<table>
<thead>
<tr>
<th>Title</th>
<th>A necessary and sufficient condition for a kernel to be a weak potential kernel of a recurrent Markov chain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ôshima, Yoichi</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 6(1) P.29–P.37</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1969</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/10624">https://doi.org/10.18910/10624</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/10624</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
<tr>
<td>Note</td>
<td></td>
</tr>
</tbody>
</table>
A NECESSARY AND SUFFICIENT CONDITION FOR A KERNEL TO BE A WEAK POTENTIAL KERNEL OF A RECURRENT MARKOV CHAIN

KOICHI ŌSHIMA

(Received December 14, 1968)

1. Introduction

Let \( P \) be an irreducible recurrent transition probability on a denumerable space \( S \) with invariant measure \( \alpha \). Let \( c \) be an arbitrary (but fixed) state of \( S \). Then from the work of Kondo [3] and Orey [8], there exist the class of weak potential kernels \( A(x, y) \) defined by the property that, for every null charge \( f \), \( Af \) is bounded and satisfies the equation

\[
(I - P)Af = f.
\]

Moreover \( Af \) is represented by

\[
Af = Gf + 1(f),
\]

where \( f \) is a null charge, \( 1(\cdot) \) is an arbitrary linear functional on the space of null charges and \( G \) is defined as follow;

\[
cP(x, y) = \begin{cases} 
P(x, y) & x \neq c, y \neq c \\ 0 & \text{otherwise}, \end{cases}
\]

\[
G(x, y) = \begin{cases} 
\sum_{n=0}^{\infty} P^n(x, y) & x \neq c, y \neq c \\ 0 & \text{otherwise}. \end{cases}
\]

Moreover \( A \) satisfies the following maximum principle [4], [5]:

(RSCM)\(^1\) If \( m \) is a real number and \( f \) is a null charge then the relation that

\[
m \geq Af \quad \text{on the set } \{f > 0\}
\]

implies that

\[
m - f^- \geq Af \quad \text{everywhere},
\]

\(^1\) This is the abbreviation of "reinforced semi-complete maximum principle"; this maximum principle corresponds to the semi-complete M.P. as well as the reinforced M.P. (of Meyer) corresponds to the complete M.P.
where $f^\equiv (\bar{-}f) \vee 0$.

In the present paper we are concerned with the following construction problem. Given a positive measure $\alpha$ and a (not necessarily positive) kernel $A$ satisfying (RSCM), does there exist an irreducible recurrent transition probability which has $\alpha$ as its invariant measure, and $A$ as its weak potential kernel? This is not true in general\textsuperscript{2)}, but as Kondo [4] has proved, it is true if $\alpha$ is a finite measure. In section 2 we shall introduce another necessary condition for the weak potential kernel $A$ (referred to as condition (*)). Then we shall prove (theorem 3.1) that, if the pair $(A, \alpha)$ satisfies maximum principle (RSCM) and condition (*), $A$ is a weak potential kernel of a (unique) recurrent Markov chain with $\alpha$ as its invariant measure.

I should like to express my hearty gratitude to T. Watanabe for his kind advices.

2. Some potential theory for a kernel $A$ satisfying (RSCM)

Let $\alpha$ be a strictly positive measure and $A$, a kernel on $S$. A function $f$ on $S$ is said to be a null charge with respect to $\alpha$ if $\sum |\alpha(x)f(x)| < \infty$ and $\sum \alpha(x)f(x) = 0$. Let $N$ be the space of null charges vanishing outside a finite subset of $S$. We assume that the kernel $A$ satisfies condition (RSCM) for $f \in N$. Fix an arbitrary state $c$ and define

\begin{equation}
(cG(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)}.
\end{equation}

If $A$ is a weak potential kernel then (2.1) is clearly satisfied by taking $f$ in equation (1.2) as

\begin{equation}
f(x) = \begin{cases} 
\frac{\alpha(y)}{\alpha(c)} & x = c \\
-1 & x = y \\
0 & \text{otherwise},
\end{cases}
\end{equation}

and calculating $Af(x) - Af(c)$.

From definition (2.1) $cG(c, x) = cG(x, c) = 0$ for every $x \in S$.

**Lemma 2.1** For arbitrary elements $x$, $y$ in $S$ which are different from $c$

$I(x, y) \leq cG(x, y) \leq cG(y, y)$.

