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# Effects of an interest coverage ratio covenant on optimal capital structure and firm value<sup>☆</sup>

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## ABSTRACT

This paper develops a capital structure model with a financial covenant that imposes a lower limit on a firm's interest coverage ratio. Shareholders reduce their debt level or default whenever this ratio falls below the limit. In the model, firm value, debt repayment policy, and capital structure are derived explicitly. For low levels of the limit, shareholders prefer to reduce their debt every time the ratio reaches the limit. In other words, the covenant acts as early pressure on shareholders and eliminate their incentives to default. Then, it decreases the cost of debt but also lowers equity value by constraining shareholders. Because of this trade-off, the covenant can improve firm value. With the covenant, the firm can begin with high leverage to take advantage of the decreased cost of debt. The covenant tends to improve firm value for higher bankruptcy cost and volatility because these conditions lead to high expected default costs without a covenant. It can also improve firm value for higher growth and tax rates by easing the restriction on future debt issuance. These results are consistent with empirical evidence and support the optimal contracting hypothesis for covenants.

## 1. Introduction

Debt contracts often include covenants, such as restrictions on financing, payouts, (dis)investment activities, and financial ratios (e.g., Smith and Warner (1979), Dichev and Skinner (2002), Chava and Roberts (2008), and Bradley and Roberts (2015)). Specifically, recent studies, such as Lian and Ma (2021), Kermani and Ma (2020), Drechsel (2020), and Graham (2022) emphasize the prevalence of cash flow-based borrowing constraints and leverage targets, which can be implemented by financial covenants that limit debt-earnings and interest coverage (IC) ratios. In fact, these accounting ratios are most frequently adopted in financial covenants (e.g., Dichev and Skinner (2002) and Chava and Roberts (2008)). Although many capital structure models presume restrictions on financing and (dis)investment activities,<sup>1</sup> few papers have developed capital structure models with

such financial covenants.<sup>2</sup> The purpose of this paper is to fill the gaps by studying a capital structure model with a financial covenant that limits the IC ratio. We prove that the IC ratio covenant can act as a commitment device for future debt repayment and thereby increase firm value. We also clarify the circumstances under which the covenant enhances firm value.

Our model builds on the standard capital structure model by Leland (1994). Consider a firm that issues consol debt at the initial time and receives stochastic earnings. When earnings deteriorate, shareholders can default on their debt, where a fraction of firm value is lost as bankruptcy cost, and debt holders receive the post-bankruptcy firm value. The firm optimizes debt financing based on a trade-off between the tax benefits and potential bankruptcy costs. The model prohibits the firm to increase debt after initial debt issuance. We add the following covenant to the standard setup: The firm must maintain its IC ratio

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<sup>1</sup> For instance, Leland (1994) assumes that a firm can neither liquidate assets before bankruptcy nor increase debt. These assumptions can be interpreted as restrictive covenants.

<sup>2</sup> Notable exceptions are Gamba and Mao (2020) and Xiang (2024). They develop discrete-time models with covenants and show positive effects of covenants on firm value.

over an initially determined lower limit. When the IC ratio reaches the limit due to deteriorating earnings, shareholders must repay part of the debt to comply with the covenant, or they default on their debt. They cannot reissue debt reduced due to covenant violation. Our model also includes a Poisson shock that causes unexpected bankruptcy.

In the model, we derive equity, debt, firm values, capital structure, and debt repayment policy in closed-form expressions. When the covenant tightness (i.e., the limit) is sufficiently low such that the IC ratio does not reach the limit before shareholders' optimal default time, the covenant does not bind shareholders. This case agrees with [Leland \(1994\)](#)'s unprotected debt model (denoted by the no-covenant model), where default occurs at a threshold endogenously determined by equity value maximization.

Next, suppose that the covenant tightness is high enough to bind shareholders. Shareholders decide whether to default or reduce their debt whenever the ratio falls below the limit. As shareholders reduce their debt, the tax shield of debt decreases. They optimize the default threshold by balancing the trade-off between the tax shield and default loss. Notably, the solution becomes either of two corner solutions: defaulting at the first hitting time to the limit or avoiding default until bankruptcy occurs due to a Poisson shock. The former solution arises when the covenant tightness is below a certain threshold. In this case, the firm's IC ratio is already low at the covenant violation time, and hence, shareholders prefer default over reducing debt to comply with the covenant and continuing operation. The covenant violation triggers default earlier than in the no-covenant model, decreasing equity, debt, and firm values.

The latter solution emerges when the covenant tightness is above the threshold. In this case, the firm's IC ratio is still high whenever the ratio reaches the limit, motivating shareholders to reduce their debt to comply with the covenant and continue operation. In other words, the covenant renders shareholders' default option worthless by accelerating their choice between default and repayment. Then, the IC ratio covenant acts as a commitment device for debt reductions and decreases the cost of debt. However, the covenant decreases equity value by binding shareholders. The trade-off between the increased debt value and decreased equity value determines whether the covenant improves initial firm value (i.e., ex ante shareholders' value). The firm initially issues debt as much as the covenant allows to take advantage of the reduced cost of debt.

The covenant tends to improve firm value for higher bankruptcy cost and volatility. Note that in the no-covenant model, higher bankruptcy cost and volatility increase the expected cost of default. Then, the covenant that decreases the default probability is more valuable in these environments. The covenant can increase firm value by relaxing the restriction on future debt issuance particularly for higher growth and tax rates. Indeed, the firm can receive more tax benefits by increasing debt in response to improving earnings in these situations. With such an efficient covenant, the firm tends to start with higher leverage than that of the no-covenant model to benefit from the reduced cost of debt. Our theoretical results can account for empirical relations between firm characteristics and debt covenants observed in [Bradley and Roberts \(2015\)](#). The results also support the optimal contracting hypothesis; that is, firms design optimal covenants based on their benefits and costs (e.g., [Smith and Warner \(1979\)](#), [Reisel \(2014\)](#), and [Bradley and Roberts \(2015\)](#)).

Finally, we briefly review the technically related literature. [Gamba and Mao \(2020\)](#) and [Xiang \(2024\)](#) are most closely related to our paper. [Gamba and Mao \(2020\)](#) develop a capital structure model incorporating multiple covenants, such as debt and asset sweeps and debt-earnings covenants, and show that debt-earnings covenants can act as a commitment device for leverage reductions and enhance firm value. [Xiang \(2024\)](#) examines a capital structure model with a financial covenant that allows lenders to accelerate debt repayment upon a covenant viola-

tion, albeit at a cost. He shows that the covenant can reduce the default probability and increase firm value, with the latter exhibiting a hump-shaped relation with covenant tightness. Although these studies rely on numerical analysis in discrete-time models, our paper incorporates an IC ratio covenant into [Leland \(1994\)](#) and derives closed-form solutions for equity, debt, firm value, and leverage dynamics. Specifically, we explicitly derive the region in which the covenant, as early pressure on shareholders, eliminates their incentives to default and increase firm value.

[Strebulaev \(2007\)](#) develops a continuous-time capital structure model, where shareholders must repay part of the debt by asset sales when the IC ratio falls to a given threshold. In his model, after reducing debt, shareholders default at an endogenous threshold as in [Goldstein et al. \(2001\)](#). [Sarkar \(2023\)](#) examines a debt financing model with a growth investment option and shows that a covenant that imposes a lower limit on debt issuance in the growth stage can enhance firm value by mitigating the agency problem. These models' one-time covenants differ substantially from our model's permanent covenant, which requires shareholders to reduce their debt every time the IC ratio falls below the limit. The main contribution of our paper is the first closed-form continuous-time analysis of a permanent IC ratio covenant within the [Leland \(1994\)](#) framework, together with an explicit characterization of the parameter region in which the covenant generates minimal-risk debt and enhances firm value.

The remainder of this paper is organized as follows. Section 2 introduces the model setup. Section 3 derives equity, debt, firm values, repayment policy, capital structure, and the condition under which the IC ratio covenant increases firm value. Section 3 also shows numerical examples and comparative statics. Section 4 discusses model limitations. Section 5 concludes.

## 2. Model setup

### 2.1. Cash flows

A firm receives a continuous stream of earnings before interest and taxes (EBIT)  $X(t)$ , which follows a geometric Brownian motion with killing probability  $\lambda dt$ :

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t) \quad (t > 0), \quad X(0) = x,$$

where  $B(t)$  denotes the standard Brownian motion defined in a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$  and  $\mu, \sigma (> 0)$  and  $x (> 0)$  are constants. To capture unexpected downside shocks and exogenous credit events (e.g., liquidity freezes and covenant-triggered renegotiations documented in [Roberts and Sufi \(2009\)](#)), we assume that  $X(t)$  falls to 0 (i.e., bankruptcy) at killing probability  $\lambda dt$  over infinitesimal time interval  $dt$ , where  $\lambda$  denotes the Poisson jump hazard rate.<sup>3</sup> For tractability, we assume the independence between  $B(t)$  and the Poisson jump. For convergence,  $r + \lambda > \mu$  is assumed, where a positive constant  $r$  denotes the risk-free interest rate. At time  $t$ , the firm receives cash flows  $(1 - \tau)(X(t) - C(t))$  after interest and tax payments, where  $C(t)$  and  $\tau$  denote a coupon of the existing debt and corporate tax rate, respectively. As in [Leland \(1994\)](#), shareholders can freely use equity issues or retained earnings to pay coupons.

<sup>3</sup> Although our main results remain unchanged without Poisson risk, we included it in our model to ensure positive default probabilities for firms with tight covenants (i.e., Case (iii) in [Proposition 1](#) of Section 3). The model that includes downward jumps by a fraction  $\Delta\%$  is not tractable, but Section 4 examine its results approximately.

