system $[X, k, \pi]$.

Here

$$T_t^0 f(x) = E_x^x \left[e^{-\int_0^t k(X_s) ds} f(X_t); t < \eta \right]$$

$$K(x; ds dy) = T_s^0 (k I_{(dy)})(x) ds,^{11}$$

where η is the first hitting time of the set B . This paper primarily concerns itself with discussing the explosion problem for such processes.¹²

Before moving on to the main results of this paper, we first make some general comments. The problem we are concerned with is the following; is it possible to produce an infinite number of particles in a finite amount of time? As we shall soon see (Lemma 2.1), it suffices to ask the question: starting from one particle, is it possible to produce an infinite number of particles in a finite amount of time? More precisely, is $P_x(\xi_t = \infty \text{ for some } t \geq 0) > 0$, or equivalently, is $e(x) = P_x(e_\Delta = \infty) < 1$? Recall that $e_t = T_t^1 \hat{1} \downarrow e$ and e_t is the minimal solution of the S -equation with initial value $f = 1$:

$$(1.3) \quad u_t(x) = T_t^0 1(x) + \int_0^t \int_S \Psi(x; ds dy) \hat{u}_{t-s}(y).$$

9. Think of B as the boundary of a domain S in R^n . We call B the absorbing set for X .

10. For fixed $x \in S$, $\pi(x, \cdot)$ is a probability on $\mathcal{B}(\hat{S})$ and for fixed $\Lambda \in \mathcal{B}(\hat{S})$, $\pi(\cdot, \Lambda)$ is $\mathcal{B}(S)$ -measurable.

11. I_A is the indicator function of the set A .

12. For a more rigorous treatment of these processes, see [5].

The only case in which the problem is interesting is when $P_x(X_\tau = \Delta; \tau < \infty) = 0$ and so we shall always assume this.¹³ Note then that $u_t \equiv 1$ is also a solution of (1.3). Hence we are interested in the uniqueness and non-uniqueness of certain integral equations; in fact, we have

(1.4) **Proposition.** $P_x[e_\Delta = +\infty] = 1$ for every $x \in S$ iff $u(t, x) \equiv 1$ is the unique solution of (1.3) (unique within the class of all solutions v such that $0 \leq v \leq 1$).

(1.5) **Corollary.** Let X possess a regular fundamental system $[X, k, \pi]$ such that $\|k\| < \infty$ and suppose that $\sup_{x \in S} \sum_{n=0}^{\infty} n\pi(x; S^n) < \infty$. Then $P_x(e_\Delta = +\infty) = 1$ for every $x \in S$.

The proof of the corollary follows from the fact that F is Lipschitz continuous in this case.

We should also remark that in many cases, the S -equation has a differential analogue. For example, if X possesses a sufficiently "nice" regular fundamental system $[X, k, \pi]$, then the differential equation analogue of (1.3) is the non-linear evolution equation

$$\begin{aligned} \frac{d}{dt} u_t &= Au_t + k[F(\cdot; u_t) - u_t] \\ u(0+, x) &= 1 \\ u(t, x)|_{x \rightarrow B} &= 1, \end{aligned}$$

where A is the infinitesimal generator of the process X . H. Fujita and S. Watanabe [2] considered such problems of uniqueness and non-uniqueness.

2. A probabilistic approach

In this section we shall always assume that S is compact. So let X be a bmp on S . Recall the functions e_t and e defined in §1: $e_t(x) = T_t \hat{1}(x) = P_x(e_\Delta > t) \downarrow$, $e(x) = P_x(e_\Delta = \infty)$. Thus, we can say that explosion happens starting from x iff $e(x) < 1$. Our first aim will be to show that under suitable conditions $e \equiv 1$ or $e \equiv 0$ on $S \setminus \{\partial\}$. Moreover, the former is true iff $E[e_\Delta]$ is everywhere infinite there.

As a first step we observe

(2.1) **Lemma.**

- (i) $\widehat{e|_S} = e$
- (ii) $T_t e = e$ for all $t \geq 0$.

13. When X possesses the fundamental system (T_t^0, K, π) , this amounts to assuming that $\pi(x; \{\Delta\}) = 0$, all $x \in S$.

Proof. Since $\widehat{e_t|_S} = e_t$ all $t \geq 0$, the first assertion is clear.

Also

$$T_t e(x) = \lim_{s \rightarrow \infty} T_t T_s \hat{1}(x) = \lim_{s \rightarrow \infty} T_{t+s} \hat{1}(x) = e(x).$$

We now impose the following set of assumptions $[A]$.

- (A₁) $P_x[X_\tau = \partial; \tau < \infty] = 0$ for all $x \in S$.
 (2.2) (A₂) e_t and e are upper semi-continuous.
 (A₃) For every $t > 0$, all $x \in S$, and every non-empty open $U \subset S$, there exists a $V \in \mathcal{B}(\hat{S})$ such that $P_x[X_t \in V] > 0$ and for every $y \in V$, say $y = [y_1, \dots, y_m]$, some $y_i \in U$.

(A₁) is the assumption of no death; (A₂) is a regularity condition on X ; (A₃) is some type of communication assumption. Roughly, (A₃) states that for every $t > 0$ and open $U \subset S$, at least one particle is in U at time t with positive probability.

(2.3) **Theorem.** $P_x[e_\Delta = \infty] \equiv 1$ or $\equiv 0$ on S .

Proof. Note that (A₁) implies $P(t, x, \{\partial\}) = 0$ for all $t \geq 0$, $x \in S$, where P is the transition function for X . Let $\beta = \sup_{x \in S} e(x)$. Then $0 \leq \beta \leq 1$. From (A₂) and the assumption of compactness it follows that there exists some $x_0 \in S$ with $e(x_0) = \beta$. If $\beta = 0$ we are through. So suppose not. Then we claim that $\beta = 1$. For otherwise $0 < \beta < 1$. By Lemma 2.1 and (A₁) we can write for any $t \geq 0$

$$\begin{aligned} (2.4) \quad \beta &= e(x_0) = E_{x_0}[\widehat{e|_S}(X_t)] = \int_{\hat{S}} \widehat{e|_S}(y) P(t, x_0, dy) \\ &= \sum_{n=1}^{\infty} \int_{S^n} \widehat{e|_S}(y) P(t, x_0, dy) \leq \sum_{n=1}^{\infty} \beta^n P(t, x_0, S^n). \end{aligned}$$

Now if $P(t, x_0, S) = 1$ for all $t \geq 0$, it would imply by right-continuity that $P_{x_0}[X_t \in S, \text{ all } t \geq 0] = 1$, contradicting the assumption that $\beta < 1$. Thus, there exists some t_0 such that $P(t_0, x_0, S) < 1$. For this t_0 it would follow from (2.4) that $\beta < \beta$.

