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THE EXPLOSION PROBLEM FOR BRANCHING
MARKOV PROCESS

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0. Introduction

Consider a single-type branching process. Then a well-known result of Dynkin is the following: explosion happens (i.e., the number of particles will be infinite in a finite time with positive probability) if $\int_{1^-}^{1^+} \frac{du}{u - h(u)}$ converges for every $\varepsilon > 0$, where $h$ is the generating function of new-born particles (see, e.g., [3, p. 106]). N. Ikeda [4] has also given an interesting proof of this fact using probabilistic techniques. Indeed he shows that the convergence of $\int_{1^-}^{1^+} \frac{du}{u - h(u)}$ is equivalent to the finiteness of the expected value of $e^{\alpha t}$, the time of explosion (i.e., the first time when the number of particles is infinite).

The purpose of this paper is to investigate the explosion problem for a more general class of branching processes: branching Markov process (see Ikeda, Nagasawa and Watanabe [5]). For a large class of bmp. we are able to show that a sufficient condition for explosion (non-explosion) is the convergence (divergence) of a particular integral. In many cases of interest, this condition is also necessary and sufficient.

In §1 we introduce the necessary terminology and notation; in §2 we generalize the methods of Ikeda and thus treat the problem from a probabilistic viewpoint; in §3, we consider the explosion problem from the analytical viewpoint. These results are of a more local character than those of §2 and hence give stronger results in some sense. Section 4 is devoted to applications. In particular, we consider branching diffusion processes with absorbing boundary. Another interesting application is that of branching Brownian motion whose splits occur only on a "fat" Cantor set.

It should be remarked that the explosion problem is intimately related to the uniqueness (or non-uniqueness) of solution of certain semi-linear parabolic equations. Such questions have been considered by Fujita and Watanabe [2].

1. We usually abbreviate this as bmp.
Acknowledgments

I wish to take this opportunity to thank N. Ikeda and S. Watanabe for their constant guidance and encouragement during the writing of my dissertation [7] from which the contents of this paper were extracted.

1. Definitions and statement of problem

Let \( S \) be a locally compact, second-countable, Hausdorff topological space. Form the \( n \)-fold direct-product topological space \( S^{(n)} \). Let \( S^n = S^{(n)}/\sim \) be the quotient topological space induced by the equivalence relation \( \sim \) of permutation: \((x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)\) iff there exists a permutation \( \pi \) on \( \{1, \ldots, n\} \) such that \( x_i = y_{\pi(i)} \), all \( i = 1, \ldots, n \). The topological sum \( \bigcup_{n=0}^{\infty} S^n \) is denoted by \( S \), where \( S^0 = \{\partial\} \), \( \partial \) being an isolated point. Since \( S \) is locally compact (but not compact) we let \( \hat{S} = S \cup \{\Delta\} \) be its one-point compactification.

In order to define a branching Markov process, it is convenient to introduce the mapping \( \Lambda : B_1(S) \to B(S) \) defined by

\[
\Lambda(x) = \begin{cases} 
1 & \text{if } x = \partial , \\
\prod_{i=1}^{n} f(x_i) & \text{if } x = [x_1, \ldots, x_n] \in S^n , \\
0 & \text{if } x = \Delta .
\end{cases}
\]

Another mapping that we shall have occasion to use is the following: given \( f, g \in B_1(S) \), we define the \( B(\hat{S}) \)-measurable function \( \langle f \mid g \rangle \) by

\[
\langle f \mid g \rangle (x) = \begin{cases} 
\sum_{i=1}^{n} g(x_i) \prod_{j \neq i} f(x_j) & \text{if } x = [x_1, \ldots, x_n] \in S^n , \\
0 & \text{if } x = \partial \text{ or } \Delta .
\end{cases}
\]

Now let \( X = (\Omega, \mathcal{F}, P, X_t, \theta_t) \) be a Markov process on \( S^3 \), and let \( T_t \) be the semi-group on \( B(\hat{S}) \) induced by \( X \); i.e., \( T_t f(x) = E_x[f(X_t)] \). Following Ikeda, Nagasawa, and Watanabe, we say that \( X \) is a branching Markov process\(^4\) (on \( S \)) if

\[
T_t \hat{f}(x) = (T_t f)_{\mid S}(x)^3
\]

2. For any topological space \( E \), \( \mathcal{B}(E) \) is the Borel sets, \( B(E) \) the space of all (real-valued) bounded Borel-measurable functions, and \( B_1(E) = \{ f \in B(E) : \|f\| = \sup_{x \in E} |f(x)| \leq 1 \} \).

3. We refer the reader to Dynkin [1] for the relevant definitions and properties concerning Markov processes.

4. For a clear and detailed exposition of such processes, see Ikeda, Nagasawa, and Watanabe [5].

5. For \( f \in B(S) \), \( f \mid S \) means the restriction of \( f \) to \( S \).
for all $t \geq 0, x \in \hat{S}$, and $f \in B_1(S)$. We shall always assume that $X$ is right-
continuous, strong Markov, and $\mathcal{B}_t = \mathcal{B}_t$, $\mathcal{B}_{t+} = \mathcal{B}_t$, all $t \geq 0$.

One easily sees that $\Delta$ is a trap, and if $e_\Delta$ is the first hitting time of $\Delta$, then $P_x(e_\Delta > t) = T_t^x(\Delta)$. This representation will play an important role in §2. We shall call $e_\Delta$ the explosion time. Furthermore, letting $e_t(x) = P_x(e_\Delta > t)$ it follows that $e_t \downarrow e$ as $t \to \infty$, where $e(x) = P_x(e_\Delta = \infty)$.

Let $\xi_t$ be the number of particles at time $t$; i.e., $\xi_t(\omega) = n$ if $X_t(\omega) \in S^n$, $n=0$, $\ldots$, $\infty$, where $S^n = \{\Delta\}$. Then the first splitting time $\tau$ is defined by

$$\tau(\omega) = \inf \{ t: \xi_t(\omega) + e_\Delta(\omega) \} \quad (\inf \phi = \infty).$$

The successive splitting times $\tau_n$ are defined inductively by $\tau_0 \equiv 0$ and $\tau_{n+1} = \tau_n + \tau_0 e_{\tau_n}$. Let $\tau_\infty = \lim_{n \to \infty} \tau_n$. We shall always assume that a bmp $X$ satisfies the conditions

(i) $P_x[\tau_\infty = e_\Delta; \tau_\infty < \infty] = P_x[\tau_\infty < \infty]$,
(ii) $P_x[\tau = \infty] = 0$

for every $x \in S$ and $s \geq 0$.

Given a bmp $X$, we call $X_0$ the non-branching part, where

$$X_0^0(\omega) = \begin{cases} X_t(\omega) & \text{if } t < \tau(\omega) \\ \Delta & \text{otherwise.} \end{cases}$$

We have the following important property for a bmp $X$. For every $f \in B_1(S)$, $u(t, x) = T_t f(x)$ $(t \geq 0, x \in S)$ is a solution of the $S$-equation with initial value $f$:

$$u(t, x) = T_t^x f(x) + \int_0^t \int_S \Psi(x; dsdy) u(t-s, \cdot)(y),$$

where $T_t^x f(x) = E_x[f(X_t); t < \tau]$ and $\Psi(x; dsdy) = P_x[\tau < ds, X_{\tau} \in dy]$. Moreover, it is the minimal solution in the sense that when $0 \leq f \leq 1$ and if $0 \leq u \leq 1$ also satisfies (1.1), then $u \geq v$.

