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A NECESSARY AND SUFFICIENT CONDITION FOR
A KERNEL TO BE A WEAK POTENTIAL KERNEL
OF A RECURRENT MARKOV CHAIN

YOICHI ŌSHIMA

(Received December 14, 1968)

1. Introduction

Let \( P \) be an irreducible recurrent transition probability on a denumerable space \( S \) with invariant measure \( \alpha \). Let \( c \) be an arbitrary (but fixed) state of \( S \). Then from the work of Kondo [3] and Orey [8], there exist the class of weak potential kernels \( A(x, y) \) defined by the property that, for every null charge \( f \), \( Af \) is bounded and satisfies the equation

\[
(I - P)Af = f.
\]  

Moreover \( Af \) is represented by

\[
Af = \theta Gf + 1(f),
\]

where \( f \) is a null charge, \( 1(\cdot) \) is an arbitrary linear functional on the space of null charges and \( \theta G \) is defined as follow;

\[
\theta P(x, y) = \begin{cases} 
P(x, y) & x \neq c, y \neq c \\
0 & \text{otherwise},
\end{cases}
\]

\[
\theta G(x, y) = \begin{cases} 
\sum_{n=0}^{\infty} \theta P^n(x, y) & x \neq c, y \neq c \\
0 & \text{otherwise}.
\end{cases}
\]

Moreover \( A \) satisfies the following maximum principle [4], [5]: (RSCM)\(^1\) If \( m \) is a real number and \( f \) is a null charge then the relation that

\[
m \geq Af \quad \text{on the set } \{ f > 0 \}
\]

implies that

\[
m - f^- \geq Af \quad \text{everywhere},
\]

\(^1\) This is the abbreviation of "reinforced semi-complete maximum principle"; this maximum principle corresponds to the semi-complete M.P. as well as the reinforced M.P. (of Meyer) corresponds to the complete M.P.
where \( f^- = (-f) \vee 0 \).

In the present paper we are concerned with the following construction problem. Given a positive measure \( \alpha \) and a (not necessarily positive) kernel \( A \) satisfying (RSCM), does there exist an irreducible recurrent transition probability which has \( \alpha \) as its invariant measure, and \( A \) as its weak potential kernel? This is not true in general, but as Kondo [4] has proved, it is true if \( \alpha \) is a finite measure. In section 2 we shall introduce another necessary condition for the weak potential kernel \( A \) (referred to as condition (*)). Then we shall prove (theorem 3.1) that, if the pair \((A, \alpha)\) satisfies maximum principle (RSCM) and condition (*), \( A \) is a weak potential kernel of a (unique) recurrent Markov chain with \( \alpha \) as its invariant measure.

I should like to express my hearty gratitude to T. Watanabe for his kind advices.

2. Some potential theory for a kernel \( A \) satisfying (RSCM)

Let \( \alpha \) be a strictly positive measure and \( A \), a kernel on \( S \). A function \( f \) on \( S \) is said to be a null charge with respect to \( \alpha \) if \( \sum \alpha(x)|f(x)| < \infty \) and \( \sum \alpha(x)f(x) = 0 \). Let \( N \) be the space of null charges vanishing outside a finite subset of \( S \). We assume that the kernel \( A \) satisfies condition (RSCM) for \( f \in N \). Fix an arbitrary state \( c \) and define

\[
(2.1) \quad \kappa G(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)}.
\]

If \( A \) is a weak potential kernel then (2.1) is clearly satisfied by taking \( f \) in equation (1.2) as

\[
(2.2) \quad f(x) = \begin{cases} \frac{\alpha(y)}{\alpha(c)} & x = c \\ -1 & x = y \\ 0 & \text{otherwise}, \end{cases}
\]

and calculating \( Af(x) - Af(c) \).

From definition (2.1) \( \kappa G(x, c) = \kappa G(x, c) = 0 \) for every \( x \in S \).

**Lemma 2.1** For arbitrary elements \( x, y \) in \( S \) which are different from \( c \)

\[
I(x, y) \leq \kappa G(x, y) \leq \kappa G(y, y).
\]

**Proof.** By taking \( f \) as (2.2) we have

\[
Af(c) = A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y)
\]

2) A counter example was given by Kondo and T. Watanabe.
Hence, if we write \( f^+ = f \lor 0, f^- = (-f) \lor 0 \), by (RSCM)

\[
A(y, c) \frac{\alpha(y)}{\alpha(c)} - A(y, y) + f^+(x) \leq A(x) \leq A(c, c) \frac{\alpha(y)}{\alpha(c)} - A(c, y) - f^-(x),
\]

so that

\[
\frac{\alpha(y)}{\alpha(c)} \leq \frac{\alpha(x)}{\alpha(c)} - A(c, y) - I(x, y),
\]

which proves the lemma.