## 2.2. IC ratio covenant

At time 0, the firm issues consol debt with the following covenant based on the IC ratio:

$$X(t)/C(t) \geq \phi, \quad (1)$$

where  $\phi$  denotes a lower limit (i.e., covenant tightness) of the IC ratio in the covenant. The initial coupon  $C(0)$  is chosen to maximize the firm value. When the IC ratio  $X(t)/C(t)$  falls below  $\phi$ , shareholders of the firm must choose either to default or reduce their debt level to ensure compliance with the covenant. Shareholders must repay the principal value  $(C - C')/(r + \lambda)$  to debt holders when they reduce their coupon level from  $C$  to  $C'$ .<sup>4</sup> In the baseline model, they cannot reissue the debt reduced due to covenant violation, while Section 3.3 and Appendix F discuss the possibility of reissuing debt. Shareholders are allowed to reduce their debt anytime (even when  $X(t)/C(t)$  is higher than  $\phi$ ), but they do not reduce their debt voluntarily because it decreases their value (see Section VIII of Leland (1994)).

Although a variety of financial ratios can be capped by financial covenants, IC and debt-earnings ratios are most frequently adopted (e.g., Dichev and Skinner (2002), Chava and Roberts (2008), Greenwald (2019), and Lian and Ma (2021)). Note that covenant (1) can also be identified as a covenant that sets a upper limit on the firm's debt-earnings ratio (i.e.,  $C(t)/((r + \lambda)X(t)) \leq 1/((r + \lambda)\phi)$ ).<sup>5</sup> In reality, a firm can be allowed to violate and renegotiate a financial covenant. However, to capture the roles of the covenant as simple as possible, our baseline model assumes the covenant that cannot be renegotiated. Section 4 discusses the possibility of covenant violation and renegotiation.

## 2.3. Bankruptcy and agency problems after debt issuance

For debt in place, shareholders choose debt repayment and default policies in their best interests, although debt holders prefer a debt reduction outcome. This is how the agency problem arises between equity and debt holders. Shareholders can declare default anytime. At default time  $t$ , shareholders receive nothing, and debt holders receive post-bankruptcy firm value  $(1 - \alpha)(1 - \tau)X(t)/(r + \lambda - \mu)$ , where  $\alpha \in (0, 1)$  represents the bankruptcy cost, and  $(1 - \tau)X(t)/(r + \lambda - \mu)$  is the unlevered firm value.<sup>6</sup> These assumptions are standard (e.g., Leland (1994) and Goldstein et al. (2001)).

Debt is fairly priced based on the rational expectation of shareholders' debt repayment and default policies. The ex post agency problem can reduce firm value at time 0. If shareholders' debt reduction commitment were credible, it would decrease the cost of debt and increase firm value at time 0. However, without a covenant, the debt reduction policy is not credible because the debt reduction policy is not optimal for shareholders after time 0 (e.g., Leland (1994) and Demarzo and He (2021)). As shown in Section 3, with a sufficiently tight covenant, the debt reduction policy becomes optimal for shareholders after time 0. In other words, by the covenant, shareholders can credibly commit to the debt reduction policy and decrease the cost of debt.

<sup>4</sup> Shareholders use retained earnings or equity issues to repay part of the debt. Our main results will remain unchanged whether shareholders repay the principal or market value of debt when reducing debt. As shown in Section 3, in the most interesting region (i.e., Case (iii) in Section 3), the market value of debt agrees with its principal value  $C(t)/(r + \lambda)$ .

<sup>5</sup> Nishihara et al. (2023) study an investment and financing model with an earnings-based borrowing constraint, where a debt-earnings ratio covenant is imposed only at debt issuance. In contrast, this paper assumes constraint (1) that is valid ever since debt issuance in order to reveal the effects of the covenant on dynamic debt policy and firm value.

<sup>6</sup> At bankruptcy caused by a Poisson shock, the post-bankruptcy firm value becomes 0 due to  $X(t) = 0$ .

An alternative way to resolve the ex post agency problem is debt renegotiation, where equity and debt holders alter the debt contract to avoid default and share the surplus. Although many papers (e.g., Sundaresan and Wang (2007), Morellec et al. (2015), and Shibata and Nishihara (2015)) examine debt renegotiation, few papers examine a covenant as a device to resolve the agency problem. The covenant which is contracted at time 0 is quite different from debt renegotiation that occurs after time 0. To focus on the role of covenants, our model does not consider the possibility of debt renegotiation.

## 3. Model solution

### 3.1. Debt, equity, and firm values for exogenous coupon level

We solve shareholders' debt reduction and default problem for given  $(X(0), C(0)) = (x, c)$ . Shareholders are obligated to default or reduce debt whenever  $X(t)$  falls below  $\phi C(t)$ . They do not reduce their debt more (or earlier) than is required by the covenant as the tax shield decreases as they reduce their debt. They gradually decrease debt to comply with the covenant until default threshold  $x^d(c)$ . By solving the shareholders' problem of finding the optimal default threshold  $x^d(c)$ , we explicitly derive the equity, debt, firm values, denoted by  $E(x, c)$ ,  $D(x, c)$ ,  $F(x, c)$ , coupon dynamics  $C(t)$ , and default threshold  $x^d(c)$  for given  $(X(0), C(0)) = (x, c)$  in the following proposition, where  $\gamma$ ,  $\phi_1$ , and  $\phi_2$  are defined by

$$\begin{aligned} \gamma &= \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(r + \lambda)}{\sigma^2}} (< 0), \\ \phi_1 &= \frac{\gamma}{\gamma - 1} \frac{r + \lambda - \mu}{r + \lambda}, \quad \phi_2 = \frac{\gamma - 1/(1 - \tau)}{\gamma - 1} \frac{r + \lambda - \mu}{r + \lambda}, \end{aligned} \quad (2)$$

Note that  $\phi_2 > \phi_1 > 0$  follows from  $\gamma < 0$ ,  $\tau \in (0, 1)$ , and  $r - \mu > 0$ . For proof, see Appendix A.

**Proposition 1.** Suppose that initial values  $(X(0), C(0)) = (x, c)$  satisfy covenant (1) (i.e.,  $x/c \geq \phi$ ). There are several cases of interest depending on covenant parameter  $\phi$  (illustrated in Fig. 1).

Case (i):  $\phi \leq \phi_1$ . The covenant is not binding, and the solution agrees with that of the no-covenant model (i.e., Leland (1994)). See Appendix A for the detailed expressions.

Case (ii):  $\phi_1 < \phi < \phi_2$ . Shareholders default when the IC ratio  $X(t)/c$  falls below  $\phi$ .

$$C(t) = c, \quad x^d(c) = \phi c, \quad (3)$$

$$E(x, c) = (1 - \tau) \left( \frac{x}{r + \lambda - \mu} - \frac{c}{r + \lambda} - \left( \frac{x}{\phi c} \right)^\gamma \left( \frac{\phi c}{r + \lambda - \mu} - \frac{c}{r + \lambda} \right) \right), \quad (4)$$

$$D(x, c) = \frac{c}{r + \lambda} - \left( \frac{x}{\phi c} \right)^\gamma \left( \frac{c}{r + \lambda} - \frac{(1 - \alpha)(1 - \tau)\phi c}{r + \lambda - \mu} \right), \quad (5)$$

$$F(x, c) = \frac{(1 - \tau)x}{r + \lambda - \mu} + \frac{\tau c}{r + \lambda} - \left( \frac{x}{\phi c} \right)^\gamma \left( \frac{\alpha(1 - \tau)\phi c}{r + \lambda - \mu} + \frac{\tau c}{r + \lambda} \right). \quad (6)$$

Case (iii):  $\phi \geq \phi_2$ . Shareholders reduce their debt every time the IC ratio  $X(t)/C(t)$  falls below  $\phi$  until EBIT  $X(t)$  falls to 0 by a Poisson shock.

$$C(t) = \min\{c, \min_{0 \leq s \leq t} X(s)/\phi\}, \quad x^d(c) = 0, \quad (7)$$

$$E(x, c) = (1 - \tau) \left( \frac{x}{r + \lambda - \mu} - \frac{c}{r + \lambda} \right) - \left( \frac{x}{\phi c} \right)^\gamma \frac{\tau c}{(r + \lambda)(1 - \gamma)}, \quad (8)$$

$$D(x, c) = c/(r + \lambda), \quad (9)$$

$$F(x, c) = \frac{(1 - \tau)x}{r + \lambda - \mu} + \frac{\tau c}{r + \lambda} - \left( \frac{x}{\phi c} \right)^\gamma \frac{\tau c}{(r + \lambda)(1 - \gamma)}. \quad (10)$$

<sup>7</sup> This means that  $X(t)/C(t)$  has a reflecting barrier  $\phi$ .

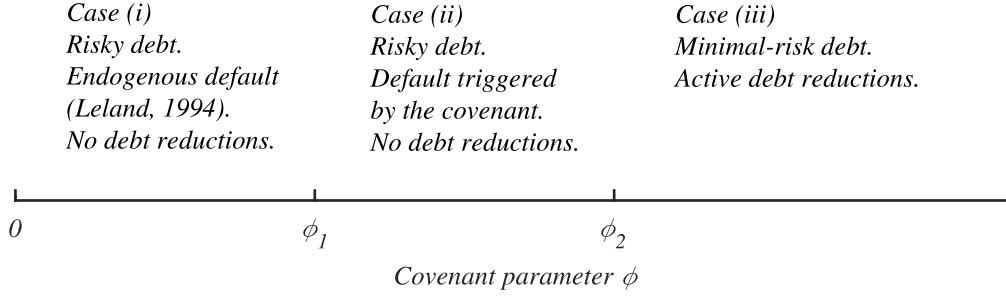


Fig. 1. Cases (i), (ii), and (iii) depending on covenant parameter  $\phi$ .

