



Title	A necessary and sufficient condition for a kernel to be a weak potential kernel of a recurrent Markov chain
Author(s)	Ôshima, Yoichi
Citation	Osaka Journal of Mathematics. 1969, 6(1), p. 29-37
Version Type	VoR
URL	<a href="https://doi.org/10.18910/10624">https://doi.org/10.18910/10624</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## A NECESSARY AND SUFFICIENT CONDITION FOR A KERNEL TO BE A WEAK POTENTIAL KERNEL OF A RECURRENT MARKOV CHAIN

YOICHI ŌSHIMA

(Received December 14, 1968)

### 1. Introduction

Let  $P$  be an irreducible recurrent transition probability on a denumerable space  $S$  with invariant measure  $\alpha$ . Let  $c$  be an arbitrary (but fixed) state of  $S$ . Then from the work of Kondō [3] and Orey [8], there exist the class of *weak potential kernels*  $A(x, y)$  defined by the property that, for every null charge  $f$ ,  $Af$  is bounded and satisfies the equation

$$(1.1) \quad (I - P)Af = f.$$

Moreover  $Af$  is represented by

$$(1.2) \quad Af = {}^cGf + 1(f),$$

where  $f$  is a null charge,  $1(\cdot)$  is an arbitrary linear functional on the space of null charges and  ${}^cG$  is defined as follow;

$$(1.3) \quad {}^cP(x, y) = \begin{cases} P(x, y) & x \neq c, y \neq c \\ 0 & \text{otherwise,} \end{cases}$$

$$(1.4) \quad {}^cG(x, y) = \begin{cases} \sum_{n=0}^{\infty} {}^cP^n(x, y) & x \neq c, y \neq c \\ 0 & \text{otherwise.} \end{cases}$$

Moreover  $A$  satisfies the following maximum principle [4], [5]:

(RSCM)<sup>1)</sup> If  $m$  is a real number and  $f$  is a null charge then the relation that

$$(1.5) \quad m \geq Af \quad \text{on the set } \{f > 0\}$$

implies that

$$(1.6) \quad m - f^- \geq Af \quad \text{everywhere,}$$

---

1) This is the abbreviation of "reinforced semi-complete maximum principle"; this maximum principle corresponds to the semi-complete M.P. as well as the reinforced M.P. (of Meyer) corresponds to the complete M.P.

where  $f^- = (-f) \vee 0$ .

In the present paper we are concerned with the following construction problem. Given a positive measure  $\alpha$  and a (not necessarily positive) kernel  $A$  satisfying (RSCM), does there exist an irreducible recurrent transition probability which has  $\alpha$  as its invariant measure, and  $A$  as its weak potential kernel? This is not true in general<sup>2)</sup>, but as Kondō [4] has proved, it is true if  $\alpha$  is a finite measure. In section 2 we shall introduce another necessary condition for the weak potential kernel  $A$  (referred to as condition (\*)). Then we shall prove (theorem 3.1) that, if the pair  $(A, \alpha)$  satisfies maximum principle (RSCM) and condition (\*),  $A$  is a weak potential kernel of a (unique) recurrent Markov chain with  $\alpha$  as its invariant measure.

I should like to express my hearty gratitude to T. Watanabe for his kind advices.

## 2. Some potential theory for a kernel $A$ satisfying (RSCM)

Let  $\alpha$  be a strictly positive measure and  $A$ , a kernel on  $S$ . A function  $f$  on  $S$  is said to be a *null charge with respect to  $\alpha$*  if  $\sum \alpha(x)|f(x)| < \infty$  and  $\sum \alpha(x)f(x) = 0$ . Let  $N$  be the space of null charges vanishing outside a finite subset of  $S$ . We assume that *the kernel  $A$  satisfies condition (RSCM) for  $f \in N$* . Fix an arbitrary state  $c$  and define

$$(2.1) \quad {}^cG(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)}.$$

If  $A$  is a weak potential kernel then (2.1) is clearly satisfied by taking  $f$  in equation (1.2) as

$$(2.2) \quad f(x) = \begin{cases} \frac{\alpha(y)}{\alpha(c)} & x = c \\ -1 & x = y \\ 0 & \text{otherwise,} \end{cases}$$

and calculating  $Af(x) - Af(c)$ .