**Proof.** By taking $f$ as (2.2) we have

$Af(c) = A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y)$

\textsuperscript{2)} A counter example was given by Kondo and T. Watanabe.
Hence, if we write \( f^+ = f \vee 0, f^- = (-f) \vee 0 \), by (RSCM)

\[
A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) + f^+(x) \leq Af(x) \leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - f^-(x),
\]

so that

\[
(2.3) \quad A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) \leq A(x, c) \frac{\alpha(y)}{\alpha(c)} - A(x, y)
\]

\[
\leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - I(x, y),
\]

which proves the lemma.

**Corollary.** For every \( x \in S \) there exists a constant \( C \) such that

\[
(2.4) \quad cG(x, y) \leq C \cdot \alpha(y) \quad \text{for every } y \in S.
\]

**Proof.** Exchanging \( c \) and \( x \) in the second inequality in (2.3), it follows that

\[
cG(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)}
\]

\[
\leq \left( \frac{-A(x, c)}{\alpha(c)} - \frac{A(c, x)}{\alpha(x)} + \frac{A(c, c)}{\alpha(c)} + \frac{A(x, x)}{\alpha(x)} \right) \alpha(y).
\]

Let \( \mathcal{S} \) be the set \( S - \{c\} \), and \( \mathcal{M} \) be the space of all functions on \( \mathcal{S} \) vanishing outside a finite subset of \( \mathcal{S} \). Let \( \mathcal{M}^+ \) be the space of all non-negative functions in \( \mathcal{M} \).

**Theorem 2.1.** The kernel \( cG \) satisfies the reinforced maximum principle [7]:

\( (\text{RM}) \) If \( a \) is a non-negative constant and if \( c g \) and \( c f \) are two elements of \( \mathcal{M}^+ \), then the relation that

\[
a + cGc \geq cG c \quad \text{on the set } \{g > 0\} \text{ implies that}
\]

\[
a + cGc \geq cG c \quad \text{everywhere on } c \mathcal{S}.
\]

**Proof.** Let \( f \) be the function on \( S \) such that \( f \in \mathcal{N} \) and \( f|_{\mathcal{S}} = f \). Such \( f \) is obviously unique. The function \( g \in \mathcal{N} \) is defined similarly. Then inequality (2.5) implies that

\[
a + A(g - f)(c) \geq A(g - f) \quad \text{on the set } \{g - f > 0\}.
\]

For, since \( c f \) and \( c g \) are non-negative, the set \( \{g - f > 0\} \) is contained in the union of \( c \) and \( \{g > 0\} \). Hence by (RSCM)
\[ a + A(g-f)(c) - (g-f)^- \geq A(g-f) \quad \text{everywhere.} \]

Since the function \((g-f)^-\) is equal to \(^c f\) on \(^c S \cap \{g=0\}\), the above inequality, combined with (2.5), proves the theorem.

A non-negative function \(^c h\) on \(^c S\) is said to be quasi-excessive\(^3\) if, for every \(^c g \in ^c M\), the inequality

\[ ^c h \geq ^c G^c g \quad \text{on the set } \{^c g > 0\} \]

implies that

\[ ^c h - ^c g^- \geq ^c G^c g \quad \text{everywhere.} \]

Moreover Meyer introduced the notion of the pseudo-réduit \(^c H^c E^c h\) for every quasi-excessive function \(^c h\) and every subset \(E\) of \(^c S\). This function \(^c H^c E^c h\) satisfies the following four conditions.

(2.7) \(^c H^c E^c h\) is quasi-excessive.

(2.8) \(^c H^c E^c h \leq ^c h\) on \(^c S\) and \(^c H^c E^c h = ^c h\) on \(E\).

(2.9) If \(^c h_1\) and \(^c h_2\) are two quasi-excessive functions such that \(^c h_1 \leq ^c h_2\) on \(E\), then \(^c H^c E^c h_1 \leq ^c H^c E^c h_2\).

(2.10) If \(^c f \in ^c M^+\) vanishes outside of \(E\) then \(^c H^c E^c G^c f = ^c G^c f\).

For example, the function \(^c G^c f\), \(^c f \in ^c M^+\), and every positive constant are quasi-excessive ([7] see also [5]).

Now we introduce a condition.

Condition (*): There exists a sequence of finite sets \(\{E_n\}_{n=1,2,\ldots}\) increasing to \(S\) such that \(c \in E_n\) for each \(n\), and a sequence \(\{h_n\}_{n=1,2,\ldots}\) of function on \(S\) satisfying the following conditions.

(i) \(0 \leq h_n \leq 1\), \(h_n(c) = 0\), \(h_n = 1\) on \(F_n = S - E_n\), and \(\lim h_n = 0\).