In Case (i), the covenant does not affect shareholders' default decision because EBIT  $X(t)$  falls to the no-covenant default threshold before a covenant violation. In Cases (ii) and (iii), a covenant violation occurs before endogenous default (i.e.,  $\phi c > x_N^d(c)$ ), influencing shareholders' default decision. Considering the trade-off between the default loss and tax shield, shareholders determine default threshold  $x^d(c)$ , above which they comply with the covenant. Notably, as shown in Appendix A, the solution to the shareholders' default timing problem (i.e., (28) in Appendix A) results in a corner solution (i.e.,  $x^d(c) = \phi c$  or 0). If covenant parameter  $\phi$  is low (i.e., Case (ii)), shareholders default without reducing debt at threshold  $x^d(c) = \phi c$  (i.e., the first hitting time to the limit). Case (ii) includes Leland (1994)'s protected debt model as a special case by setting  $\phi = 1$  (i.e.,  $x^d(c) = c$ ).

If  $\phi$  is high (i.e., Case (iii)), shareholders will not default until  $X(t)$  hits threshold  $x^d(c) = 0$  by a Poisson shock.<sup>8</sup> Shareholders indeed reduce their coupon level from  $C(t)$  to the limit  $X(t)/\phi$  every time the IC ratio  $X(t)/C(t)$  falls below  $\phi$ . This happens because shareholders can receive a nonnegative value by reducing debt and continuing operation. Indeed, in (8),  $E(X(t), C(t)) \geq E(X(t), X(t)/\phi) \geq 0$  holds whenever  $(X(t), C(t))$  satisfies covenant (1). Debt is priced at the minimal-risk debt value  $c/(r+\lambda)$  because debt holders rationally expect shareholders' debt reductions. In (8), the first two terms represent the values of receiving cash flows  $X(t)$  and paying coupon  $c$  constantly. The tax benefits of fixed coupon  $c$  is equal to  $\tau c/(r+\lambda)$ , but shareholders cannot fully obtain this value because they reduce their debt as  $X(t)$  decreases. The last term in (8) stands for the gradual loss in the tax benefits. Firm value (10) can be interpreted similarly. In fact, the first, second, and third terms in (10) represent the unlevered firm value, tax benefits of constant debt, and loss in the tax benefits, respectively. Case (iii) illustrates how a tight covenant can act as early pressure on shareholders and eliminate their default option value. Although the existing discrete-time models (e.g., Gamba and Mao (2020) and Xiang (2024)) do not identify this minimal-risk region, our continuous-time model derives this interesting region in a closed form and shows its economic importance of the region (see the efficient covenant region in Section 3.3).<sup>9</sup>

It follows from (8) and (10) that

$$\underbrace{\frac{(1-\tau)x}{r+\lambda-\mu} - \frac{c}{r+\lambda}}_{=: E_{NTB}(x,c)} < (1-\tau) \underbrace{\left( \frac{x}{r+\lambda-\mu} - \frac{c}{r+\lambda} \right)}_{=: E_{PTB}(x,c)} - \underbrace{\left( \frac{x}{\phi c} \right)^\gamma \frac{\tau c}{r+\lambda}}_{=: F_{CTB}(x,c)}$$

<sup>8</sup> Several papers, including Mella-Barral and Perraudin (1997), Lambrecht and Myers (2008), Shibata and Nishihara (2018), Nishihara and Shibata (2021), and Shibata and Nishihara (2023) assume a constant component of the post-bankruptcy value (e.g., the constant scrap value). The presence of constant liquidation value can also generate riskless debt. Indeed, shareholders could retire the principal of debt by part of the constant liquidation value and obtain the residual value at liquidation. In contrast, in our model, the covenant leads shareholders to reduce debt and continue operation.

<sup>9</sup> The minimal-risk region can arise in the discrete versions of our model due to truncation of the tails of the distribution. For details, see Appendix E.

$$\begin{aligned} &< E(x, c) < (1-\tau) \underbrace{\left( \frac{x}{r+\lambda-\mu} - \frac{c}{r+\lambda} \right)}_{=: E_{CTB}(x,c)}, \\ &\underbrace{\frac{(1-\tau)x}{r+\lambda-\mu}}_{=: F_{NTB}(x,c)} < \underbrace{\frac{(1-\tau)x}{r+\lambda-\mu} + \frac{\tau c}{r+\lambda}}_{=: F_{PTB}(x,c)} - \underbrace{\left( \frac{x}{\phi c} \right)^\gamma \frac{\tau c}{r+\lambda}}_{=: F_{CTB}(x,c)} < F(x, c) < \underbrace{\frac{(1-\tau)x}{r+\lambda-\mu} + \frac{\tau c}{r+\lambda}}_{=: F_{CTB}(x,c)}. \end{aligned}$$

Subscripts  $NTB$ ,  $PTB$ , and  $CTB$  stand for the cases with no tax benefits, partial tax benefits where shareholders retire their debt all at once when  $X(t)/c$  falls below  $\phi$ , and constant tax benefits, respectively. Note that  $E(x, c)$  in (8) and  $F(x, c)$  in (10) are higher than  $E_{PTB}(x, c)$  and  $F_{PTB}(x, c)$  because of  $\tau c/((r+\lambda)(1-\gamma)) < \tau c/(r+\lambda)$ . The denominator  $(1-\gamma)$  measures how slowly rather than all at once shareholders will reduce debt and lose the tax benefits.

In contrast to the results in the no-covenant model,  $E(x, c)$  and  $F(x, c)$  in Case (iii) decrease in volatility  $\sigma$  because of  $\partial \gamma / \sigma > 0$ . By  $\lim_{\sigma \rightarrow \infty} \gamma = 0$ , we have  $\lim_{\sigma \rightarrow \infty} E(x, c) = E_{NTB}(x, c)$  and  $\lim_{\sigma \rightarrow \infty} F(x, c) = F_{NTB}(x, c)$ , which imply that the tax benefits of debt disappear as  $\sigma \rightarrow \infty$ . This is because, in this limit case, EBIT  $X(t)$  immediately falls close to 0, which causes coupon  $C(t)$  to fall to 0. Then, the tax benefits of debt is removed immediately. By  $\lim_{\sigma \rightarrow 0} \gamma = -\infty$  for  $\mu \geq 0$  and  $\lim_{\sigma \rightarrow 0} \gamma = (r+\lambda)/\mu$  for  $\mu < 0$ , we have

$$\begin{aligned} \lim_{\sigma \rightarrow 0} E(x, c) &= \begin{cases} E_{CTB}(x, c) & (\mu \geq 0), \\ E_{CTB}(x, c) - \left( \frac{x}{\phi c} \right)^{(r+\lambda)/\mu} \frac{\tau c}{(r+\lambda)(1-(r+\lambda)/\mu)} & (\mu < 0), \end{cases} \\ \lim_{\sigma \rightarrow 0} F(x, c) &= \begin{cases} F_{CTB}(x, c) & (\mu \geq 0), \\ F_{CTB}(x, c) - \left( \frac{x}{\phi c} \right)^{(r+\lambda)/\mu} \frac{\tau c}{(r+\lambda)(1-(r+\lambda)/\mu)} & (\mu < 0), \end{cases} \end{aligned}$$

which imply that the firm can extract the constant tax benefits for  $\sigma \rightarrow 0$  and  $\mu \geq 0$ . In this limit case,  $X(t)$  grows constantly in a deterministic manner, and the firm will keep  $C(t) = c$  constant.

Let us look at the firm's debt repayment policy in Case (iii) in terms of leverage dynamics. It follows from (9) and (10) that market leverage  $LV(X(t), C(t)) = D(X(t), C(t))/F(X(t), C(t))$  satisfies

$$\begin{aligned} LV(X(t), C(t)) &= \frac{1}{\frac{(1-\tau)(r+\lambda)X(t)}{(r+\lambda-\mu)C(t)} + \tau - \left( \frac{X(t)}{\phi C(t)} \right)^\gamma \frac{\tau}{1-\gamma}} \\ &\leq LV(X(t), X(t)/\phi) = \frac{1}{\frac{(1-\tau)(r+\lambda)\phi}{r+\lambda-\mu} + \frac{\gamma\tau}{\gamma-1}}. \end{aligned} \quad (11)$$

Leverage  $LV(X(t), C(t))$  decreases in  $X(t)$ , but the upper bound  $LV(X(t), X(t)/\phi)$  is independent of  $X(t)$ . Shareholders do not change leverage as long as it lies below the bound. They pull back leverage every time it exceeds the bound. We will later show a trajectory of  $LV(X(t), C(t))$  in Fig. 5 of Section 3.3.

Proposition 1 also shows the comparative statics with respect to levels of  $\phi$ . Clearly, in Case (i), where the covenant does not bind shareholders,  $\phi$  does not affect the results. In Case (ii), where a covenant violation causes default, a tighter covenant (i.e., higher  $\phi$ ) accelerates default (i.e., increases default threshold  $x^d(c)$ ). Then, all values

**Table 1**  
Baseline parameter values.

| $r$  | $\lambda$ | $\mu$ | $\sigma$ | $\tau$ | $\alpha$ | $x$ |
|------|-----------|-------|----------|--------|----------|-----|
| 0.05 | 0.005     | 0.01  | 0.2      | 0.15   | 0.5      | 1   |

$E(x, c)$ ,  $D(x, c)$ , and  $F(x, c)$  decrease with a tighter covenant. At  $\phi = \phi_2$ , Case (iii) holds, and hence, debt value  $D(x, c)$  increases to minimal-risk debt value  $c/(r + \lambda)$ . In other words, the covenant acts as a commitment device for future debt repayment and decreases the cost of debt. On the other hand, in Case (iii), a tighter covenant causes shareholders to reduce their debt more and earlier. This negative effect decreases equity value  $E(x, c)$ . Hence,  $D(x, c)$  is higher in Case (iii) than in Cases (i) and (ii), although  $E(x, c)$  is lower in Case (iii) than in Cases (i) and (ii). This trade-off determines whether  $F(x, c)$  is higher in Case (iii) than in Cases (i) and (ii). The covenant increase firm value when its positive commitment effect on debt value dominates its negative effect on equity value.

