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A NECESSARY AND SUFFICIENT CONDITION FOR
A KERNEL TO BE A WEAK Potential KERNEL
OF A RECURRENT MARKOV CHAIN

YOICHI ÖSHIMA

(Received December 14, 1968)

1. Introduction

Let $P$ be an irreducible recurrent transition probability on a denumerable
space $S$ with invariant measure $\alpha$. Let $c$ be an arbitrary (but fixed) state of $S$.
Then from the work of Kondo [3] and Orey [8], there exist the class of weak
potential kernels $A(x, y)$ defined by the property that, for every null charge $f$, $Af$
is bounded and satisfies the equation

$$ (I - P)Af = f. $$

Moreover $Af$ is represented by

$$ Af = cGf + 1(f), $$

where $f$ is a null charge, $1(\cdot)$ is an arbitrary linear functional on the space of
null charges and $cG$ is defined as follow;

$$
(1.3) \quad cP(x, y) = \begin{cases} P(x, y) & x \neq c, y \neq c \\ 0 & \text{otherwise,} \end{cases}
$$

$$
(1.4) \quad cG(x, y) = \begin{cases} \sum_{n=0}^{\infty} cP^n(x, y) & x \neq c, y \neq c \\ 0 & \text{otherwise.} \end{cases}
$$

Moreover $A$ satisfies the following maximum principle [4], [5]:

(RSCM)" If $m$ is a real number and $f$ is a null charge then the relation that

$$ m \geq Af \quad \text{on the set } \{f > 0\} $$

implies that

$$ m - f^- \geq Af \quad \text{everywhere}, $$

1) This is the abbreviation of "reinforced semi-complete maximum principle"; this
maximum principle corresponds to the semi-complete M.P. as well as the reinforced M.P.
(of Meyer) corresponds to the complete M.P.
where \( f^- = (-f) \vee 0 \).

In the present paper we are concerned with the following construction problem. Given a positive measure \( \alpha \) and a (not necessarily positive) kernel \( A \) satisfying (RSCM), does there exist an irreducible recurrent transition probability which has \( \alpha \) as its invariant measure, and \( A \) as its weak potential kernel? This is not true in general, but as Kondo [4] has proved, it is true if \( \alpha \) is a finite measure. In section 2 we shall introduce another necessary condition for the weak potential kernel \( A \) (referred to as condition (*)). Then we shall prove (theorem 3.1) that, if the pair \((A, \alpha)\) satisfies maximum principle (RSCM) and condition (*), \( A \) is a weak potential kernel of a (unique) recurrent Markov chain with \( \alpha \) as its invariant measure.

I should like to express my hearty gratitude to T. Watanabe for his kind advices.

2. Some potential theory for a kernel \( A \) satisfying (RSCM)

Let \( \alpha \) be a strictly positive measure and \( A \), a kernel on \( S \). A function \( f \) on \( S \) is said to be a null charge with respect to \( \alpha \) if \( \sum \alpha(x) |f(x)| < \infty \) and \( \sum \alpha(x)f(x) = 0 \). Let \( N \) be the space of null charges vanishing outside a finite subset of \( S \). We assume that the kernel \( A \) satisfies condition (RSCM) for \( f \in N \). Fix an arbitrary state \( c \) and define

\[
\psi G(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c)) - \frac{\alpha(y)}{\alpha(c)}.
\]

If \( A \) is a weak potential kernel then (2.1) is clearly satisfied by taking \( f \) in equation (1.2) as

\[
f(x) = \begin{cases} \frac{\alpha(y)}{\alpha(c)} & x = c \\ -1 & x = y \\ 0 & \text{otherwise}, \end{cases}
\]

and calculating \( Af(x) - Af(c) \).

From definition (2.1) \( \psi G(c, x) = \psi G(x, c) = 0 \) for every \( x \in S \).