Two other properties enjoyed by a bmp which we shall have need of are

$$T_t^x f(x) = (T_t^0 f)|_{S^n}(x)$$

(i) if $x \in S^n$,

$$\int_0^t \int_{S^n} \Psi(x; dsdy) f(y) = \begin{cases} \int_0^t \int_{S^n} \Psi(x; dsdy) f(y) > (x) & \text{provided } m+n, m \geq n-1 \\ 0 & \text{otherwise} \end{cases}$$

for $f \in B_1(S)$.

6. $P_x[X_t = \Delta = e_\Delta, \forall t \geq t] = 1$, all $x \in \hat{S}$.

7. For most cases of interest, this constitutes no loss of generality. See [5] for more detail. There the conditions are labelled as (c. 1) and (c. 2) respectively.

8. When restricting our attention to $x \in S$, we often write $x$ instead of $x$. 
A large class of bmp may be described in the following intuitive manner. Let \( X^\circ=(X^\circ, P^\circ) \) be a Markov process on \( S \cup B \cup \{\nabla\} \), \( \nabla \) an isolated point (\( B \) may be empty). Let \( \zeta \) be the first hitting time of the set \( B \cup \{\nabla\} \). Then, a particle moves on \( S \) according to \( X^\circ \) up to time \( \zeta \). If at time \( \zeta \), \( X^\circ_\zeta \in B \), the particle is absorbed into \( \partial \); otherwise, it splits into \( n \)-particles starting at \( y \in S^* \) with probability \( \pi(X^\circ_\zeta, dy) \), where \( \pi \) is a given stochastic kernel on \( S \times \mathcal{B}(S) \) such that \( \pi(x, S)=0 \), all \( x \in S \). Each newborn particle then exhibits the same motion as the original independent of one another. The \( S \)-equation then becomes
\[
\begin{align*}
\frac{\partial u(t, x)}{\partial t} &= \mathcal{T}^0_t f(x) + h(t, x) + \int_S \int_S K(x; ds, dy)F[y; u(t-s, \cdot)], \\
&= \mathcal{E}^0_t[f(X^\circ_t)]; \ t<\zeta, \\
h(t, x) &= \mathcal{P}^0_\zeta \mathbf{1} \{\zeta \leq t \}; \ X^\circ_\zeta \in B, \\
K(x; ds, dy) &= \mathcal{P}^0_\zeta \mathbf{1} \{\zeta \in ds, \ X^\circ_\zeta \in dy \cap S \}, \\
\text{and } F[y; g] &= \int_S \pi(y; dz) \hat{g}(z); \\
\text{furthermore, we have the relation } h(t, x) &= 1 - \mathcal{T}^0_t(1) - K(x; [0, t] \times S).
\end{align*}
\]
In this case we say that \( X \) possesses the fundamental system \( (\mathcal{T}^0_t, K, \pi) \). In particular, if \( X^\circ \) is obtained from a conservative Markov process \( X=(X_t, P_x) \) by first absorbing it into \( \partial \) (an isolated point) when it hits \( B \) and then killing this process with a non-negative measurable function \( k (k=0 \text{ on } \partial) \), we say that the fundamental system \( (\mathcal{T}^0_t, K, \pi) \) is determined by \( [X, k, \pi] \), or briefly, that \( X \) possesses the regular fundamental system \( [X, k, \pi] \).

Here
\[
\begin{align*}
\mathcal{T}^0_t f(x) &= \mathcal{E}^0_{\zeta}[e^{-\int_0^{\zeta} k(X_s)ds} f(X_s)]; \ t<\eta, \\
K(x; ds, dy) &= \mathcal{T}^0_{\zeta}(k \mathbf{1}_{(dy)})(x) ds, \quad 11
\end{align*}
\]
where \( \eta \) is the first hitting time of the set \( B \). This paper primarily concerns itself with discussing the explosion problem for such processes. 12

Before moving on to the main results of this paper, we first make some general comments. The problem we are concerned with is the following: is it possible to produce an infinite number of particles in a finite amount of time? As we shall soon see (Lemma 2.1), it suffices to ask the question: starting from one particle, is it possible to produce an infinite number of particles in a finite amount of time? More precisely, is \( P_x(\tau_t=\infty \text{ for some } t \geq 0)>0 \), or equivalently, \( e(x)=P_x(e_{\tau_t}=\infty)<1 \)? Recall that \( e_t=\mathcal{T}_t \mathbf{1} \downarrow e \) and \( e_t \) is the minimal solution of the \( S \)-equation with initial value \( f=1 \):
\begin{equation}
(1.3) \quad u_t(x) = \mathcal{T}_t^1(x) + \int_0^t \int_S \Psi(x; ds, dy) u_{t-s}(y).
\end{equation}

9. Think of \( B \) as the boundary of a domain \( S \) in \( \mathbb{R}^n \). We call \( B \) the absorbing set for \( X \).
10. For fixed \( x \in S, \pi(x, \cdot) \) is a probability on \( \mathcal{B}(S) \) and for fixed \( \Lambda \in \mathcal{B}(S) \), \( \pi(\cdot, \Lambda) \) is \( \mathcal{B}(S) \)-measurable.
11. \( I_A \) is the indicator function of the set \( A \).
12. For a more rigorous treatment of these processes, see [5].
The only case in which the problem is interesting is when $P_x(X_i = \Delta; \tau < \infty) = 0$ and so we shall always assume this.\(^{13}\) Note then that $u_t \equiv 1$ is also a solution of (1.3). Hence we are interested in the uniqueness and non-uniqueness of certain integral equations; in fact, we have

\begin{equation}
(1.4) \quad \text{Proposition.} \quad P_x[e_\Delta = +\infty] = 1 \text{ for every } x \in S \text{ iff } u(t, x) \equiv 1 \text{ is the unique solution of (1.3) (unique within the class of all solutions } v \text{ such that } 0 \leq v \leq 1).\end{equation}

\begin{equation}
(1.5) \quad \text{Corollary.} \quad \text{Let } X \text{ possess a regular fundamental system } [X, k, \pi] \text{ such that } ||k|| < \infty \text{ and suppose that } \sup \sum_{s \in S} n\pi(x; S^s) < \infty. \text{ Then } P_x(e_\Delta = +\infty) = 1 \text{ for every } x \in S.\end{equation}

The proof of the corollary follows from the fact that $F$ is Lipschitz continuous in this case.

We should also remark that in many cases, the $S$-equation has a differential analogue. For example, if $X$ possesses a sufficiently “nice” regular fundamental system $[X, k, \pi]$, then the differential equation analogue of (1.3) is the non-linear evolution equation

\[
\frac{d}{dt}u_t = Au_t + k[F(\cdot ; u_t) - u_t] \\
u(0+, x) = 1 \\
u(t, x)|_{x \to 0} = 1,
\]

where $A$ is the infinitesimal generator of the process $X$. H. Fujita and S. Watanabe [2] considered such problems of uniqueness and non-uniqueness.

2. A probabilistic approach

\emph{In this section we shall always assume that $S$ is compact.} So let $X$ be a bmp on $S$. Recall the functions $e_t$ and $e$ defined in §1: $e_t(x) = T_t^1(x) = P_x(e_{\Delta > t}) \downarrow e(x) = P_x(e_{\Delta} = \infty)$. Thus, we can say that explosion happens starting from $x$ iff $e(x) < 1$. Our first aim will be to show that under suitable conditions $e \equiv 1$ or $e \equiv 0$ on $S \setminus \partial$. Moreover, the former is true iff $E[e_\Delta]$ is everywhere infinite there.