Fig. 2 shows the comparative statics in numerical examples with  $c = 0.3$ , where  $CS(x, c) = c/D(x, c) - r$  are credit spreads and measure the cost of debt. The baseline parameter values are set as in Table 1, where the values of  $r, \mu, \sigma, \tau$ , and  $\alpha$  are standard in dynamic corporate finance literature and reflect a typical S&P firm (e.g., Morellec (2001), Arnold (2014), and Nishihara et al. (2023)), and the initial EBIT is normalized as  $x = X(0) = 1$ . We set  $\lambda = 0.005$  to yield plausible levels of credits spreads. For comparison, the figure also plots the values in the no-covenant model with dashed lines. We will explain  $\Delta = 25\%$  and 50% (dotted and chained lines) in Section 4.

For the baseline parameter values, we have  $\phi_1 = 0.48$  and  $\phi_2 = 0.88$ , implying Case (i):  $\phi \leq 0.48$ , Case (ii):  $0.48 < \phi < 0.88$ , and Case (iii):  $\phi \geq 0.88$ . The center-left panel of Fig. 2 shows that  $F(x, c) > F_N(x, c)$  holds for intermediate levels of  $\phi$  in Case (iii) because the positive effect of the covenant on  $D(x, c)$  dominates its negative effect on  $E(x, c)$ . This result is consistent with the optimal contracting hypothesis and its empirical evidence (e.g., Smith and Warner (1979), Reisel (2014), and Bradley and Roberts (2015)). For instance, many empirical papers examine bonds with restrictive covenants and show their positive role in reducing credit spreads (e.g., Reisel (2014)). Bradley and Roberts (2015), who examine private loans with a variety of covenants including financial covenants, also show that these covenants reduce credit spreads. These papers argue that covenants can increase firm value by reducing the cost of debt and easing debt issuance.

In Case (iii) of the center-right panel of Fig. 2,  $LV(x, c) > LV_N(x, c)$  holds because the covenant decreases  $E(x, c)$  and increases  $D(x, c)$  compared to those in the no-covenant case. This result can explain a positive relation between the leverage and probability of including covenants in Bradley and Roberts (2015).

### 3.2. Optimal capital structure

This subsection endogenizes coupon  $C(0)$  so that it maximizes firm value  $F(x, C(0))$  at time 0. We denote this optimal coupon by  $c(x)$  and also use notations  $x^d(x) = x^d(c(x))$ ,  $E(x) = E(x, c(x))$ ,  $D(x) = D(x, c(x))$ , and  $F(x) = F(x, c(x))$ . We can derive these values explicitly by solving  $\max_{c \in [0, x/\phi]} F(x, c)$  in each case of Proposition 1. For proof, see Appendix B.

**Proposition 2.** Case (i):  $\phi \leq \phi_1$ . The covenant is not binding, and the solution agrees with that of the no-covenant model (i.e., Leland (1994)). See Appendix B for the detailed expressions.

Case (ii):  $\phi_1 < \phi < \phi_2$ . Shareholders default when EBIT  $X(t)$  falls below default threshold  $x^d(x)$ .

$$c(x) = x/(\phi g), \quad x^d(x) = x/g, \quad (12)$$

$$E(x) = \left( \frac{1 - g^{\gamma-1}}{r + \lambda - \mu} - \frac{1 - g^{\gamma}}{(r + \lambda)\phi g} \right) (1 - \tau)x, \quad (13)$$

$$D(x) = \left( \frac{1 - g^{\gamma}}{(r + \lambda)\phi} + \frac{(1 - \alpha)(1 - \tau)g^{\gamma}}{r + \lambda - \mu} \right) \frac{x}{g}, \quad (14)$$

$$F(x) = \left( \frac{1 - \tau}{r + \lambda - \mu} + \frac{\gamma \tau}{(\gamma - 1)(r + \lambda)\phi g} \right) x, \quad (15)$$

where

$$g = \left( (1 - \gamma) \left( 1 + \frac{\alpha(1 - \tau)(r + \lambda)\phi}{\tau(r + \lambda - \mu)} \right) \right)^{-1/\gamma} (> 1). \quad (16)$$

Case (iii):  $\phi \geq \phi_2$ . Shareholders reduce their debt every time the IC ratio  $X(t)/C(t)$  falls below  $\phi$  until EBIT  $X(t)$  falls to 0 by a Poisson shock.

$$c(x) = x/\phi, \quad x^d(x) = 0, \quad (17)$$

$$E(x) = \left( \frac{1 - \tau}{r + \lambda - \mu} - \frac{\gamma(1 - \tau) - 1}{(\gamma - 1)(r + \lambda)\phi} \right) x, \quad (18)$$

$$D(x) = x/(\phi(r + \lambda)), \quad (19)$$

$$F(x) = \left( \frac{1 - \tau}{r + \lambda - \mu} + \frac{\gamma \tau}{(\gamma - 1)(r + \lambda)\phi} \right) x, \quad (20)$$

Note that in all the cases,  $c(x)$ ,  $E(x)$ ,  $D(x)$ , and  $F(x)$  are linear functions of  $x$ . In Case (i), these values are independent of covenant parameter  $\phi$  because the covenant does not bind the firm. In Case (ii), where a covenant violation leads to default, the results nests Leland (1994)'s protected debt model. The firm chooses an optimal capital structure (i.e.,  $c(x) = x/(\phi g)$ ) based on the trade-off between the tax benefits and bankruptcy costs of debt. In this case, all the values depend on  $\phi$ . It follows from (12)–(16) that  $c(x)$ ,  $x^d(x)$ , and  $F(x)$  decrease with  $\phi$ . If the effects of  $\phi$  on  $g^{\gamma}$  and  $g^{\gamma-1}$  are negligible,  $E(x)$  increases with  $\phi$ , whereas  $D(x)$  decreases with  $\phi$ . These sensitivities will be later verified numerically in Fig. 3. Note that  $c(x)$ ,  $x^d(x)$ ,  $E(x)$ ,  $D(x)$ , and  $F(x)$  are continuous at  $\phi = \phi_1$ . Hence,  $F(x)$  in Case (ii) is lower than in Case (i).

In Case (iii), the firm does not go bankrupt until  $X(t)$  falls to 0. Initially, the firm increases debt to its maximum level (i.e.,  $c(x) = x/\phi$ ) to take advantage of the reduced cost of debt. In standard trade-off models, such as Leland (1994), shareholders choose the default time without consideration of debt holders' interests. The default loss caused by this agency problem limits the tax benefits of debt. In contrast, in our model, a firm needs a sufficiently tight covenant (i.e., Case (iii):  $\phi \geq \phi_2$ ) to remove the agency cost, and this covenant tightness constrains the tax benefits.

From (17)–(20),  $c(x)$ ,  $D(x)$ , and  $F(x)$  decrease with  $\phi$ , whereas  $E(x)$  increases with  $\phi$ . From  $g > 1$  and (17)–(20), we have

$$c(x; \phi_2) > \lim_{\phi \uparrow \phi_2} c(x; \phi), \quad D(x; \phi_2) > \lim_{\phi \uparrow \phi_2} D(x; \phi), \quad F(x; \phi_2) > \lim_{\phi \uparrow \phi_2} F(x; \phi), \quad (21)$$

where the discontinuities arise because debt turns riskier at  $\phi = \phi_2$ . From (21),  $F(x)$  can be higher in Case (iii) than in Cases (i) and (ii). In other words, the covenant can improve firm value through its commitment effect in Case (iii). Section 3.3 will derive the explicit condition under which the covenant increases firm value.

Fig. 3 shows the comparative static results with respect to  $\phi$  for the baseline parameter values in Table 1, where Case (i):  $\phi \leq 0.48$ , Case (ii):  $0.48 < \phi < 0.88$ , and Case (iii):  $\phi \geq 0.88$ . For comparison, the figure also plots the values in the no-covenant model with dashed lines. We will explain  $\Delta = 25\%$  and 50% (dotted and chained lines) in Section 4.

As in Fig. 2, the center-left panel of Fig. 3 shows that the covenant increases firm value (i.e.,  $F(x) > F_N(x)$ ) for intermediate levels of  $\phi$  in Case (iii) because the positive commitment effect on debt value dominates the negative effect on equity value. In the center-right panel of Fig. 3, leverage  $LV(x)$  tends to be higher than the no-covenant level  $LV_N(x)$  in the region where the covenant improves firm value. The firm can choose high leverage at the initial time by using the IC ratio covenant as a commitment device for future debt repayment. This commitment effect is similar to the commitment effect of short debt maturity in Dangi and Zechner (2021). Indeed, they show that a firm

