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ASYMPTOTIC BEHAVIOR AND AREA GROWTH OF MINIMAL SURFACES IN $\mathbf{H}^n$

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Let $\mathbf{H}^n$ be the hyperbolic n-space of constant curvature -1. We identify $\mathbf{H}^n$ with the unit ball in the Euclidean n-space $\mathbf{R}^n$ via the Poincare model, i.e., $\mathbf{H}^n=(B^n(1), \frac{4ds^2}{(1-r^2)^2})$, where ds^2 is the Euclidean metric and r is the Euclidean distance from the origin. The sphere $\partial B^n(1)$ is called the sphere at infinity and denoted by $S^{n-1}(\infty)$. It represents the asymptotic classes of geodesics in $\mathbf{H}^n$. Let M be an immersed complete minimal surface in $\mathbf{H}^n$. The intersection of the closure of M in the Euclidean topology with $S^{n-1}(\infty)$ can be seen as the asymptotic boundary of M. In [1], M.T.Anderson established the general existence theorem of complete area-minimizing current with prescribed asymptotic boundary, which enclosed the following result as a special case:

Let C be a smooth closed curve (or more generally, closed 1-current) in $S^{n-1}(\infty)$, then there is a complete area-minimizing smooth surface M with asymptotic boundary C.

This paper is concerned with the asymptotic behaviour of complete minimal surfaces in $\mathbf{H}^n$. This is partially suggested by the "*good*" behaviour at infinity of complete minimal surfaces in $\mathbf{R}^n$ with finite total Gaussian curvature, namely, the tangent cone at infinity of a such minimal surface is uniquely a collection of 2-plane with multiplicities (see [2],[3],[5]). But the situation is different in $\mathbf{H}^n$. One of the differences is the above mentioned result of Anderson; another is that the total Gaussian curvature of complete minimal surface in $\mathbf{H}^n$ is infinite (this can be seen by the Gauss equation). We will study a class of minimal surfaces in $\mathbf{H}^n$ with minimal area growth (see §1 of the definition). We present some geometric descriptions of such surfaces, and find that the "*length*" of their asymptotic boundary in $S^{n-1}(\infty)$ are finite(see Theorem 3.2). A corollary of our theorems is (Corollary 3.3)

Let M be a properly immersed complete and oriented minimal surface in $\mathbf{H}^n$ with Gaussian curvature K. Suppose M has finite topological type and

$$-\int_M (1+K) < +\infty,$$

then the asymptotic boundary of M is a rectifiable 1-varifold with finite mass.

This paper is organized as follows. In section1, we introduces some basic properties of minimal surfaces in $\mathbf{H}^n$. In section 2 we present Theorem 2.1 which establishes a relation between the area growth of a minimal surface in $\mathbf{H}^n$and some weighted L^2-norm of its second fundamental form. The corresponding result of minimal surfaces in $\mathbf{R}^n$was obtained in a previous paper [2]. In section 3, we discuss the Euclidean area and boundary structure of a minimal surface in $\mathbf{H}^n$. Since the asymptotic boundary of minimal surface in $\mathbf{H}^n$is not a smooth curve in general. Some concepts from geometric measure theory should be employed in the proof of the theorem. We refer to [8] for a reference of definitions and terminology of geometric measure theory.

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1. Preliminaries

Throughout this paper $\mathbf{H}^n$denotes the hyperbolic n-space of constant curvature -1, and M an oriented and properly immersed complete minimal surface in $\mathbf{H}^n$. We first recall some well-known facts.

Proposition 1.1. (*Monotonicity formula, Theorem 1 of* [1]) *Let $B(t)$ be the geodesic ball of $\mathbf{H}^n$ centered at a fixed point and radius t, and set $M(t) = M \cap B(t)$ and $v(t) = \mathrm{Area} M(t)$. Then the function*

$$\frac{v(t)}{\cosh t - 1}$$

is monotone non-decreasing in t, particularly,

$$v'(t)\cosh t - v(t)\sinh t \geq v'(t).$$

DEFINITION **1.2.** *M is said to have minimal area growth if*

$$\sup \frac{v(t)}{\cosh t - 1} < +\infty.$$

The following proposition is a direct consequence of the definitions of the *Hessian* and the second fundamental form, for the proof see [6].