As a first step we observe

\begin{equation}
(2.1) \quad \text{Lemma.} \end{equation}

\begin{enumerate}
  \item $\widehat{e}|_S = e$
  \item $T_t e = e$ for all $t \geq 0$.
\end{enumerate}

\(^{13}\) When $X$ possesses the fundamental system $(T^t, K, \pi)$, this amounts to assuming that $\pi(x; \Delta) = 0$, all $x \in S$. 

Proof. Since \( e_t|_S = e_t \) all \( t \geq 0 \), the first assertion is clear.

Also

\[
T_t e(x) = \lim_{t \to \infty} T_t T_s e(x) = \lim_{t \to \infty} T_{t+s} e(x) = e(x).
\]

We now impose the following set of assumptions \([A]\).

\[
(A_1) \quad P_x[X_t = \partial; \tau < \infty] = 0 \quad \text{for all} \quad x \in S.
\]

\[
(A_2) \quad e_t \text{ and } e \text{ are upper semi-continuous.}
\]

\[
(A_3) \quad \text{For every } t > 0, \text{ all } x \in S, \text{ and every non-empty open } U \subset S, \text{ there exists a } V \in \mathcal{B}(S) \text{ such that } P_x[X_t \in V] > 0 \text{ and for every } y \in V, \text{ say } y = [y_1, \ldots, y_m], \text{ some } y_i \in U.
\]

\((A_1)\) is the assumption of no death; \((A_2)\) is a regularity condition on \( X \); \((A_3)\) is some type of communication assumption. Roughly, \((A_3)\) states that for every \( t > 0 \) and open \( U \subset S \), at least one particle is in \( U \) at time \( t \) with positive probability.

\[
(2.3) \quad \textbf{Theorem. } \quad P_x[e_u = \infty] \equiv 1 \quad \text{or } \equiv 0 \quad \text{on } S.
\]

Proof. Note that \((A_1)\) implies \( P(t, x, [\beta]) = 0 \) for all \( t \geq 0, \ x \in S \), where \( P \) is the transition function for \( X \). Let \( \beta = \sup_{x \in S} e(x) \). Then \( 0 \leq \beta \leq 1 \). From \((A_2)\) and the assumption of compactness it follows that there exists some \( x_0 \in S \) with \( e(x_0) = \beta \). If \( \beta = 0 \) we are through. So suppose not. Then we claim that \( \beta = 1 \). For otherwise \( 0 < \beta < 1 \). By Lemma 2.1 and \((A_1)\) we can write for any \( t \geq 0 \)

\[
(2.4) \quad \beta = e(x_0) = E_{x_0}[e|_S(X_t)] = \int_S e|_S(y)P(t, x_0, dy)
= \sum_{n=1}^{\infty} e|_S(y)P(t, x_0, dy) \leq \sum_{n=1}^{\infty} \beta^n P(t, x_0, S^n).
\]

Now if \( P(t, x_0, S) = 1 \) for all \( t \geq 0 \), it would imply by right-continuity that \( P_{x_0}[X_t \in S, \ \text{all } t \geq 0] = 1 \), contradicting the assumption that \( \beta < 1 \). Thus, there exists some \( t_0 \) such that \( P(t_0, x_0, S) < 1 \). For this \( t_0 \) it would follow from (2.4) that \( \beta < \beta \).

We will now show that \( e|_S \equiv 1 \) if \( \beta = 1 \). Suppose not. Then there exists an \( \varepsilon > 0 \) and open \( U \subset S \) such that \( e|_U \leq 1 - \varepsilon \). Fix any \( t > 0 \). Let \( V \) be a set corresponding to \( U \) in \((A_3)\). Then

\[
1 = e(x_0) \leq \left( \int_V + \int_{S \setminus V} \right) e|_S(y) P(t, x_0, dy)
\leq (1 - \varepsilon) P(t, x_0, V) + P(t, x_0, S \setminus V) < 1.
\]

Contradiction.
Theorem 2.3 states that $e \equiv 1$ or $\equiv 0$ on $S$. Clearly if $e \mid_S \equiv 1$ then $E_x[e_\Delta] \equiv \infty$ on $S$. An interesting and useful fact, however, is that the converse is also true.

(2.5) **Lemma.** If $e \equiv 0$ on $S$, then for all $t > 0$, $||e_t \mid_S|| < 1$.

Proof. Suppose there exists some $t_0 > 0$ such that $||e_{t_0} \mid_S|| = 1$. Let $y_0 \in S$ be such that $e_{t_0}(y_0) = 1$ and choose $h > 0$ such that $t = t_0 - h > 0$. Then

$$1 = e_{t_0}(y_0) = T_t T_{t_1} \hat{1}(y_0) = T_{t_1} e_{t_1} \mid_S(y_0).$$

By the same reasoning as in Theorem 2.3, we conclude that $e_{t_1} \mid_S \equiv 1$. Hence for every $n$,

$$e_{nt_1}(y_0) = T_{nt_1} \hat{1}(y_0) = T_{(n-1)t_1} T_{t_1} \hat{1}(y_0) = T_{(n-2)t_1} e_{t_1} \mid_S(y_0) = \cdots = T_{t_1} \hat{1}(y_0) = 1,$$

and so $e(y_0) = \lim_{n \to \infty} e_{nt_1}(y_0) = 1$. Contradiction.

(2.6) **Theorem.** $P_x[e_\Delta = +\infty] = 1$ iff $E_x[e_\Delta] = \infty$.

Proof. We need only prove sufficiency as necessity is clear. Applying Dynkin's formula to $g = R \hat{1} = \int_0^\infty e^{-t} T_t \hat{1} dt$,

$$E_x[g(X_{e_\Delta \wedge \omega})] = g(x) = E_x[\int_{e_\Delta \wedge \omega} (g-\hat{1})(X_t) dt]$$

for every $M > 0$.

So suppose $P_x(e_\Delta = \infty) = 0$. Applying Lemma 2.5 we conclude that there exists some $\alpha > 0$ such that $0 \leq g(y) \leq 1 - \alpha$ for all $y \not= \partial$. But from the right-continuity of the process and the assumption of no dying we have $P_x[\forall t \geq 0, X_t \not= \partial] = 1$. Consequently,

$$\alpha E_x[e_\Delta \wedge M] \leq 2 ||g||_S \leq 2.$$

Letting $M \uparrow \infty$, $E_x[e_\Delta] \leq \frac{2}{\alpha}$ (independent of $x$).

Combining Theorems 2.3 and 2.6 we have

(2.7) **Theorem.** Let $X$ be a bmp on a compact space $S$ satisfying $[A]$. Then $P_x(e_\Delta = +\infty) \equiv 1$ or $\equiv 0$ accordingly as $E_x[e_\Delta] \equiv \infty$ or uniformly bounded on $S$.

(2.8) **Corollary.** Let $X$ possess a regular fundamental system $[X, k, \pi]$ with no absorbing set (i.e., $B = \phi$) and such that

(i) $\pi(x; \{\partial\}) = 0$ all $x \in S$,

(ii) $||k|| < \infty$,
(iii) $T^x_t$ strongly Feller* and

(iv) for every $t > 0$, $x \in S$, and non-empty open $U \subset S$,

$$P^x(t, x, U) \equiv T^x_t I_U(x) > 0.$$  

Then the conclusions of Theorem 2.7 are valid.