**Corollary.** For every \( x \in S \) there exists a constant \( C \) such that

\[
^cG(x, y) \leq C \cdot \alpha(y) \quad \text{for every } y \in S.
\]

**Proof.** Exchanging \( c \) and \( x \) in the second inequality in (2.3), it follows that

\[
^cG(x, y) = A(x, y) = A(c, y) - (A(x, c) - A(c, c)) \frac{\alpha(y)}{\alpha(c)}
\leq \frac{A(x, c)}{\alpha(c)} - \frac{A(c, c)}{\alpha(c)} + A(x, y) \frac{\alpha(y)}{\alpha(c)}.
\]

Let \(^cS\) be the set \( S - \{c\} \), and \(^cM\) be the space of all functions on \(^cS\) vanishing outside a finite subset of \(^cS\). Let \(^cM^+\) be the space of all non-negative functions in \(^cM\).

**Theorem 2.1.** The kernel \(^cG\) satisfies the reinforced maximum principle [7]:

(RM) If \( a \) is a non-negative constant and if \(^c f\) and \(^c g\) are two elements of \(^cM^+\), then the relation that

\[
a + ^cG^c f - ^c f \geq ^cG^c g \quad \text{on the set } \{^c g > 0\} \quad \text{implies that}
\]

\[
a + ^cG^c f - ^c f \geq ^cG^c g \quad \text{everywhere on } ^c S.
\]

**Proof.** Let \( f \) be the function on \( S \) such that \( f \in N \) and \( f|_S = f \). Such \( f \) is obviously unique. The function \( g \in N \) is defined similarly. Then inequality (2.5) implies that

\[
a + A(g - f)(c) \geq A(g - f) \quad \text{on the set } \{g - f > 0\}.
\]

For, since \(^c f\) and \(^c g\) are non-negative, the set \( \{g - f > 0\} \) is contained in the union of \( c \) and \( \{^c g > 0\} \). Hence by (RSCM)
Since the function \((g-f)^-\) is equal to \(f\) on \(S \cap \{g=0\}\), the above inequality, combined with (2.5), proves the theorem.

A non-negative function \(\mathcal{h}\) on \(S\) is said to be \textit{quasi-excessive}\(^3\) if, for every \(\mathcal{g}\in \mathcal{M}\), the inequality
\[
\mathcal{h} \geq \mathcal{g} \quad \text{on the set} \{\mathcal{g} > 0\}
\]
implies that
\[
\mathcal{h} - \mathcal{g}^- \geq \mathcal{g}^- \quad \text{everywhere.}
\]

Moreover Meyer introduced the notion of the \textit{pseudo-réduit} \(\mathcal{H}_E \mathcal{h}\) for every quasi-excessive function \(\mathcal{h}\) and every subset \(E\) of \(S\). This function \(\mathcal{H}_E \mathcal{h}\) satisfies the following four conditions.

(2.7) \(\mathcal{H}_E \mathcal{h}\) is quasi-excessive.

(2.8) \(\mathcal{H}_E \mathcal{h}\leq \mathcal{h}\) on \(S\) and \(\mathcal{H}_E \mathcal{h} = \mathcal{h}\) on \(E\).

(2.9) If \(\mathcal{h}_1\) and \(\mathcal{h}_2\) are two quasi-excessive functions such that \(\mathcal{h}_1 \leq \mathcal{h}_2\) on \(E\), then \(\mathcal{H}_E \mathcal{h}_1 \leq \mathcal{H}_E \mathcal{h}_2\).

(2.10) If \(\mathcal{f} \in \mathcal{M}^+\) vanishes outside of \(E\) then \(\mathcal{H}_E \mathcal{G}\mathcal{f} = \mathcal{G}\mathcal{f}\).

For example, the function \(\mathcal{G}\mathcal{f}, \mathcal{f} \in \mathcal{M}^+, \) and every positive constant are quasi-excessive ([7] see also [5]).

Now we introduce a condition.

\textbf{Condition (*):} There exists a sequence of finite sets \(\{E_n\}_{n=1,2,\ldots}\) increasing to \(S\) such that \(\mathcal{c} \in E_n\) for each \(n\), and a sequence \(\{h_n\}_{n=1,2,\ldots}\) of function on \(S\) satisfying the following conditions.

