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Osaka University
A NECESSARY AND SUFFICIENT CONDITION FOR A KERNEL TO BE A WEAK POTENTIAL KERNEL OF A RECURRENT MARKOV CHAIN

YOICHI OSHIMA

(Received December 14, 1968)

1. Introduction

Let \( P \) be an irreducible recurrent transition probability on a denumerable space \( S \) with invariant measure \( \alpha \). Let \( c \) be an arbitrary (but fixed) state of \( S \). Then from the work of Kondo [3] and Orey [8], there exist the class of weak potential kernels \( A(x, y) \) defined by the property that, for every null charge \( f \), \( Af \) is bounded and satisfies the equation

\[
(I - P)Af = f.
\]

Moreover \( Af \) is represented by

\[
Af = ^cGf + 1(f),
\]

where \( f \) is a null charge, \( 1(\cdot) \) is an arbitrary linear functional on the space of null charges and \( ^cG \) is defined as follow;

\[
\begin{align*}
^cP(x, y) &= \begin{cases} 
P(x, y) & x \neq c, \; y \neq c \\
0 & \text{otherwise,}
\end{cases} \\
^cG(x, y) &= \begin{cases} 
\sum_{n=0}^\infty ^cP^n(x, y) & x \neq c, \; y \neq c \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Moreover \( A \) satisfies the following maximum principle [4], [5]:

(RSCM)\(^1\) If \( m \) is a real number and \( f \) is a null charge then the relation that

\[
m \geq Af
\]

on the set \( \{f > 0\} \)

implies that

\[
m - f^- \geq Af \quad \text{everywhere,}
\]

\(^1\) This is the abbreviation of "reinforced semi-complete maximum principle"; this maximum principle corresponds to the semi-complete M.P. as well as the reinforced M.P. (of Meyer) corresponds to the complete M.P.
where \( f^- = (-f) \land 0 \).

In the present paper we are concerned with the following construction problem. Given a positive measure \( \alpha \) and a (not necessarily positive) kernel \( A \) satisfying (RSCM), does there exist an irreducible recurrent transition probability which has \( \alpha \) as its invariant measure, and \( A \) as its weak potential kernel? This is not true in general\(^2\), but as Kondo \([4]\) has proved, it is true if \( \alpha \) is a finite measure. In section 2 we shall introduce another necessary condition for the weak potential kernel \( A \) (referred to as condition (*)). Then we shall prove (theorem 3.1) that, if the pair \((A, \alpha)\) satisfies maximum principle (RSCM) and condition (*), \( A \) is a weak potential kernel of a (unique) recurrent Markov chain with \( \alpha \) as its invariant measure.

I should like to express my hearty gratitude to T. Watanabe for his kind advices.

2. Some potential theory for a kernel \( A \) satisfying (RSCM)

Let \( \alpha \) be a strictly positive measure and \( A \), a kernel on \( S \). A function \( f \) on \( S \) is said to be a null charge with respect to \( \alpha \) if \( \sum \alpha(x) |f(x)| < \infty \) and \( \sum \alpha(x)f(x) = 0 \). Let \( N \) be the space of null charges vanishing outside a finite subset of \( S \). We assume that the kernel \( A \) satisfies condition (RSCM) for \( f^N \). Fix an arbitrary state \( c \) and define

\[
(2.1) \quad \mathcal{G}(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)}.
\]

If \( A \) is a weak potential kernel then (2.1) is clearly satisfied by taking \( f \) in equation (1.2) as

\[
(2.2) \quad f(x) = \begin{cases} \frac{\alpha(y)}{\alpha(c)} & x = c \\ -1 & x = y \\ 0 & \text{otherwise,} \end{cases}
\]

and calculating \( Af(x) - Af(c) \).

From definition (2.1) \( \mathcal{G}(c, x) = \mathcal{G}(x, c) = 0 \) for every \( x \in S \).

**Lemma 2.1** For arbitrary elements \( x, y \) in \( S \) which are different from \( c \)

\[
I(x, y) \preceq \mathcal{G}(x, y) \preceq \mathcal{G}(y, y).
\]

Proof. By taking \( f \) as (2.2) we have

\[
Af(c) = A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y)
\]

\(^2\) A counter example was given by Kondo and T. Watanabe.
\[ Af(y) = A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y). \]

Hence, if we write \( f^+ = f \lor 0, f^- = (-f) \lor 0 \), by (RSCM)

\[ A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) + f^+(x) \leq Af(x) \leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - f^-(x), \]

so that

\[ (2.3) \quad A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) \leq A(x, c) \frac{\alpha(y)}{\alpha(c)} - A(x, y) \]

\[ \leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - I(x, y), \]

which proves the lemma.

