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**A NECESSARY AND SUFFICIENT CONDITION FOR
 A KERNEL TO BE A WEAK POTENTIAL KERNEL
 OF A RECURRENT MARKOV CHAIN**

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1. Introduction

Let P be an irreducible recurrent transition probability on a denumerable space S with invariant measure α . Let c be an arbitrary (but fixed) state of S . Then from the work of Kondō [3] and Orey [8], there exist the class of *weak potential kernels* $A(x, y)$ defined by the property that, for every null charge f , Af is bounded and satisfies the equation

$$(1.1) \quad (I - P)Af = f.$$

Moreover Af is represented by

$$(1.2) \quad Af = {}^cGf + 1(f),$$

where f is a null charge, $1(\cdot)$ is an arbitrary linear functional on the space of null charges and cG is defined as follow;

$$(1.3) \quad {}^cP(x, y) = \begin{cases} P(x, y) & x \neq c, y \neq c \\ 0 & \text{otherwise,} \end{cases}$$

$$(1.4) \quad {}^cG(x, y) = \begin{cases} \sum_{n=0}^{\infty} {}^cP^n(x, y) & x \neq c, y \neq c \\ 0 & \text{otherwise.} \end{cases}$$

Moreover A satisfies the following maximum principle [4], [5]:
 (RSCM)¹⁾ If m is a real number and f is a null charge then the relation that

$$(1.5) \quad m \geq Af \quad \text{on the set } \{f > 0\}$$

implies that

$$(1.6) \quad m - f^- \geq Af \quad \text{everywhere,}$$

1) This is the abbreviation of “reinforced semi-complete maximum principle”; this maximum principle corresponds to the semi-complete M.P. as well as the reinforced M.P. (of Meyer) corresponds to the complete M.P.

where $f^- = (-f) \vee 0$.

In the present paper we are concerned with the following construction problem. Given a positive measure α and a (not necessarily positive) kernel A satisfying (RSCM), does there exist an irreducible recurrent transition probability which has α as its invariant measure, and A as its weak potential kernel? This is not true in general²⁾, but as Kondō [4] has proved, it is true if α is a finite measure. In section 2 we shall introduce another necessary condition for the weak potential kernel A (referred to as condition (*)). Then we shall prove (theorem 3.1) that, if the pair (A, α) satisfies maximum principle (RSCM) and condition (*), A is a weak potential kernel of a (unique) recurrent Markov chain with α as its invariant measure.

I should like to express my hearty gratitude to T. Watanabe for his kind advices.

2. Some potential theory for a kernel A satisfying (RSCM)

Let α be a strictly positive measure and A , a kernel on S . A function f on S is said to be a *null charge with respect to α* if $\sum \alpha(x) |f(x)| < \infty$ and $\sum \alpha(x) f(x) = 0$. Let N be the space of null charges vanishing outside a finite subset of S . We assume that *the kernel A satisfies condition (RSCM) for $f \in N$* . Fix an arbitrary state c and define

$$(2.1) \quad {}^cG(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)}.$$

If A is a weak potential kernel then (2.1) is clearly satisfied by taking f in equation (1.2) as

$$(2.2) \quad f(x) = \begin{cases} \frac{\alpha(y)}{\alpha(c)} & x = c \\ -1 & x = y \\ 0 & \text{otherwise,} \end{cases}$$

and calculating $Af(x) - Af(c)$.

From definition (2.1) ${}^cG(c, x) = {}^cG(x, c) = 0$ for every $x \in S$.

Lemma 2.1 *For arbitrary elements x, y in S which are different from c*

$$I(x, y) \leq {}^cG(x, y) \leq {}^cG(y, y).$$

Proof. By taking f as (2.2) we have

$$Af(c) = A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y)$$

2) A counter example was given by Kondō and T. Watanabe.

$$Af(y) = A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y).$$

Hence, if we write $f^+ = f \vee 0$, $f^- = (-f) \vee 0$, by (RSCM)

$$A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) + f^+(y) \leq Af(y) \leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - f^-(y),$$

so that

$$\begin{aligned} (2.3) \quad A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) &\leq A(x, c) \frac{\alpha(y)}{\alpha(c)} - A(x, y) \\ &\leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - I(x, y), \end{aligned}$$

which proves the lemma.