Proof. $(A_1)$ follows from (i). Since $e_t(x)$ is a solution of the $S$-equation

$$u(t, x) = T^x_t 1(x) + \int_0^t T^x_{t-s}[k(\cdot)F(\cdot; u_s)](x) ds,$$

(ii) and (iii) imply that $e_t$ is continuous for all $t$. Thus $e$ is upper semi-continuous. $(A_2)$ follows easily from (iv). Now apply Theorem 2.7.

In the remainder of this section we assume that $X$ possesses a fundamental system $(\Omega^0, K, \pi)$ with no absorbing set such that $\pi(x, \{\emptyset\}) = \pi(x, \{\Delta\}) = 0$ on $S$. Our aim here is to derive a condition for explosion similar to that of E.B. Dynkin. We shall only sketch the details. In section 3 we are able to derive essentially much stronger results.  

Consider

$$(2.9) \quad E_x[e_\Delta] = E_x[\tau_m] = \sum_{n=0}^\infty E_x[\tau_n \theta_{n}; \tau_n < \infty]$$

$$= \sum_{n=0}^\infty E_x[EX_n[\tau; \tau_n < \infty]]$$

$$= \sum_{n=0}^\infty \sum_{\nu_n = n} E_x[EX_n[\tau]; X_{\tau_n} \in S^\nu_n, \ldots, \tau_n < \infty]$$

by the S.M.P. Again by the repeated use of the S.M.P., we can write

$$E_x[EX_n[\tau]; X_{\tau_n} \in S^\nu_n, \ldots, X_{\tau_n} \in S^\nu_n; \tau_n < \infty]$$

$$= E_x[EX_n[\tau]; X_{\tau_n} \in S^\nu_n, \tau < \infty; \ldots, \tau < \infty]$$

$$= \int_{S^\nu_n} \cdots \int_{S^\nu_n} E_x[\tau] P_{\nu_1} (X_{\tau_n} \in dy_n) \cdots P_{\nu_1} (X_{\tau_n} \in dy_n) P_{\nu_1} (X_{\tau_n} \in dy_1).$$

Furthermore, for $z \in \Delta$

$$E_x[\tau] = E_x[\int_0^\tau dt] = \int_0^\tau E_x[1; t < \tau] dt$$

$$= \int_0^\tau T^x_{t}(x) dt = \int_0^\tau (T^x_{t})_{1}(x) dt.$$  

14. That is, $T^x_t: B(S) \rightarrow C(S) = \langle$bounded continuous functions on $S\rangle$, all $t \geq 0.$

15. We need assume there, however, that $K(x; dsdy) = j(x, s; dy)ds$. 


making use of (1.2). Now define the following for \( t \geq 0, 0 \leq \xi \leq 1 \):

\[
\alpha(t) = \inf_{x \in S} T^0_t(x) \quad F_* (\xi) = \inf_{x \in S} \sum_{v=2}^{\infty} q_v(x) \xi^v
\]

\[
\beta(t) = \sup_{x \in S} T^0_t(x) \quad F^* (\xi) = \sup_{x \in S} \sum_{v=0}^{\infty} q_v(x) \xi^v,
\]

where \( q_v(x) = \pi(x; S^v) \).

Observe that if \( q_v(x) \) is independent of \( z \), all \( v \), then \( F_* = F^* \). Continuing, we estimate for \( y \in S^u \)

\[
\sum_{v=\mu+1}^{\infty} \int_{S^v} \mathbb{E}_z [\pi] P_z \left[ X_t = dz \right]
\]

\[
= \int_0^\infty \sum_{v=\mu+1}^{\infty} \int_{S^v} (T^0_t)^v \Psi(y; ds dz) dt
\]

\[
\geq \int_0^\infty \sum_{v=\mu+1}^{\infty} \alpha^v(t) \int_{S^v} \int_0^\infty \hat{1}(z) \Psi(y; ds dz) dt
\]

\[
= \int_0^\infty \sum_{v=\mu+1}^{\infty} \alpha^v(t) \int_0^\infty <T^0_t 1 | \int_{S^v} K(\cdot; ds dz) \sum_{v=\mu+1}^{\infty} \alpha^v(t) \pi(z, S^v) \Psi(y; ds dz) > (y) dt
\]

\[
\geq \int_0^\infty \alpha^v(t) \left( F_* [\alpha(t)] \right) \int_0^\infty <T^0_t 1 | \int_{S^v} K(\cdot; ds dz) 1(z) > (y) dt
\]

using (1.2) and the fact that \( \Psi(x; ds dz) = \int_S K(x; ds dy) \pi(y, dz) \). If we assume that \( \lim_{t \to \infty} T^0_t (x) = 0 \) for all \( x \in S \), then

\[
\int_0^\infty <T^0_t 1 | \int_S K(\cdot; ds dz) 1(z) > (y)
\]

\[
= 1 - \lim_{t \to \infty} (T^0_t | S(y) = 1.
\]

Hence

\[
\sum_{v=\mu+1}^{\infty} \int_{S^v} \mathbb{E}_z [\pi] P_z \left[ X_t = dz \right] \geq \int_0^\infty \alpha^v(t) \left( F_* [\alpha(t)] \right) dt.
\]

Iterating this in (2.9) one obtains the estimate

\[
\mathbb{E}_z [e_\omega] \supseteq \int_0^\infty \alpha(t) \sum_{o=0}^{\omega} \left( F_* [\alpha(t)] \right)^o dt
\]

\[
= \int_0^\infty \alpha^2(t) dt \frac{\alpha(t) - F_* [\alpha(t)]}{\alpha(t)}.
\]

Although the intermediate calculations in the case \( \alpha(t) = 0 \) are not valid, the
end result is provided we interpret the integrand to be zero for such \( t \). A similar calculation yields
\[
E_x[e_\Delta] \leq \int_0^{\infty} \frac{\beta^2(t) dt}{\beta(t) - F^*[\beta(t)]}.
\]
So under the assumptions \([B]\),

\[
(B_i) \quad X \text{ possesses a fundamental system } (T_i, K, \pi) \text{ with no absorbing set},
\]

\[
(B_\pi) \quad \pi(x, \{\Delta\}) = \pi(x, \{\Delta\}) = 0 \quad \text{on } S,
\]

\[
(B_\infty) \quad \lim_{t \to \infty} T_i \pi(x) = 0 \quad \text{on } S,
\]

we have the following

(2.12) **Proposition.**

\[
\int_0^{\infty} \frac{\alpha^2(t) dt}{\alpha(t) - F^*}[\alpha(t)] \leq E_x[e_\Delta] \leq \int_0^{\infty} \frac{\beta^2(t) dt}{\beta(t) - F^*[\beta(t)]}
\]

for every \( x \in S \).