Proposition 1.3. *Let A be the second fundamental form of M, ρ the distance function of $\mathbf{H}^n$ from a fixed point and ∇ the covariant derivative of M, then*

$$(\nabla^2\rho)(e,e) = \coth\rho(1 - \langle e, \nabla\rho\rangle^2) + \langle A(e,e), \nabla^\perp\rho\rangle,$$

for any unit tangent vector e of M, where $\nabla^\perp\rho$ is the normal projection of $Grad_{\mathbf{H}^n}\rho$ to M.

The restriction of distance ρ on M is smooth. By Sard's theorem, for almost all $t > 0$, $\overline{M(t)}$ is a compact surface with boundary $\partial M(t)$ being an immersed closed curve. Denote the geodesic curvature of $\partial M(t)$ by k_g^t. Then the Gauss-Bonnet formula is

$$\int_{\partial M(t)} k_g^t + \int_{M(t)} K = 2\pi\chi(M(t)), \tag{1.1}$$

where K is the Gaussian curvature of M, and $\chi(M(t))$ is the Euler characteristic of $M(t)$. Substituting the Gauss equation $K = -1 - \frac{1}{2}|A|^2$ into (1.1), and putting $R(t) = \int_{M(t)} |A|^2$, then

$$v(t) + \frac{1}{2}R(t) + 2\pi\chi(M(t)) = \int_{\partial M(t)} k_g^t. \tag{1.2}$$

Suppose e is tangent to $\partial M(t)$. Then, $\frac{\nabla\rho}{|\nabla\rho|}$ being normal to $\partial M(t)$ in M, the expression of k_g^t is

$$\begin{aligned} k_g^t &= -\langle\nabla_e e, \frac{\nabla\rho}{|\nabla\rho|}\rangle \\ &= \frac{1}{|\nabla\rho|}(\nabla^2\rho)(e,e) \\ &= \frac{1}{|\nabla\rho|}\left(\coth\rho + \langle A(e,e), \nabla^\perp\rho\rangle\right), \end{aligned} \tag{1.3}$$

where the last equality is followed by Proposition 1.2.

By the co-area formula (see [8]), $v'(t) = \int_{\partial M(t)} \frac{1}{|\nabla\rho|}$. Substituting (1.3) into (1.2), we obtain

Proposition 1.4. *For almost all $t > 0$,*

$$v(t) + \frac{1}{2}R(t) + 2\pi\chi(M(t)) = v'(t)\coth t - \int_{\partial M(t)} \langle A(\frac{\nabla\rho}{|\nabla\rho|}, \frac{\nabla\rho}{|\nabla\rho|}), \frac{\nabla^\perp\rho}{|\nabla\rho|}\rangle.$$

2. Area growth estimate

In this section we prove the following theorem, which is analogous to Theorem 1 of [2].

Theorem 2.1. *Let M be a properly immersed complete minimal surface in $\mathbf{H}^n$. Suppose M is of finite topological type. Then M has minimal area growth if and only if*

$$\int_M e^{-\rho(x)}|A|^2(x)dx < +\infty,$$

where A is the second fundamental form of M and ρ is the distance function of $\mathbf{H}^n$ to a fixed point.

Lemma 2.2. *For $t > s > t_0 \geq 0$,*

$$\frac{\int_{M(t)-M(t_0)} \cosh \rho}{\cosh^2 t} - \frac{\int_{M(s)-M(t_0)} \cosh \rho}{\cosh^2 s} = \int_{M(t)-M(s)} \frac{1+|\nabla^\perp \rho|^2 \sinh^2 \rho}{\cosh^3 \rho} + \int_s^t \frac{\sinh t_0 \sinh u}{\cosh^3 u} \int_{\partial M(t_0)} |\nabla \rho|.$$

Proof. By the minimality of M and Proposition 1.3, we observe that

$$\Delta \rho = (2 - |\nabla \rho|^2) \coth \rho,$$

where Δ is the Laplacian of M. It yields

$$\Delta \cosh \rho = 2 \cosh \rho. \tag{2.1}$$

Integrating (2.1) over $M(t) - M(t_0)$ and by using Green's formula, we have

$$2 \int_{M(t)-M(t_0)} \cosh \rho = \int_{\partial M(t)} |\nabla \rho| \sinh \rho - \int_{\partial M(t_0)} |\nabla \rho| \sinh \rho. \tag{2.2}$$