(ii) For every \(f \in N\) and every real number \(m \geq Af(c)\) the relation that

\[ m + h_n \geq Af \quad \text{on the set } \{f > 0\} \]

implies that

\[ m + h_n - f^- \geq Af \quad \text{everywhere on } ^c S. \]

In section 3 we shall show that if \(A\) is a weak potential kernel of an irreducible recurrent Markov chain, it satisfies condition (*).

**Theorem 2.2** Condition (*) is equivalent to the condition that, there exists a sequence of finite sets \(\{E_n\}_{n=1,2,\ldots}\) increasing to \(^c S\) such that

\[ \lim \frac{c H_{c S - c E_n}}{c h_n} \cdot 1 = 0. \]

Proof. Suppose that condition (*) holds and let \(^c h_n\) be the restriction of \(h_n\) to \(^c S\) and \(^c E_n = S \cap E_n\). Obviously \(^c S - ^c E_n = F_n\), \(0 \leq c h_n \leq 1\), and \(^c h_n = 1\) on \(F_n\). It then follows that \(^c h_n\) is a quasi-excessive function for every \(n\). In fact,

\(^3\) This definition is slightly different from Meyer's one; this is the discrete version of Meyer's.
let \( f \) be in \( \mathcal{M} \) and \( f \), the extension of \( f \) to \( S \) such that \( f \in \mathcal{N} \). If
\[
\mathcal{E}_n \geq \mathcal{E} \mathcal{G}_f \quad \text{on the set } \{ f > 0 \}
\]
then
\[
h_n + Af(c) \geq Af \quad \text{on the set } \{ f > 0 \},
\]
since \( \{ f > 0 \} \) is contained in \( \{ f > 0 \} \cup \{ c \} \). Hence from condition (*),
\[
h_n + Af(c) - f \geq Af \quad \text{everywhere on } \mathcal{E} S,
\]
that is,
\[
\mathcal{E}_n - \mathcal{E} - \mathcal{G}_f \quad \text{everywhere on } \mathcal{E} S.
\]
Since \( \mathcal{E} \mathcal{H}_n \cdot 1 \leq \mathcal{E}_n \) by definition,
\[
\lim \mathcal{E} \mathcal{H}_n \cdot 1 = 0.
\]
Conversely, if (2.11) holds, set \( \mathcal{E} E_n \cup \{ c \} = \mathcal{E} E_n \), \( F_n = S - E_n \) and
\[
h_n = \begin{cases} 
\mathcal{E} \mathcal{H}_n \cdot 1 & \text{on } \mathcal{E} S \\
0 & \text{at } c.
\end{cases}
\]
It is enough to show the property (ii) of condition (*). Suppose that, for some \( f \in \mathcal{N} \) and some real number \( m \) (\( \geq Af(c) \))
\[
m + h_n \geq Af \quad \text{on } \{ f > 0 \}.
\]
Then one has
\[
m - Af(c) + \mathcal{E} \mathcal{H}_n \cdot 1 \geq \mathcal{E} \mathcal{G}_f \quad \text{on } \{ f > 0 \},
\]
where \( f \) is the restriction of \( f \) to \( \mathcal{E} S \). The fact that \( m - Af(c) + \mathcal{E} \mathcal{H}_n \cdot 1 \) is a quasi-

excessive function implies that
\[
m - Af(c) + \mathcal{E} \mathcal{H}_n \cdot 1 - f \geq \mathcal{E} \mathcal{G}_f \quad \text{everywhere on } \mathcal{E} S,
\]
which is nothing but condition (*).

**Note.** If \( \alpha \) is a finite measure, then condition (*) is satisfied.

Let \( I_F \) be the indicator function of a set \( F \), then from lemma 2.1 \( \mathcal{E} I_F \geq 1 \) on \( F \). Hence from (2.8) and (2.9) \( \mathcal{E} F_n \cdot 1 \leq \mathcal{E} I_F \). Hence if \( F_n \) decrease to empty set, inequality
\[
\mathcal{E} F_n \cdot 1(x) \leq \mathcal{E} I_F(x) \leq \sum_{y \in F_n} C \cdot \alpha(y),
\]
implies that
\[
\lim \mathcal{E} F_n \cdot 1(x) = 0.
\]
Where the second inequality follows from the corollary of lemma 2.1.

3. **Main result**

Let \( A \) be a weak potential kernel of an irreducible recurrent transition
probability \( P \) with invariant measure \( \alpha \). We shall now prove that \( A \) satisfies condition (\*) of section 2.