We will now show that $e|_S \equiv 1$ if $\beta = 1$. Suppose not. Then there exists an $\varepsilon > 0$ and open $U \subset S$ such that $e|_U \leq 1 - \varepsilon$. Fix any $t > 0$. Let V be a set corresponding to U in (A₃). Then

$$\begin{aligned} 1 &= e(x_0) = \left(\int_V + \int_{\hat{S} \setminus V} \right) \widehat{e|_S}(y) P(t, x_0, dy) \\ &\leq (1 - \varepsilon) P(t, x_0, V) + P(t, x_0, \hat{S} \setminus V) < 1. \end{aligned}$$

Contradiction.

Theorem 2.3 states that $e \equiv 1$ or $\equiv 0$ on S . Clearly if $e|_S \equiv 1$ then $E_x[e_\Delta] \equiv \infty$ on S . An interesting and useful fact, however, is that the converse is also true.

(2.5) **Lemma.** *If $e \equiv 0$ on S , then for all $t > 0$, $\|e_t|_S\| < 1$.*

Proof. Suppose there exists some $t_0 > 0$ such that $\|e_{t_0}|_S\| = 1$. Let $y_0 \in S$ be such that $e_{t_0}(y_0) = 1$ and choose $h > 0$ such that $t_1 = t_0 - h > 0$. Then

$$1 = e_{t_0}(y_0) = T_h T_{t_1} \hat{1}(y_0) = T_h \widehat{e_{t_1}|_S}(y_0).$$

By the same reasoning as in Theorem 2.3, we conclude that $e_{t_1}|_S \equiv 1$. Hence for every n ,

$$\begin{aligned} e_{nt_1}(y_0) &= T_{nt_1} \hat{1}(y_0) = T_{(n-1)t_1} T_{t_1} \hat{1}(y_0) = T_{(n-1)t_1} \widehat{e_{t_1}|_S}(y_0) \\ &= T_{(n-1)t_1} \hat{1}(y_0) = \cdots = T_{t_1} \hat{1}(y_0) = 1, \end{aligned}$$

and so $e(y_0) = \lim_{n \rightarrow \infty} e_{nt_1}(y_0) = 1$. Contradiction.

(2.6) **Theorem.** $P_x[e_\Delta = +\infty] = 1$ iff $E_x[e_\Delta] = \infty$.

Proof. We need only prove sufficiency as necessity is clear. Applying Dynkin's formula to $g = R_1 \hat{1} = \int_0^\infty e^{-t} T_t \hat{1} dt$,

$$E_x[g(X_{e_\Delta \wedge M})] - g(x) = E_x\left[\int_0^{e_\Delta \wedge M} (g - \hat{1})(X_t) dt\right] \quad \text{for every } M > 0.$$

So suppose $P_x(e_\Delta = \infty) = 0$. Applying Lemma 2.5 we conclude that there exists some $\alpha > 0$ such that $0 \leq g(y) \leq 1 - \alpha$ for all $y \neq \partial$. But from the right-continuity of the process and the assumption of no dying we have $P_x[\forall t \geq 0, X_t \neq \partial] = 1$. Consequently,

$$\alpha E_x[e_\Delta \wedge M] \leq 2\|g\|_S \leq 2.$$

Letting $M \uparrow \infty$, $E_x[e_\Delta] \leq \frac{2}{\alpha}$ (independent of x).

Combining Theorems 2.3 and 2.6 we have

(2.7) **Theorem.** *Let X be a bmp on a compact space S satisfying $[A]$. Then $P_x(e_\Delta = +\infty) \equiv 1$ or $\equiv 0$ accordingly as $E_x[e_\Delta] \equiv \infty$ or uniformly bounded on S .*

(2.8) **Corollary.** *Let X possess a regular fundamental system $[X, k, \pi]$ with no absorbing set (i.e., $B = \emptyset$) and such that*

- (i) $\pi(x; \{\partial\}) = 0 \quad \text{all } x \in S,$
- (ii) $\|k\| < \infty,$

- (iii) T_t^0 strongly Feller¹⁴, and
 (iv) for every $t > 0$, $x \in S$, and non-empty open $U \subset S$,

$$P^0(t, x, U) \equiv T_t^0 I_U(x) > 0.$$

Then the conclusions of Theorem 2.7 are valid.

Proof. (A_1) follows from (i). Since $e_t(x)$ is a solution of the S -equation

$$u(t, x) = T_t^0 1(x) + \int_0^t T_{t-s}^0 [k(\cdot)F(\cdot; u_s)](x) ds,$$

(ii) and (iii) imply that e_t is continuous for all t . Thus e is upper semi-continuous. (A_3) follows easily from (iv). Now apply Theorem 2.7.

In the remainder of this section we assume that X possesses a fundamental system (T_t^0, K, π) with no absorbing set such that $\pi(x, \{\partial\}) = \pi(x, \{\Delta\}) = 0$ on S . Our aim here is to derive a condition for explosion similar to that of E.B. Dynkin. We shall only sketch the details. In section 3 we are able to derive essentially much stronger results.¹⁵

Consider

$$\begin{aligned} (2.9) \quad E_x[e_\Delta] &= E_x[\tau_\infty] = \sum_{n=0}^{\infty} E_x[\tau \circ \theta_{\tau_n}; \tau_n < \infty] \\ &= \sum_{n=0}^{\infty} E_x[E_{X_{\tau_n}}[\tau]; \tau_n < \infty] \\ &= \sum_{n=0}^{\infty} \sum_{v_1=2}^{\infty} \sum_{v_2=v_1+1}^{\infty} \cdots \sum_{v_n=v_{n-1}+1}^{\infty} E_x[E_{X_{\tau_n}}[\tau]; X_{\tau_1} \in S^{v_1}, \dots, \\ &\quad X_{\tau_n} \in S^{v_n}; \tau_n < \infty] \end{aligned}$$

by the S.M.P.. Again by the repeated use of the S.M.P., we can write

$$\begin{aligned} E_x[E_{X_{\tau_n}}[\tau]; X_{\tau_1} \in S^{v_1}, \dots, X_{\tau_n} \in S^{v_n}; \tau_n < \infty] \\ &= E_x[E_{X_\tau}[\dots[E_{X_\tau}[\tau]; X_\tau \in S^{v_n}, \tau < \infty] \dots]; X_\tau \in S^{v_1}; \tau < \infty] \\ &= \int_{S^{v_1}} \cdots \int_{S^{v_n}} E_{y_n}[\tau] P_{y_{n-1}}(X_\tau \in dy_n) \cdots P_{y_1}(X_\tau \in dy_2) P_x(X_\tau \in dy_1). \end{aligned}$$