**Corollary.** For every \( x \in S \) there exists a constant \( C \) such that

\[ ^cG(x, y) \leq C \cdot \alpha(y) \quad \text{for every } y \in S. \]

Proof. Exchanging \( c \) and \( x \) in the second inequality in (2.3), it follows that

\[ ^cG(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)} \]

\[ \leq \left( - \frac{A(x, c)}{\alpha(c)} - \frac{A(c, c)}{\alpha(x)} + \frac{A(c, c)}{\alpha(c)} + \frac{A(x, x)}{\alpha(x)} \right) \alpha(y). \]

Let \( ^cS \) be the set \( S - \{c\} \), and \( ^cM \) be the space of all functions on \( ^cS \) vanishing outside a finite subset of \( ^cS \). Let \( ^cM^+ \) be the space of all non-negative functions in \( ^cM \).

**Theorem 2.1.** The kernel \( ^cG \) satisfies the reinforced maximum principle [7]:

(RM) If \( a \) is a non-negative constant and if \( ^c f \) and \( ^c g \) are two elements of \( ^cM^+ \), then the relation that

\[ (2.5) \quad a + ^cG \cdot \overline{f} - ^cG \cdot \overline{g} \geq ^cG \cdot \overline{g} \]

on the set \( \{^c g > 0\} \) implies that

\[ (2.6) \quad a + ^cG \cdot \overline{f} - ^cG \cdot \overline{g} \geq ^cG \cdot \overline{g} \quad \text{everywhere on } ^cS. \]

Proof. Let \( f \) be the function on \( S \) such that \( f \in N \) and \( f \mid \_S = ^c f \). Such \( f \) is obviously unique. The function \( g \in N \) is defined similarly. Then inequality (2.5) implies that

\[ a + A(g - f)(c) \geq A(g - f) \quad \text{on the set } \{g - f > 0\}. \]

For, since \( ^c f \) and \( ^c g \) are non-negative, the set \( \{g - f > 0\} \) is contained in the union of \( c \) and \( \{^c g > 0\} \). Hence by (RSCM)
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\[ a + A(g-f)(c)-(g-f) \geq A(g-f) \]  everywhere.

Since the function \((g-f)^{\ast}\) is equal to \(\epsilon f\) on \(\epsilon S \cap \{g=0\}\), the above inequality, combined with (2.5), proves the theorem.

A non-negative function \(\epsilon h\) on \(\epsilon S\) is said to be quasi-excessive\(^3\) if, for every \(\epsilon g \in \epsilon M\), the inequality

\[ \epsilon h \geq \epsilon G^c g \]  on the set \(\{\epsilon g > 0\}\)

implies that

\[ \epsilon h - \epsilon g \geq \epsilon G^c g \]  everywhere.

Moreover Meyer introduced the notion of the pseudo-réduite \(\epsilon H^c G^c h\) for every quasi-excessive function \(\epsilon h\) and every subset \(E\) of \(\epsilon S\). This function \(\epsilon H^c G^c h\) satisfies the following four conditions.

(2.7) \(\epsilon H^c G^c h\) is quasi-excessive.

(2.8) \(\epsilon H^c G^c h \leq \epsilon h\) on \(\epsilon S\) and \(\epsilon H^c G^c h = \epsilon h\) on \(E\).

(2.9) If \(\epsilon h_1\) and \(\epsilon h_2\) are two quasi-excessive functions such that \(\epsilon h_1 \leq \epsilon h_2\) on \(E\), then \(\epsilon H^c G^c h_1 \leq \epsilon H^c G^c h_2\).

(2.10) If \(\epsilon f \in \epsilon M^+\) vanishes outside of \(E\) then \(\epsilon H^c G^c f = \epsilon G^c f\).

For example, the function \(\epsilon G^c f\), \(\epsilon f \in \epsilon M^+\), and every positive constant are quasi-excessive ([7] see also [5]).

Now we introduce a condition.

Condition (*): There exists a sequence of finite sets \(\{\epsilon E_n\}_{n=1,2,\ldots}\) increasing to \(\epsilon S\) such that \(\epsilon c \in \epsilon E_n\) for each \(n\), and a sequence \(\{\epsilon h_n\}_{n=1,2,\ldots}\) of function on \(\epsilon S\) satisfying the following conditions.