(2.13) **Remark.** If \( \alpha \) is integrable (on \([0, \infty)\)), then \( \frac{\alpha^2(t)}{\alpha(t) - F^*[\alpha(t)]} \) is integrable iff it is locally integrable at 0. Similarly for \( \beta \). In particular, if \((T_i, K, \pi)\) is determined by \([X, k, \pi]\) such that \(0 < k_1 \leq k \leq k_2\) for some constants \( k \), then

(i) \[
\int_{1-\tau}^{t} \frac{d\xi}{\xi - F^*[\xi]} \leq \infty \quad \text{imply } E_x[e_\Delta] < \infty \quad \text{for all } x \in S,
\]

(ii) \[
\int_{1-\tau}^{t} \frac{d\xi}{\xi - F^*[\xi]} = \infty \quad \text{imply } E_x[e_\Delta] = \infty \quad \text{for all } x \in S.
\]

By combining Theorem 2.7 and Proposition 2.12 we obtain

(2.14) **Theorem.** Let \( X \) be a bmp on compact \( S \) satisfying \([A]\) and \([B]\). Then

(i) \[
\int_0^{\infty} \frac{\beta^2(t) dt}{\beta(t) - F^*[\beta(t)]} < \infty \quad \text{implies } P_x(e_{\Delta} = \infty) = 0 \quad \text{on } S.
\]

(ii) \[
\int_0^{\infty} \frac{\alpha^2(t) dt}{\alpha(t) - F^*[\alpha(t)]} = \infty \quad \text{implies } P_x(e_{\Delta} = \infty) = 1 \quad \text{on } S.
\]

We conclude this section with the following theorem. These results were first obtained by N. Ikeda [4] for the single-type branching process.

(2.15) **Theorem.** Let \( X \) be a bmp on compact \( S \). Suppose it possesses a regular fundamental system \([X, k, \pi]\) with no absorbing set such that
(i) \( \tau(\cdot, S^n) = q_n \text{(constant), } n = 0, 1, \ldots, +\infty \) and

\( q_0 = q_1 = q_m = 0 \),

(ii) there exist constants \( k_1, k_2 \) with \( 0 < k_1 \leq k \leq k_2 \),

(iii) \( T^o \) strongly Feller and

(iv) for every non-empty open \( U \subset S \), \( T^o I_U(x) > 0 \), for all \( t > 0, x \in S \).

Then the following statements are equivalent:

\begin{align*}
(1) & \quad P_x[\tau = \infty] = 1 \text{ on } S \\
(2) & \quad E_x[\tau] = \infty \text{ on } S \\
(3) & \quad \int_{1-\tau}^{\infty} \frac{d\xi}{F[\xi]} = \infty
\end{align*}

where \( F[\xi] = \sum_{n=1}^{\infty} q_n \xi^n \).

3. An analytic approach

Recall that in §1 we pointed out that \( e_t(x) = T_t \hat{f}(x) \) is the minimal solution of the \( S \)-equation with initial value \( f=1 \). We shall exploit this fact here.

We shall suppose that \( X \) possesses a fundamental system \((T^o, K, \pi)\) such that

\begin{equation}
K(x; dsdy) = J(x, s; dy)ds.
\end{equation}

In particular this is true if \( X \) possesses a regular fundamental system. Then \( \psi(t, x) = 1 - e_t(x) \) is the maximal solution of

\begin{equation}
u(t, x) = \int_0^t ds \int S J(x, t-s; dy) G[y, u(s, \cdot)] ,
\end{equation}

where \( G[x; f] = 1 - F[x; 1-f] \). The idea now is to compare \( \psi \) with a solution of a related integral equation. A key lemma in this direction is the following:

\begin{equation}
\textbf{Lemma.} \quad \text{Let } g \text{ be a non-negative non-decreasing function on } [0, 1], \text{ and let } \tau \text{ be a non-negative integrable function on } [0, \delta], \text{ some } \delta > 0. \text{ Consider the integral equation}
\end{equation}

\begin{equation}
\psi(t) = \int_0^t \tau(s) g[\psi(s)] ds .
\end{equation}

Then

16. Maximal in the sense that if \( \psi \) is also a solution, \( 0 \leq \psi \leq 1 \), then \( \psi \leq \bar{\psi} \).
(i) If \( \int_0^\delta \frac{d\xi}{g(\xi)} = \infty \), any solution \( v \) of (3.4) defined on \([0, \eta]\) such that 
\[ 0 \leq v \leq 1 \] 
is identically zero on \([0, \delta \wedge \eta]\).

(ii) If \( \int_0^\delta \frac{d\xi}{g(\xi)} < \infty \) and \( \tau \) is (essentially) locally positive at 0, then there 
exists an increasing solution \( v \) of (3.4) on \([0, \eta]\), some \( \eta > 0 \), such 
that \( 0 \leq v \leq 1 \); moreover \( v(t) > 0 \) for \( t > 0 \).

Proof.
(i) Let \( 0 \leq v \leq 1 \) be a solution of (3.4) on \([0, \eta]\). Without loss of generality, 
we may assume \( \eta \leq \delta \). Clearly \( v \) is absolutely continuous and increasing. Set 
\[ \mu = \sup \{ t : 0 < t \leq \eta \text{ and } g[v(t)] = 0 \} \] (sup \( \phi = 0 \)).
If \( \mu = \eta \), then \( g \circ v = 0 \) on \([0, \eta]\). Consequently \( v = 0 \) on \([0, \eta]\) and we are 
done. So suppose \( \mu < \eta \). Then \( g[v(t)] > 0 \) for \( \mu < t \leq \eta \). Now from (3.4) it follows that 
\[ v'(t) = \tau(t)g[v(t)] \quad \text{a.e.} \]
Consequently for every \( \varepsilon > 0 \), 
\[ \int_\mu^\eta \frac{v'(s)}{g[v(s)]} ds = \int_\mu^\eta \tau(s) ds , \]
or, by a change of variables, 
\[ \int_{g[v(\mu)]}^{g[v(\eta)]} \frac{d\xi}{g(\xi)} = \int_\mu^\eta \tau(s) ds . \]
Letting \( \varepsilon \downarrow 0 \) we obtain 
\[ \int_0^{g[v(\eta)]} \frac{d\xi}{g(\xi)} = \int_\mu^\eta \tau(s) ds < \infty . \]
Contradiction.
(ii) Define on \([0, 1] \times [0, \delta]\) the function 
\[ A(v, t) = \int_0^v \frac{d\xi}{g(\xi)} - \int_t^\eta \tau(s) ds . \]
Note that for fixed \( t \), \( A \) is strictly increasing and continuous in \( v \), and that \( A(0, 0) = 0 \). Set

---

17. By \( \int_0^\delta \frac{d\xi}{g(\xi)} = +\infty \), we mean that \( \int_0^\varepsilon \frac{d\xi}{g(\xi)} = +\infty \) for every sufficiently small \( \varepsilon > 0 \); 
i.e., \( \left( \frac{1}{g} \right) \) is not locally integrable at 0.

18. That is, for every sufficiently small \( r > 0 \), \( \int_0^r \tau > 0 \).
Clearly \( \eta > 0 \). So, for every \( t \) with \( 0 \leq t \leq \eta \), there exists a unique \( \nu \) such that \( 0 \leq \nu \leq 1 \) and \( A(\nu, t) = 0 \). Denote it by \( \nu = \nu(t) \). It is not hard to show that \( \nu \) has all the required properties.

In order to apply this lemma it is convenient to make the following set up. Let \( \mathcal{S} = B_1([0, \infty) \times S) \) and define the operator \( \Phi \) by

\[
(\Phi \nu)(t, x) = \int_0^t ds \int_S f(x, t - s; dy) G[y, v(s, \cdot)]
\]

It is clear that \( \Phi \) has the properties

\[
\begin{align*}
(i) & \quad \Phi \mathcal{S} \subset \mathcal{S} \\
(ii) & \quad \Phi \nu \leq \Phi \nu \quad \text{if} \quad \nu \leq \nu
\end{align*}
\]

(3.6) DEFINITION. \( \nu \) is a solution of \( \Phi \nu = \nu \) if \( \nu \in \mathcal{S} \) and \( \Phi \nu = \nu \); \( \nu \) is a maximal solution if \( \nu \) is a solution and if \( \nu \) is also a solution, then \( \nu \geq \nu \).