From definition (2.1)  ${}^cG(c, x) = {}^cG(x, c) = 0$  for every  $x \in S$ .

**Lemma 2.1** *For arbitrary elements  $x, y$  in  $S$  which are different from  $c$*

$$I(x, y) \leq {}^cG(x, y) \leq {}^cG(y, y).$$

*Proof.* By taking  $f$  as (2.2) we have

$$Af(c) = A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y)$$

---

2) A counter example was given by Kondō and T. Watanabe.

$$Af(y) = A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y).$$

Hence, if we write  $f^+ = f \vee 0$ ,  $f^- = (-f) \vee 0$ , by (RSCM)

$$A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) + f^+(x) \leq Af(x) \leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - f^-(x),$$

so that

$$(2.3) \quad \begin{aligned} A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) &\leq A(x, c) \frac{\alpha(y)}{\alpha(c)} - A(x, y) \\ &\leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - I(x, y), \end{aligned}$$

which proves the lemma.

**Corollary.** *For every  $x \in S$  there exists a constant  $C$  such that*

$$(2.4) \quad {}^cG(x, y) \leq C \cdot \alpha(y) \quad \text{for every } y \in S.$$

Proof. Exchanging  $c$  and  $x$  in the second inequality in (2.3), it follows that

$$\begin{aligned} {}^cG(x, y) &= A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)} \\ &\leq \left( -\frac{A(x, c)}{\alpha(c)} - \frac{A(c, x)}{\alpha(x)} + \frac{A(c, c)}{\alpha(c)} + \frac{A(x, x)}{\alpha(x)} \right) \alpha(y). \end{aligned}$$

Let  ${}^cS$  be the set  $S - \{c\}$ , and  ${}^cM$  be the space of all functions on  ${}^cS$  vanishing outside a finite subset of  ${}^cS$ . Let  ${}^cM^+$  be the space of all non-negative functions in  ${}^cM$ .

**Theorem 2.1.** *The kernel  ${}^cG$  satisfies the reinforced maximum principle [7]: (RM) If  $a$  is a non-negative constant and if  ${}^cf$  and  ${}^cg$  are two elements of  ${}^cM^+$ , then the relation that*

$$(2.5) \quad a + {}^cG^c f - {}^cf \geq {}^cG^c g \quad \text{on the set } \{{}^cg > 0\} \text{ implies that}$$

$$(2.6) \quad a + {}^cG^c f - {}^cf \geq {}^cG^c g \quad \text{everywhere on } {}^cS.$$

Proof. Let  $f$  be the function on  $S$  such that  $f \in N$  and  $f|_{{}^cS} = {}^cf$ . Such  $f$  is obviously unique. The function  $g \in N$  is defined similarly. Then inequality (2.5) implies that

$$a + A(g - f)(c) \geq A(g - f) \quad \text{on the set } \{g - f > 0\}.$$

For, since  ${}^cf$  and  ${}^cg$  are non-negative, the set  $\{g - f > 0\}$  is contained in the union of  $c$  and  $\{{}^cg > 0\}$ . Hence by (RSCM)

$$a + A(g-f)(c) - (g-f)^- \geq A(g-f) \quad \text{everywhere.}$$

Since the function  $(g-f)^-$  is equal to  ${}^c f$  on  ${}^c S \cap \{g=0\}$ , the above inequality, combined with (2.5), proves the theorem.

A non-negative function  ${}^c h$  on  ${}^c S$  is said to be *quasi-excessive*<sup>3)</sup> if, for every  ${}^c g \in {}^c M$ , the inequality

$${}^c h \geq {}^c G^c g \quad \text{on the set } \{{}^c g > 0\}$$

implies that

$${}^c h - {}^c g^- \geq {}^c G^c g \quad \text{everywhere.}$$

Moreover Meyer introduced the notion of the *pseudo-réduite*  ${}^c H_E {}^c h$  for every quasi-excessive function  ${}^c h$  and every subset  $E$  of  ${}^c S$ . This function  ${}^c H_E {}^c h$  satisfies the following four conditions.