**Lemma 2.1** For arbitrary elements \( x, y \) in \( S \) which are different from \( c \)

\[
I(x, y) \leq \psi G(x, y) \leq \psi G(y, y).
\]

Proof. By taking \( f \) as (2.2) we have

\[
Af(c) = A(c, c) - \frac{\alpha(y)}{\alpha(c)} - A(c, y)
\]

2) A counter example was given by Kondô and T. Watanabe.
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Hence, if we write \( f^+ = f \vee 0 \), \( f^- = (-f) \vee 0 \), by (RSCM)

\[
A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) + f^+(x) \leq A(x) \leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - f^-(x),
\]

so that

\[
(2.3) \quad A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) \leq A(x, c) \frac{\alpha(y)}{\alpha(c)} - A(x, y) \leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - I(x, y),
\]

which proves the lemma.

**Corollary.** For every \( x \in S \) there exists a constant \( C \) such that

\[
(2.4) \quad G(x, y) \leq C \cdot \alpha(y) \quad \text{for every } y \in S.
\]

**Proof.** Exchanging \( c \) and \( x \) in the second inequality in (2.3), it follows that

\[
G(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)} \leq \left( -\frac{A(x, c)}{\alpha(c)} + \frac{A(c, c)}{\alpha(c)} + \frac{A(x, x)}{\alpha(x)} \right) \alpha(y).
\]

Let \( S \) be the set \( S - \{c\} \), and \( M^+ \) be the space of all functions on \( S \) vanishing outside a finite subset of \( S \). Let \( M^+ \) be the space of all non-negative functions in \( M^+ \).

**Theorem 2.1.** The kernel \( ^cG \) satisfies the reinforced maximum principle [7]:

(RM) If \( a \) is a non-negative constant and if \( ^c f \) and \( ^c g \) are two elements of \( M^+ \), then the relation that

\[
a + ^cG f - ^c f \geq ^cG g \quad \text{on the set } \{g > 0\} \quad \text{implies that}
\]

\[
a + ^cG f - ^c f \geq ^cG g \quad \text{everywhere on } S.
\]

**Proof.** Let \( f \) be the function on \( S \) such that \( f \in M \) and \( f|_S = ^c f \). Such \( f \) is obviously unique. The function \( g \in M \) is defined similarly. Then inequality (2.5) implies that

\[
a + A(g - f)(c) \geq A(g - f) \quad \text{on the set } \{g - f > 0\}.
\]

For, since \( ^c f \) and \( ^c g \) are non-negative, the set \( \{g - f > 0\} \) is contained in the union of \( c \) and \( \{g > 0\} \). Hence by (RSCM)
\[ a + A(g - f)(c) - (g - f)^- \geq A(g - f) \quad \text{everywhere.} \]

Since the function \((g - f)^-\) is equal to \(c\) on \(cS \cap \{g = 0\}\), the above inequality, combined with (2.5), proves the theorem.

A non-negative function \(c^h\) on \(cS\) is said to be quasi-excessive\(^3\) if, for every \(c^g \in cM\), the inequality
\[ c^h \geq c^g \quad \text{on the set } \{c^g > 0\} \]
implies that
\[ c^h - c^g \geq c^g \quad \text{everywhere.} \]

Moreover Meyer introduced the notion of the pseudo-réduit \(c^H_E^ch\) for every quasi-excessive function \(c^h\) and every subset \(E\) of \(cS\). This function \(c^H_E^ch\) satisfies the following four conditions.

(2.7) \(c^H_E^ch\) is quasi-excessive.

(2.8) \(c^H_E^ch \leq c^h\) on \(cS\) and \(c^H_E^ch = c^h\) on \(E\).

(2.9) If \(c^h_1\) and \(c^h_2\) are two quasi-excessive functions such that \(c^h_1 \leq c^h_2\) on \(E\), then \(c^H_E^ch_1 \leq c^H_E^ch_2\).

(2.10) If \(c^f \in cM^+\) vanishes outside of \(E\) then \(c^H_E^cG^cf = c^G^cf\).

For example, the function \(c^G^cf\), \(c^f \in cM^+\), and every positive constant are quasi-excessive ([7] see also [5]).

Now we introduce a condition.

Condition (*): There exists a sequence of finite sets \(\{E_n\}_{n=1,2,\ldots}\) increasing to \(cS\) such that \(c \in E_n\) for each \(n\), and a sequence \(\{h_n\}_{n=1,2,\ldots}\) of function on \(cS\) satisfying the following conditions.