(i) \(0 \leq h_n \leq 1, \ h_n(c) = 0, \ h_n = 1\) on \(F_n = S - E_n\), and \(\lim h_n = 0\).

(ii) For every \(\mathcal{f} \in \mathcal{N}\) and every real number \(m (\geq Af(c))\) the relation that
\[
m + h_n \geq Af \quad \text{on the set} \{f > 0\}.
\]
implies that
\[
m + h_n - f^- \geq Af \quad \text{everywhere on} \ S.
\]

In section 3 we shall show that if \(A\) is a weak potential kernel of an irreducible recurrent Markov chain, it satisfies condition (*).

\begin{theorem}
Condition (*) is equivalent to the condition that, there exists a sequence of finite sets \(\{E_n\}_{n=1,2,\ldots}\) increasing to \(S\) such that
\end{theorem}

\[
\lim \mathcal{H}_{E_{n} - \varepsilon_{n}} = 0
\]

Proof. Suppose that condition (*) holds and let \(\mathcal{h}_n\) be the restriction of \(h_n\) to \(S\) and \(\mathcal{E}_n = S \cap E_n\). Obviously \(\mathcal{S} \setminus \mathcal{E}_n = F_n, \ 0 \leq h_n \leq 1, \) and \(h_n = 1\) on \(F_n\). It then follows that \(\mathcal{h}_n\) is a quasi-excessive function for every \(n\). In fact,

\(^{3}\) This definition is slightly different from Meyer's one; this is the discrete version of Meyer's,
let $f$ be in $\mathcal{C}$ and $f$, the extent of $f$ to $S$ such that $f \in \mathcal{N}$. If

$$c_h \geq \mathcal{C} \mathcal{G} f$$
onumber

on the set $\{f>0\}$

then

$$h_n + A f(c) \geq Af$$
onumber

on the set $\{f>0\}$,

since $\{f>0\}$ is contained in $\{f>0\} \cup \{c\}$. Hence from condition (*),

$$h_n + A f(c) - f \geq Af$$
onumber

everywhere on $\mathcal{C} S$,

that is,

$$c h_n - f \geq \mathcal{C} \mathcal{G} f$$

everywhere on $\mathcal{C} S$.

Since $\mathcal{H} F_n \cdot 1 \leq c h_n$ by definition,

$$\lim c H_{F_n} \cdot 1 = 0.$$\nonumber

Conversely, if (2.11) holds, set $\mathcal{E}_n \cup \{c\} = E_n$, $F_n = S - E_n$ and

$$h_n = \begin{cases} \mathcal{H} F_n \cdot 1 & \text{on } \mathcal{C} S \\ 0 & \text{at } c. \end{cases}$$

It is enough to show the property (ii) of condition (*). Suppose that, for some $f \in \mathcal{N}$ and some real number $m (\geq A f(c))$

$$m + h_n \geq Af$$
onumber

on $\{f>0\}$.

Then one has

$$m - A f(c) + \mathcal{C} H_{F_n} \cdot 1 \geq \mathcal{C} \mathcal{G} f$$

on $\{f>0\}$,

where $f$ is the restriction of $f$ to $\mathcal{C} S$. The fact that $m - A f(c) + \mathcal{C} H_{F_n} \cdot 1$ is a quasi-excessive function implies that

$$m - A f(c) + \mathcal{C} H_{F_n} \cdot 1 - f \geq \mathcal{C} \mathcal{G} f$$

everywhere on $\mathcal{C} S$,

which is nothing but condition (*).

**Note.** If $\alpha$ is a finite measure, then condition (*) is satisfied.

Let $I_F$ be the indicator function of a set $F$, then from lemma 2.1 $\mathcal{C} G I_F \geq 1$ on $F$. Hence from (2.8) and (2.9) $\mathcal{C} H_F \cdot 1 \leq \mathcal{C} G I_F$. Hence if $F_n$ decrease to empty set, inequality

$$\mathcal{C} H_{F_n} \cdot 1(x) \leq \mathcal{C} G I_{F_n}(x) \leq \sum_{y \in F_n} C \cdot \alpha(y),$$

implies that

$$\lim \mathcal{C} H_{F_n} \cdot 1(x) = 0.$$\nonumber

Where the second inequality follows from the corollary of lemma 2.1.