The co-area formula ([8]) leads

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\int_{M(t)-M(t_0)} \cosh \rho}{\cosh^2 t}\right) \\ &= \frac{1}{\cosh^3 t}\left(\cosh t \int_{\partial M(t)} \frac{\cosh \rho}{|\nabla \rho|} - 2 \sinh t \int_{M(t)-M(t_0)} \cosh \rho\right) \\ &= \frac{1}{\cosh^3 t}\left(\int_{\partial M(t)} \left(\frac{\cosh^2 t}{|\nabla \rho|} - |\nabla \rho| \sinh^2 t\right) + \sinh t_0 \sinh t \int_{\partial M(t_0)} |\nabla \rho|\right) \\ &= \frac{1}{\cosh^3 t}\left(\int_{\partial M(t)} \frac{1}{|\nabla \rho|}(1 + |\nabla^\perp \rho|^2 \sinh^2 t) + \sinh t_0 \sinh t \int_{\partial M(t_0)} |\nabla \rho|\right). \end{aligned} \tag{2.3}$$

The lemma is then proved by integrating (2.3) from s to t and the co-area formula.

Proof of Theorem 2.1. By the co-area formula,

$$\begin{aligned}\int_{M(t)} e^{-\rho}|A|^2 &= \int_0^t e^{-s}R'(s)ds \\ &= e^{-t}R(t) + \int_0^t e^{-s}R(s)ds.\end{aligned} \tag{2.4}$$

We rewrite Proposition 1.4 as

$$\begin{aligned}\frac{v'(t)\cosh t - v(t)\sinh t}{\sinh t} &= \frac{1}{2}R(t) + 2\pi\chi(M(t)) \\ &\quad + \int_{\partial M(t)} \langle A(\frac{\nabla\rho}{|\nabla\rho|}, \frac{\nabla\rho}{|\nabla\rho|}), \frac{\nabla^\perp\rho}{|\nabla\rho|}\rangle.\end{aligned} \tag{2.5}$$

From now on, $C_i (i = 1, 2, \cdots)$ will be denoted as constants independent of t.

If $\int_M e^{-\rho(x)}|A|^2(x)dx < +\infty$, by (2.5) we have

$$\frac{d}{dt}\frac{v(t)}{\cosh t} \le \frac{\sinh t}{\cosh^2 t}\left(\frac{1}{2}R(t) + 2\pi\chi(M(t))\right) + \int_{\partial M(t)} \frac{|A|}{|\nabla\rho|}\frac{|\nabla^\perp\rho|\sinh t}{\cosh^2 t}. \tag{2.6}$$

Integrating (2.6) from 0 to t and by the co-area formula,

$$\frac{v(t)}{\cosh t} \le 2\int_0^t (\frac{1}{2}R(s) + 2\pi\chi(M(s))e^{-s}ds + \int_{M(t)} |A|\frac{|\nabla^\perp\rho|\sinh\rho}{\cosh^2\rho} \tag{2.7}$$

Since $\chi(M(t)) \le 1$, by using the Schwarz inequality, (2.7) and the hypothesis, we have

$$\begin{aligned}\frac{v(t)}{\cosh t} &\le C_1 + \left(\int_{M(t)} \frac{|A|^2}{\cosh\rho}\right)^{\frac{1}{2}} \left(\int_{M(t)} \frac{|\nabla^\perp\rho|^2\sinh^2\rho}{\cosh^3\rho}\right)^{\frac{1}{2}} \\ &\le C_1 + C_2\left(\frac{\int_{M(t)}\cosh\rho}{\cosh^2 t}\right)^{\frac{1}{2}} \quad \text{(by Lemma 2.2)} \\ &\le C_1 + C_2\left(\frac{v(t)}{\cosh t}\right)^{\frac{1}{2}}.\end{aligned} \tag{2.8}$$

Thus by the monotonicity of $\frac{v(t)}{\cosh t}$ we see either $\sup\frac{v(t)}{\cosh t} \le C_1^2$ or, when t is large enough, $C_1^2 < \frac{v(t)}{\cosh t}$, so

$$\frac{v(t)}{\cosh t} \le \left(\frac{v(t)}{\cosh t}\right)^{\frac{1}{2}} + C_2\left(\frac{v(t)}{\cosh t}\right)^{\frac{1}{2}}.$$

It follows

$$\sup \frac{v(t)}{\cosh t} \leq \max\{C_1^2, (1+C_2)^2\},$$

this proves that M has minimal area growth.