(2.7)  ${}^c H_E {}^c h$  is quasi-excessive.

(2.8)  ${}^c H_E {}^c h \leq {}^c h$  on  ${}^c S$  and  ${}^c H_E {}^c h = {}^c h$  on  $E$ .

(2.9) If  ${}^c h_1$  and  ${}^c h_2$  are two quasi-excessive functions such that  ${}^c h_1 \leq {}^c h_2$  on  $E$ , then  ${}^c H_E {}^c h_1 \leq {}^c H_E {}^c h_2$ .

(2.10) If  ${}^c f \in {}^c M^+$  vanishes outside of  $E$  then  ${}^c H_E {}^c G^c f = {}^c G^c f$ .

For example, the function  ${}^c G^c f$ ,  ${}^c f \in {}^c M^+$ , and every positive constant are quasi-excessive ([7] see also [5]).

Now we introduce a condition.

Condition (\*): There exists a sequence of finite sets  $\{E_n\}_{n=1,2,\dots}$  increasing to  $S$  such that  $c \in E_n$  for each  $n$ , and a sequence  $\{h_n\}_{n=1,2,\dots}$  of function on  $S$  satisfying the following conditions.

(i)  $0 \leq h_n \leq 1$ ,  $h_n(c) = 0$ ,  $h_n = 1$  on  $F_n = S - E_n$ , and  $\lim h_n = 0$ .

(ii) For every  $f \in N$  and every real number  $m$  ( $\geq Af(c)$ ) the relation that

$$m + h_n \geq Af \quad \text{on the set } \{f > 0\}.$$

implies that

$$m + h_n - f^- \geq Af \quad \text{everywhere on } {}^c S.$$

In section 3 we shall show that if  $A$  is a weak potential kernel of an irreducible recurrent Markov chain, it satisfies condition (\*).

**Theorem 2.2** Condition (\*) is equivalent to the condition that, there exists a sequence of finite sets  $\{E_n\}_{n=1,2,\dots}$  increasing to  ${}^c S$  such that

$$(2.11) \quad \lim {}^c H_{c_S - c_{E_n}} \cdot 1 = 0.$$

Proof. Suppose that condition (\*) holds and let  ${}^c h_n$  be the restriction of  $h_n$  to  ${}^c S$  and  ${}^c E_n = {}^c S \cap E_n$ . Obviously  ${}^c S - {}^c E_n = F_n$ ,  $0 \leq {}^c h_n \leq 1$ , and  ${}^c h_n = 1$  on  $F_n$ . It then follows that  ${}^c h_n$  is a quasi-excessive function for every  $n$ . In fact,

3) This definition is slightly different from Meyer's one; this is the discrete version of Meyer's,

let  ${}^c f$  be in  ${}^c M$  and  $f$ , the extension of  ${}^c f$  to  $S$  such that  $f \in N$ . If

$${}^c h_n \geq {}^c G {}^c f \quad \text{on the set } \{f > 0\}$$

then

$$h_n + Af(c) \geq Af \quad \text{on the set } \{f > 0\},$$

since  $\{f > 0\}$  is contained in  $\{f > 0\} \cup \{c\}$ . Hence from condition (\*),

$$h_n + Af(c) - f^- \geq Af \quad \text{everywhere on } {}^c S,$$

that is,

$${}^c h_n - {}^c f^- \geq {}^c G {}^c f \quad \text{everywhere on } {}^c S.$$

Since  ${}^c H_{F_n} \cdot 1 \leq {}^c h_n$  by definition,

$$\lim {}^c H_{F_n} \cdot 1 = 0.$$

Conversely, if (2.11) holds, set  ${}^c E_n \cup \{c\} = E_n$ ,  $F_n = S - E_n$  and

$$h_n = \begin{cases} {}^c H_{F_n} \cdot 1 & \text{on } {}^c S \\ 0 & \text{at } c. \end{cases}$$