Furthermore, for $z \neq \Delta$

$$\begin{aligned} E_z[\tau] &= E_z\left[\int_0^\tau dt\right] = \int_0^\infty E_z[1; t < \tau] dt \\ &= \int_0^\infty T_t^0 \hat{1}(z) dt = \int_0^\infty (\widehat{T_t^0 \hat{1}}|_S)(z) dt \end{aligned}$$

14. That is, $T_t^0: \mathbf{B}(S) \rightarrow \mathbf{C}(S) = \{\text{bounded continuous functions on } S\}$, all $t \geq 0$.

15. We need assume there, however, that $K(x: ds dy) = J(x, s; dy) ds$.

making use of (1.2). Now define the following for $t \geq 0$, $0 \leq \xi \leq 1$:

$$\begin{aligned}\alpha(t) &= \inf_{x \in S} T_t^0 \hat{1}(x) & F_*(\xi) &= \inf_{x \in S} \sum_{\nu=2}^{\infty} q_{\nu}(x) \xi^{\nu} \\ \beta(t) &= \sup_{x \in S} T_t^0 \hat{1}(x) & F^*(\xi) &= \sup_{x \in S} \sum_{\nu=2}^{\infty} q_{\nu}(x) \xi^{\nu},\end{aligned}$$

where $q_{\nu}(z) = \pi(z; S^{\nu})$.

Observe that if $q_{\nu}(z)$ is independent of z , all ν , then $F_* = F^*$. Continuing, we estimate for $y \in S^{\mu}$

$$\begin{aligned}& \sum_{\nu=\mu+1}^{\infty} \int_{S^{\nu}} E_z[\tau] P_y[X_{\tau} \in dz] \\&= \int_0^{\infty} \int_0^{\infty} \sum_{\nu=\mu+1}^{\infty} \int_{S^{\nu}} (\widehat{T_t^0 1})|_S(z) \Psi(y; ds dz) dt \\&\geq \int_0^{\infty} \sum_{\nu=\mu+1}^{\infty} \alpha^{\nu}(t) \int_0^{\infty} \int_{S^{\nu}} \hat{1}(z) \Psi(y; ds dz) dt \\&= \int_0^{\infty} \sum_{\nu=\mu+1}^{\infty} \alpha^{\nu}(t) \int_0^{\infty} \langle T_s^0 1 | \int_{S^{\nu-\mu+1}} \hat{1}(z) \Psi(\cdot; ds dz) \rangle(y) dt \\&= \int_0^{\infty} \int_0^{\infty} \langle T_s^0 1 | \int_S K(\cdot; ds dz) \sum_{\nu=\mu+1}^{\infty} \alpha^{\nu}(t) \pi(z, S^{\nu-\mu+1}) \rangle(y) dt \\&\geq \int_0^{\infty} \alpha^{\mu}(t) \left(\frac{F_*[\alpha(t)]}{\alpha(t)} \right) \int_0^{\infty} \langle T_s^0 1 | \int_S K(\cdot; ds dz) 1(z) \rangle(y) dt\end{aligned}$$

using (1.2) and the fact that $\Psi(x; ds dz) = \int_S K(x; ds dy) \pi(y, dz)$. If we assume that $\lim_{t \uparrow \infty} T_t^0 \hat{1}(x) = 0$ for all $x \in S$, then

$$\begin{aligned}& \int_0^{\infty} \langle T_s^0 1 | \int_S K(\cdot; ds dz) 1(z) \rangle(y) \\&= 1 - \lim_{t \uparrow \infty} (\widehat{T_t^0 1})|_S(y) = 1.\end{aligned}$$

Hence

$$\sum_{\nu=\mu+1}^{\infty} \int_{S^{\nu}} E_z[\tau] P_y[X_{\tau} \in dz] \geq \int_0^{\infty} \alpha^{\mu}(t) \left(\frac{F_*[\alpha(t)]}{\alpha(t)} \right) dt.$$

Iterating this in (2.9) one obtains the estimate

$$\begin{aligned}E_x[e_{\Delta}] &\geq \int_0^{\infty} \alpha(t) \sum_{n=0}^{\infty} \left(\frac{F_*[\alpha(t)]}{\alpha(t)} \right)^n dt \\&= \int_0^{\infty} \frac{\alpha^2(t) dt}{\alpha(t) - F_*[\alpha(t)]}.\end{aligned}$$

Although the intermediate calculations in the case $\alpha(t) = 0$ are not valid, the

end result is provided we interpret the integrand to be zero for such t . A similar calculation yields

$$E_x[e_\Delta] \leq \int_0^\infty \frac{\beta^2(t) dt}{\beta(t) - F^*[\beta(t)]}.$$

So under the assumptions $[B]$,

$$(2.11) \quad \begin{aligned} (B_1) \quad & X \text{ possesses a fundamental system } (T_t^0, K, \pi) \text{ with no absorbing set,} \\ (B_2) \quad & \pi(x, \{\partial\}) = \pi(x, \{\Delta\}) = 0 \quad \text{on } S, \\ (B_3) \quad & \lim_{t \uparrow \infty} T_t^0 \hat{1}(x) = 0 \quad \text{on } S, \end{aligned}$$

we have the following

(2.12) **Proposition.**

$$\int_0^\infty \frac{\alpha^2(t) dt}{\alpha(t) - F_*[\alpha(t)]} \leq E_x[e_\Delta] \leq \int_0^\infty \frac{\beta^2(t) dt}{\beta(t) - F^*[\beta(t)]}$$

for every $x \in S$.