Conversely, when $\sup \frac{v(t)}{\cosh t} < \infty$, it suffices to show $\int_0^\infty e^{-t}R(t)dt < +\infty$. Indeed, this implies that there is a sequence $\{t_i\}$ tending to infinity such that $e^{-t_i}R(t_i) \to 0$ as $i \to \infty$, taking $t = t_i$ in (2.4) then letting i tending to infinity, the theorem follows.

we derive from (2.5) that

$$\text{(2.9)}\quad \frac{1}{2}R(t)e^{-t} \leq -2\pi\chi(M)e^{-t} + \frac{e^{-t}\cosh^2 t}{\sinh t}\frac{d}{dt}\left(\frac{v(t)}{\cosh t}\right) + \int_{\partial M(t)} \frac{e^{-t}|A||\nabla^\perp \rho|}{|\nabla \rho|},$$

and we integrate above inequality from 0 to t. Then, since the integrals of the first two terms in the right hand side of (2.9) is bounded above, we have

$$\text{(2.10)}\quad \begin{aligned} \frac{1}{2}\int_0^t e^{-s}R(s)ds &\leq C_3 + \int_{M(t)} e^{-\rho}|A||\nabla^\perp \rho| \\ &\leq C_3 + \sqrt{\int_{M(t)} e^{-\rho}|A|^2 \int_{M(t)} e^{-\rho}|\nabla^\perp \rho|^2} \\ &\leq C_3 + C_4\sqrt{\int_{Mt} e^{-\rho}|A|^2}, \end{aligned}$$

where the last inequality is followed by Lemma 2.2. For the convenience we set

$$f(t) = \int_0^t e^{-s}R(s)ds.$$

Combining (2.10) with (2.4) we obtain

$$\text{(2.11)}\quad \frac{1}{2}f(t) \leq C_3 + C_4\sqrt{f(t) + f'(t)}.$$

We claim that either of the following holds:
(a): $\int_0^\infty e^{-t}R(t)dt = \sup f(t) \leq 2C_3$,
(b): there is a sequence $\{t_i\}$ tending to infinity such that $f'(t_i) < f(t_i)$,
otherwise, there is a t_0 sufficient large such that $f(t) < f'(t)$ and $f(t) > 2C_3$ when $t \geq t_0$, then by (2.11)

$$\frac{8C_4^2 f'(t)}{(f(t) - 2C_3)^2} \geq 1,$$

integrating this from t_0 to t, we get

$$8C_4^2\left(\frac{1}{f(t_0) - 2C_3} - \frac{1}{f(t) - 2C_3}\right) \geq t - t_0,$$

which contradicts with the fact that t is unbounded.

It remains to prove the theorem when (b) holds. Taking $t = t_i$ in (2.11), one has

$$\frac{1}{2} f(t_i) \leq C_3 + C_4\sqrt{2f(t_i)}. \tag{2.12}$$

If $f(t_i) \to \infty$ as $i \to \infty$, then dividing (2.12) by $f(t_i)$ and letting $i \to \infty$ would lead to an obvious contradiction. Hence $\sup f(t) = \sup f(t_i) < \infty$ and the theorem follows.

3. Boundary behaviour of minimal surfaces

We regard $\mathbf{H}^n$ as the Poincaré model, that is $\mathbf{H}^n = (B^n(1), ds_H^2)$, where $B^n(1)$ is the unit ball of $\mathbf{R}^n$and $ds_H^2 = \frac{4}{(1-r^2)^2} ds_E^2$ with r being the Euclidean distance function from the origin. Here and after, the subscripts H and E indicate, respectively, the notations with respect to the hyperbolic metric and the Euclidean metric.

Theorem 3.1. *Suppose $M \mapsto (B^n(1), ds_H)$ is a properly immersed complete minimal surface. If* $\sup \frac{v_H(t)}{\cosh t} < \infty$ *then* $Area_E(M) < \infty$.