Corollary. *For every $x \in S$ there exists a constant C such that*

$$(2.4) \quad {}^cG(x, y) \leq C \cdot \alpha(y) \quad \text{for every } y \in S.$$

Proof. Exchanging c and x in the second inequality in (2.3), it follows that

$$\begin{aligned} {}^cG(x, y) &= A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)} \\ &\leq \left(-\frac{A(x, c)}{\alpha(c)} - \frac{A(c, x)}{\alpha(x)} + \frac{A(c, c)}{\alpha(c)} + \frac{A(x, x)}{\alpha(x)} \right) \alpha(y). \end{aligned}$$

Let cS be the set $S - \{c\}$, and cM be the space of all functions on cS vanishing outside a finite subset of cS . Let ${}^cM^+$ be the space of all non-negative functions in cM .

Theorem 2.1. *The kernel cG satisfies the reinforced maximum principle [7]: (RM) If a is a non-negative constant and if ${}^c f$ and ${}^c g$ are two elements of ${}^cM^+$, then the relation that*

$$(2.5) \quad a + {}^cG{}^c f - {}^c f \geq {}^cG{}^c g \quad \text{on the set } \{{}^c g > 0\} \text{ implies that}$$

$$(2.6) \quad a + {}^cG{}^c f - {}^c f \geq {}^cG{}^c g \quad \text{everywhere on } {}^cS.$$

Proof. Let f be the function on S such that $f \in N$ and $f|_{{}^cS} = {}^c f$. Such f is obviously unique. The function $g \in N$ is defined similarly. Then inequality (2.5) implies that

$$a + A(g - f)(c) \geq A(g - f) \quad \text{on the set } \{g - f > 0\}.$$

For, since ${}^c f$ and ${}^c g$ are non-negative, the set $\{g - f > 0\}$ is contained in the union of c and $\{{}^c g > 0\}$. Hence by (RSCM)

$$a + A(g-f)(c) - (g-f)^- \geq A(g-f) \quad \text{everywhere.}$$

Since the function $(g-f)^-$ is equal to $^c f$ on $^c S \cap \{g=0\}$, the above inequality, combined with (2.5), proves the theorem.

A non-negative function $^c h$ on $^c S$ is said to be *quasi-excessive*³⁾ if, for every $^c g \in {}^c M$, the inequality

$$^c h \geq {}^c G^c g \quad \text{on the set } \{^c g > 0\}$$

implies that

$$^c h - {}^c g^- \geq {}^c G^c g \quad \text{everywhere.}$$

Moreover Meyer introduced the notion of the *pseudo-réduite* ${}^c H_E {}^c h$ for every quasi-excessive function $^c h$ and every subset E of $^c S$. This function ${}^c H_E {}^c h$ satisfies the following four conditions.

(2.7) ${}^c H_E {}^c h$ is quasi-excessive.

(2.8) ${}^c H_E {}^c h \leq {}^c h$ on $^c S$ and ${}^c H_E {}^c h = {}^c h$ on E .

(2.9) If $^c h_1$ and $^c h_2$ are two quasi-excessive functions such that ${}^c h_1 \leq {}^c h_2$ on E , then ${}^c H_E {}^c h_1 \leq {}^c H_E {}^c h_2$.

(2.10) If $^c f \in {}^c M^+$ vanishes outside of E then ${}^c H_E {}^c G^c f = {}^c G^c f$.

For example, the function ${}^c G^c f$, $^c f \in {}^c M^+$, and every positive constant are quasi-excessive ([7] see also [5]).

Now we introduce a condition.

Condition (*): There exists a sequence of finite sets $\{E_n\}_{n=1,2,\dots}$ increasing to S such that $c \in E_n$ for each n , and a sequence $\{h_n\}_{n=1,2,\dots}$ of functions on S satisfying the following conditions.

(i) $0 \leq h_n \leq 1$, $h_n(c) = 0$, $h_n = 1$ on $F_n = S - E_n$, and $\lim h_n = 0$.