(i) \(0 \leq \epsilon h_n \leq 1\), \(\epsilon h_n(c) = 0\), \(\epsilon h_n = 1\) on \(\epsilon F_n = \epsilon S - \epsilon E_n\), and \(\lim \epsilon h_n = 0\).

(ii) For every \(\epsilon f \in \epsilon N\) and every real number \(m \geq Af(c)\) the relation that

\[ m + \epsilon h_n \geq Af \]  on the set \(\{\epsilon f > 0\}\),

implies that

\[ m + \epsilon h_n - f \geq Af \]  everywhere on \(\epsilon S\).

In section 3 we shall show that if \(A\) is a weak potential kernel of an irreducible recurrent Markov chain, it satisfies condition (*).

**Theorem 2.2** Condition (*) is equivalent to the condition that, there exists a sequence of finite sets \(\{\epsilon E_n\}_{n=1,2,\ldots}\) increasing to \(\epsilon S\) such that

\[ \lim \epsilon H_{S - \epsilon E_n} 1 = 0 . \]  (2.11)

Proof. Suppose that condition (*) holds and let \(\epsilon h_n\) be the restriction of \(\epsilon h\) to \(\epsilon S\) and \(\epsilon E_n = \epsilon S \cap \epsilon E_n\). Obviously \(\epsilon S - \epsilon E_n = \epsilon F_n\), \(0 \leq \epsilon h_n \leq 1\), and \(\epsilon h_n = 1\) on \(\epsilon F_n\). It then follows that \(\epsilon h_n\) is a quasi-excessive function for every \(n\). In fact,

\[^3\) This definition is slightly different from Meyer's one; this is the discrete version of Meyer's,
let \( f \) be in \( \mathcal{M} \) and \( f \), the extension of \( f \) to \( S \) such that \( f \in \mathcal{N} \). If
\[
\mathcal{C} h_n \geq \mathcal{C} G f \quad \text{on the set } \{ f > 0 \}
\]
then
\[
h_n + Af(c) \geq Af \quad \text{on the set } \{ f > 0 \},
\]
since \( \{ f > 0 \} \) is contained in \( \{ f > 0 \} \cup \{ c \} \). Hence from condition \((*)\),
\[
h_n + Af(c) - f \geq Af \quad \text{everywhere on } \mathcal{C} S,
\]
that is,
\[
\mathcal{C} h_n - f \geq \mathcal{C} G f \quad \text{everywhere on } \mathcal{C} S.
\]
Since \( \mathcal{C} H_{F_n} \cdot 1 \leq \mathcal{C} h_n \) by definition,
\[
\lim \mathcal{C} H_{F_n} \cdot 1 = 0.
\]
Conversely, if \((2.11)\) holds, set \( \mathcal{E} E_n \cup \{ c \} = E_n, \ F_n = S - E_n \) and
\[
h_n = \begin{cases} \mathcal{C} H_{F_n} \cdot 1 & \text{on } \mathcal{C} S \\ 0 & \text{at } c. \end{cases}
\]
It is enough to show the property \((ii)\) of condition \((*)\). Suppose that, for some \( f \in \mathcal{N} \) and some real number \( m (\geq Af(c)) \)
\[
m + h_n \geq Af \quad \text{on } \{ f > 0 \}.
\]
Then one has
\[
m - Af(c) + \mathcal{C} H_{F_n} \cdot 1 \geq \mathcal{C} G f \quad \text{on } \{ f > 0 \},
\]
where \( f \) is the restriction of \( f \) to \( \mathcal{C} S \). The fact that \( m - Af(c) + \mathcal{C} H_{F_n} \cdot 1 \) is a quasi-

Note. If \( \alpha \) is a finite measure, then condition \((*)\) is satisfied.

Let \( I_F \) be the indicator function of a set \( F \), then from lemma 2.1 \( \mathcal{C} GI_F \geq 1 \) on \( F \). Hence from \((2.8)\) and \((2.9)\) \( \mathcal{C} H_{F} \cdot 1 \leq \mathcal{C} GI_F \). Hence if \( F_n \) decrease to empty
set, inequality
\[
\mathcal{C} H_{F_n} \cdot 1(x) \leq \mathcal{C} GI_{F_n}(x) \leq \sum_{y \in F_n} C \cdot \alpha(y),
\]
implies that
\[
\lim \mathcal{C} H_{F_n} \cdot 1(x) = 0.
\]
Where the second inequality follows from the corollary of lemma 2.1.