Define \( ^c P \) and \( ^c G \) as (1.3) and (1.4) respectively. Let \( ^c H_F \) be the réduit defined by \( ^c P \). Since \( ^c H_F \cdot 1 \) is the pseudo-réduit associated with the above \( ^c G \) (see [5] P. 37, theorem 1.3), it is enough to show that for a sequence of finite sets \( \{ E_n \} \) increasing to \( ^c S \), \( \lim ^c H_{F_n} \cdot 1 = 0 \) \((F_n = ^c S - E_n)\) by theorem 2.2. One can easily seen that the function \( ^c h(x) = \lim ^c H_{F_n} \cdot 1(x) \) is an invariant function for \( ^c P \) (i.e. \( ^c P^c h = ^c h \)) and bounded by 1. On the other hand,

\[
1 = ^c G(1 - ^c P \cdot 1)(x) + \lim ^c P^n \cdot 1(x)
\]

implies that 1 is a potential of non-gengative function (where \( \sigma\{c\} \) is the hitting time of the Markov chain with transition probability \( P \)). Hence \( ^c h \) is also a potential. The fact that \( ^c h \) is an invariant function and also a potential shows that \( ^c h = 0 \).

The main result of the present paper is this.

**Theorem 3.1.** Given a positive measure \( \alpha \) and a kernel \( A \) satisfying maximum principle (RSCM) and condition (\*), there exists a unique irreducible recurrent transition probability \( P \) which has \( \alpha \) as its invariant measure, and \( A \) as its weak potential kernel.

Uniqueness was proved by Kondō [4]. We shall divide the proof of existence into several lemmas. In the following we shall use the notation of section 2 with no further reference.

**Lemma 3.1.** There exists a sub-Markov transition probability \( ^c P(x, y) \) on \( ^c S \) such that

\[
^c G(x, y) = \sum_{n=0}^\infty ^c P^n(x, y) \quad \text{for every } x, y \in ^c S.
\]


**Lemma 3.2.** For every \( y \in ^c S \), \( \sum_{x \in c} \alpha(x)^c P(x, y) \leq \alpha(y) \).

Proof. To the contrary, suppose that there exists some state \( y \in ^c S \) such that

\[
\sum_{x \in c} \alpha(x)^c P(x, y) - \alpha(y) > 0.
\]

Then there exists a finite subset \( F \) of \( ^c S \) containing \( y \) and satisfying

\[
\sum_{s \in F} \alpha(x)^c P(x, y) - \alpha(y) = a > 0.
\]

Define a function \( f \in N \) by
\[
\begin{align*}
f(x) &= \begin{cases} 
\epsilon P(x, y) - I(x, y) & x \in F \\
-\frac{a}{\alpha(c)} & x = c \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Since \(Af+f^-\) attains its maximum on the set \(\{f>0\}\) and since \(f(c)<0\), there exists a state \(x_0 \in F\) such that,

\[Af(x_0) \geq Af+f^- \quad \text{everywhere on } S.\]

In particular,

\[Af(x_0) \geq Af(c) + \frac{a}{\alpha(c)}.\]

Hence,

\[0 > -\frac{a}{\alpha(c)} \geq Af(c) - Af(x_0) = \epsilon G(-\epsilon f)(x_0),\]

where \(\epsilon f\) is the restriction of \(f\) to \(\epsilon S\). On the other hand,

\[
\epsilon G(-\epsilon f)(x_0) = \epsilon G(x_0, y) - \sum_{z \in F} \epsilon G(x_0, z) \epsilon P(z, y) \\
\geq \epsilon G(x_0, y) - (\epsilon G(x_0, y) - I(x_0, y)) = I(x_0, y) \geq 0.
\]

This leads us to a contradiction.

**Lemma 3.3.** \(\epsilon G(1-\epsilon P \cdot 1)=1\) on \(\epsilon S\).

**Proof.** For any positive integer \(n\), we have

\[1 = \sum_{i=0}^{n} \epsilon P^i(1-\epsilon P \cdot 1)(x) + \epsilon P^{n+1} \cdot 1(x).\]

Passing to the limit we obtain

\[1 = \epsilon G(1-\epsilon P \cdot 1)(x) + r(x),\]

where \(r(x)=\lim \epsilon P^{n+1} \cdot 1(x)\). It remains to show that \(r(x)=0\). From condition \((*)\) for arbitrary \(\varepsilon>0\) there exists a number \(M\) such that for any integer \(n \geq M,\)

\[\epsilon H_{F_m} \cdot 1(x) < \varepsilon.\]

Hence

\[\sum_{P \in E} \epsilon P^{m+1}(x, y) = \epsilon P^{m+1}I_{F_m}(x) + \epsilon P^{m+1}I_{E_m}(x) \leq \epsilon H_{F_m} \cdot 1(x) + \epsilon P^{m+1}I_{E_m}(x),\]

where \(I_F\) is the indicator function of \(F\). Tending \(n\) to infinity we obtain \(r(x) \leq \varepsilon\).