It is enough to show the property (ii) of condition (\*). Suppose that, for some  $f \in N$  and some real number  $m (\geq Af(c))$

$$m + h_n \geq Af \quad \text{on } \{f > 0\}.$$

Then one has

$$m - Af(c) + {}^c H_{F_n} \cdot 1 \geq {}^c G {}^c f \quad \text{on } \{f > 0\},$$

where  ${}^c f$  is the restriction of  $f$  to  ${}^c S$ . The fact that  $m - Af(c) + {}^c H_{F_n} \cdot 1$  is a quasi-excessive function implies that

$$m - Af(c) + {}^c H_{F_n} \cdot 1 - {}^c f^- \geq {}^c G {}^c f \quad \text{everywhere on } {}^c S,$$

which is nothing but condition (\*).

**Note.** If  $\alpha$  is a finite measure, then condition (\*) is satisfied.

Let  $I_F$  be the indicator function of a set  $F$ , then from lemma 2.1  ${}^c G I_F \geq 1$  on  $F$ . Hence from (2.8) and (2.9)  ${}^c H_F \cdot 1 \leq {}^c G I_F$ . Hence if  $F_n$  decrease to empty set, inequality

$${}^c H_{F_n} \cdot 1(x) \leq {}^c G I_{F_n}(x) \leq \sum_{y \in F_n} C \cdot \alpha(y),$$

implies that

$$\lim {}^c H_{F_n} \cdot 1(x) = 0.$$

Where the second inequality follows from the corollary of lemma 2.1.

### 3. Main result

Let  $A$  be a weak potential kernel of an irreducible recurrent transition

probability  $P$  with invariant measure  $\alpha$ . We shall now prove that  $A$  satisfies condition (\*) of section 2.

Define  ${}^cP$  and  ${}^cG$  as (1.3) and (1.4) respectively. Let  ${}^cH_F$  be the réduite defined by  ${}^cP$ . Since  ${}^cH_F \cdot 1$  is the pseudo-réduite associated with the above  ${}^cG$  (see [5] P. 37, theorem 1.3), it is enough to show that for a sequence of finite sets  $\{{}^cE_n\}_{n=1,2,\dots}$  increasing to  ${}^cS$ ,  $\lim {}^cH_{F_n} \cdot 1 = 0$  ( $F_n = {}^cS - {}^cE_n$ ) by theorem 2.2. One can easily see that the function  ${}^ch(x) = \lim {}^cH_{F_n} \cdot 1(x)$  is an invariant function for  ${}^cP$  (i.e.  ${}^cP^c h = {}^ch$ ) and bounded by 1. On the other hand,

$$1 = {}^cG(1 - {}^cP \cdot 1)(x) + \lim {}^cP^n \cdot 1(x)$$

and

$$\lim {}^cP^n \cdot 1(x) = \lim P_x[\sigma_{(c)} > n] = 0,$$

implies that 1 is a potential of non-generative function (where  $\sigma_{(c)}$  is the hitting time of the Markov chain with transition probability  $P$ ). Hence  ${}^ch$  is also a potential. The fact that  ${}^ch$  is an invariant function and also a potential shows that  ${}^ch = 0$ .

The main result of the present paper is this.

**Theorem 3.1.** *Given a positive measure  $\alpha$  and a kernel  $A$  satisfying maximum principle (RSCM) and condition (\*), there exists a unique irreducible recurrent transition probability  $P$  which has  $\alpha$  as its invariant measure, and  $A$  as its weak potential kernel.*

Uniqueness was proved by Kondō [4]. We shall divide the proof of existence into several lemmas. In the following we shall use the notation of section 2 with no further reference.

**Lemma 3.1.** *There exists a sub-Markov transition probability  ${}^cP(x, y)$  on  ${}^cS$  such that*

$${}^cG(x, y) = \sum_{n=0}^{\infty} {}^cP^n(x, y) \quad \text{for every } x, y \text{ in } {}^cS.$$

Proof. See Meyer [7] P. 238 lemma 10.