3. Main result

Let \( A \) be a weak potential kernel of an irreducible recurrent transition
probability $P$ with invariant measure $\alpha$. We shall now prove that $A$ satisfies condition (*) of section 2.

Define $^cP$ and $^cG$ as (1.3) and (1.4) respectively. Let $^cH_F$ be the réduit defined by $^cP$. Since $^cH_F\cdot 1$ is the pseudo-réduit associated with the above $^cG$ (see [5] P. 37, theorem 1.3), it is enough to show that for a sequence of finite sets $\{E_n\}_{n=1}^{\infty}$ increasing to $^cS$, $\lim^cH_{F_n}\cdot 1=0 \ (F_n=^cS-E_n)$ by theorem 2.2. One can easily see that the function $^c\eta(x)=\lim^cH_{F_n}\cdot 1(x)$ is an invariant function for $^cP$ (i.e. $^cP^c\eta=^c\eta$) and bounded by 1. On the other hand,

$$1 = {^cG}(1-^cP\cdot 1)(x) + \lim^cP^n\cdot 1(x)$$

implies that 1 is a potential of non-gengative function (where $\sigma_{(c)}$ is the hitting time of the Markov chain with transition probability $P$). Hence $^c\eta$ is also a potential. The fact that $^c\eta$ is an invariant function and also a potential shows that $^c\eta=0$.

The main result of the present paper is this.

**Theorem 3.1.** Given a positive measure $\alpha$ and a kernel $A$ satisfying maximum principle (RSCM) and condition (*), there exists a unique irreducible recurrent transition probability $P$ which has $\alpha$ as its invariant measure, and $A$ as its weak potential kernel.

Uniqueness was proved by Kondō [4]. We shall divide the proof of existence into several lemmas. In the following we shall use the notation of section 2 with no further reference.

**Lemma 3.1.** There exists a sub-Markov transition probability $^cP(x, y)$ on $^cS$ such that

$$^cG(x, y) = \sum_{n=0}^{\infty}^cP^n(x, y) \quad \text{for every } x, y \text{ in } ^cS.$$ 


**Lemma 3.2.** For every $y \in ^cS$, $\sum_{x \in ^cF} \alpha(x)^cP(x, y) \leq \alpha(y)$.

Proof. To the contrary, suppose that there exists some state $y \in ^cS$ such that

$$\sum_{x \in ^cF} \alpha(x)^cP(x, y) - \alpha(y) > 0.$$ 

Then there exists a finite subset $F$ of $^cS$ containing $y$ and satisfying

$$\sum_{x \in F} \alpha(x)^cP(x, y) - \alpha(y) = a > 0.$$ 

Define a function $f \in N$ by
WEAK POTENTIAL KERNEL OF A RECURRENT MARKOV CHAIN

\[
f(x) = \begin{cases} 
  \varepsilon P(x, y) - I(x, y) & x \in F \\
  -\frac{a}{\alpha(c)} & x = c \\
  0 & \text{otherwise.}
\end{cases}
\]

Since \( A\phi + f^- \) attains its maximum on the set \( \{f > 0\} \) and since \( f(c) < 0 \), there exists a state \( x_0 \in F \) such that,

\[ A\phi(x_0) \geq A\phi + f^- \quad \text{everywhere on } S. \]

In particular,

\[ A\phi(x_0) \geq A\phi(c) - \frac{a}{\alpha(c)}. \]

Hence,

\[ 0 > -\frac{a}{\alpha(c)} \geq A\phi(c) - A\phi(x_0) = \varepsilon G(-\varepsilon f)(x_0), \]

where \( \varepsilon f \) is the restriction of \( f \) to \( \varepsilon S \). On the other hand,

\[
\varepsilon G(-\varepsilon f)(x_0) = \varepsilon G(x_o, y) - \sum_{x \in F} \varepsilon G(x_o, z)^\varepsilon P(z, y) \\
\geq \varepsilon G(x_o, y) - (\varepsilon G(x_o, y) - I(x_o, y)) = I(x_o, y) \geq 0.
\]

This lead us to a contradiction.

