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Osaka University
ASYMPTOTIC BEHAVIOR AND AREA GROWTH 
OF MINIMAL SURFACES IN $H^n$

QING CHEN and YI CHENG

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Let $H^n$ be the hyperbolic $n$-space of constant curvature $-1$. We identify $H^n$ with the unit ball in the Euclidean $n$-space $\mathbb{R}^n$ via the Poincare model, i.e., $H^n = (B^n(1), \frac{ds^2}{(1-r^2)^2})$, where $ds^2$ is the Euclidean metric and $r$ is the Euclidean distance from the origin. The sphere $\partial B^n(1)$ is called the sphere at infinity and denoted by $S^{n-1}(\infty)$. It represents the asymptotic classes of geodesics in $H^n$. Let $M$ be an immersed complete minimal surface in $H^n$. The intersection of the closure of $M$ in the Euclidean topology with $S^{n-1}(\infty)$ can be seen as the asymptotic boundary of $M$. In [1], M.T. Anderson established the general existence theorem of complete area-minimizing current with prescribed asymptotic boundary, which enclosed the following result as a special case:

Let $C$ be a smooth closed curve (or more generally, closed 1-current) in $S^{n-1}(\infty)$, then there is a complete area-minimizing smooth surface $M$ with asymptotic boundary $C$.

This paper is concerned with the asymptotic behaviour of complete minimal surfaces in $H^n$. This is partially suggested by the "good" behaviour at infinity of complete minimal surfaces in $\mathbb{R}^n$ with finite total Gaussian curvature, namely, the tangent cone at infinity of a such minimal surface is uniquely a collection of 2-plane with multiplicities (see [2],[3],[5]). But the situation is different in $H^n$. One of the differences is the above mentioned result of Anderson; another is that the total Gaussian curvature of complete minimal surface in $H^n$ is infinite (this can be seen by the Gauss equation). We will study a class of minimal surfaces in $H^n$ with minimal area growth (see §1 of the definition). We present some geometric descriptions of such surfaces, and find that the "length" of their asymptotic boundary in $S^{n-1}(\infty)$ are finite (see Theorem 3.2). A corollary of our theorems is (Corollary 3.3)

Let $M$ be a properly immersed complete and oriented minimal surface in $H^n$ with Gaussian curvature $K$. Suppose $M$ has finite topological type and

$$-\int_M (1 + K) < +\infty,$$
Then the asymptotic boundary of $M$ is a rectifiable 1-varifold with finite mass.

This paper is organized as follows. In section 1, we introduce some basic properties of minimal surfaces in $\mathbb{H}^n$. In section 2 we present Theorem 2.1 which establishes a relation between the area growth of a minimal surface in $\mathbb{H}^n$ and some weighted $L^2$-norm of its second fundamental form. The corresponding result of minimal surfaces in $\mathbb{R}^n$ was obtained in a previous paper [2]. In section 3, we discuss the Euclidean area and boundary structure of a minimal surface in $\mathbb{H}^n$. Since the asymptotic boundary of minimal surface in $\mathbb{H}^n$ is not a smooth curve in general. Some concepts from geometric measure theory should be employed in the proof of the theorem. We refer to [8] for a reference of definitions and terminology of geometric measure theory.

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1. Preliminaries

Throughout this paper $\mathbb{H}^n$ denotes the hyperbolic $n$-space of constant curvature $-1$, and $M$ an oriented and properly immersed complete minimal surface in $\mathbb{H}^n$. We first recall some well-known facts.

**Proposition 1.1.** (Monotonicity formula, Theorem 1 of [1]) Let $B(t)$ be the geodesic ball of $\mathbb{H}^n$ centered at a fixed point and radius $t$, and set $M(t) = M \cap B(t)$ and $v(t) = \text{Area}M(t)$. Then the function

$$\frac{v(t)}{\cosh t - 1}$$

is monotone non-decreasing in $t$, particularly,

$$v'(t) \cosh t - v(t) \sinh t \geq v'(t).$$

**Definition 1.2.** $M$ is said to have minimal area growth if

$$\sup \frac{v(t)}{\cosh t - 1} < +\infty.$$ 

The following proposition is a direct consequence of the definitions of the Hessian and the second fundamental form, for the proof see [6].
Proposition 1.3. Let $A$ be the second fundamental form of $M$, $\rho$ the distance function of $H^n$ from a fixed point and $\nabla$ the covariant derivative of $M$, then

$$(\nabla^2 \rho)(e, e) = \coth \rho (1 - (e, \nabla \rho)^2) + \langle A(e, e), \nabla^\perp \rho \rangle,$$

for any unit tangent vector $e$ of $M$, where $\nabla^\perp \rho$ is the normal projection of $\text{Grad}_\rho \rho$ to $M$.