(i) \(0 \leq h_n \leq 1\), \(h_n(c) = 0\), \(h_n = 1\) on \(F_n = cS - E_n\), and \(\lim h_n = 0\).

(ii) For every \(c^f \in cN\) and every real number \(m \geq Af(c)\) the relation that
\[ m + h_n \geq Af \quad \text{on the set } \{c^f > 0\}. \]
implies that
\[ m + h_n - f \geq Af \quad \text{everywhere on } cS. \]

In section 3 we shall show that if \(A\) is a weak potential kernel of an irreducible recurrent Markov chain, it satisfies condition (*).

**Theorem 2.2** Condition (*) is equivalent to the condition that, there exists a sequence of finite sets \(\{E_n\}_{n=1,2,\ldots}\) increasing to \(cS\) such that
\[ \lim c^H_{cS - E_n}1 = 0. \]

Proof. Suppose that condition (*) holds and let \(c^h_n\) be the restriction of \(h_n\) to \(cS\) and \(c^S = cS \cap E_n\). Obviously \(cS - cE_n = F_n\), \(0 \leq c^h_n \leq 1\), and \(c^h_n = 1\) on \(F_n\). It then follows that \(c^h_n\) is a quasi-excessive function for every \(n\). In fact,

\(^3\) This definition is slightly different from Meyer's one; this is the discrete version of Meyer's,
let \( \phi \) be in \( \mathcal{M} \) and \( f \), the extension of \( \phi \) to \( S \) such that \( f \in \mathcal{N} \). If
\[
\phi h_n \geq \phi f \quad \text{on the set } \{ f > 0 \}
\]
then
\[
h_n + \phi f(c) \geq \phi f \quad \text{on the set } \{ f > 0 \},
\]
since \( \{ f > 0 \} \) is contained in \( \{ f > 0 \} \cup \{ c \} \). Hence from condition (\( * \)),
\[
h_n + \phi f(c) - f \geq \phi f \quad \text{everywhere on } \phi S,
\]
that is,
\[
\phi h_n - \phi f - \phi G f \quad \text{everywhere on } \phi S.
\]
Since \( \phi H_{\phi} \cdot 1 \leq \phi h_n \) by definition,
\[
\lim \phi H_{\phi} \cdot 1 = 0.
\]

Conversely, if (2.11) holds, set \( \phi E_n \cup \{ c \} = E_n, F_n = S - E_n \) and
\[
h_n = \begin{cases} 
\phi H_{\phi} \cdot 1 & \text{on } \phi S \\
0 & \text{at } c.
\end{cases}
\]
It is enough to show the property (ii) of condition (\( * \)). Suppose that, for some \( f \in \mathcal{N} \) and some real number \( m \geq \phi f(c) \)
\[
m + h_n \geq \phi f \quad \text{on } \{ f > 0 \}.
\]
Then one has
\[
m - \phi f(c) + \phi H_{\phi} \cdot 1 \geq \phi G f \quad \text{on } \{ f > 0 \},
\]
where \( \phi f \) is the restriction of \( f \) to \( \phi S \). The fact that \( m - \phi f(c) + \phi H_{\phi} \cdot 1 \) is a quasi-
excessive function implies that
\[
m - \phi f(c) + \phi H_{\phi} \cdot 1 - \phi f \geq \phi G f \quad \text{everywhere on } \phi S,
\]
which is nothing but condition (\( * \)).

**Note.** If \( \alpha \) is a finite measure, then condition (\( * \)) is satisfied.

Let \( I_F \) be the indicator function of a set \( F \), then from lemma 2.1 \( \phi GI_F \geq 1 \) on \( F \). Hence from (2.8) and (2.9) \( \phi H_{\phi} \cdot 1 \leq \phi GI_F \). Hence if \( F_n \) decrease to empty set, inequality
\[
\phi H_{\phi} \cdot 1(x) \leq \phi GI_{F_n}(x) \leq \sum_{y \in F_n} C \cdot \alpha(y),
\]
implies that
\[
\lim \phi H_{\phi} \cdot 1(x) = 0.
\]
Where the second inequality follows from the corollary of lemma 2.1.

3. **Main result**

Let \( A \) be a weak potential kernel of an irreducible recurrent transition
probability $P$ with invariant measure $\alpha$. We shall now prove that $A$ satisfies condition (*) of section 2.