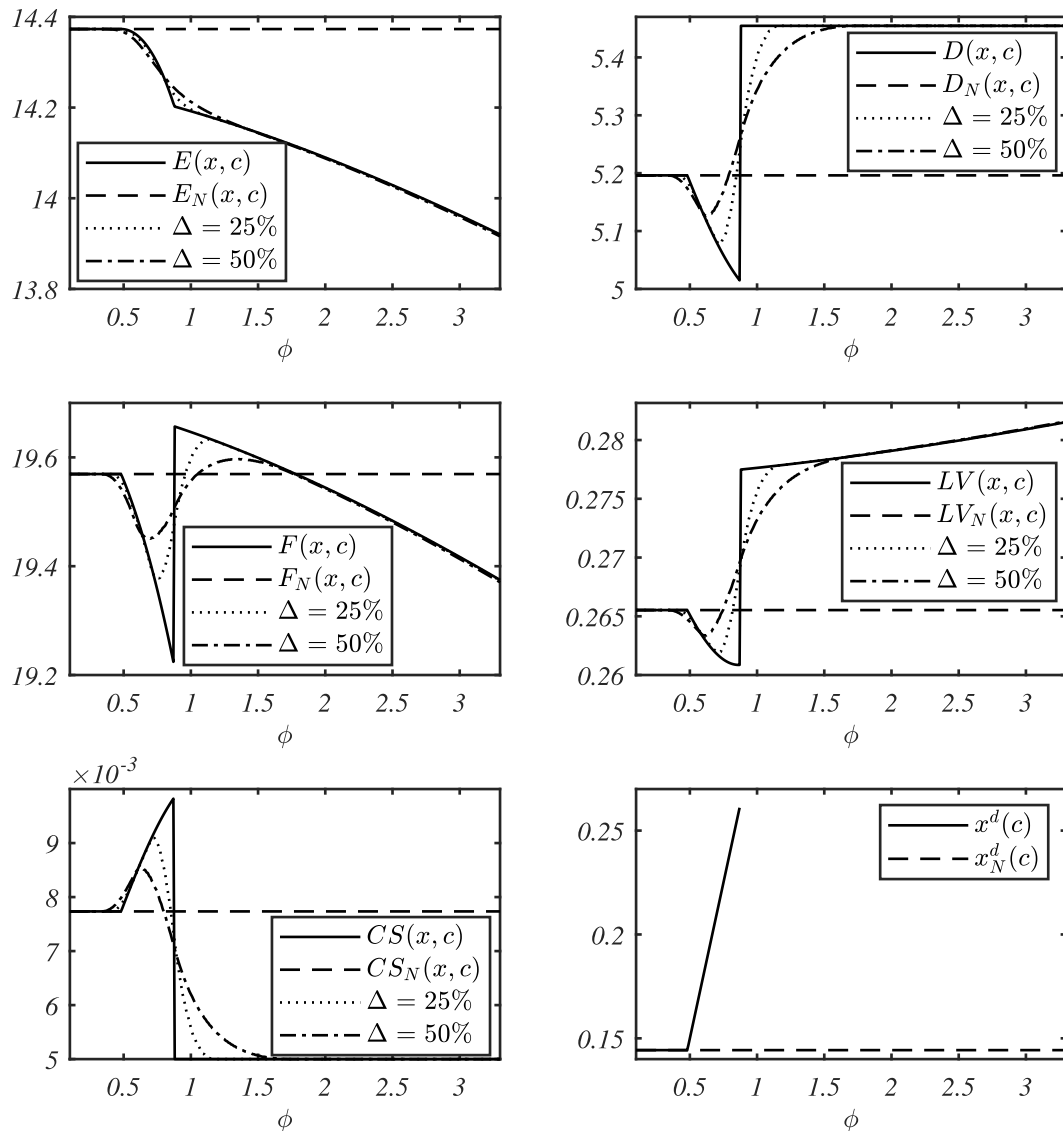


**Fig. 2.** Results with fixed coupon  $c = 0.3$  for covenant parameter  $\phi \in [0.1, 3.3]$ . The figure depicts equity value  $E(x, c)$ , debt value  $D(x, c)$ , firm value  $F(x, c)$ , leverage  $LV(x, c)$ , credit spreads  $CS(x, c)$ , and default threshold  $x^d(c)$ . Note that  $\phi_1 = 0.48$  and  $\phi_2 = 0.88$ . The figure also depicts the values in the no-covenant model with dashed lines and the weighted average values for the covenant tightness, following a symmetric triangular distribution in  $[(1 - \Delta)\phi, (1 + \Delta)\phi]$ , with dotted and chain lines. The other parameter values are set as in Table 1.

can choose high leverage by issuing debt with short maturity because short maturity acts as a commitment device for leverage reductions and decreases the cost of debt.

Proposition 2 also shows the comparative statics of  $E(x)$  and  $F(x)$  with respect to volatility  $\sigma$  in Case (iii). From (18), (20), and  $\partial\gamma/\sigma > 0$ ,  $E(x)$  and  $F(x)$  decrease in  $\sigma$ . As in the limiting results of  $E(x, c)$  and  $F(x, c)$ ,  $E(x)$  and  $F(x)$  approach the values with no tax benefits for  $\sigma \rightarrow \infty$ , and they approach the values with constant tax benefits for  $\sigma \rightarrow 0$  and  $\mu \geq 0$ .

### 3.3. Comparison of firm values

We now compare firm values with and without the covenant. Before doing this, we discuss the possibility to issue new debt. It is known as the leverage ratchet effect (e.g., Admati et al. (2018) and Demarzo and He (2021)) that in the no-covenant model with no restriction on new debt issuance, shareholders do not reduce their debt in response to deteriorating cash flows but issue new debt in response to ameliorating cash flows. The leverage ratchet effect (i.e., the ex post agency problem)

decreases firm value at time 0. In fact, as shown in Demarzo and He (2021), the rational expectation of shareholders' new debt issuance increases the cost of debt and fully offsets the tax benefits of debt at time 0. In this circumstance, prohibition of new debt issuance,<sup>10</sup> as assumed in Leland (1994), plays a role in preventing the leverage ratchet effect and improving firm value.

The commitment problem does not arise with an IC ratio covenant that ensures debt reductions (i.e., Case (iii)) because the value of existing debt does not depend on shareholders' new debt issuance within the covenant.<sup>11</sup> This implies that a tight IC ratio covenant can ease the restriction on future debt issuance to improve firm value. Considering this, we introduce a benchmark model (hereafter called the additional

<sup>10</sup> This can be implemented by a restrictive covenant against future financing.

<sup>11</sup> In reality, time lags and downward jumps may cause shareholders to default, even under a tight covenant (for details, see Section 4). Even in such scenarios, a tight covenant that leads to debt reductions can decrease the default probability and partially mitigate the leverage ratchet effect.

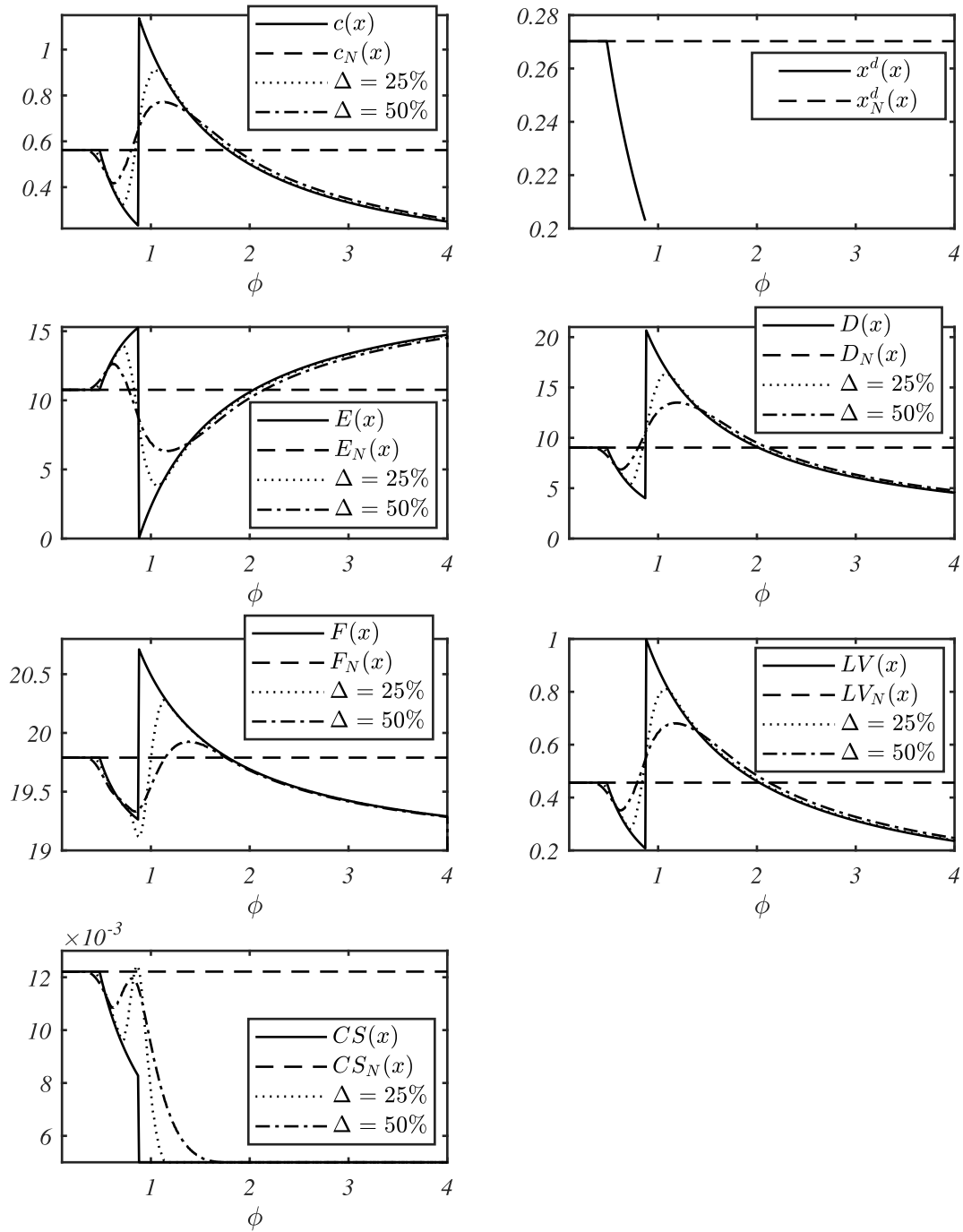


**Fig. 3.** Results with optimal coupon  $c(x)$  for covenant parameter  $\phi \in [0.1, 4]$ . The figure depicts coupon  $c(x)$ , default threshold  $x^d(x)$ , equity value  $E(x)$ , debt value  $D(x)$ , firm value  $F(x)$ , leverage  $LV(x) = D(x)/F(x)$ , and credit spreads  $CS(x) = c/D(x) - r$ . Note that  $\phi_1 = 0.48$  and  $\phi_2 = 0.88$ . The figure also depicts the values in the no-covenant model with dashed lines and the weighted average values for the covenant tightness, following a symmetric triangular distribution in  $[(1 - \Delta)\phi, (1 + \Delta)\phi]$ , with dotted and chain lines. The other parameter values are set as in Table 1.

debt model) where shareholders are allowed to freely increase debt within the covenant. In the optimum, shareholders always align debt with its limit (i.e.,  $C_{AD}(t) = X(t)/\phi$ ) to maximize the tax benefits of debt. Hereafter subscript  $AD$  stands for the additional debt model. The next proposition shows equity, debt, and firm values, denoted by  $E_{AD}(x)$ ,  $D_{AD}(x)$ , and  $F_{AD}(x)$ , respectively, in the additional debt model,

where  $\phi_{2AD}$  is defined by

$$\phi_{2AD} = \frac{(1 - \tau)(r + \lambda) - \mu}{(1 - \tau)(r + \lambda)}. \quad (22)$$

For proof, see Appendix C.