**Lemma 3.4.** \(\sum_{x \in \epsilon} \alpha(x)(1-\epsilon P \cdot 1)(x) \leq \alpha(c).\)

**Proof.** Let \(F\) be an arbitrary finite subset of \(\epsilon S\), and define
\[ f(x) = \begin{cases} 1 - \varepsilon P \cdot 1(x) & x \in F \\ -\frac{\alpha(y)}{\alpha(c)}(1 - \varepsilon P \cdot 1(y)) & x = c \\ 0 & \text{otherwise.} \end{cases} \]

As noted in the proof of lemma 3.2, there exists a state \( x_0 \in F \) such that

\[ Af(x_0) \geq Af + f^- \quad \text{on } S. \]

In particular,

\[ \varepsilon G^c f(x_0) = Af(x_0) - Af(c) \geq f^-(c) = \sum_{y \in F} \alpha(y)(1 - \varepsilon P \cdot 1)/\alpha(c), \]

and by lemma 3.3, the left side of the above inequality is bounded by 1.

Now we can define the desired transition probability \( P \).

\[ P(x, y) = \begin{cases} \varepsilon P(x, y) & x \neq c, y \neq c \\ 1 - \varepsilon P \cdot 1(x) & x \neq c, y = c \\ (\alpha(y) - \alpha^c P(y))/\alpha(c) & x = c, y \neq c \\ 1 - \sum_{z \neq c} P(c, z) & x = c, y = c. \end{cases} \]

From lemmas 3.2 and 3.4, \( P \) is a transition probability on \( S \).

**Lemma 3.5.** \( \alpha P = \alpha \) and \( (I-P)Af = f^- \) for any \( f \in N \).

**Proof.** If \( x \neq c \), then

\[ \alpha P(x) = \sum_{y \neq c} \alpha(y)\varepsilon P(y, x) + \alpha(x) - \alpha^c P(x) = \alpha(x) \]

and

\[ (I-P)Af(x) = (I-P)(Af(c) + \varepsilon Gf)(x) = f(x). \]

By the same argument for \( x = c \), lemma follows.

**Lemma 3.6.** The transition probability \( P \) is recurrent and irreducible.

**Proof.** Let \( \sigma_{(x)} \) be the hitting time for \( x \) of the Markov chain with transition probability \( P \). Then for every \( x \neq c \),

\[ P_x[\sigma_{(x)} < \infty] = \sum_{y \neq c} \varepsilon G(x, y)P(y, c) = \varepsilon G(1 - \varepsilon P - 1)(x) = 1, \]

by lemma 3.3. Hence,

\[ P_c[\sigma^+_c < \infty] = \sum_{x \in S} P(c, x)P_x[\sigma_{(x)} < \infty] = 1, \]

where \( \sigma^+_c \) is the positive hitting time for state \( c \). Thus \( c \) is a recurrent state for \( P \) and hence also for \( \hat{P} \), where \( \hat{P} \) is defined by,

\[ \hat{P}(x, y) = \frac{\alpha(y)}{\alpha(x)} P(y, x). \]
Moreover,

\[ \hat{P}^n(c, x) = \frac{\alpha(x)}{\alpha(c)} P^n(x, c), \] and \( P_x[\sigma(x) < \infty] = 1 \),

shows that

\[ \hat{P}_x[\sigma(x) < \infty] > 0 \] for all \( x \in S \).

Hence \( x \) is a recurrent state for \( \hat{P} \) and hence for \( P \). Since \( x \) is recurrent and \( P_x[\sigma(x) < \infty] = 1 \), it follows that \( P_x[\sigma(x) < \infty] = 1 \) for all \( x \in S \). Irreducibility follows from the fact that, \( P_x[\sigma(x) < \infty] = 1 \) and \( P_x[\sigma(y) < \infty] = 1 \) for every \( x, y \) in \( S \).

OSAKA CITY UNIVERSITY

Bibliography