(ii) For every $f \in N$ and every real number m ($\geq A f(c)$) the relation that

$$m + h_n \geq A f \quad \text{on the set } \{f > 0\}.$$

implies that

$$m + h_n - f^- \geq A f \quad \text{everywhere on } {}^c S.$$

In section 3 we shall show that if A is a weak potential kernel of an irreducible recurrent Markov chain, it satisfies condition (*).

Theorem 2.2 *Condition (*) is equivalent to the condition that, there exists a sequence of finite sets $\{E_n\}_{n=1,2,\dots}$ increasing to ${}^c S$ such that*

$$(2.11) \quad \lim {}^c H_{{}^c S - {}^c E_n} \cdot 1 = 0.$$

Proof. Suppose that condition (*) holds and let $^c h_n$ be the restriction of h_n to ${}^c S$ and ${}^c E_n = {}^c S \cap E_n$. Obviously ${}^c S - {}^c E_n = F_n$, $0 \leq {}^c h_n \leq 1$, and ${}^c h_n = 1$ on F_n . It then follows that $^c h_n$ is a quasi-excessive function for every n . In fact,

3) This definition is slightly different from Meyer's one; this is the discrete version of Meyer's,

let ${}^c f$ be in ${}^c M$ and f , the extention of ${}^c f$ to S such that $f \in N$. If

$${}^c h_n \geq {}^c G {}^c f \quad \text{on the set } \{{}^c f > 0\}$$

then

$$h_n + Af(c) \geq Af \quad \text{on the set } \{f > 0\},$$

since $\{{}^c f > 0\}$ is contained in $\{f > 0\} \cup \{c\}$. Hence from condition (*),

$$h_n + Af(c) - f^- \geq Af \quad \text{everywhere on } {}^c S,$$

that is,

$${}^c h_n - {}^c f^- \geq {}^c G {}^c f \quad \text{everywhere on } {}^c S.$$

Since ${}^c H_{F_n} \cdot 1 \leq {}^c h_n$ by definition,

$$\lim {}^c H_{F_n} \cdot 1 = 0.$$

Conversely, if (2.11) holds, set ${}^c E_n \cup \{c\} = E_n$, $F_n = S - E_n$ and

$$h_n = \begin{cases} {}^c H_{F_n} \cdot 1 & \text{on } {}^c S \\ 0 & \text{at } c. \end{cases}$$

It is enough to show the property (ii) of condition (*). Suppose that, for some $f \in N$ and some real number m ($\geq Af(c)$)

$$m + h_n \geq Af \quad \text{on } \{f > 0\}.$$

Then one has

$$m - Af(c) + {}^c H_{F_n} \cdot 1 \geq {}^c G {}^c f \quad \text{on } \{{}^c f > 0\},$$

where ${}^c f$ is the restriction of f to ${}^c S$. The fact that $m - Af(c) + {}^c H_{F_n} \cdot 1$ is a quasi-excessive function implies that

$$m - Af(c) + {}^c H_{F_n} \cdot 1 - {}^c f^- \geq {}^c G {}^c f \quad \text{everywhere on } {}^c S,$$

which is nothing but condition (*).

Note. If α is a finite measure, then condition (*) is satisfied.

Let I_F be the indicator function of a set F , then from lemma 2.1 ${}^c G I_F \geq 1$ on F . Hence from (2.8) and (2.9) ${}^c H_F \cdot 1 \leq {}^c G I_F$. Hence if F_n decrease to empty set, inequality

$${}^c H_{F_n} \cdot 1(x) \leq {}^c G I_{F_n}(x) \leq \sum_{y \in F_n} C \cdot \alpha(y),$$

implies that

$$\lim {}^c H_{F_n} \cdot 1(x) = 0.$$

Where the second inequality follows from the corollary of lemma 2.1.

3. Main result

Let A be a weak potential kernel of an irreducible recurrent transition

probability P with invariant measure α . We shall now prove that A satisfies condition $(*)$ of section 2.