The restriction of distance $\rho$ on $M$ is smooth. By Sard’s theorem, for almost all $t > 0$, $M(t)$ is a compact surface with boundary $\partial M(t)$ being an immersed closed curve. Denote the geodesic curvature of $\partial M(t)$ by $k_g^t$. Then the Gauss-Bonnet formula is

$$(1.1) \quad \int_{\partial M(t)} k_g^t + \int_{M(t)} K = 2\pi \chi(M(t)),$$

where $K$ is the Gaussian curvature of $M$, and $\chi(M(t))$ is the Euler characteristic of $M(t)$. Substituting the Gauss equation $K = -1 - \frac{1}{2} |A|^2$ into (1.1), and putting $R(t) = \int_{M(t)} |A|^2$, then

$$(1.2) \quad v(t) + \frac{1}{2} R(t) + 2\pi \chi(M(t)) = \int_{\partial M(t)} k_g^t.$$

Suppose $e$ is tangent to $\partial M(t)$. Then, $\frac{\nabla \rho}{|\nabla \rho|}$ being normal to $\partial M(t)$ in $M$, the expression of $k_g^t$ is

$$(1.3) \quad k_g^t = -\langle \nabla e, \frac{\nabla \rho}{|\nabla \rho|} \rangle = \frac{1}{|\nabla \rho|} (\nabla^2 \rho)(e, e) = \frac{1}{|\nabla \rho|} (\coth \rho + \langle A(e, e), \nabla^\perp \rho \rangle),$$

where the last equality is followed by Proposition 1.2.

By the co-area formula (see [8]), $v'(t) = \int_{\partial M(t)} |\nabla \rho|$. Substituting (1.3) into (1.2), we obtain

Proposition 1.4. For almost all $t > 0$,

$$v(t) + \frac{1}{2} R(t) + 2\pi \chi(M(t)) = v'(t) \coth t - \int_{\partial M(t)} \langle A\frac{\nabla \rho}{|\nabla \rho|}, \frac{\nabla \rho}{|\nabla \rho|}, \nabla^\perp \rho \rangle.$$
2. Area growth estimate

In this section we prove the following theorem, which is analogous to Theorem 1 of [2].

**Theorem 2.1.** Let $M$ be a properly immersed complete minimal surface in $H^n$. Suppose $M$ is of finite topological type. Then $M$ has minimal area growth if and only if

$$\int_M e^{-\rho(x)} |A|^2(x)dx < +\infty,$$

where $A$ is the second fundamental form of $M$ and $\rho$ is the distance function of $H^n$ to a fixed point.

**Lemma 2.2.** For $t > s > t_0 \geq 0$,

$$\frac{\int_{M(t)} \cosh \rho - \int_{M(s)} \cosh \rho}{\cosh^2 t} = \int_{M(t)} \frac{1 + |\nabla \rho|^2 \sinh^2 \rho}{\cosh^3 \rho} + \int_s^t \frac{\sinh t_0 \sinh u}{\cosh^3 u} \int_{\partial M(t_0)} |\nabla \rho|.$$

Proof. By the minimality of $M$ and Proposition 1.3, we observe that

$$\Delta \rho = (2 - |\nabla \rho|^2) \coth \rho,$$

where $\Delta$ is the Laplacian of $M$. It yields

(2.1) \hspace{1cm} \Delta \cosh \rho = 2 \cosh \rho.

Integrating (2.1) over $M(t) - M(t_0)$ and by using Green’s formula, we have

(2.2) \hspace{1cm} 2 \int_{M(t) - M(t_0)} \cosh \rho = \int_{\partial M(t)} |\nabla \rho| \sinh \rho - \int_{\partial M(t_0)} |\nabla \rho| \sinh \rho.