Define $^cP$ and $^cG$ as (1.3) and (1.4) respectively. Let $^cH_p$ be the réduit defined by $^cP$. Since $^cH_p\cdot1$ is the pseudo-réduit associated with the above $^cG$ (see [5] P. 37, theorem 1.3), it is enough to show that for a sequence of finite sets $\{E_n\}_{n=1,2,\ldots}$ increasing to $^cS$, $\lim^cH_{F_n}\cdot1=0$ ($F_n=^cS-^cE_n$) by theorem 2.2. One can easily seen that the function $^c\theta(x)=\lim^cH_{F_n}\cdot1(x)$ is an invariant function for $^cP$ (i.e. $^cP^c\theta=^c\theta$) and bounded by 1. On the other hand,

$$1=^cG(1-^cP\cdot1)(x)+\lim^cP^n\cdot1(x)$$

and

$$\lim^cP^n\cdot1(x)=\lim P_n[\sigma(\cdot)>n]=0,$$

implies that 1 is a potential of non-generative function (where $\sigma(\cdot)$ is the hitting time of the Markov chain with transition probability $P$). Hence $^c\theta$ is also a potential. The fact that $^c\theta$ is an invariant function and also a potential shows that $^c\theta=0$.

The main result of the present paper is this.

**Theorem 3.1.** Given a positive measure $\alpha$ and a kernel $A$ satisfying maximum principle (RSCM) and condition (*), there exists a unique irreducible recurrent transition probability $P$ which has $\alpha$ as its invariant measure, and $A$ as its weak potential kernel.

Uniqueness was proved by Kondo [4]. We shall divide the proof of existence into several lemmas. In the following we shall use the notation of section 2 with no further reference.

**Lemma 3.1.** There exists a sub-Markov transition probability $^cP(x, y)$ on $^cS$ such that

$$^cG(x, y) = \sum_{n=0}^{\infty} ^cP^n(x, y) \quad \text{for every } x, y \text{ in } ^cS.$$


**Lemma 3.2.** For every $y \in ^cS$, $\sum_{x \in ^cS} \alpha(x)^cP(x, y) \leq \alpha(y)$.

Proof. To the contrary, suppose that there exists some state $y \in ^cS$ such that

$$\sum_{x \in ^cS} \alpha(x)^cP(x, y) - \alpha(y) > 0.$$

Then there exists a finite subset $F$ of $^cS$ containing $y$ and satisfying

$$\sum_{x \in F} \alpha(x)^cP(x, y) - \alpha(y) = a > 0.$$

Define a function $f \in N$ by
Since $Af + f^-$ attains its maximum on the set $\{f > 0\}$ and since $f(c) < 0$, there exists a state $x_0 \in F$ such that,

$$Af(x_0) \geq Af + f^- \quad \text{everywhere on } S.$$ 

In particular,

$$Af(x_0) \geq Af(c) + \frac{a}{\alpha(c)}.$$ 

Hence,

$$0 > -\frac{a}{\alpha(c)} \geq Af(c) - Af(x_0) = cG(-f)(x_0),$$

where $^c f$ is the restriction of $f$ to $^c S$. On the other hand,

$$^c G(-^c f)(x_0) = ^c G(x_0, y) - \sum_{x \in F} ^c G(x_0, z)^c P(z, y)$$

$$\geq ^c G(x_0, y) - (^c G(x_0, y) - I(x_0, y)) = I(x_0, y) \geq 0.$$ 

This lead us to a contradiction.

**Lemma 3.3.** $^c G(1 - ^c P \cdot 1) = 1$ on $^c S$.

**Proof.** For any positive integer $n$, we have

$$1 = \sum_{i=0}^n ^c P^i(1 - ^c P \cdot 1)(x) + ^c P^{n+1} \cdot 1(x).$$

Passing to the limit we obtain

$$1 = ^c G(1 - ^c P \cdot 1)(x) + r(x),$$

where $r(x) = \lim ^c P^{n+1} \cdot 1(x)$. It remains to show that $r(x) = 0$. From condition $(*)$ for arbitrary $\varepsilon > 0$ there exists a number $M$ such that for any integer $m \geq M$,

$$^c H_{F_m} \cdot 1(x) < \varepsilon.$$ 

Hence

$$\sum_{y \in c} ^c P^{n+1}(x, y) = ^c P^{n+1} I_{F_m}(x) + ^c P^{n+1} I_{E_m}(x) \leq ^c H_{F_m} \cdot 1(x) + ^c P^{n+1} I_{E_m}(x),$$

where $I_F$ is the indicator function of $F$. Tending $n$ to infinity we obtain $r(x) \leq \varepsilon$.