**Proposition 3.** Suppose that  $\phi \geq \phi_{2AD}$  holds. In the additional debt model, shareholders continuously adjust coupon  $C_{AD}(t) = X(t)/\phi$  at  $t \geq 0$  until

EBIT  $X(t)$  falls to 0 by a Poisson shock.

$$E_{AD}(x) = \left(1 - \tau + \frac{\mu - (1 - \tau)(r + \lambda)}{\phi(r + \lambda)}\right) \frac{x}{r + \lambda - \mu}, \quad (23)$$

$$D_{AD}(x) = x/(\phi(r + \lambda)), \quad (24)$$

$$F_{AD}(x) = \left(1 - \tau + \frac{\tau}{\phi}\right) \frac{x}{r + \lambda - \mu}. \quad (25)$$

If  $\phi \geq \phi_{2AD}$  holds, the right-hand side of (23) is nonnegative; hence, shareholders prefer to continue operation.<sup>12</sup> The debt reduction region is larger than Case (iii) because of  $\phi_{2AD} < \phi_2$  (see the last paragraph of Appendix C). All the claim values  $E_{AD}(x)$ ,  $D_{AD}(x)$ , and  $F_{AD}(x)$  are linear functions of  $x$ . They decrease with  $\phi$  and do not depend on volatility  $\sigma$ . Leverage  $LV_{AD}(X(t)) = D_{AD}(X(t))/F_{AD}(X(t))$  is constant, equal to  $(r + \lambda - \mu)/((r + \lambda)((1 - \tau)\phi + \tau))$ . From (18)–(20), (23)–(25), and  $\lim_{\sigma \rightarrow 0} \gamma = (r + \lambda)/\mu$  for  $\mu < 0$ ,  $E(x)$ ,  $D(x)$ , and  $F(x)$  coincide with  $E_{AD}(x)$ ,  $D_{AD}(x)$ , and  $F_{AD}(x)$ , respectively for  $\sigma \rightarrow 0$  and  $\mu < 0$ . The reason is that in the limit case, EBIT  $X(t)$  decreases deterministically over time, and shareholders never issue new debt.

Note that a similar outcome to Proposition 3 arises in Malenko and Tsoy (2025), who study optimal time-consistent leverage policies which shareholders prefer over deviating and losing credibility ex-post. In their model, shareholders maintain a constant IC ratio by continuously adjusting their debt level (as in our additional debt model) during the normal regime (i.e., the high IC ratio region). However, in the distress regime (i.e., the low IC ratio region) triggered by a large downward jump, shareholders temporarily abandon the constant IC policy until the IC ratio either recovers to the normal region or declines to the default threshold. Transitions between the normal and distress regimes make their model more insightful and applicable than our benchmark model. Unlike our paper, they do not assume a covenant but rely on a self-sustained reputation mechanism to address the commitment problem.

Now, we derive the explicit condition under which the covenant increases firm value beyond the no-covenant firm value  $F_N(x)$ . As shown in Proposition 2, the covenant is inefficient in Cases (i) and (ii). Thus, we focus on Case (iii). By comparing (20), (25), and (39), we readily have the following proposition, where  $\phi^*$  and  $\phi_{AD}^*$  are defined by

$$\phi^* = \frac{\gamma}{\gamma - 1} \frac{(r + \lambda - \mu)h}{r + \lambda}, \quad \phi_{AD}^* = h, \quad (26)$$

where  $h$  is defined by (40) in Appendix B. Note that  $\phi^* < \phi_{AD}^*$  follows from  $(r + \lambda)/\mu < \gamma < 0$  for  $\mu < 0$  and by  $\gamma < 0$  for  $\mu \geq 0$ .

**Proposition 4.** In Case (iii) (i.e.,  $\phi \geq \phi_2$ ), the initial firm values satisfy the following relations.  $F(x) < F_{AD}(x)$  holds.  $F(x) > F_N(x)$  and  $c(x) > c_N(x)$  hold if and only if  $\phi < \phi^*$ .  $F_{AD}(x) > F_N(x)$  holds if and only if  $\phi < \phi_{AD}^*$ .

For  $\phi \in [\phi_2, \phi^*)$ , the positive effect of the covenant on debt value dominates its negative effect on equity value, and, hence, the covenant improves firm value overall. We call the interval  $[\phi_2, \phi^*)$  the efficient covenant region. In contrast, for  $\phi > \phi^*$ , the covenant decreases firm value. The efficient covenant region in the additional debt model,  $[\phi_{2AD}, \phi_{AD}^*)$ , is larger than in the baseline model.

While Gamba and Mao (2020) and Xiang (2024) also show that covenants can increase firm value through their commitment effects, our findings differ in several ways. In their models, which allow new debt issuance, the positive value primarily arises from the role of covenants in suppressing the leverage ratchet effect (as in our additional debt model). In contrast, Proposition 4 shows that even in our baseline model with no new debt issuance, a covenant can increase

firm value (i.e.,  $F(x) > F_N(x)$ ) solely through its role in eliminating shareholders' incentive to default. Moreover, we show that  $F(x) > F_N(x)$  holds if and only if  $c(x) (= x/\phi)$  is higher than  $c_N(x)$ . In other words, whether the covenant improves firm value is equivalent to whether it allows more debt issuance than the optimal debt level in the no-covenant model. As noted in Sections 3.2 and 3.3, this result is consistent with empirical findings that covenants can increase firm value by reducing the cost of debt and increasing the debt capacity (e.g., Bradley and Roberts (2015)).

From (2), (22), and (26), we obtain the efficient covenant regions  $[\phi_2, \phi^*) = [0.88, 1.79]$  and  $[\phi_{2AD}, \phi_{AD}^*) = [0.79, 3.7]$  for the baseline parameter values in Table 1. Fig. 4 shows how the efficient covenant regions change with varying levels of growth rate  $\mu$ , volatility  $\sigma$ , bankruptcy cost  $\alpha$ , and corporate tax rate  $\tau$ .<sup>13</sup> First, we explain the effects of  $\sigma$  and  $\alpha$ . The corresponding panels of Fig. 4 show that  $(\phi^*, \phi_2]$  and  $(\phi_{AD}^*, \phi_{2AD}]$  expand with higher  $\sigma$  and  $\alpha$ . From  $\partial\gamma/\partial\sigma > 0$ , we can prove that  $\partial\phi_2/\partial\sigma > 0$ ,  $\partial\phi_{AD}^*/\partial\sigma > 0$ ,  $\partial\phi_{2AD}/\partial\sigma > 0$ ,  $\partial\phi^*/\partial\alpha > 0$ ,  $\partial\phi_2/\partial\alpha = 0$ ,  $\partial\phi_{AD}^*/\partial\alpha > 0$ , and  $\partial\phi_{2AD}/\partial\alpha = 0$ . These results are intuitive. With no covenant, higher  $\sigma$  and  $\alpha$  increase the expected cost of default. Then, the covenant, which leads to shareholders' debt reduction policy, becomes more efficient. Our results are consistent with the empirical findings in Bradley and Roberts (2015). In fact, they show a negative relation between tangible assets and the probability of including covenants, where more tangible assets imply lower bankruptcy costs. Bradley and Roberts (2015) also show that firms with higher volatility and default risk are more likely to include covenants.

Next, we turn to the effects of growth rate  $\mu$ . It follows from  $\partial\gamma/\partial\mu < 0$ , (2), (22), and (26) that  $\partial\phi_2/\partial\mu < 0$ ,  $\partial\phi_{AD}^*/\partial\mu < 0$ , and  $\partial\phi_{2AD}/\partial\mu < 0$ . In the upper-left panel of Fig. 4, lower  $\mu$  shifts  $[\phi_2, \phi^*)$  upward. Indeed, lower  $\mu$  leads to higher default risk; hence, the firm must limit the IC ratio more strictly to make debt minimal-risk and improve firm value. Additional debt weakens this effect, and  $[\phi_{2AD}, \phi_{AD}^*)$  is much larger than  $[\phi_2, \phi^*)$  for higher  $\mu$ . This implies that the additional debt channel can greatly improve a high-growth firm's value. Bradley and Roberts (2015) observe that high-growth firms are less likely to include covenants that restrict external financing in the future but are more likely to include the other types of covenants. Our model can account for their empirical findings. Indeed, we argue that high-growth firms optimally design such covenants; that is, they increase firm values by relaxing future financing covenants in exchange for adding IC ratio covenants.