3. **Main result**

Let $A$ be a weak potential kernel of an irreducible recurrent transition
probability $P$ with invariant measure $\alpha$. We shall now prove that $A$ satisfies condition (*) of section 2.

Define $^cP$ and $^cG$ as (1.3) and (1.4) respectively. Let $^cH_n$ be the réduite defined by $^cP$. Since $^cH_n\cdot 1$ is the pseudo-réduite associated with the above $^cG$ (see [5] P. 37, theorem 1.3), it is enough to show that for a sequence of finite sets $\{E_n\}_{n=1,2,\ldots}$ increasing to $^cS$, $\lim^cH_n \cdot 1 = 0 \ (F_n = ^cS - E_n)$ by theorem 2.2. One can easily see that the function $^c\eta = \lim^cH_n \cdot 1(x)$ is an invariant function for $^cP$ (i.e. $^cP^\infty \eta = \eta$) and bounded by 1. On the other hand,

$$1 = ^cG(1 - ^cP \cdot 1)(x) + \lim^cP^n \cdot 1(x)$$

implies that 1 is a potential of non-gengative function (where $\sigma_{(c)}$ is the hitting time of the Markov chain with transition probability $P$). Hence $^c\eta$ is also a potential. The fact that $^c\eta$ is an invariant function and also a potential shows that $^c\eta = 0$.

The main result of the present paper is this.

**Theorem 3.1.** *Given a positive measure $\alpha$ and a kernel $A$ satisfying maximum principle (RSCM) and condition (*), there exists a unique irreducible recurrent transition probability $P$ which has $\alpha$ as its invariant measure, and $A$ as its weak potential kernel.*

Uniqueness was proved by Kondō [4]. We shall divide the proof of existence into several lemmas. In the following we shall use the notation of section 2 with no further reference.

**Lemma 3.1.** *There exists a sub-Markov transition probability $^cP(x, y)$ on $^cS$ such that*

$$^cG(x, y) = \sum_{n=0}^{\infty}^cP^n(x, y) \quad \text{for every } x, y \in ^cS.$$


**Lemma 3.2.** *For every $y \in ^cS$, $\sum_{x \in e} \alpha(x)^cP(x, y) \leq \alpha(y)$.*

**Proof.** To the contrary, suppose that there exists some state $y \in ^cS$ such that

$$\sum_{x \in e} \alpha(x)^cP(x, y) - \alpha(y) > 0.$$

Then there exists a finite subset $F$ of $^cS$ containing $y$ and satisfying

$$\sum_{s \in F} \alpha(x)^cP(x, y) - \alpha(y) = a > 0.$$

Define a function $f \in N$ by
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\[
f(x) = \begin{cases} 
  \varepsilon P(x, y) - I(x, y) & x \in F \\
  -\frac{a}{\alpha(c)} & x = c \\
  0 & \text{otherwise}
\end{cases}
\]

Since \(Af + f^-\) attains its maximum on the set \(\{f > 0\}\) and since \(f(c) < 0\), there exists a state \(x_0 \in F\) such that,

\[Af(x_0) \geq Af + f^-\] everywhere on \(S\).

In particular,

\[Af(x_0) \geq Af(c) + \frac{a}{\alpha(c)}.
\]

Hence,

\[0 > -\frac{a}{\alpha(c)} \geq Af(c) - Af(x_0) = \varepsilon G(-\varepsilon f)(x_0),
\]

where \(\varepsilon f\) is the restriction of \(f\) to \(\varepsilon S\). On the other hand,

\[
\varepsilon G(-\varepsilon f)(x_0) = \varepsilon G(x_0, y) - \sum_{z \in F} \varepsilon G(x_0, z)\varepsilon P(z, y) \\
\geq \varepsilon G(x_0, y) - (\varepsilon G(x_0, y) - I(x_0, y)) = I(x_0, y) \geq 0.
\]

This leads us to a contradiction.