**Lemma 3.4.** $\sum_{x \in c} \alpha(x)(1 - ^c P \cdot 1)(x) \leq \alpha(c)$.

**Proof.** Let $F$ be an arbitrary finite subset of $^c S$, and define
As noted in the proof of lemma 3.2, there exists a state \( x_0 \in F \) such that
\[
Af(x_0) \geq Af^+ \quad \text{on } S.
\]
In particular,
\[
^cG^f(x_0) = Af(x_0) - Af(c) \geq f^-(c) = \sum_{y \in F} \alpha(y)(1 - ^cP \cdot 1(y))/\alpha(c),
\]
and by lemma 3.3, the left side of the above inequality is bounded by 1.

Now we can define the desired transition probability \( P \).

\[
P(x, y) = \begin{cases} 
  ^cP(x, y) & x \neq c, y \neq c \\
  1 - ^cP \cdot 1(x) & x \neq c, y = c \\
  (\alpha(y) - \alpha^cP(y))/\alpha(c) & x = c, y \neq c \\
  1 - \sum_{z \neq c} P(c, z) & x = c, y = c.
\end{cases}
\]

From lemmas 3.2 and 3.4, \( P \) is a transition probability on \( S \).

**Lemma 3.5.** \( \alpha P = \alpha \) and \( (I - P)Af = f \) for any \( f \in \mathbb{N} \).

**Proof.** If \( x \neq c \), then
\[
\alpha P(x) = \sum_{y \neq c} \alpha(y)^cP(y, x) + \alpha(x) - \alpha^cP(x) = \alpha(x)
\]
and
\[
(I - P)Af(x) = (I - P)(Af(c) + ^cGf)(x) = f(x).
\]
By the same argument for \( x = c \), lemma follows.

**Lemma 3.6.** The transition probability \( P \) is recurrent and irreducible.

**Proof.** Let \( \sigma_{(x)} \) be the hitting time for \( x \) of the Markov chain with transition probability \( P \). Then for every \( x \neq c \),
\[
P_x[\sigma_{(x)} < \infty] = \sum_{y \neq c} ^cG(x, y)P(y, c) = ^cG(1 - ^cP \cdot 1)(x) = 1,
\]
by lemma 3.3. Hence,
\[
P_c[\sigma_{(c)} < \infty] = \sum_{x \in S} P(c, x)P_x[\sigma_{(c)} < \infty] = 1,
\]
where \( \sigma_{(c)}^+ \) is the positive hitting time for state \( c \). Thus \( c \) is a recurrent state for \( P \) and hence also for \( \hat{P} \), where \( \hat{P} \) is defined by,
\[
\hat{P}(x, y) = \frac{\alpha(y)}{\alpha(x)} P(y, x).
\]
Moreover,
\[ \hat{P}^n(c, x) = \frac{\alpha(x)}{\alpha(c)} P^n(x, c), \text{ and } P_x[\sigma_{\{x\}} < \infty] = 1, \]
shows that
\[ \hat{P}_x[\sigma_{\{x\}} < \infty] > 0 \text{ for all } x \in \mathcal{S}. \]
Hence \( x \) is a recurrent state for \( \hat{P} \) and hence for \( P \). Since \( x \) is recurrent and \( P_x[\sigma_{\{x\}} < \infty] = 1 \), it follows that \( P_x[\sigma_{\{x\}} < \infty] = 1 \) for all \( x \in \mathcal{S} \). Irreducibility follows from the fact that, \( P_x[\sigma_{\{x\}} < \infty] = 1 \) and \( P_x[\sigma_{\{y\}} < \infty] = 1 \) for every \( x, y \) in \( \mathcal{S} \).

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