**Lemma 3.2.** *For every  $y \in {}^cS$ ,  $\sum_{x \neq c} \alpha(x) {}^cP(x, y) \leq \alpha(y)$ .*

Proof. To the contrary, suppose that there exists some state  $y \in {}^cS$  such that

$$\sum_{x \neq c} \alpha(x) {}^cP(x, y) - \alpha(y) > 0.$$

Then there exists a finite subset  $F$  of  ${}^cS$  containing  $y$  and satisfying

$$\sum_{x \in F} \alpha(x) {}^cP(x, y) - \alpha(y) = a > 0.$$

Define a function  $f \in \mathcal{N}$  by

$$f(x) = \begin{cases} {}^cP(x, y) - I(x, y) & x \in F \\ -\frac{a}{\alpha(c)} & x = c \\ 0 & \text{otherwise.} \end{cases}$$

Since  $Af + f^-$  attains its maximum on the set  $\{f > 0\}$  and since  $f(c) < 0$ , there exists a state  $x_0 \in F$  such that,

$$Af(x_0) \geq Af + f^- \quad \text{everywhere on } S.$$

In particular,

$$Af(x_0) \geq Af(c) + \frac{a}{\alpha(c)}.$$

Hence,

$$0 > -\frac{a}{\alpha(c)} \geq Af(c) - Af(x_0) = {}^cG(-^cf)(x_0),$$

where  ${}^cf$  is the restriction of  $f$  to  ${}^cS$ . On the other hand,

$$\begin{aligned} {}^cG(-^cf)(x_0) &= {}^cG(x_0, y) - \sum_{z \in F} {}^cG(x_0, z) {}^cP(z, y) \\ &\geq {}^cG(x_0, y) - ({}^cG(x_0, y) - I(x_0, y)) = I(x_0, y) \geq 0. \end{aligned}$$

This lead us to a contradiction.

**Lemma 3.3.**  ${}^cG(1 - {}^cP \cdot 1) = 1$  on  ${}^cS$ .

*Proof.* For any positive integer  $n$ , we have

$$1 = \sum_{k=0}^n {}^cP^k(1 - {}^cP \cdot 1)(x) + {}^cP^{n+1} \cdot 1(x).$$

Passing to the limit we obtain

$$1 = {}^cG(1 - {}^cP \cdot 1)(x) + r(x),$$

where  $r(x) = \lim {}^cP^{n+1} \cdot 1(x)$ . It remains to show that  $r(x) = 0$ . From condition (\*) for arbitrary  $\varepsilon > 0$  there exists a number  $M$  such that for any integer  $m \geq M$ ,

$${}^cH_{F_m} \cdot 1(x) < \varepsilon.$$

Hence

$$\sum_{y \neq c} {}^cP^{n+1}(x, y) = {}^cP^{n+1}I_{F_m}(x) + {}^cP^{n+1}I_{E_m}(x) \leq {}^cH_{F_m} \cdot 1(x) + {}^cP^{n+1}I_{E_m}(x),$$

where  $I_F$  is the indicator function of  $F$ . Tending  $n$  to infinity we obtain  $r(x) \leq \varepsilon$ .

**Lemma 3.4.**  $\sum_{x \neq c} \alpha(x)(1 - {}^cP \cdot 1)(x) \leq \alpha(c)$ .

*Proof.* Let  $F$  be an arbitrary finite subset of  ${}^cS$ , and define

$$f(x) = \begin{cases} 1 - {}^cP \cdot 1(x) & x \in F \\ -\sum_{y \in F} \frac{\alpha(y)}{\alpha(c)} (1 - {}^cP \cdot 1(y)) & x = c \\ 0 & \text{otherwise.} \end{cases}$$

As noted in the proof of lemma 3.2, there exists a state  $x_0 \in F$  such that

$$Af(x_0) \geq Af + f^- \quad \text{on } S.$$

In particular,

$${}^cGf(x_0) = Af(x_0) - Af(c) \geq f^-(c) = \sum_{y \in F} \alpha(y)(1 - {}^cP \cdot 1(y))/\alpha(c),$$

and by lemma 3.3, the left side of the above inequality is bounded by 1.