The co-area formula ([8]) leads

$$\frac{d}{dt} \left( \frac{\int_{M(t)} \cosh \rho}{\cosh^2 t} \right)$$

$$= \frac{1}{\cosh^3 t} \left( \cosh t \int_{\partial M(t)} \frac{\cosh \rho}{|\nabla \rho|} - 2 \sinh t \int_{M(t) - M(t_0)} \cosh \rho \right)$$

$$= \frac{1}{\cosh^3 t} \left( \int_{\partial M(t)} \left( \frac{\cosh^2 \rho}{|\nabla \rho|} - |\nabla \rho| \sinh^2 t \right) + \sinh t_0 \sinh t \int_{\partial M(t_0)} |\nabla \rho| \right)$$

$$= \frac{1}{\cosh^3 t} \left( \int_{\partial M(t)} \frac{1}{|\nabla \rho|} (1 + |\nabla \rho|^2 \sinh^2 t) + \sinh t_0 \sinh t \int_{\partial M(t_0)} |\nabla \rho| \right).$$
The lemma is then proved by integrating (2.3) from \( s \) to \( t \) and the co-area formula.

Proof of Theorem 2.1. By the co-area formula,

\[
\int_{\mathcal{M}(t)} e^{-\rho}|A|^2 = \int_0^t e^{-s} R'(s) ds
\]

(2.4)

\[
= e^{-t} R(t) + \int_0^t e^{-s} R(s) ds.
\]

We rewrite Proposition 1.4 as

\[
v'(t) \cosh t - v(t) \sinh t = \frac{1}{2} R(t) + 2\pi \chi(M(t))
\]

(2.5)

\[
+ \int_{\partial \mathcal{M}(t)} \left( A \left( \frac{\nabla \rho}{|\nabla \rho|^2} \right) \cdot \frac{\nabla \rho}{|\nabla \rho|} \right).
\]

From now on, \( C_i (i = 1, 2, \ldots) \) will be denoted as constants independent of \( t \).

If \( \int_M e^{-\rho(x)} |A|^2(x) dx < +\infty \), by (2.5) we have

\[
\frac{d}{dt} \frac{v(t)}{\cosh t} \leq \frac{\sinh t}{\cosh^2 t} \left( \frac{1}{2} R(t) + 2\pi \chi(M(t)) \right) + \int_{\partial \mathcal{M}(t)} |A| \frac{|\nabla \rho| \sinh \rho}{\cosh^2 \rho}.
\]

(2.6)

Integrating (2.6) from 0 to \( t \) and by the co-area formula,

\[
\frac{v(t)}{\cosh t} \leq 2 \int_0^t \left( \frac{1}{2} R(s) + 2\pi \chi(M(s)) e^{-s} ds + \int_{\mathcal{M}(t)} |A| \frac{|\nabla \rho| \sinh \rho}{\cosh^2 \rho}
\]

(2.7)

Since \( \chi(M(t)) \leq 1 \), by using the Schwarz inequality, (2.7) and the hypothesis, we have

\[
\frac{v(t)}{\cosh t} \leq C_1 + \left( \int_{\mathcal{M}(t)} \frac{|A|^2}{\cosh \rho} \right)^{\frac{1}{2}} \left( \int_{\mathcal{M}(t)} \frac{|\nabla \rho|^2 \sinh^2 \rho}{\cosh^4 \rho} \right)^{\frac{1}{2}}
\]

(2.8)

\[
\leq C_1 + C_2 \left( \frac{\int_{\mathcal{M}(t)} \cosh \rho}{\cosh^2 t} \right)^{\frac{1}{2}} \quad \text{(by Lemma 2.2)}
\]

\[
\leq C_1 + C_2 \left( \frac{v(t)}{\cosh t} \right)^{\frac{1}{2}}.
\]

Thus by the monotonicity of \( \frac{v(t)}{\cosh t} \) we see either \( \sup \frac{v(t)}{\cosh t} \leq C_2^2 \) or, when \( t \) is large enough, \( C_2^2 < \frac{v(t)}{\cosh t} \), so

\[
\frac{v(t)}{\cosh t} \leq \left( \frac{v(t)}{\cosh t} \right)^{\frac{1}{2}} + C_2 \left( \frac{v(t)}{\cosh t} \right)^{\frac{1}{2}}.
\]
It follows
\[ \sup \frac{v(t)}{\cosh t} \leq \max\{C_1^2, (1 + C_2)^2\}, \]
this proves that \( M \) has minimal area growth.