We already know, of course, the maximal solution \( \nu \) in terms of the semigroup \( T_t \) induced from the bmp \( X \). This appears to be difficult to work with directly, however. It is more convenient to use the subterfuge of an approximating sequence.

(3.8) Proposition. There exists a sequence \( \nu_n \), \( 1 \leq n \leq \infty \), with \( 0 \leq \nu_n \leq 1 \) such that \( \nu_0 = 1 \), \( \nu_n = \nu \), and \( \nu_n \downarrow \nu \).

Proof. Set \( \nu_0 = 1 \) and define inductively for \( n \geq 1 \), \( \nu_n = \Phi \nu_{n-1} \). Since \( \nu_0 \in \mathcal{S} \) it follows from (3.6) that \( \nu_n \in \mathcal{S} \) and \( \nu_n \downarrow \). Set \( \nu_n = \lim \nu_n \), which clearly exists. By the dominated convergence theorem, \( \nu_n = \Phi \nu_n \).

Now suppose \( \nu \) is any other solution. But \( \nu \leq \nu_n \). So suppose \( \nu \leq \nu_n \). Then

\[
u = \Phi \nu \leq \Phi \nu_n = \nu_{n+1}.
\]

Hence \( \nu \leq \nu_n \). By the uniqueness of the maximal solution, we have then that \( \nu_n = \nu \).

(3.9) DEFINITION. The sequence \( \nu_0 = 1 \), \( \nu_n = \Phi \nu_{n-1} \) for \( n \geq 1 \) is called the defining sequence for the maximal solution \( \nu \).

We are now ready to reap the main results of this section.

(3.10) Theorem. Let \( \delta > 0 \) be fixed and set

\[
\tau^*(s) = \sup_{x \in \delta} \{ \sup_{y \in \delta} J(x, t-s; S) \} , \quad 0 \leq s \leq \delta
\]
Suppose \( \int_0^t \tau^* < \infty \). Define \( G^*(\xi) = \sup_{x \in S} G[x; \xi] \), for \( 0 \leq \xi \leq 1 \). Then if \( \int_0^t \frac{d\xi}{G^*(\xi)} = \infty \), \( \vartheta \equiv 0 \) (i.e., no explosion).

Proof. Let \( G^*_t \) be the right-continuous version of \( G^* \); i.e., \( G^*_t(\xi) = \lim_{\eta \to t} G^*(\eta) \), \( 0 \leq \xi \leq 1 \), and \( G^*_t(1) = G^*(1) \). Then \( G^*_t \) is monotone increasing, \( G^*_t \geq G^* \) and \( G^*_t = G^* \) a.e. Let \( \langle v_n \rangle \) be the defining sequence for \( v \). Take \( u_0 \equiv 1 \) and define \( u_n \) iteratively by

\[
u_n(t) = \int_0^t \tau^*(s) G^*_t[u_{n-1}(s)] ds, \quad n \geq 1.
\]

Set \( \eta = \sup \{ t: G^*_t(1) \int_0^t \tau^*(s) ds \leq 1 \} \). Then \( \eta > 0 \). Also, since \( 0 \leq u_{n+1} \leq u_n \leq 1 \), we have \( u_n \downarrow u_\infty \) exists on \([0, \eta]\) with \( 0 \leq u_\infty \leq 1 \).

Now \( v_0 = 1 = u_0 \); so suppose \( v_n(t, x) \leq u_n(t)(x) \) on \([0, \eta] \times S\). Then for \((t, x) \in [0, \eta] \times S\)

\[
v_{n+1}(t, x) = \int_0^t ds \int_S J(x, t-s; dy) G[y; v_n(s, \cdot)]
\]

\[
\leq \int_0^t ds \int_S J(x, t-s; dy) G[y; u_n(s) 1(\cdot)]
\]

\[
\leq \int_0^t G^*[u_n(s)] J(x, t-s; S) ds
\]

\[
\leq \int_0^t \tau^*(s) G^*[u_n(s)] ds = u_{n+1}(t).
\]

Consequently, \( v \leq u_\infty \). But \( u_\infty \) satisfies

\[
u_\infty(t) = \int_0^t \tau^*(s) G^*_t[u_\infty(s)] ds.
\]

From Lemma 3.3 we conclude that \( u_\infty \equiv 0 \) on \([0, \eta]\); hence \( \vartheta \equiv 0 \) on \([0, \eta] \times S\).

Now set

\[
\sigma = \sup \{ t: \vartheta(s, x) = 0 \quad \text{on} \quad [0, t] \times S \}.
\]

If \( \sigma = \infty \), we are done; so suppose not. Then \( \sigma \geq \eta > 0 \). Now set \( u(t, x) = \vartheta(t+\sigma, x) \). Then \( u \) satisfies the equation

\[
u(t, x) = \vartheta(t+\sigma, x) = \int_0^{t+\sigma} ds \int_S J(x, t+\sigma-s; dy) G[y; \vartheta(s, \cdot)]
\]

\[
= \int_\sigma^{t+\sigma} ds \int_S J(x, t+\sigma-s; dy) G[y; \vartheta(s, \cdot)]
\]

since from the condition \( \int_0^t \frac{d\xi}{G^*(\xi)} = \infty \), we must have \( G(0) = 0 \).
Then

\[ u(t, x) = \int_0^t ds \int_S J(x, t-s; dy) G[y; u(s, \cdot)] , \]

and so \( u \) is a solution of (3.2). Consequently, \( \varphi \geq u \). But \( \varphi = 0 \) on \([0, \sigma] \times S\) which implies that \( \varphi = 0 \) on \([0, 2\sigma] \times S\). Contradiction.

(3.12) **Theorem.** Let \( \Gamma \in \mathcal{B}(S) \) and \( \delta > 0 \). Set

\[ \tau_*(s) = \inf \{ t-s, x \in \Gamma \} , \quad 0 \leq s \leq \delta . \]

Suppose \( \tau_* \) is locally positive at 0 (cf. footnote 18). Define \( G_\tau(x; \xi | \Gamma) \), \( 0 \leq \xi \leq 1 \) and suppose \( \int_0^\delta G_\tau(x; \xi | \Gamma) < \infty \). Then \( \varphi > 0 \) on \((0, \infty) \times \Gamma \) (i.e., explosion happens starting from \( \Gamma \)).

**Proof.** Since \( \tau_* \) is integrable, it follows from Lemma 3.3 that there exists a function \( u \) defined on \([0, \eta] \) for some \( \eta > 0 \), such that \( 0 < u \leq 1 \) on \([0, \eta] \) and satisfies the integral equation

\[ u(t) = \int_0^t \tau_*(s) G_\tau[u(s)] ds , \quad 0 \leq t \leq \eta . \]

Let \( v_n \) be the defining sequence for \( \varphi \). Then \( v_n(t, x) \equiv 1 \geq u(t) I_\Gamma(x) \) on \([0, \eta] \times S\). Suppose \( v_n \geq u I_\Gamma \) on \([0, \eta] \times S\). Then for \((t, x) \in [0, \eta] \times \Gamma \), we have

\[ v_{n+1}(t, x) = \int_0^t ds \int_S J(x, t-s; dy) G[y; v_n(s, \cdot)] \]
\[ \geq \int_0^t ds \int_S J(x, t-s; dy) G[y; u(s) I_\Gamma] \]
\[ \geq \int_0^t ds \int_S J(x, t-s; dy) G[y; u(s)] \]
\[ \geq \int_0^t \tau_*(s) G_\tau[u(s)] = u(t) . \]

Consequently \( \varphi \geq u I_\Gamma \) on \([0, \eta] \times S\). But \( \varphi(t, x) \) is an increasing function of \( t \) and so \( \varphi > 0 \) on \((0, \infty) \times \Gamma \).