(2.13) **REMARK.** If α is integrable (on $[0, \infty)$), then $\frac{\alpha^2(t)}{\alpha(t) - F_*[\alpha(t)]}$ is integrable iff it is locally integrable at 0. Similarly for β . In particular, if $(T_t^0 \hat{1}, K, \pi)$ is determined by $[X, k, \pi]$ such that $0 < k_1 \leq k \leq k_2$ for some constants k_i then

$$(i) \quad \int_{1-\varepsilon}^1 \frac{d\xi}{\xi - F_*[\xi]} < \infty \text{ implies } E_x[e_\Delta] < \infty \quad \text{for all } x \in S,$$

$$(ii) \quad \int_{1-\varepsilon}^1 \frac{d\xi}{\xi - F_*[\xi]} = \infty \text{ implies } E_x[e_\Delta] = \infty \quad \text{for all } x \in S.$$

By combining Theorem 2.7 and Proposition 2.12 we obtain

(2.14) **Theorem.** Let X be a bmp on compact S satisfying $[A]$ and $[B]$. Then

$$(i) \quad \int_0^\infty \frac{\beta^2(t) dt}{\beta(t) - F^*[\beta(t)]} < \infty \text{ implies } P_x(e_\Delta = \infty) = 0 \quad \text{on } S.$$

$$(ii) \quad \int_0^\infty \frac{\alpha^2(t) dt}{\alpha(t) - F_*[\alpha(t)]} = \infty \text{ implies } P_x(e_\Delta = \infty) = 1 \quad \text{on } S.$$

We conclude this section with the following theorem. These results were first obtained by N. Ikeda [4] for the single-type branching process.

(2.15) **Theorem.** Let X be a bmp on compact S . Suppose it possesses a regular fundamental system $[X, k, \pi]$ with no absorbing set such that

- (i) $\pi(\cdot, S^n) = q_n(\text{constant})$, $n = 0, 1, \dots, +\infty$ and $q_0 = q_1 = q_\infty = 0$,
- (ii) there exist constants k_1, k_2 with $0 < k_1 \leq k \leq k_2$,
- (iii) T_t^0 strongly Feller and
- (iv) for every non-empty open $U \subset S$, $T_t^0 I_U(x) > 0$, for all $t > 0, x \in S$.

Then the following statements are equivalent :

$$\left(\begin{array}{ll} (1) & P_x[e_\Delta = \infty] = 1 \text{ on } S \\ (2) & E_x[e_\Delta] = \infty \text{ on } S \\ (3) & \int_{1-\varepsilon}^1 \frac{d\xi}{\xi - F[\xi]} = \infty \end{array} \right) \quad \left(\begin{array}{ll} (1)' & P_x[e_\Delta < \infty] = 1 \text{ on } S \\ (2)' & E_x[e_\Delta] < \infty \text{ on } S \\ (3)' & \int_{1-\varepsilon}^1 \frac{d\xi}{\xi - F[\xi]} < \infty \end{array} \right)$$

where $F[\xi] = \sum_{n=2}^{\infty} q_n \xi^n$.

3. An analytic approach

Recall that in §1 we pointed out that $e_t(x) = T_t \hat{1}(x)$ is the minimal solution of the S -equation with initial value $f=1$. We shall exploit this fact here.

We shall suppose that X possesses a fundamental system (T_t^0, K, π) such that

$$(3.1) \quad K(x; ds dy) = J(x, s; dy) ds.$$

In particular this is true if X possesses a regular fundamental system. Then $v(t, x) = 1 - e_t(x)$ is the maximal solution¹⁶ of

$$(3.2) \quad u(t, x) = \int_0^t ds \int_S J(x, t-s; dy) G[y; u(s, \cdot)],$$

where $G[x; f] = 1 - F[x; 1-f]$. The idea now is to compare v with a solution of a related integral equation. A key lemma in this direction is the following:

(3.3) **Lemma.** Let g be a non-negative non-decreasing function on $[0, 1]$, and let τ be a non-negative integrable function on $[0, \delta]$, some $\delta > 0$. Consider the integral equation

$$(3.4) \quad v(t) = \int_0^t \tau(s) g[v(s)] ds.$$

Then

16. Maximal in the sense that if v is also a solution, $0 \leq v \leq 1$, then $v \leq \bar{v}$.

- (i) if $\int_0^{\delta} \frac{d\xi}{g(\xi)} = \infty$,¹⁷ any solution v of (3.4) defined on $[0, \eta]$ such that $0 \leq v \leq 1$ is identically zero on $[0, \delta \wedge \eta]$.
- (ii) if $\int_0^{\delta} \frac{d\xi}{g(\xi)} < \infty$ and τ is (essentially) locally positive at 0,¹⁸ then there exists an increasing solution v of (3.4) on $[0, \eta]$, some $\eta > 0$, such that $0 \leq v \leq 1$; moreover $v(t) > 0$ for $t > 0$.

Proof.

(i) Let $0 \leq v \leq 1$ be a solution of (3.4) on $[0, \eta]$. Without loss of generality, we may assume $\eta \leq \delta$. Clearly v is absolutely continuous and increasing. Set

$$\mu = \sup \{t: 0 \leq t \leq \eta \text{ and } g[v(t)] = 0\} \quad (\sup \phi = 0).$$

If $\mu = \eta$, then $g \circ v = 0$ on $[0, \eta]$. Consequently $v = 0$ on $[0, \eta]$ and we are done. So suppose $\mu < \eta$. Then $g[v(t)] > 0$ for $\mu < t \leq \eta$. Now from (3.4) it follows that

$$v'(t) = \tau(t)g[v(t)] \quad \text{a.e.}$$

Consequently for every $\varepsilon > 0$,

$$\int_{\mu+\varepsilon}^{\eta} \frac{v'(s)}{g[v(s)]} ds = \int_{\mu+\varepsilon}^{\eta} \tau(s) ds,$$

or, by a change of variables,

$$\int_{v(\mu+\varepsilon)}^{v(\eta)} \frac{d\xi}{g(\xi)} = \int_{\mu+\varepsilon}^{\eta} \tau(s) ds.$$

Letting $\varepsilon \downarrow 0$ we obtain

$$\int_0^{v(\eta)} \frac{d\xi}{g(\xi)} = \int_{\mu}^{\eta} \tau(s) ds < \infty.$$

Contradiction.