Conversely, when \( \sup \frac{v(t)}{\cosh t} < \infty \), it suffices to show \( \int_0^\infty e^{-t} R(t) dt < +\infty \).

Indeed, this implies that there is a sequence \( \{t_i\} \) tending to infinity such that 
\[ e^{-t_i} R(t_i) \to 0 \text{ as } i \to \infty, \]
taking \( t = t_i \) in (2.4) then letting \( i \) tending to infinity, the theorem follows.

we derive from (2.5) that
\[ (2.9) \quad \frac{1}{2} R(t)e^{-t} \leq -2\pi \chi(M) e^{-t} + \frac{e^{-t} \cosh^2 t}{\sinh t} \frac{d}{dt} \left( \frac{v(t)}{\cosh t} \right) + \int_{\partial M(t)} \frac{e^{-t} |A| |\nabla \rho|}{|\nabla \rho|}, \]
and we integrate above inequality from 0 to \( t \). Then, since the integrals of the first two terms in the right hand side of (2.9) is bounded above, we have
\[ \frac{1}{2} \int_0^t e^{-s} R(s) ds \leq C_3 + \int_{M(t)} e^{-\rho} |A| |\nabla \rho| \]
\[ \leq C_3 + \sqrt{\int_{M(t)} e^{-\rho} |A|^2} \sqrt{\int_{M(t)} e^{-\rho} |\nabla \rho|^2} \]
\[ \leq C_3 + C_4 \sqrt{\int_{M} e^{-\rho} |A|^2}, \]
where the last inequality is followed by Lemma 2.2. For the convenience we set
\[ f(t) = \int_0^t e^{-s} R(s) ds. \]
Combining (2.10) with (2.4) we obtain
\[ (2.11) \quad \frac{1}{2} f(t) \leq C_3 + C_4 \sqrt{f(t) + f'(t)}. \]
We claim that either of the following holds:
(a): \( \int_0^\infty e^{-t} R(t) dt = \sup f(t) \leq 2C_3 \),
(b): there is a sequence \( \{t_i\} \) tending to infinity such that \( f'(t_i) < f(t_i) \),
otherwise, there is a \( t_0 \) sufficient large such that \( f(t) \) \( f(t) > 2C_3 \) when
\( t \geq t_0 \), then by (2.11)
\[ \frac{8C_4^2 f'(t)}{(f(t) - 2C_3)^2} \geq 1, \]
integrating this from \( t_0 \) to \( t \), we get
\[ 8C_4^2 \left( \frac{1}{f(t_0) - 2C_3} - \frac{1}{f(t) - 2C_3} \right) \geq t - t_0, \]
which contradicts with the fact that $t$ is unbounded.

It remains to prove the theorem when (b) holds. Taking $t = t_i$ in (2.11), one has

$$
\frac{1}{2} f(t_i) \leq C_3 + C_4 \sqrt{2f(t_i)}.
$$

If $f(t_i) \to \infty$ as $i \to \infty$, then dividing (2.12) by $f(t_i)$ and letting $i \to \infty$ would lead to an obvious contradiction. Hence $\sup f(t) = \sup f(t_i) < \infty$ and the theorem follows.

3. Boundary behaviour of minimal surfaces

We regard $\mathbb{H}^n$ as the Poincaré model, that is $\mathbb{H}^n = (B^n(1), ds_H^2)$, where $B^n(1)$ is the unit ball of $\mathbb{R}^n$ and $ds_H^2 = \frac{4}{(1-r^2)^2} ds_E^2$ with $r$ being the Euclidean distance function from the origin. Here and after, the subscripts $H$ and $E$ indicate, respectively, the notations with respect to the hyperbolic metric and the Euclidean metric.

**Theorem 3.1.** Suppose $M \to (B^n(1), ds_H)$ is a properly immersed complete minimal surface, then $\text{Area}_E(M) < \infty$.