**Lemma 3.3.** \( \varepsilon G(1 - \varepsilon P \cdot 1) = 1 \) on \( \varepsilon S \).

**Proof.** For any positive integer \( n \), we have

\[ 1 = \sum_{k=0}^n \varepsilon P^k(1 - \varepsilon P \cdot 1)(x) + \varepsilon P^{n+1} \cdot 1(x). \]

Passing to the limit we obtain

\[ 1 = \varepsilon G(1 - \varepsilon P \cdot 1)(x) + r(x), \]

where \( r(x) = \lim \varepsilon P^{n+1} \cdot 1(x) \). It remains to show that \( r(x) = 0 \). From condition (*) for arbitrary \( \varepsilon > 0 \) there exists a number \( M \) such that for any integer \( m \geq M \),

\[ \varepsilon H_{F_m} \cdot 1(x) < \varepsilon. \]

Hence

\[ \sum_{x \in \varepsilon} \varepsilon P^{n+1}(x, y) = \varepsilon P^{n+1} I_{F_m} + \varepsilon P^{n+1} I_{E_m}(x) \leq \varepsilon H_{F_m} \cdot 1(x) + \varepsilon P^{n+1} I_{E_m}(x), \]

where \( I_F \) is the indicator function of \( F \). Tending \( n \) to infinity we obtain \( r(x) \leq \varepsilon \).

**Lemma 3.4.** \( \sum_{x \in \varepsilon} \alpha(x)(1 - \varepsilon P \cdot 1)(x) \leq \alpha(c) \).

**Proof.** Let \( F \) be an arbitrary finite subset of \( \varepsilon S \), and define
As noted in the proof of lemma 3.2, there exists a state \( x_0 \in F \) such that
\[
Af(x_0) \geq Af + f^-
\]
on \( S \).

In particular,
\[
\mathcal{C}G f(x_0) = Af(x_0) - Af(c) \geq f^-(c) = \sum_{y \in F} \alpha(y)(1 - \mathcal{C}P \cdot 1(y))/\alpha(c),
\]
and by lemma 3.3, the left side of the above inequality is bounded by 1.

Now we can define the desired transition probability \( P \).

\[
P(x, y) = \begin{cases} 
\mathcal{C}P(x, y) & x \neq c, y \neq c \\
1 - \mathcal{C}P \cdot 1(x) & x \neq c, y = c \\
(\alpha(y) - \alpha^c P(y))/\alpha(c) & x = c, y \neq c \\
1 - \sum_{z \neq c} P(c, z) & x = c, y = c.
\end{cases}
\]

From lemmas 3.2 and 3.4, \( P \) is a transition probability on \( S \).

**Lemma 3.5.** \( \alpha P = \alpha \) and \((I - P)Af = f \) for any \( f \in N \).

**Proof.** If \( x \neq c \), then
\[
\alpha P(x) = \sum_{y \neq c} \alpha(y)\mathcal{C}P(y, x) + \alpha(x) - \alpha^c P(x) = \alpha(x)
\]
and
\[
(I - P)Af(x) = (I - P)(Af(c) + \mathcal{C}f)(x) = f(x).
\]

By the same argument for \( x = c \), lemma follows.

**Lemma 3.6.** The transition probability \( P \) is recurrent and irreducible.

**Proof.** Let \( \sigma_{(x)} \) be the hitting time for \( x \) of the Markov chain with transition probability \( P \). Then for every \( x \neq c \),
\[
P_x[\sigma_{(x)} < \infty] = \sum_{y \neq c} \mathcal{C}G(x, y)P(y, c) = \mathcal{C}G(1 - \mathcal{C}P \cdot 1) (x) = 1,
\]
by lemma 3.3. Hence,
\[
P_c[\sigma^+_c < \infty] = \sum_{x \in S} P(c, x)P_x[\sigma_{(x)} < \infty] = 1,
\]
where \( \sigma^+_c \) is the positive hitting time for state \( c \). Thus \( c \) is a recurrent state for \( P \) and hence also for \( \hat{P} \), where \( \hat{P} \) is defined by,
\[
\hat{P}(x, y) = \frac{\alpha(y)}{\alpha(x)} P(y, x).
\]
Moreover,
\[
\hat{P}^n(x, c) = \frac{\alpha(x)}{\alpha(c)} P^n(x, c), \text{ and } P_x[\sigma_{\{c\}} < \infty] = 1,
\]
shows that
\[
\hat{P}_x[\sigma_{\{x\}} < \infty] > 0 \text{ for all } x \in S.
\]
Hence \(x\) is a recurrent state for \(\hat{P}\) and hence for \(P\). Since \(x\) is recurrent and \(P_x[\sigma_{\{x\}} < \infty] = 1\), it follows that \(P_x[\sigma_{\{x\}} < \infty] = 1\) for all \(x \in S\). Irreducibility follows from the fact that, \(P_x[\sigma_{\{c\}} < \infty] = 1\) and \(P_x[\sigma_{\{y\}} < \infty] = 1\) for every \(x, y\) in \(S\).

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