(ii) Define on $[0, 1] \times [0, \delta]$ the function

$$A(v, t) = \int_0^v \frac{d\xi}{g(\xi)} - \int_0^t \tau(s) ds.$$

Note that for fixed t , A is strictly increasing and continuous in v , and that $A(0, 0) = 0$. Set

17. By $\int_0^{\delta} \frac{d\xi}{g(\xi)} = +\infty$, we mean that $\int_0^{\varepsilon} \frac{d\xi}{g(\xi)} = +\infty$ for every sufficiently small $\varepsilon > 0$; i.e., $\left(\frac{1}{g}\right)$ is not locally integrable at 0.

18. That is, for every sufficiently small $r > 0$, $\int_0^r \tau > 0$.

$$\eta = \sup_{0 \leq t \leq \delta} \{t: A(1, t) \geq 0\}.$$

Clearly $\eta > 0$. So, for every t with $0 \leq t \leq \eta$, there exists a unique v such that $0 \leq v \leq 1$ and $A(v, t) = 0$. Denote it by $v = v(t)$. It is not hard to show that v has all the required properties.

In order to apply this lemma it is convenient to make the following set up. Let $\mathfrak{S} = B_1([0, \infty) \times S)$ and define the operator Φ by

$$(3.5) \quad (\Phi v)(t, x) = \int_0^t ds \int_S J(x, t-s; dy) G[y: v(s, \cdot)].$$

It is clear that Φ has the properties

$$(3.6) \quad \begin{aligned} & \text{(i)} \quad \Phi \mathfrak{S} \subset \mathfrak{S} \\ & \text{(ii)} \quad \Phi u \leq \Phi v \quad \text{if} \quad u \leq v \end{aligned}$$

(3.7) **DEFINITION.** v is a solution of $\Phi u = u$ if $v \in \mathfrak{S}$ and $\Phi v = v$; v is a maximal solution if v is a solution and if v is also a solution, then $v \geq v$.

We already know, of course, the maximal solution v in terms of the semigroup T_t induced from the bmp X . This appears to be difficult to work with directly, however. It is more convenient to use the subterfuge of an approximating sequence.

(3.8) **Proposition.** *There exists a sequence v_n , $1 \leq n \leq \infty$, with $0 \leq v_n \leq 1$ such that $v_0 = 1$, $v_\infty = v$, and $v_n \downarrow v_\infty$.*

Proof. Set $v_0 \equiv 1$ and define inductively for $n \geq 1$, $v_n = \Phi v_{n-1}$. Since $v_0 \in \mathfrak{S}$ it follows from (3.6) that $v_n \in \mathfrak{S}$ and $v_n \downarrow$. Set $v_\infty = \lim v_n$, which clearly exists. By the dominated convergence theorem, $v_\infty = \Phi v_\infty$.

Now suppose u is any other solution. But $u \leq 1 = v_0$. So suppose $u \leq v_n$. Then

$$u = \Phi u \leq \Phi v_n = v_{n+1}.$$

Hence $u \leq v_\infty$. By the uniqueness of the maximal solution, we have then that $v_\infty = v$.

(3.9) **DEFINITION.** The sequence $v_0 = 1$, $v_n = \Phi v_{n-1}$ for $n \geq 1$ is called the defining sequence for the maximal solution v .

We are now ready to reap the main results of this section.

(3.10) **Theorem.** *Let $\delta > 0$ be fixed and set*

$$(3.11) \quad \tau^*(s) = \sup_{\substack{x \in S \\ t \in [s, \delta]}} J(x, t-s; S), \quad 0 \leq s \leq \delta.$$

Suppose $\int_0^\delta \tau^* < \infty$. Define $G^*(\xi) = \sup_{x \in S} G[x; \xi 1]$, $0 \leq \xi \leq 1$. Then if $\int_0 \frac{d\xi}{G^*(\xi)} = \infty$, $\vartheta \equiv 0$ (i.e., no explosion).

Proof. Let G_+^* be the right-continuous version of G^* ; i.e., $G_+^*(\xi) = \lim_{\eta \downarrow \xi} G^*(\eta)$, $0 \leq \xi < 1$, and $G_+^*(1) = G^*(1)$. Then G_+^* is monotone increasing, $G_+^* \geq G^*$ and $G_+^* = G^*$ a.e. Let $\langle v_n \rangle$ be the defining sequence for ϑ . Take $u_0 \equiv 1$ and define u_n iteratively by

$$u_n(t) = \int_0^t \tau^*(s) G_+^*[u_{n-1}(s)] ds, \quad n \geq 1.$$

Set $\eta = \sup_{0 \leq t \leq \delta} \{t: G_+^*(1) \int_0^t \tau^*(s) ds \leq 1\}$. Then $\eta > 0$. Also, since $0 \leq u_{n+1} \leq u_n \leq 1$, we have $u_n \downarrow u_\infty$ exists on $[0, \eta]$ with $0 \leq u_\infty \leq 1$.

Now $v_0 = 1 \leq u_0$; so suppose $v_n(t, x) \leq u_n(t) 1(x)$ on $[0, \eta] \times S$. Then for $(t, x) \in [0, \eta] \times S$

$$\begin{aligned} v_{n+1}(t, x) &= \int_0^t ds \int_S J(x, t-s; dy) G[y; v_n(s, \cdot)] \\ &\leq \int_0^t ds \int_S J(x, t-s; dy) G[y; u_n(s) 1(\cdot)] \\ &\leq \int_0^t G^*[u_n(s)] J(x, t-s; S) ds \\ &\leq \int_0^t \tau^*(s) G_+^*[u_n(s)] ds = u_{n+1}(t). \end{aligned}$$

Consequently, $\vartheta \leq u_\infty$. But u_∞ satisfies

$$u_\infty(t) = \int_0^t \tau^*(s) G_+^*[u_\infty(s)] ds.$$

From Lemma 3.3 we conclude that $u_\infty \equiv 0$ on $[0, \eta]$; hence $\vartheta \equiv 0$ on $[0, \eta] \times S$.

Now set

$$\sigma = \sup \{t: \vartheta(s, x) = 0 \text{ on } [0, t] \times S\}.$$

If $\sigma = \infty$, we are done; so suppose not. Then $\sigma \geq \eta > 0$. Now set $u(t, x) = \vartheta(t + \sigma, x)$. Then u satisfies the equation

$$\begin{aligned} u(t, x) &= \vartheta(t + \sigma, x) = \int_0^{t+\sigma} ds \int_S J(x, t+\sigma-s; dy) G[y; \vartheta(s, \cdot)] \\ &= \int_\sigma^{t+\sigma} ds \int_S J(x, t+\sigma-s; dy) G[y; \vartheta(s, \cdot)] \end{aligned}$$

since from the condition $\int_0 \frac{d\xi}{G^*(\xi)} = \infty$, we must have $G(0) = 0$.