Proof. For $p \in M$, the orthonormal basis $e_1, e_2$ (or $\tilde{e}_1, \tilde{e}_2$) of $T_pM$ with respect to the Euclidean metric (or the hyperbolic metric) is related by

$$
\tilde{e}_i = \frac{1 - r^2(p)}{2} e_i, \quad i = 1, 2.
$$

Since $r = \tanh \frac{\rho}{2}$, we have

$$
\nabla_E r(p) = \sum_{i=1}^{2} e_i(r) e_i
$$

$$
= \sum_{i=1}^{2} \frac{4}{(1 - r^2)^2} \tilde{e}_i(r) \tilde{e}_i
$$

$$
= (1 + \cosh \rho)^2 \sum_{i=1}^{2} \tilde{e}_i(\tanh \frac{\rho}{2}) \tilde{e}_i
$$

$$
= (1 + \cosh \rho) \nabla_H \rho(p).
$$
It follows $|\nabla_E r|_E = |\nabla_H \rho|_H$. Then the co-area formula yields

$$v'_E(t) = \int_{\partial(M \cap B_H(t))} \frac{1}{|\nabla_H \rho|_H} ds_H$$

$$= \int_{\partial(M \cap B_E(\tanh \frac{t}{2}))} \frac{1}{|\nabla_E r|_E \left(1 - \tanh^2 \frac{t}{2}\right)} ds_E$$

$$= (\cosh t + 1)v'_E(\tanh \frac{t}{2}),$$

hence, by Proposition 1.1 we have

$$+\infty > \int_0^\infty \frac{v'_E(t)}{\cosh t} dt$$

$$\geq \int_0^\infty \frac{v'_E(t)}{\cosh^2 t} dt$$

$$= \int_0^\infty \frac{(1 + \cosh t)^2}{\cosh^2 t} v'_E(\tanh \frac{t}{2}) d(\tanh \frac{t}{2})$$

$$\geq \int_0^\infty v'_E(\tanh \frac{t}{2}) d(\tanh \frac{t}{2})$$

$$= Area_E(M).$$

This completes the proof.

Next we discuss the boundary behaviour of minimal surfaces. Following Anderson [1], the asymptotic boundary $\partial M$ of a complete minimal surface $M$ in $H^n$ is defined by

$$\partial M = \text{closure}(M) \cap S^{n-1}(\infty),$$

where the closure is taken in the Euclidean topology. When $M$ is properly immersed, then $\partial M$ is just the boundary of $M$ in the Euclidean space $R^n$.

**Theorem 3.2.** Let $M$ be an immersed complete minimal surface in $H^n$ with minimal area growth and finite topological type. Then the asymptotic boundary $\partial M$ of $M$ is a rectifiable 1-varifold with finite mass when $\partial M$ is considered as a subset of $S^{n-1}(\infty) \subset R^n$.

Proof. We claim that $M$ is properly immersed, which can be proved in the same way as the proof of Lemma 3 in [2]. Hence the boundary $\partial M(t)$ of $M(t) = M \cap B_H(t)$ is a smooth closed curve, for almost all $t > 0$. Since

$$\text{length}_E(\partial M(t)) = \int_{\partial M(t)} ds_E = \frac{1}{1 + \cosh t} \text{length}_H(\partial M(t)).$$

By the minimal growth of the area, there is a sequence $\{t_i\}$ tending to infinity such that

$$\sup \text{length}_E(\partial M(t_i)) = \sup \frac{\text{length}_H(\partial M(t_i))}{\cosh t_i + 1} < \infty.$$
By the compactness theorem of current (Theorem 27.3 of [8]), there is a sub-
sequence of \( \partial M(t_i) \), denoted again by \( \{ \partial M(t_i) \} \), converges to an integer multipli-
city 1-current as currents in \( \mathbb{R}^n \). If we regard \( \partial M(t_i) \) as a rectifiable 1-varifold in \( B^n(1) \subset \mathbb{R}^n \), then \( \{ \partial M(t_i) \} \) also converges to a rectifiable 1-varifold \( V \) with integer multipli-
city.

Suppose \( V = \varepsilon(\Sigma, \theta) \), where \( \Sigma = \text{support}(V) \) and \( \theta \) is the multiplicity function of \( V \). It is obvious that \( \Sigma \subset \partial M \). In the following we show actually \( V = \varepsilon(\partial M, \theta) \), then the theorem follows.