**Lemma 3.3.** \(\varepsilon G(1 - \varepsilon P \cdot 1) = 1\) on \(\varepsilon S\).

**Proof.** For any positive integer \(n\), we have

\[1 = \sum_{s=0}^{n} \varepsilon P^s(1 - \varepsilon P \cdot 1)(x) + \varepsilon P^{n+1} \cdot 1(x).
\]

Passing to the limit we obtain

\[1 = \varepsilon G(1 - \varepsilon P \cdot 1)(x) + r(x),
\]

where \(r(x) = \lim \varepsilon P^{n+1} \cdot 1(x)\). It remains to show that \(r(x) = 0\). From condition (\(\ast\)) for arbitrary \(\varepsilon > 0\) there exists a number \(M\) such that for any integer \(m \geq M\),

\[\varepsilon H_{F_m} \cdot 1(x) < \varepsilon.
\]

Hence

\[\sum_{y \in c} \varepsilon P^{n+1}(x, y) = \varepsilon P^{n+1}I_{F_m}(x) + \varepsilon P^{n+1}I_{E_m}(x) \leq \varepsilon H_{F_m} \cdot 1(x) + \varepsilon P^{n+1}I_{E_m}(x),
\]

where \(I_F\) is the indicator function of \(F\). Tending \(n\) to infinity we obtain \(r(x) \leq \varepsilon\).

**Lemma 3.4.** \(\sum_{x \in c} \alpha(x)(1 - \varepsilon P \cdot 1)(x) \leq \alpha(c).

**Proof.** Let \(F\) be an arbitrary finite subset of \(\varepsilon S\), and define
As noted in the proof of lemma 3.2, there exists a state \( x_0 \in F \) such that
\[
Af(x_0) \geq Af + f^-
\]
on \( S \).

In particular,
\[
\alpha^f(x_0) = Af(x_0) - Af(c) \geq f^-(c) = \sum_{y \in F} \alpha(y)(1 - \alpha^f_y) / \alpha(c),
\]
and by lemma 3.3, the left side of the above inequality is bounded by 1.

Now we can define the desired transition probability \( P \).

\[
P(x, y) = \begin{cases} 
\alpha^f(x, y) & x \neq c, y \neq c \\
1 - \alpha^f \cdot 1(x) & x \neq c, y = c \\
(\alpha(y) - \alpha^f \cdot P(y)) / \alpha(c) & x = c, y \neq c \\
1 - \sum_{z \in c} P(c, z) & x = c, y = c.
\end{cases}
\]

From lemmas 3.2 and 3.4, \( P \) is a transition probability on \( S \).

**Lemma 3.5.** \( \alpha P = \alpha \) and \((1 - P)Af = f\) for any \( f \in N \).

Proof. If \( x \neq c \), then
\[
\alpha P(x) = \sum_{y \neq c} \alpha(y) \alpha^f(x, y) + \alpha(x) - \alpha^f(x) = \alpha(x)
\]
and
\[
(1 - P)Af(x) = (1 - P)(Af(c) + \alpha^f(x)) = f(x).
\]
By the same argument for \( x = c \), lemma follows.

**Lemma 3.6.** The transition probability \( P \) is recurrent and irreducible.

Proof. Let \( \sigma_{(x)} \) be the hitting time for \( x \) of the Markov chain with transition probability \( P \). Then for every \( x \neq c \),
\[
P_x[\sigma_{(c)} < \infty] = \sum_{x \neq c} \alpha^f(x, y) \alpha^f(y, c) = \alpha^f(1 - \alpha^f \cdot 1) (x) = 1,
\]
by lemma 3.3. Hence,
\[
P_c[\sigma_{(c)} < \infty] = \sum_{x \in S} P(c, x) P_x[\sigma_{(c)} < \infty] = 1,
\]
where \( \sigma_{(c)}^+ \) is the positive hitting time for state \( c \). Thus \( c \) is a recurrent state for \( P \) and hence also for \( \hat{P} \), where \( \hat{P} \) is defined by,
\[
\hat{P}(x, y) = \frac{\alpha(y)}{\alpha(x)} P(y, x).
\]
Moreover,
\[ \dot{P}^n(c, x) = \frac{\alpha(x)}{\alpha(c)} P^n(x, c), \] and \( P_x[\sigma_{\{c\}} < \infty] = 1 \),
shows that
\[ \dot{P}_x[\sigma_{\{c\}} < \infty] > 0 \] for all \( x \in S \).

Hence \( x \) is a recurrent state for \( \dot{P} \) and hence for \( P \). Since \( x \) is recurrent and \( P_x[\sigma_{\{c\}} < \infty] = 1 \), it follows that \( P_x[\sigma_{\{x\}} < \infty] = 1 \) for all \( x \in S \). Irreducibility follows from the fact that, \( P_{\{c\}}[\sigma_{\{x\}} < \infty] = 1 \) and \( P_{\{c\}}[\sigma_{\{y\}} < \infty] = 1 \) for every \( x, y \) in \( S \).

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