Then

$$u(t, x) = \int_0^t ds \int_S J(x, t-s; dy) G[y; u(s, \cdot)],$$

and so u is a solution of (3.2). Consequently, $v \geq u$. But $v=0$ on $[0, \sigma] \times S$ which implies that $v=0$ on $[0, 2\sigma] \times S$. Contradiction.

(3.12) **Theorem.** Let $\Gamma \in \mathcal{B}(S)$ and $\delta > 0$. Set

$$(3.13) \quad \tau_*(s) = \inf_{\substack{x \in \Gamma \\ t \in [s, \delta]}} J(t-s, x; \Gamma), \quad 0 \leq s \leq \delta.$$

Suppose τ_* is locally positive at 0 (cf. footnote 18). Define $G_*(\xi) = \inf_{x \in \Gamma} G[x; \xi I_\Gamma]$, $0 \leq \xi \leq 1$ and suppose $\int_0^1 \frac{d\xi}{G_*(\xi)} < \infty$. Then $v > 0$ on $(0, \infty) \times \Gamma$ (i.e., explosion happens starting from Γ).

Proof. Since τ_* is integrable, it follows from Lemma 3.3 that there exists a function u defined on $[0, \eta]$, some $\eta > 0$, such that $0 < u \leq 1$ on $(0, \eta]$ and satisfies the integral equation

$$u(t) = \int_0^t \tau_*(s) G_*[u(s)] ds, \quad 0 \leq t \leq \eta.$$

Let v_n be the defining sequence for v . Then $v_0(t, x) \equiv 1 \geq u(t) I_\Gamma(x)$ on $[0, \eta] \times S$. Suppose $v_n \geq u I_\Gamma$ on $[0, \eta] \times S$. Then for $(t, x) \in [0, \eta] \times \Gamma$, we have

$$\begin{aligned} v_{n+1}(t, x) &= \int_0^t ds \int_S J(x, t-s; dy) G[y; v_n(s, \cdot)] \\ &\geq \int_0^t ds \int_S J(x, t-s; dy) G[y; u(s) I_\Gamma] \\ &\geq \int_0^t ds \int_\Gamma J(x, t-s; dy) G[y; u(s) I_\Gamma] \\ &\geq \int_0^t \tau_*(s) G_*[u(s)] ds = u(t). \end{aligned}$$

Consequently $v \geq u I_\Gamma$ on $[0, \eta] \times S$. But $v(t, x)$ is an increasing function of t and so $v > 0$ on $(0, \infty) \times \Gamma$.

(3.14) **Corollary.** Let Γ be as in Theorem 3.12. If there exists a $\Lambda \in \mathcal{B}(S)$ such that for every $x \in \Lambda$ and for every sufficiently small $r > 0$, $\int_0^r J(x, r-s; \Gamma) ds > 0$, then under the assumptions of the above theorem, $v > 0$ on $(0, \infty) \times \Lambda$.

(3.15) **REMARK.** Let (T_t^0, K, π) be determined by $[X, k, \pi]$. Then $J(x, s; dy) = P^0(s, x, dy) k(y)$, where P^0 is the transition function corresponding to T_t^0 . Thus

- (i) if $\|k\| < \infty$, then τ^* is integrable on $[0, \delta]$.
(ii) if $k|\Gamma \geq k_1 > 0$ for some $\Gamma \in \mathcal{B}(S)$, then τ_* is locally positive at 0
if

$$\inf_{\substack{s \in \Gamma \\ 0 \leq s \leq \delta}} P^0(s, x, \Gamma) > 0.$$

- (iii) if $\|k\| < \infty$ and $k|\Gamma \geq k_1 > 0$ for some $\Gamma \in \mathcal{B}(S)$,

then τ_* is locally positive at 0 if

$$\inf_{\substack{x \in \Gamma \\ 0 \leq s \leq \delta}} P_x^X(X_s \in \Gamma; s > \eta) > 0,$$

where η is the first hitting time of B .

4. Applications

EXAMPLE 1 (multi-type bmp).

Let $S = \{a_1, \dots, a_N\}$. Then a bmp X on S is called an N -type bmp. In particular, let X be a (π_{ij}, b_i) -Markov chain on S , where $0 < b_i < \infty$ and $0 \leq \pi_{ij} \leq 1$, $\pi_{ii} = 0$, $\sum_{j=1}^N \pi_{ij} = 1$, $i, j = 1, \dots, N$; i.e., X is the Markov chain on S such that

$$b_i = (E_{a_i}^X[\sigma])^{-1} \quad \text{and} \quad \pi_{ij} = P_{a_i}^X[X_\sigma = a_j],$$

where σ is the first jump time. Let k be defined on S such that $k(a_i) = k_i > 0$ and $q_n (n \geq 2)$ non-negative constants such that $\sum_{n=2}^{\infty} q_n = 1$. Define the stochastic kernel π on $S \times \mathcal{B}(\hat{S})$ by

$$(4.1) \quad \pi(x, d\mathbf{y} \cap S^n) = q_n \delta_{(\underbrace{[x, \dots, x]}_n)}(d\mathbf{y}),$$

where we set $q_0 = q_1 = q_\infty = 0$. Then there exists a bmp X on S with $[X, k, \pi]$ as its regular fundamental system. Theorem 2.15 says that explosion happens with probability one independent of the starting point iff $\int_0^1 \frac{d\xi}{\xi - F[\xi]} < \infty$, $F[\xi] = \sum_{n=2}^{\infty} q_n \xi^n$.

EXAMPLE 2. (Branching diffusion with reflecting boundary)

Let D be a bounded domain in $E = \mathbf{R}^d$ and set $S = \bar{D}$. We assume that D has a sufficiently smooth boundary, say $C^{(2)}$. Consider the operator

$$(4.2) \quad Af(x) = \sum_{i,j=1}^l a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^l b_i(x) \frac{\partial f}{\partial x_i}$$

where the a_{ij}, b_i are bounded and satisfy a Hölder condition on S . We also assume that A is uniformly elliptic. Then it is known (cf., Itô [6]) that there exists a conservative diffusion process X on S such that for f sufficiently smooth, $u(t, x) = E_x^X[f(X_t)]$ satisfies

$$(4.3) \quad \begin{aligned} \frac{\partial u}{\partial t} &= Au \\ \frac{\partial u}{\partial n} \Big|_{\partial D} &= 0. \end{aligned}$$

Furthermore, X is strongly Feller and if $p(t, x, y)$ is the fundamental solution of (4.3), it is strictly positive for $t > 0$ and $x, y \in S$.