Denote \( \mathcal{H}^1 \) the 1-dimensional Hausdorff measure of \( \mathbb{R}^n \), and \( \mu_v \) the weight measure of \( V \). By the convergence,

\[
\mathcal{H}^1|_{\partial M(t_i)} \to \mu_v \ (i \to \infty)
\]
as the Radon measures. Suppose \( p \in \partial M - \Sigma \neq \emptyset \), there is a neighbourhood \( O \) of \( p \) in \( \mathbb{R}^n \)such that \( \Sigma \cap O = \emptyset \). We can choose \( O \) to be a ball in \( \mathbb{R}^n \)such that \( \partial O \cap B^n(1) \) is a hyperplane of \( \mathbb{H}^n \). Then

\[
\text{(3.4)} \quad \text{length}_E(O \cap \partial M(t_i)) = \mathcal{H}^1(O \cap \partial M(t_i)) \to \mu_v(B_E(p, \varepsilon)) = 0.
\]

Since \( M \) has finite topological type, \( p \) represents an end \( V \) of \( M \), which is
topologically an annulus. Let \( C_i = V \cap O \cap \partial M(t_i) \). If \( C_i \) is not a closed curve, taking \( p_i \in C_i \) such that \( p_i \to p \), then

\[
\text{length}_E(C_i) \geq \text{dist}_E(p_i, \partial O) \geq \text{dist}_E(p, \partial O) - \text{dist}_E(p_i, p).
\]

This implies by (3.4) that \( C_i \) is a closed curve when \( i \) is sufficiently large. By the
convex hull property of minimal surface in \( \mathbb{H}^n \) (Lemma 5 of [1]), when \( i \geq i_0, \)

\[
V(t_i) := V \cap (M(t_i) - M(t_{i_0})) \subset O.
\]

Applying Lemma 2.2 to \( V(t_i) \),

\[
\int_{V(t_i)} \cosh \rho ds^2_H \geq \cosh^2 t_i \int_{V(t_i)} \frac{1 + |\nabla \rho|^2 \sinh^2 \rho}{\cosh^3 \rho} ds^2_H.
\]

Now (2.2) implies

\[
\mathcal{H}^1(C_i) = \frac{\text{length}_H(C_i)}{1 + \cosh t_i} \geq \frac{1}{\sinh t_i(1 + \cosh t_i)} \int_{V(t_i)} \cosh \rho ds^2_H \geq \frac{\cosh^2 t_i}{\sinh t_i(1 + \cosh t_i)} \int_{V(t_i)} \frac{1 + |\nabla \rho|^2 \sinh^2 \rho}{\cosh^3 \rho} ds^2_H,
\]

which contradicts (3.4). This completes the proof of the theorem.
Corollary 3.3. Let \( M \) be a properly immersed complete and oriented minimal surface in \( \mathbb{H}^n \) with Gaussian curvature \( K \). Suppose \( M \) has finite topological type and
\[
- \int_M (1 + K) < +\infty,
\]
then the asymptotic boundary of \( M \) is a rectifiable 1-varifold with finite mass.

Proof. By the hypothesis and the Gauss equation,
\[
\int_M |A|^2 < \infty.
\]
Therefore the corollary is followed by Theorem 2.2 and Theorem 3.2.

Remark 3.4. 1. Let \( M \) be a properly immersed complete minimal surface in \( \mathbb{H}^n \) with minimal area growth and finite topological type. By Theorem 3.1, \( M \) is a 2-current with finite mass when \( M \) is considered as a current in \( \mathbb{R}^n \). Then Theorem 3.2 implies readily that asymptotic boundary \( \partial M \) coincides with the boundary current of \( M \) in \( \mathbb{R}^n \), which generalizes Proposition 6 of [1].

2. It would be an interesting question whether the following equality holds for the properly immersed complete minimal surfaces in \( \mathbb{H}^n \) with minimal area growth:
\[
\sup \frac{v_H(t)}{\cosh t - 1} = \int_{\partial M} \theta d\mathcal{H}^1 \quad (\text{mass of } \mathcal{V}).
\]

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