Define cP and cG as (1.3) and (1.4) respectively. Let cH_F be the réduite defined by cP . Since ${}^cH_F \cdot 1$ is the pseudo-réduite associated with the above cG (see [5] P. 37, theorem 1.3), it is enough to show that for a sequence of finite sets $\{{}^cE_n\}_{n=1,2,\dots}$ increasing to cS , $\lim {}^cH_{F_n} \cdot 1 = 0$ ($F_n = {}^cS - {}^cE_n$) by theorem 2.2. One can easily see that the function ${}^ch(x) = \lim {}^cH_{F_n} \cdot 1(x)$ is an invariant function for cP (i.e. ${}^cP {}^ch = {}^ch$) and bounded by 1. On the other hand,

$$1 = {}^cG(1 - {}^cP \cdot 1)(x) + \lim {}^cP^n \cdot 1(x)$$

and

$$\lim {}^cP^n \cdot 1(x) = \lim P_x[\sigma_{\{c\}} > n] = 0,$$

implies that 1 is a potential of non-negative function (where $\sigma_{\{c\}}$ is the hitting time of the Markov chain with transition probability P). Hence ch is also a potential. The fact that ch is an invariant function and also a potential shows that ${}^ch = 0$.

The main result of the present paper is this.

Theorem 3.1. *Given a positive measure α and a kernel A satisfying maximum principle (RSCM) and condition $(*)$, there exists a unique irreducible recurrent transition probability P which has α as its invariant measure, and A as its weak potential kernel.*

Uniqueness was proved by Kondō [4]. We shall divide the proof of existence into several lemmas. In the following we shall use the notation of section 2 with no further reference.

Lemma 3.1. *There exists a sub-Markov transition probability ${}^cP(x, y)$ on cS such that*

$${}^cG(x, y) = \sum_{n=0}^{\infty} {}^cP^n(x, y) \quad \text{for every } x, y \text{ in } {}^cS.$$

Proof. See Meyer [7] P. 238 lemma 10.

Lemma 3.2. *For every $y \in {}^cS$, $\sum_{x \neq c} \alpha(x) {}^cP(x, y) \leq \alpha(y)$.*

Proof. To the contrary, suppose that there exists some state $y \in {}^cS$ such that

$$\sum_{x \neq c} \alpha(x) {}^cP(x, y) - \alpha(y) > 0.$$

Then there exists a finite subset F of cS containing y and satisfying

$$\sum_{x \in F} \alpha(x) {}^cP(x, y) - \alpha(y) = a > 0.$$

Define a function $f \in N$ by

$$f(x) = \begin{cases} {}^cP(x, y) - I(x, y) & x \in F \\ -\frac{a}{\alpha(c)} & x = c \\ 0 & \text{otherwise.} \end{cases}$$

Since $Af + f^-$ attains its maximum on the set $\{f > 0\}$ and since $f(c) < 0$, there exists a state $x_0 \in F$ such that,

$$Af(x_0) \geq Af + f^- \quad \text{everywhere on } S.$$

In particular,

$$Af(x_0) \geq Af(c) + \frac{a}{\alpha(c)}.$$

Hence,

$$0 > -\frac{a}{\alpha(c)} \geq Af(c) - Af(x_0) = {}^cG(-{}^c f)(x_0),$$

where ${}^c f$ is the restriction of f to ${}^c S$. On the other hand,

$$\begin{aligned} {}^cG(-{}^c f)(x_0) &= {}^cG(x_0, y) - \sum_{z \in F} {}^cG(x_0, z) {}^cP(z, y) \\ &\geq {}^cG(x_0, y) - ({}^cG(x_0, y) - I(x_0, y)) = I(x_0, y) \geq 0. \end{aligned}$$

This lead us to a contradiction.

Lemma 3.3. ${}^cG(1 - {}^cP \cdot 1) = 1$ on ${}^c S$.

Proof. For any positive integer n , we have

$$1 = \sum_{k=0}^n {}^cP^k(1 - {}^cP \cdot 1)(x) + {}^cP^{n+1} \cdot 1(x).$$

Passing to the limit we obtain

$$1 = {}^cG(1 - {}^cP \cdot 1)(x) + r(x),$$

where $r(x) = \lim {}^cP^{n+1} \cdot 1(x)$. It remains to show that $r(x) = 0$. From condition (*) for arbitrary $\varepsilon > 0$ there exists a number M such that for any integer $m \geq M$,

$${}^cH_{F_m} \cdot 1(x) < \varepsilon.$$

Hence

$$\sum_{y \neq c} {}^cP^{n+1}(x, y) = {}^cP^{n+1}I_{F_m}(x) + {}^cP^{n+1}I_{E_m}(x) \leq {}^cH_{F_m} \cdot 1(x) + {}^cP^{n+1}I_{E_m}(x),$$

where I_F is the indicator function of F . Tending n to infinity we obtain $r(x) \leq \varepsilon$.