(3.14) **Corollary.** Let \( \Gamma \) be as in Theorem 3.12. If there exists a \( \Lambda \in \mathcal{B}(S) \) such that for every \( x \in \Lambda \) and for every sufficiently small \( r > 0 \), \( \int_0^r J(x, r-s; \Gamma) ds > 0 \), then under the assumptions of the above theorem, \( \varphi > 0 \) on \((0, \infty) \times \Lambda \).

(3.15) **Remark.** Let \((T^\circ, K, \pi)\) be determined by \([X, k, \pi] \). Then \( J(x, s; dy) = P^\circ(s, x, dy) k(y) \), where \( P^\circ \) is the transition function corresponding to \( T^\circ \). Thus
(i) if \( ||k|| < \infty \), then \( \tau^* \) is integrable on \([0, \delta] \).

(ii) if \( k|\Gamma \geq k_i > 0 \) for some \( \Gamma \in \mathcal{B}(S) \), then \( \tau^* \) is locally positive at 0 if

\[
\inf_{s \in \Gamma} P^x(s, x, \Gamma) > 0.
\]

(iii) if \( ||k|| < \infty \) and \( k|\Gamma \geq k_i > 0 \) for some \( \Gamma \in \mathcal{B}(S) \), then \( \tau^* \) is locally positive at 0 if

\[
\inf_{s \in \Gamma} P^x(X_s \in \Gamma; s > \eta) > 0,
\]

where \( \eta \) is the first hitting time of \( B \).

4. Applications

**Example 1.** (multi-type bmp).

Let \( S = \{a_1, \ldots, a_N\} \). Then a bmp \( X \) on \( S \) is called an \( N \)-type bmp. In particular, let \( X \) be a \((\pi_{ij}, b_i)\)-Markov chain on \( S \), where \( 0 < b_i < \infty \) and \( 0 \leq \pi_{ij} \leq 1 \), \( \pi_{ii} = 0 \), \( \sum_{j=1}^{N} \pi_{ij} = 1 \), \( i, j = 1, \ldots, N \); i.e., \( X \) is the Markov chain on \( S \) such that

\[
b_i = (E_x^{a_i}[\sigma])^{-1} \quad \text{and} \quad \pi_{ij} = P^{x}_{a_i}[X_\sigma = a_j],
\]

where \( \sigma \) is the first jump time. Let \( k \) be defined on \( S \) such that \( k(a_i) = k_i > 0 \) and \( q_n(n \geq 2) \) non-negative constants such that \( \sum_{n=2}^{\infty} q_n = 1 \). Define the stochastic kernel \( \pi \) on \( S \times \mathcal{B}(\hat{S}) \) by

\[
\pi(x, dy \cap S^n) = q_n \delta_{\{\ldots, x, \ldots\}}(dy),
\]

where we set \( q_0 = q_1 = q_\infty = 0 \). Then there exists a bmp \( X \) on \( S \) with \([X, k, \pi] \) as its regular fundamental system. Theorem 2.15 says that explosion happens with probability one independent of the starting point iff

\[
\int_0^1 \frac{d\xi}{\xi - F[\xi]} < \infty,
\]

\[
F[\xi] = \sum_{n=2}^\infty q_n \tau^n.
\]

**Example 2.** (Branching diffusion with reflecting boundary)

Let \( D \) be a bounded domain in \( E = \mathbb{R}^l \) and set \( S = \overline{D} \). We assume that \( D \) has a sufficiently smooth boundary, say \( C^{(2)} \). Consider the operator

\[
Af(x) = \sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{l} b(x) \frac{\partial f}{\partial x_i}
\]
where the $a_{ij}, b_i$ are bounded and satisfy a Hölder condition on $S$. We also assume that $A$ is uniformly elliptic. Then it is known (cf., Itô [6]) that there exists a conservative diffusion process $X$ on $S$ such that for $f$ sufficiently smooth, $u(t, x) = E_x^t[f(X_t)]$ satisfies

$$\frac{\partial u}{\partial t} = Au$$

(4.3)

$$\frac{\partial u}{\partial n} = 0.$$

Furthermore, $X$ is strongly Feller and if $p(t, x, y)$ is the fundamental solution of (4.3), it is strictly positive for $t > 0$ and $x, y \in S$.

Let $k$ be a non-negative measurable function on $S$ such that there exists constants $k_1, k_2$ with $0 < k_1 \leq k \leq k_2$ and let $\pi$ be as in (4.1). The bmp $X$ on $S$ which has $[X, k, \pi]$ as its regular fundamental system will be called a branching diffusion process with reflecting boundary. We again conclude from Theorem 2.15 that explosion happens with probability one (independent of the starting position) iff

$$\int_0^t \frac{d\xi}{\xi - F[\xi]} < \infty.$$
The absorbed process $\hat{X} = (\hat{X}_t, \hat{P}_x)$ on $S \cup \{\delta\}$ with $\delta$ as trap is given by

$$
\begin{align*}
\hat{X}_t &= \begin{cases} X_t, & t < \eta, \\ \delta, & \text{otherwise,} \end{cases} \\
\hat{P}_x &= P_x,
\end{align*}
$$

where $\eta$ is the first hitting time of $B$. Given a bounded, non-negative, $\mathcal{B}(S)$-measurable function $k$ and a stochastic kernel $\pi$ on $S \times \mathcal{B}(S)$ such that $\pi(x, S) \equiv 0$, we let $X$ be the bnp on $S$ possessing the regular fundamental system $[X, k, \pi]$ and absorbing set $B$. Since this process has the property that whenever a particle hits the boundary of $S$ it is absorbed into $\{\delta\}$ we call $X$ a branching diffusion process with absorbing boundary. Note that $X^\circ$ is the $e^{-\int_0^t k(X_s) \, ds}$ subprocess of $\hat{X}$, where we extend $k$ as a function on $S \cup \{\delta\}$ by setting $k=0$ on $\delta$.

In order to apply the results of §3 for the exploding case we must show that the conditions of Theorem 3.12 are satisfied. According to Remark 3.15, assuming $k|\Gamma \geq k_0 > 0$, it suffices to show that

$$
(4.4) \inf_{x \in \Gamma} \mathbb{P}_x^t (X_s \in \Gamma; s < \eta) > 0
$$

for some $\delta > 0$. Since $\Gamma \in \mathcal{B}(S)$, $P_x^t (X_s \in \Gamma; s < \eta) = \hat{P}_x (\hat{X}_s \in \Gamma)$. Let $p$ and $\dot{p}$ be the transition density for $X$ and $\hat{X}$ respectively. Then we have the relation

$$
\dot{p}(t, x, y) = p(t, x, y) - \int_0^t \int_B p(t-s, z, y) \mu_x (ds \, dz)
$$

for all $t > 0$, $x$ and $y \in S$. Here $\mu_x (ds \, dz) = \mathbb{P}_x^t (\eta \leq ds, X_\eta \in dz)$. Integrating over $\Gamma$, we obtain

$$
\dot{P}(t, x, \Gamma) = P(t, x, \Gamma) - \int_0^t \int_B P(t-s, z, \Gamma) \mu_x (ds \, dz)
$$

$$
\geq P(t, x, \Gamma) - \mathbb{P}_x^t (\eta \leq t).
$$

But we have the lower estimate for $p$

$$
p(t, x, y) \geq M_1 t^{-t/2} \exp \left[ -\alpha_1 \frac{|x-y|^2}{t} \right] - M_2 t^{-(t/2) + \lambda} \exp \left[ -\alpha_2 \frac{|x-y|^2}{t} \right]
$$

where $M_1$, $M_2$, $\alpha_1$, $\alpha_2$, and $\lambda$ are positive constants (cf. Dynkin [1: Theorem 0.5]). Furthermore from a result of Varadhan [8] we obtain the estimate: for every compact subset $K \subset S$, there exists a $\rho > 0$ such that for all $x \in K$.