Let k be a non-negative measurable function on S such that there exists constants k_i with $0 < k_1 \leq k \leq k_2$ and let π be as in (4.1). The bmp X on S which has $[X, k, \pi]$ as its regular fundamental system will be called a branching diffusion process with reflecting boundary. We again conclude from Theorem 2.15 that explosion happens with probability one (independent of the starting position) iff $\int_0^1 \frac{d\xi}{\xi - F[\xi]} < \infty$.

EXAMPLE 3.

Let X be Brownian motion on \mathbf{R} , and let X^0 be the $e^{-\varphi_t}$ -subprocess, where φ_t is local time at the origin. Given a kernel π on $S \times \mathcal{B}(\hat{S})$, let X be the (T_t^0, K, π) bmp on $S = \mathbf{R}$. We assume, of course, that $\pi(x, S) \equiv 0$. Here

$$\begin{aligned} K(x; ds dy) &= (-d_s E_x^X[e^{-\varphi_s}]) \delta_{\{0\}}(dy) \\ &= J(x, s; dy) ds. \end{aligned}$$

In particular,

$$J(0, s; dy) = \frac{1}{\sqrt{\pi}} \left\{ \frac{1}{\sqrt{2s}} - e^{s/2} \int_{\sqrt{s/2}}^{\infty} e^{-z^2} dz \right\} \delta_{\{0\}}(dy).$$

It is easy to see that for $\Gamma = \{0\}$ and sufficiently small $\delta > 0$, τ_* is locally positive at zero. So by Theorem 3.12 we conclude that if $\int_0^\infty \frac{d\xi}{G[0; \xi I_{\{0\}}]} < \infty$, then explosion happens starting from zero. But since $J(x, s; \{0\}) > 0$ for every $x \in S, s > 0$ we can conclude from Corollary 3.14 that explosion happens starting from any $x \in S$.

EXAMPLE 4. (Branching diffusion with absorbing boundary).

Let A be as in (4.2) except that we assume it to be defined on all of \mathbf{R}^d for simplicity. Let $X = (X_t, P_x)$ be the corresponding conservative diffusion on E . Let S be a bounded domain in E with sufficiently smooth boundary $B = \partial S$

(e.g., $C^{(2)}$ -boundary). The absorbed process $\hat{X}=(\hat{X}_t, \hat{P}_x)$ on $S \cup \{\delta\}$ ¹⁹ with δ as trap is given by

$$\hat{X}_t = \begin{cases} X_t, & t < \eta, \\ \delta, & \text{otherwise,} \end{cases}$$

$$\hat{P}_x = P_x,$$

where η is the first hitting time of B . Given a bounded, non-negative, $\mathcal{B}(S)$ -measurable function k and a stochastic kernel π on $S \times \mathcal{B}(\hat{S})$ such that $\pi(x, S) \equiv 0$, we let X be the bmp on S possessing the regular fundamental system $[X, k, \pi]$ and absorbing set B . Since this process has the property that whenever a particle hits the boundary of S it is absorbed into $\{\partial\}$ we call X a branching diffusion process with absorbing boundary. Note that X^0 is the $e^{-\int_0^t k(\hat{X}_s) ds}$ -subprocess of \hat{X} , where we extend k as a function on $S \cup \{\delta\}$ by setting $k=0$ on δ .

In order to apply the results of §3 for the exploding case we must show that the conditions of Theorem 3.12 are satisfied. According to Remark 3.15, assuming $k|_{\Gamma} \geq k_1 > 0$, it suffices to show that

$$(4.4) \quad \inf_{\substack{x \in \Gamma \\ 0 \leq s \leq \delta}} P_x^X(X_s \in \Gamma; s < \eta) > 0$$

for some $\delta > 0$. Since $\Gamma \in \mathcal{B}(S)$, $P_x^X(X_s \in \Gamma; s < \eta) = \hat{P}_x(\hat{X}_s \in \Gamma)$. Let p and \hat{p} be the transition density for X and \hat{X} respectively. Then we have the relation

$$\hat{p}(t, x, y) = p(t, x, y) - \int_0^t \int_B p(t-s, z, y) \mu_x(ds dz)$$

for all $t > 0$, x and $y \in S$. Here $\mu_x(ds dz) = P_x^X(\eta \in ds, X_\eta \in dz)$. Integrating over Γ , we obtain

$$\begin{aligned} \hat{P}(t, x, \Gamma) &= P(t, x, \Gamma) - \int_0^t \int_B P(t-s, z, \Gamma) \mu_x(ds dz) \\ &\geq P(t, x, \Gamma) - P_x^X(\eta \leq t). \end{aligned}$$

But we have the lower estimate for p

$$p(t, x, y) \geq M_1 t^{-l/2} \exp \left[-\alpha_1 \frac{|x-y|^2}{t} \right] - M_2 t^{-(l/2)+\lambda} \exp \left[-\alpha_2 \frac{|x-y|^2}{t} \right]$$

where $M_1, M_2, \alpha_1, \alpha_2$, and λ are positive constants (cf. Dynkin [1: Theorem 0.5]). Furthermore from a result of Varadhan [8] we obtain the estimate: for every compact subset $K \subset S$, there exists a $\rho > 0$ such that for all $x \in K$

19. δ is an isolated point.

$$P_x^X(\eta \leq t) \leq e^{-\rho/t}$$

provided t is sufficiently small. Consequently, if Γ is such that $\bar{\Gamma} \subset S$, (4.4) will be valid if

$$(4.5) \quad \inf_{\substack{x \in \Gamma \\ 0 < t \leq \delta}} \int_{\Gamma} t^{-l/2} \exp \left[-\frac{|x-y|^2}{t} \right] dy > 0$$

for δ sufficiently small. But (4.5) is true iff there exists some positive constant κ such that for every ball B of sufficiently small radius and every $x \in \Gamma$, we have

$$(4.6) \quad m(\Gamma \cap B_x) \geq \kappa m(B),$$

where B_x is the ball B centered at x and m is l -dimensional Lebesgue measure.²⁰ In particular, (4.6) is true if Γ is itself a ball. We shall only outline the proof of the if statement.