Now we can define the desired transition probability  $P$ .

$$(3.1) \quad P(x, y) = \begin{cases} {}^cP(x, y) & x \neq c, y \neq c \\ 1 - {}^cP \cdot 1(x) & x \neq c, y = c \\ (\alpha(y) - \alpha({}^cP(y)))/\alpha(c) & x = c, y \neq c \\ 1 - \sum_{z \neq c} P(c, z) & x = c, y = c. \end{cases}$$

From lemmas 3.2 and 3.4,  $P$  is a transition probability on  $S$ .

**Lemma 3.5.**  $\alpha P = \alpha$  and  $(I - P)Af = f$  for any  $f \in N$ .

Proof. If  $x \neq c$ , then

$$\alpha P(x) = \sum_{y \neq c} \alpha(y) {}^cP(y, x) + \alpha(x) - \alpha({}^cP(x)) = \alpha(x)$$

and

$$(I - P)Af(x) = (I - P)(Af(c) + {}^cGf)(x) = f(x).$$

By the same argument for  $x = c$ , lemma follows.

**Lemma 3.6.** *The transition probability  $P$  is recurrent and irreducible.*

Proof. Let  $\sigma_{\{x\}}$  be the hitting time for  $x$  of the Markov chain with transition probability  $P$ . Then for every  $x \neq c$ ,

$$P_x[\sigma_{\{c\}} < \infty] = \sum_{y \neq c} {}^cG(x, y)P(y, c) = {}^cG(1 - {}^cP \cdot 1)(x) = 1,$$

by lemma 3.3. Hence,

$$P_c[\sigma_{\{c\}}^+ < \infty] = \sum_{x \in S} P(c, x)P_x[\sigma_{\{c\}} < \infty] = 1,$$

where  $\sigma_{\{c\}}^+$  is the positive hitting time for state  $c$ . Thus  $c$  is a recurrent state for  $P$  and hence also for  $\hat{P}$ , where  $\hat{P}$  is defined by,

$$(3.2) \quad \hat{P}(x, y) = \frac{\alpha(y)}{\alpha(x)} P(y, x).$$

Moreover,

$$\hat{P}^n(c, x) = \frac{\alpha(x)}{\alpha(c)} P^n(x, c), \text{ and } P_x[\sigma_{\{c\}} < \infty] = 1,$$

shows that

$$\hat{P}_c[\sigma_{\{x\}} < \infty] > 0 \text{ for all } x \in {}^c S.$$

Hence  $x$  is a recurrent state for  $\hat{P}$  and hence for  $P$ . Since  $x$  is recurrent and  $P_x[\sigma_{\{c\}} < \infty] = 1$ , it follows that  $P_c[\sigma_{\{x\}} < \infty] = 1$  for all  $x \in S$ . Irreducibility follows from the fact that,  $P_x[\sigma_{\{c\}} < \infty] = 1$  and  $P_c[\sigma_{\{y\}} < \infty] = 1$  for every  $x, y$  in  $S$ .

OSAKA CITY UNIVERSITY

---

### Bibliography

- [1] G.A. Hunt: *Markov processes and potentials* II, Illinois J. Math. **1** (1957), 316–395.
- [2] J.G. Kemeny and J.L. Snell: *Potentials for denumerable Markov chains*, J. Math. Anal. Appl. **3** (1961), 196–260.
- [3] R. Kondō: *On weak potential operators for recurrent Markov chains with continuous parameters*, Osaka J. Math. **4** (1967), 327–344.
- [4] R. Kondō: *On a construction of recurrent Markov chains*, Osaka J. Math. **6** (1969), 13–28.
- [5] R. Kondō, Y. Ōshima and T. Watanabe: *Topics in Markov Chain*, Seminar on Probability, vol. 28, 1968 (Japanese).
- [6] P.A. Meyer: *Probability and Potentials*, Blaisdell Publishing Company, 1966, 13–28.
- [7] P.A. Meyer: *Caractérisation des noyaux potentiels des semi-groupes discrets*, Ann. Inst. Fourier **16** (1966), 225–240.
- [8] S. Orey: *Potential kernels for recurrent Markov chains*, J. Math. Anal. Appl. **8** (1964), 104–132.