---

19. $\delta$ is an isolated point.
provided \( t \) is sufficiently small. Consequently, if \( \Gamma \) is such that \( \Gamma \subset S \), (4.4) will be valid if

\[
\inf_{x \in \Gamma} \int_{\Gamma} r^{-1/2} \exp \left[ -\frac{|x-y|^2}{t} \right] dy > 0
\]

for \( \delta \) sufficiently small. But (4.5) is true iff there exists some positive constant \( \kappa \) such that for every ball \( B \) of sufficiently small radius and every \( x \in \Gamma \), we have

\[
m(\Gamma \cap B_x) \geq \kappa m(B),
\]

where \( B_x \) is the ball \( B \) centered at \( x \) and \( m \) is \( l \)-dimensional Lebesgue measure.\(^{20}\) In particular, (4.6) is true if \( \Gamma \) is itself a ball. We shall only outline the proof of the if statement.

So suppose (4.6) is valid. For \( r \in \mathbb{R}^l \), \( x \in \mathbb{R}^l \), and \( A \subset \mathbb{R}^l \) set

\[
rA = \{ ry : y \in A \} \\
A_x = \{ y + x : y \in A \}.
\]

Also, let \( B \) be the unit ball centered at the origin. Consider the following.

\[
\int_{\Gamma} t^{-1/2} \exp \left[ -\frac{|x-y|^2}{t} \right] dy = \int_{\Gamma} \frac{1}{\sqrt{t}} e^{-|z|^2} dz \\
\geq \int_{\Gamma} \frac{1}{\sqrt{t}} e^{-|z|^2} dz \geq e^{-1} m \left( \frac{1}{\sqrt{t}} \Gamma_{-x} \cap B \right) \\
= e^{-1} t^{-1/2} m(\Gamma \cap (\sqrt{t} B)_x) \\
\geq e^{-1} t^{-1/2} \kappa m(\sqrt{t} B) = \kappa e^{-1} m(B)
\]

for \( x \in \Gamma \) and sufficiently small \( t \). Hence

\[
\inf_{x \in \Gamma} \int_{\Gamma} t^{-1/2} \exp \left[ -\frac{|x-y|^2}{t} \right] dy \geq \kappa e^{-1} m(B) > 0
\]

(provided \( \delta \) is sufficiently small).

Putting all this together, we obtain

\[\textbf{(4.7) Theorem. Let } X \text{ be the branching diffusion process with absorbing boundary as described above. Then}\]

\(^{20}\) The symbol \( B \) has been used to designate both a sphere in \( \mathbb{R}^l \) and the absorbing set of a bmp. This should introduce no confusion, however.
implies no explosion, where \( G^*(ξ) = \sup_{x \in S} G[x; ξ] \).

(ii) \( \int_0^1 \frac{dξ}{G^*(ξ)} < \infty \) implies explosion starting from \( Γ \)

provided \( Γ \) is such that it satisfies (4.6), \( Γ \subset S \), and \( k \mid Γ \geq k > 0 \), where \( G^*(ξ) = \inf_{x \in Γ} G[x; ξ] \).

(4.8) Remark.

1. Since \( p(t, x, y) > 0 \) for all \( x, y \in S \) and \( t > 0 \), then if explosion happens from \( Γ \), it happens from any \( x \in S \). (cf. Corollary 3.14).

2. Let \( Y = (Y_t, Q_x) \) be any diffusion on some \( S \subset E \) and let \( Ξ \) be its characteristic operator. Suppose that \( S \) contains a bounded smooth domain \( D \) such that \( Ξ \mid D = A \mid D \), where \( A \) is some operator on \( E \) satisfying the assumptions of (4.2). Since the absorbing diffusion process \( Y \) on \( D \) is the minimal process, we then have

\[
\inf_{x \in Γ} \inf_{0 \leq t \leq δ} Q(t, x, Γ) > 0
\]

for any \( Γ \) with \( Γ \subset D \) and satisfying (4.6), all \( δ \) sufficiently small. Consequently, we can conclude that for such \( Γ \), explosion happens from \( Γ \) for the bmp \( Y \) corresponding to the regular fundamental system \( [Y, k, π] \), if \( k \mid Γ \geq k > 0 \) and

\[
\int_0^1 \frac{dξ}{G^*(ξ)} < \infty, \quad G^*(ξ) = \inf_{x \in Γ} G[x; ξ]
\]

Example 5.

Let \( S = R \) and \( X \) be Brownian motion on \( S \). Let \( k = I_F \), where \( F \) is the following set. Take \( I = [0, 1] \) and \( α \in (0, \frac{1}{2}) \). Let \( E_1^α \) be the middle open interval of length \( α \) removed from \( I \). Inductively we define \( E_1^α, \ldots, E_{k+1}^α \) to be the middle open intervals of length \( α2^{-2k} \) removed from \( I \setminus \bigcup_{\nu=1}^{k} E_\nu \). Set \( F = I \setminus \bigcup_{\nu=1}^{k+1} E_\nu \). Then \( F \) is a perfect nowhere dense set of measure \( (1-2α) \); i.e., it is a "fat" Cantor set. We shall now show that \( F \) satisfies (4.6). At the \( k \)th stage, the distance between two adjacent sets \( E_\nu \), \( 1 \leq \mu \leq 2^\nu \), \( 0 \leq \nu \leq k \) is

\[
d(k) = \frac{2^k - α(2^{k+1} - 1)}{2^{2k+1}}.
\]

Let \( λ \) be given such that \( 0 < λ \leq (1-2α) \), and let \( B \) be the unit ball about the origin. Choose \( k = k(λ) \) to be the first non-negative integer such that \( d(k) \leq λ \). Then \( d(k-1) > λ \). Moreover, if \( x \in F \),
Consequently $F$ satisfies (4.6).

Now, let $\pi$ be a stochastic kernel on $S \times \mathcal{B}(\hat{S})$ defined by

$$
\pi(x, dy) = p_n \delta_{(\xi_n \cdots \xi_1 \cdots \xi)}(dy) \quad \text{if} \quad dy \in \mathcal{B}(S^n), \quad n = 0, 1, \ldots, +\infty,
$$

where $0 \leq p_n \leq 1$, $0 = p_0 = p_1 = p_\infty$, and $\sum p_n = 1$. If $X$ is the bmp on $S$ corresponding to $[X, k, \pi]$, then according to remark 4.8.2 we can say that explosion happens iff

$$
\int_1^\infty \frac{d\xi}{1 - F(\xi)} < \infty, \quad F(\xi) = \sum_{n \geq 2} p_\xi \xi^n.
$$

Note that splits only occur on the set $F$.

**References**