Lemma 3.4. $\sum_{x \neq c} \alpha(x)(1 - {}^cP \cdot 1)(x) \leq \alpha(c)$.

Proof. Let F be an arbitrary finite subset of ${}^c S$, and define

$$f(x) = \begin{cases} 1 - {}^c P \cdot 1(x) & x \in F \\ -\sum_{y \in F} \frac{\alpha(y)}{\alpha(c)} (1 - {}^c P \cdot 1(y)) & x = c \\ 0 & \text{otherwise.} \end{cases}$$

As noted in the proof of lemma 3.2, there exists a state $x_0 \in F$ such that

$$Af(x_0) \geq Af + f^- \quad \text{on } S.$$

In particular,

$${}^c G^c f(x_0) = Af(x_0) - Af(c) \geq f^-(c) = \sum_{y \in F} \alpha(y) (1 - {}^c P \cdot 1(y)) / \alpha(c),$$

and by lemma 3.3, the left side of the above inequality is bounded by 1.

Now we can define the desired transition probability P .

$$(3.1) \quad P(x, y) = \begin{cases} {}^c P(x, y) & x \neq c, y \neq c \\ 1 - {}^c P \cdot 1(x) & x \neq c, y = c \\ (\alpha(y) - \alpha^c P(y)) / \alpha(c) & x = c, y \neq c \\ 1 - \sum_{z \neq c} P(c, z) & x = c, y = c. \end{cases}$$

From lemmas 3.2 and 3.4, P is a transition probability on S .

Lemma 3.5. $\alpha P = \alpha$ and $(I - P)Af = f$ for any $f \in N$.

Proof. If $x \neq c$, then

$$\alpha P(x) = \sum_{y \neq c} \alpha(y) {}^c P(y, x) + \alpha(x) - \alpha^c P(x) = \alpha(x)$$

and

$$(I - P)Af(x) = (I - P)(Af(c) + {}^c Gf)(x) = f(x).$$

By the same argument for $x = c$, lemma follows.

Lemma 3.6. *The transition probability P is recurrent and irreducible.*

Proof. Let $\sigma_{\{x\}}$ be the hitting time for x of the Markov chain with transition probability P . Then for every $x \neq c$,

$$\mathbf{P}_x[\sigma_{\{c\}} < \infty] = \sum_{y \neq c} {}^c G(x, y) P(y, c) = {}^c G(1 - {}^c P \cdot 1)(x) = 1,$$

by lemma 3.3. Hence,

$$\mathbf{P}_c[\sigma_{\{c\}}^+ < \infty] = \sum_{x \in S} P(c, x) \mathbf{P}_x[\sigma_{\{c\}} < \infty] = 1,$$

where $\sigma_{\{c\}}^+$ is the positive hitting time for state c . Thus c is a recurrent state for P and hence also for \hat{P} , where \hat{P} is defined by,

$$(3.2) \quad \hat{P}(x, y) = \frac{\alpha(y)}{\alpha(x)} P(y, x).$$

Moreover,

$$\hat{P}^n(c, x) = \frac{\alpha(x)}{\alpha(c)} P^n(x, c), \text{ and } P_x[\sigma_{\{c\}} < \infty] = 1,$$

shows that

$$\hat{P}_c[\sigma_{\{x\}} < \infty] > 0 \text{ for all } x \in {}^c S.$$

Hence x is a recurrent state for \hat{P} and hence for P . Since x is recurrent and $P_x[\sigma_{\{c\}} < \infty] = 1$, it follows that $P_c[\sigma_{\{x\}} < \infty] = 1$ for all $x \in S$. Irreducibility follows from the fact that, $P_x[\sigma_{\{c\}} < \infty] = 1$ and $P_c[\sigma_{\{y\}} < \infty] = 1$ for every x, y in S .

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