The lower-right panel of Fig. 4 shows that the effects of corporate tax rate  $\tau$  also depend on whether the covenant allows future debt issuance. It follows from (2), (22), and (26) that  $\partial\phi^*/\partial\tau < 0$ ,  $\partial\phi_2/\partial\tau > 0$ ,  $\partial\phi_{AD}^*/\partial\tau < 0$ , and  $\partial\phi_{2AD}/\partial\tau < 0$ . The efficient covenant region  $[\phi_2, \phi^*)$  shrinks with higher  $\tau$  and extinguishes at  $\tau = 0.459$  in the baseline model, whereas  $[\phi_{2AD}, \phi_{AD}^*)$  remains large for higher  $\tau$  in the additional debt model. For higher  $\tau$ , the tax benefits of debt increase; hence, the value of the covenant that causes shareholders to reduce their debt decreases in the baseline model. In the additional debt model, with higher  $\tau$ , the firm can extract greater tax benefits by issuing new debt in response to ameliorating earnings.

So far, we have focused on firm value at time 0. Even if  $F_N(x)$  is higher than  $F(x)$  or  $F_{AD}(x)$  in Case (iii), the relation can invert at time  $t > 0$ . This is because shareholders dynamically adjust their debt level over time. The next proposition summarizes the results. For proof, see Appendix D.

**Proposition 5.** In Case (iii) (i.e.,  $\phi \geq \phi_2$ ), the firm values satisfy the following relations at time  $t > 0$ .  $F(X(t), C(t)) < F_{AD}(X(t))$  holds.

<sup>12</sup> If the condition does not hold, shareholders will default as in Cases (i) and (ii) in Proposition 2; hence, the restriction on additional debt makes sense from the firm value perspective.

<sup>13</sup> These efficient covenant regions may be overestimated. In practice, various costs and risks of covenants may exist (see Section 4). When these are considered, the efficient covenant regions shrink (or may disappear), although the comparative static results are likely to remain valid.

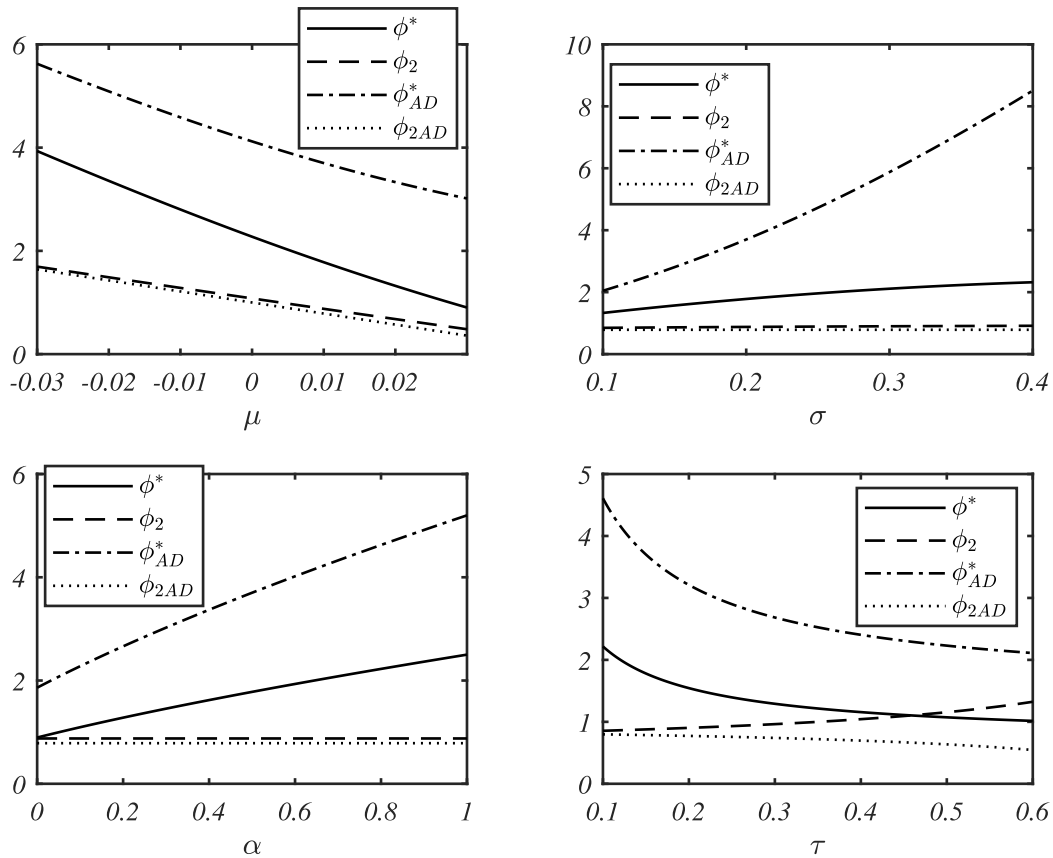


**Fig. 4.** The thresholds for the efficient covenant region,  $\phi^*$  and  $\phi_{AD}^*$ , as well as the thresholds for Case (iii),  $\phi_2$  and  $\phi_{2AD}$ , in the baseline and additional debt models, respectively. This figure shows the efficient covenant regions  $[\phi_2, \phi^*)$  and  $[\phi_{2AD}, \phi_{AD}^*)$  in the baseline and additional debt models, respectively, for growth rate  $\mu \in [-0.03, 0.03]$ , volatility  $\sigma \in [0.1, 0.4]$ , bankruptcy cost  $\alpha \in [0, 1]$ , and corporate tax rate  $\tau \in [0.1, 0.6]$ . The other parameter values are set as in Table 1.

$F(X(t), C(t))$  and  $F_{AD}(X(t))$  are higher than  $F_N(X(t), c_N(x))$  when EBIT  $X(t)$  is sufficiently low (i.e., close to  $x_N^d(x)$ ).  $F_{AD}(X(t))$  is higher than  $F_N(X(t), c_N(x))$  when EBIT  $X(t)$  is sufficiently high (i.e., close to  $\phi(r + \lambda - \mu)c_N(x)/(r + \lambda)$ ).

In the no-covenant model,  $F_N(X(t), c_N(x))$  approaches the post-bankruptcy firm value,  $(1 - \alpha)(1 - \tau)X(t)/(r + \lambda - \mu)$  when EBIT  $X(t)$  approaches default threshold  $x_N^d(x)$ . However, with the covenant,  $F(X(t), C(t))$  and  $F_{AD}(X(t))$  are always higher than the unlevered firm value  $(1 - \tau)X(t)/(r + \lambda - \mu)$  because of the existence of the tax benefits of remaining debt. Hence,  $F(X(t), C(t))$  and  $F_{AD}(X(t))$  exceed  $F_N(X(t), c_N(x))$  when  $X(t)$  approaches  $x_N^d(x)$  from above. In the additional debt model, the firm can expand debt and its tax benefits when  $X(t)$  increases. Then,  $F_{AD}(X(t))$  is higher than  $F_N(X(t), c_N(x))$  for sufficiently high  $X(t)$ .

Fig. 5 shows trajectories of  $F(X(t), C(t))$ ,  $F_{AD}(X(t))$ , and  $F_N(X(t), c_N(x))$  for a sample path of  $X(t)$ , which was generated by a simulation. The figure also shows trajectories of coupon and leverage. We set the baseline parameter values in Table 1 and  $\phi = 2(> \phi^*)$  in Case (iii). With no covenant, shareholders issue debt with  $c_N(x) = 0.561$  at time 0 and never adjust debt (see the upper-right panel of Fig. 5). They declare default when EBIT  $X(t)$  falls below default threshold  $x_N^d(x) = 0.27$  at time  $t = 6.5$ . In the lower-right panel, leverage  $LV_N(X(t), c_N(x))$  reaches 1 at default because equity value  $E_N(X(t), c_N(x))$  falls to 0. In contrast, in the baseline model, shareholders reduce coupon  $C(t)$  to the upper limit  $X(t)/\phi$  every time  $X(t)$  hits a record low (see the upper-right panel). Then, in the lower-right panel, leverage  $LV(X(t), C(t))$  has the upper bound (i.e., the right-hand side of (11)). Under the bound, shareholders do not reduce their debt, and, hence,  $LV(X(t), C(t))$  is negatively related to  $X(t)$ . In the lower-left panel,  $F(X(0), C(0)) < F_N(X(0), c_N(x))$  holds, whereas  $F(X(t), C(t)) > F_N(X(t), c_N(x))$  holds

close to default time  $t = 6.5$ . In the additional debt model, shareholders continuously adjust coupon  $C_{AD}(t) = X(t)/\phi$ , which leads to constant leverage  $LV_{AD}(X(t))$  and higher firm value  $F_{AD}(X(t)) > F(X(t), C(t))$ .

#### 4. Covenant tightness

In practice, IC ratio covenants typically specify minimum levels  $\phi$  between 1 and 4 (e.g., Palomino et al. (2019)). Our efficient covenant region  $[\phi_2, \phi^*)$  (particularly, the optimal level  $\phi_2$ ) is lower than the observed range, although  $[\phi_{2AD}, \phi_{AD}^*)$  may align more closely. Below, we consider some potential reasons for this discrepancy.

For tractability, our baseline model assumes that shareholders reduce their debt level infinitesimally or declare default as soon as the IC ratio  $X(t)/C(t)$  reaches  $\phi$ . The baseline result that the left endpoint of the minimal-risk region,  $\phi_2$ , is optimal relies on this strong enforcement assumption. In the real world, however, debt holders cannot immediately identify a covenant violation because they monitor the ratio not continuously but periodically (e.g., quarterly). Some firms may use various accounting tricks (e.g., off-balance-sheet items) to hide a covenant violation. Some firms may fail to reduce their debt to the required level (i.e., incomplete adjustment). Even if they succeed in reducing their debt by the required amount, it may take time to complete the debt repayment. These enforcement frictions can deviate the IC ratio from the specified level  $\phi$ . In addition, EBIT can jump downward by a fraction  $\Delta\%$  (rather than 100% in the baseline model), which also cause the IC ratio away from the specified level  $\phi$ . Then, even when  $\phi \geq \phi_2$ , the IC ratio can fall below  $\phi_2$ , making shareholders choose default.