Proof. For $p \in M$, the orthonormal basis e_1, e_2(or $\tilde{e}_1, \tilde{e}_2$) of T_pM with respect to the Euclidean metric (or the hyperbolic metric) is related by

$$\tilde{e}_i = \frac{1 - r^2(p)}{2} e_i, \qquad i = 1, 2.$$

Since $r = \tanh \frac{\rho}{2}$, we have

$$\begin{aligned} \nabla_E r(p) &= \sum_{i=1}^{2} e_i(r) e_i \\ &= \sum_{i=1}^{2} \frac{4}{(1-r^2)^2} \tilde{e}_i(r)\tilde{e}_i \\ &= (1+\cosh\rho)^2 \sum_{i=1}^{2} \tilde{e}_i(\tanh\frac{\rho}{2})\tilde{e}_i \\ &= (1+\cosh\rho)\nabla_H \rho(p). \end{aligned} \tag{3.1}$$

It follows $|\nabla_E r|_E = |\nabla_H \rho|_H$. Then the co-area formula yields

$$\begin{aligned} v'_H(t) &= \int_{\partial(M\cap B_H(t))} \frac{1}{|\nabla_H \rho|_H} ds_H \\ &= \int_{\partial(M\cap B_E(\tanh\frac{t}{2}))} \frac{1}{|\nabla_E r|_E} \frac{2}{1-\tanh^2\frac{t}{2}} ds_E \\ &= (\cosh t + 1) v'_E(\tanh\frac{t}{2}), \end{aligned} \tag{3.2}$$

hence, by Proposition 1.1 we have

$$\begin{aligned} +\infty &> \int_0^\infty \frac{d}{dt}(\frac{v_H(t)}{\cosh t}) \\ &\geq \int_0^\infty \frac{v'_H(t)}{\cosh^2 t} dt \\ &= \int_0^\infty \frac{(1+\cosh t)^2}{\cosh^2 t} v'_E(\tanh\frac{t}{2}) d(\tanh\frac{t}{2}) \\ &\geq \int_0^\infty v'_E(\tanh\frac{t}{2}) d(\tanh\frac{t}{2}) \\ &= \mathrm{Area}_E(M). \end{aligned} \tag{3.3}$$

This completes the proof.

Next we discuss the boundary behaviour of minimal surfaces. Following Anderson [1], the asymptotic boundary ∂M of a complete minimal surface M in $\mathbf{H}^n$ is defined by

$$\partial M = \mathrm{closure}(M) \cap S^{n-1}(\infty),$$

where the closure is taken in the Euclidean topology. When M is properly immersed, then ∂M is just the boundary of M in the Euclidean space R^n.

Theorem 3.2. *Let M be an immersed complete minimal surface in $\mathbf{H}^n$ with minimal area growth and finite topological type. Then the asymptotic boundary ∂M of M is a rectifiable 1-varifold with finite mass when ∂M is considered as a subset of $S^{n-1}(\infty) \subset R^n$.*

Proof.We claim that M is properly immersed, which can be proved in the same way as the proof of Lemma 3 in [2]. Hence the boundary $\partial M(t)$ of $M(t) = M \cap B_H(t)$ is a smooth closed curve, for almost all $t > 0$. Since

$$\mathrm{length}_E(\partial M(t)) = \int_{\partial M(t)} ds_E = \frac{1}{1+\cosh t}\mathrm{length}_H(\partial M(t)).$$

By the minimal growth of the area, there is a sequence $\{t_i\}$ tending to infinity such that

$$\sup \mathrm{length}_E(\partial M(t_i)) = \sup \frac{\mathrm{length}_H(\partial M(t_i))}{\cosh t_i + 1} < \infty.$$

By the compactness theorem of current (Theorem 27.3 of [8]), there is a subsequence of $\{\partial M(t_i)\}$, denoted again by $\{\partial M(t_i)\}$, converges to an integer multiplicity 1-current as currents in $\mathbf{R}^n$. If we regard $\partial M(t_i)$ as a rectifiable 1-varifold in $B^n(1) \subset R^n$, then $\{\partial M(t_i)\}$ also converges to a rectifiable 1-varifold $\mathcal{V}$ with integer multiplicity.