So suppose (4.6) is valid. For $r \in \mathbf{R}^1$, $x \in \mathbf{R}^l$, and $A \subset \mathbf{R}^l$ set

$$\begin{aligned} rA &= \{ry : y \in A\} \\ A_x &= \{y+x : y \in A\}. \end{aligned}$$

Also, let B be the unit ball centered at the origin. Consider the following.

$$\begin{aligned} \int_{\Gamma} t^{-l/2} \exp \left[-\frac{|x-y|^2}{t} \right] dy &= \int_{\frac{1}{\sqrt{t}}\Gamma_{-x}} e^{-|z|^2} dz \\ &\geq \int_{\frac{1}{\sqrt{t}}\Gamma_{-x} \cap B} e^{-|z|^2} dz \geq e^{-1} m \left(\frac{1}{\sqrt{t}}\Gamma_{-x} \cap B \right) \\ &= e^{-1} t^{-l/2} m(\Gamma \cap (\sqrt{t}B)_x) \\ &\geq e^{-1} t^{-l/2} \kappa m(\sqrt{t}B) = \kappa e^{-1} m(B) \end{aligned}$$

for $x \in \Gamma$ and sufficiently small t . Hence

$$\inf_{\substack{x \in \Gamma \\ 0 < t \leq \delta}} \int_{\Gamma} t^{-l/2} \exp \left[-\frac{|x-y|^2}{t} \right] dy \geq \kappa e^{-1} m(B) > 0$$

(provided δ is sufficiently small).

Putting all this together, we obtain

(4.7) **Theorem.** *Let X be the branching diffusion process with absorbing boundary as described above. Then*

20. The symbol B has been used to designate both a sphere in \mathbf{R}^l and the absorbing set of a bmp. This should introduce no confusion, however.

- (i) $\int_0 \frac{d\xi}{G^*(\xi)} = \infty$ implies no explosion, where $G^*(\xi) = \sup_{x \in \bar{S}} G[x; \xi 1]$.
- (ii) $\int_0 \frac{d\xi}{G_*(\xi)} < \infty$ implies explosion starting from Γ

provided Γ is such that it satisfies (4.6), $\bar{\Gamma} \subset S$, and $k|\Gamma| \geq k_1 > 0$, where $G_*(\xi) = \inf_{x \in \Gamma} G[x; \xi I_\Gamma]$.

(4.8) REMARK.

1. Since $\hat{p}(t, x, y) > 0$ for all $x, y \in S$ and $t > 0$, then if explosion happens from Γ , it happens from any $x \in S$. (cf. Corollary 3.14).

2. Let $Y = (Y_t, Q_x)$ be any diffusion on some $S \subset E$ and let \mathfrak{A} be its characteristic operator. Suppose that S contains a bounded smooth domain D such that $\mathfrak{A}|D = A|D$, where A is some operator on E satisfying the assumptions of (4.2). Since the absorbing diffusion process \hat{Y} on D is the minimal process, we then have

$$\inf_{\substack{x \in \Gamma \\ 0 \leq t \leq \delta}} Q(t, x, \Gamma) > 0$$

for any Γ with $\bar{\Gamma} \subset D$ and satisfying (4.6), all δ sufficiently small. Consequently, we can conclude that for such Γ , explosion happens from Γ for the bmp Y corresponding to the regular fundamental system $[Y, k, \pi]$, if $k|\Gamma| \geq k_1 > 0$ and $\int_0 \frac{d\xi}{G_*(\xi)} < \infty$, $G_*(\xi) = \inf_{x \in \Gamma} G[x; \xi I_\Gamma]$.

EXAMPLE 5.

Let $S = \mathbf{R}$ and X be Brownian motion on S . Let $k = I_F$, where F is the following set. Take $I = [0, 1]$ and $\alpha \in (0, \frac{1}{2})$. Let E_0^1 be the middle open interval of length α removed from I . Inductively we define $E_k^1, \dots, E_k^{2^k}$ to be the middle open intervals of length $\alpha 2^{-2^k}$ removed from $I \setminus \bigcup_{\nu=0}^{k-1} \bigcup_{\mu=1}^{2^\nu} E_\nu^\mu$. Set $F = I \setminus \bigcup_{\nu=0}^{\infty} \bigcup_{\mu=1}^{2^\nu} E_\nu^\mu$. Then F is a perfect nowhere dense set of measure $(1-2\alpha)$; i.e., it is a "fat" Cantor set. We shall now show that F satisfies (4.6). At the k^{th} stage, the distance between two adjacent sets E_ν^μ , $1 \leq \mu \leq 2^\nu$, $0 \leq \nu \leq k$ is

$$d(k) = \frac{2^k - \alpha(2^{k+1} - 1)}{2^{2k+1}}.$$

Let λ be given such that $0 < \lambda \leq (1-2\alpha)$, and let B be the unit ball about the origin. Choose $k = k(\lambda)$ to be the first non-negative integer such that $d(k) \leq \lambda$. Then $d(k-1) > \lambda$. Moreover, if $x \in F$,

$$\begin{aligned} \frac{m(F \cap \lambda B_x)}{m(\lambda B)} &\geq \frac{d(k) - \sum_{v=0}^{\infty} 2^v \frac{\alpha}{2^{2(v+k+1)}}}{2d(k-1)} \\ &\geq \frac{1}{4} (1-2\alpha). \end{aligned}$$

Consequently F satisfies (4.6).

Now, let π be a stochastic kernel on $S \times \mathcal{B}(\hat{S})$ defined by

$$\pi(x, d\mathbf{y}) = p_n \delta_{(\underbrace{[x, \dots, x]}_n)}(d\mathbf{y}) \text{ if } d\mathbf{y} \in \mathcal{B}(S^n), \quad n = 0, 1, \dots, +\infty,$$

where $0 \leq p_n \leq 1$, $0 = p_0 = p_1 = p_\infty$, and $\sum p_n = 1$. If X is the bmp on S corresponding to $[X, k, \pi]$, then according to remark 4.8.2 we can say that explosion happens iff $\int_0^1 \frac{d\xi}{1-F(\xi)} < \infty$, $F(\xi) = \sum_{n \geq 2} p_n \xi^n$. Note that splits only occur on the set F .

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