Suppose $\mathcal{V} = \underline{\underline{v}}(\Sigma, \theta)$, where $\Sigma = \text{support}(\mathcal{V})$ and θ is the multiplicity function of $\mathcal{V}$. It is obvious that $\Sigma \subset \partial M$. In the following we show actually $\mathcal{V} = \underline{\underline{v}}(\partial M, \theta)$, then the theorem follows.

Denote $\mathcal{H}^1$ the 1-dimensional Hausdorff measure of $\mathbf{R}^n$, and μ_v the weight measure of $\mathcal{V}$. By the convergence,

$$\mathcal{H}^1 \lfloor_{\partial M(t_i)} \to \mu_v \ (i \to \infty)$$

as the Radon measures. Suppose $p \in \partial M - \Sigma \neq \phi$, there is a neighbourhood O of p in $\mathbf{R}^n$such that $\Sigma \cap O = \phi$. We can choose O to be a ball in $\mathbf{R}^n$such that $\partial O \cap B^n(1)$ is a hyperplane of $\mathbf{H}^n$. Then

$$\text{length}_E(O \cap \partial M(t_i)) = \mathcal{H}^1(O \cap \partial M(t_i)) \to \mu_v(B_E(p, \epsilon)) = 0. \tag{3.4}$$

Since M has finite topological type, p represents an end V of M, which is topologically an annulus. Let $C_i = V \cap O \cap \partial M(t_i)$. If C_i is not a closed curve, taking $p_i \in C_i$ such that $p_i \to p$, then

$$\text{length}_E(C_i) \geq \text{dist}_E(p_i, \partial O) \geq \text{dist}_E(p, \partial O) - \text{dist}_E(p_i, p).$$

This implies by (3.4) that C_i is a closed curve when i is sufficiently large. By the convex hull property of minimal surface in H^n (Lemma 5 of [1]), when $i \geq i_0$,

$$V(t_i) := V \cap (M(t_i) - M(t_{i_0})) \subset O.$$

Applying Lemma 2.2 to $V(t_i)$,

$$\int_{V(t_i)} \cosh \rho ds_H^2 \geq \cosh^2 t_i \int_{V(t_i)} \frac{1 + |\nabla^\perp \rho|^2 \sinh^2 \rho}{\cosh^3 \rho} ds_H^2.$$

Now (2.2) implies

$$\begin{aligned}
\mathcal{H}^1(C_i) &= \frac{\text{length}_H(C_i)}{1 + \cosh t_i} \\
&\geq \frac{1}{\sinh t_i (1 + \cosh t_i)} \int_{V(t_i)} \cosh \rho ds_H^2 \\
&\geq \frac{\cosh^2 t_i}{\sinh t_i (1 + \cosh t_i)} \int_{V(t_i)} \frac{1 + |\nabla^\perp \rho|^2 \sinh^2 \rho}{\cosh^3 \rho} ds_H^2,
\end{aligned}$$

which contradicts (3.4). This completes the proof of the theorem.

Corollary 3.3. *Let M be a properly immersed complete and oriented minimal surface in $\mathbf{H}^n$ with Gaussian curvature K. Suppose M has finite topological type and*

$$-\int_M (1+K) < +\infty,$$

then the asymptotic boundary of M is a rectifiable 1-varifold with finite mass.

Proof. By the hypothesis and the Gauss equation,

$$\int_M |A|^2 < \infty.$$

Therefore the corollary is followed by Theorem 2.2 and Theorem 3.2.

REMARK 3.4. 1. Let M be a properly immersed complete minimal surface in $\mathbf{H}^n$ with minimal area growth and finite topological type. By Theorem 3.1, M is a 2-current with finite mass when M is considered as a current in $\mathbf{R}^n$. Then Theorem 3.2 implies readily that asymptotic boundary ∂M coincides with the boundary current of M in $\mathbf{R}^n$, which generalizes Proposition 6 of [1].

2: It would be an interesting question whether the following equality holds for the properly immersed complete minimal surfaces in $\mathbf{H}^n$ with minimal area growth:

$$\sup \frac{v_H(t)}{\cosh t - 1} = \int_{\partial M} \theta d\mathcal{H}^1 \text{ (mass of } \mathcal{V}).$$

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