To approximate the effects of time lags, incomplete adjustment, and jumps, in Figs. 2 and 3 we computed the weighted average values when

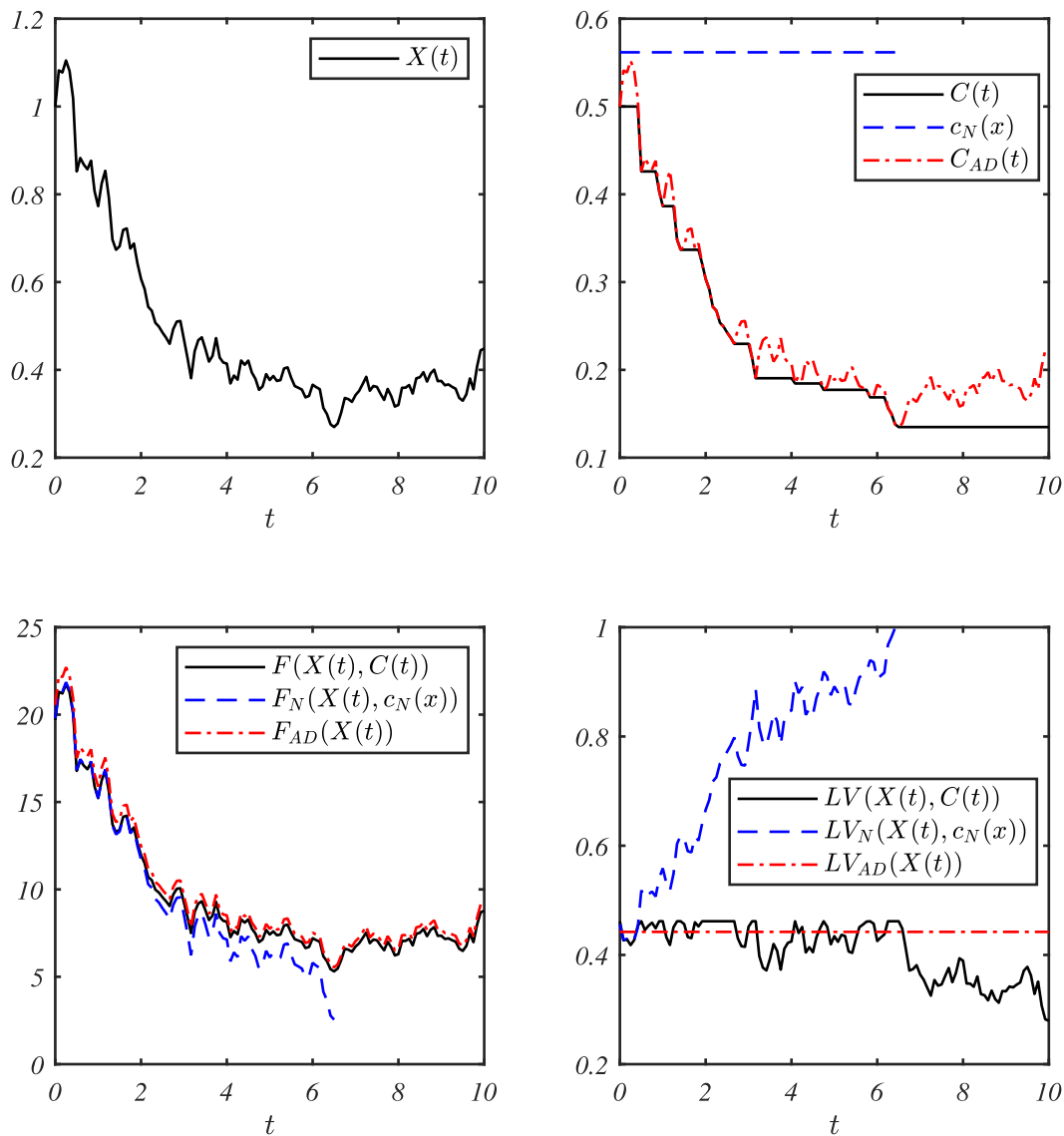


**Fig. 5.** Dynamic results. The figure depicts trajectories of coupon  $C(t)$ , firm value  $F(X(t), C(t))$ , and leverage  $LV(X(t), C(t))$  for a simulated path  $X(t)$ . A Poisson jump does not occur on this path. The figure also depicts those of the no-covenant and additional debt models by blue dashed and red chain lines, respectively. We set the IC ratio limit  $\phi = 2$  and the other parameter values as in Table 1.

the covenant tightness follows a symmetric triangular distribution in  $[(1-\Delta)\phi, (1+\Delta)\phi]$ .<sup>14</sup> For instance, with  $\Delta = 25\%$ , even if the firm specifies  $\phi_2 = 0.88$  in the covenant, realized  $\phi$  could fall within  $[0.66, 1.1]$ . The weighted average results approximate situations where, due to enforcement frictions, the IC ratio at the time shareholders decide whether to default may deviate from the specified limit.

Contrasted to the baseline graphs, graphs for  $\Delta = 25\%, 50\%$  are smooth at  $\phi = \phi_2$ . Indeed, debt has default risk due to frictions when  $\phi$  is equal to or slightly higher than  $\phi_2$  (see the top-right and bottom-left panels of Fig. 2). Considering the increased default risk, the firm chooses optimal  $\phi = 1.13, 1.34 (> \phi_2 = 0.88)$  for  $\Delta = 25\%, 50\%$ , respectively (see the center-left panel of Fig. 2). Note that higher  $\Delta$  increases optimal  $\phi$  and reduces the positive effect of the covenant on firm value.

The top-left panel of Fig. 3 shows the weighted average of optimal coupon  $c(x)$ . For the initial coupon, Fig. 3 presents the weighted average values for equity, debt, firm values, leverage, and credit spreads. Although the baseline firm value jumps at  $\phi = \phi_2 = 0.88$  and decreases for  $\phi \geq \phi_2$  (hence,  $\phi = \phi_2$  is optimal), positive  $\Delta$  produces a smooth, hump-shaped relation between firm value and  $\phi$ , where optimal  $\phi$  is higher than  $\phi_2$  (see the center-left panel of Fig. 3). The results with positive  $\Delta$  reflect a balance between default probability and debt capacity and align better with empirical observations than the baseline results.<sup>15</sup> Higher  $\Delta$  also greatly diminishes the positive effect of the covenant on firm value. In the center-right panel, higher  $\Delta$  substantially decreases the optimal leverage compared to the baseline

<sup>14</sup> We computed the results for various distributions including distributions with only downside risk and verified that the results remain robust.

<sup>15</sup> Xiang (2024) also shows a hump-shaped relation between firm value and covenant tightness by another mechanism, where it is optimal to have lenders intervene enough frequently to cure shareholders' dilution but not too much such that their interventions start to worsen ex ante incentives.

level (i.e.,  $LV(x) = 1$  at  $\phi = \phi_2$ ). In short, with the risks increased by time lags, incomplete adjustment, and jumps, the firm adopts a more conservative strategy (i.e., a tighter covenant and lower leverage) and therefore extracts less value from the covenant than it does in the baseline model under the strong enforcement assumption. Nevertheless, our main result remains qualitatively unchanged: the covenant can reduce shareholders' incentive to default and, through this channel, increase firm value. Appendix E also verifies the similar effects in the discrete-time version of the baseline model.

Furthermore, for tractability, our baseline model assumes no costs associated with debt reductions at covenant violations. In reality, shareholders may face financing costs for debt repayment in addition to transaction and adjustment costs. A looser covenant (i.e., lower  $\phi$ ) can increase these costs as firms with lower IC ratios generally face higher financing costs. Our model, based on Leland (1994), assumes that shareholders can issue equity without costs, even for low IC ratios, and thus cannot capture this effect. In practice, considering debt reduction costs, firms may opt for tighter covenants (i.e., higher  $\phi$ ) than what our model suggests.

Another reason may be the possibility of renegotiating covenants. Dichev and Skinner (2002) find that private lenders tend to set covenants tightly and renegotiate them in case of covenant violations. They argue that debt holders use covenant violations as early warning signals and increase surveillance (e.g., monitoring the ratio more frequently) in exchange for easing covenants. Taking this into account, we consider a model in which equity and debt holders can renegotiate  $\phi$  at each covenant violation.<sup>16</sup> If  $\phi > \phi_2$ , they renegotiate to set  $\phi = \phi_2$ , thereby maximizing the total firm value at the covenant violation time. The total firm value declines if they instead choose  $\phi > \phi_2$  through renegotiation. In fact,  $\phi > \phi_2$  leads to Case (i) or (ii), which reduces the firm value. Accordingly, the firm optimally sets  $\phi = \phi_2$  at time 0, and renegotiation of  $\phi$  will never occur in equilibrium. However, with incomplete information about cash flow dynamics, renegotiation could happen. For instance, suppose that all investors observe a true value of  $\mu \in \{-0.03, 0.03\}$  at a random time. In this case, equity and debt holders can initially set high  $\phi$  (see  $\phi_2$  decreasing with  $\mu$  in Fig. 4) in preparation for the worst-case scenario (i.e.,  $\mu = -0.03$ ) and decrease  $\phi$  when the true value of  $\mu$  is 0.03.<sup>17</sup> Unlike the baseline model, this may explain empirical observations that  $\phi$  decreases through renegotiation.

Finally, lender risk tolerance also influences the required IC ratio level  $\phi$ . Conservative lenders, such as traditional banks, may demand higher  $\phi$  to enhance loan safety. By setting higher  $\phi$ , they can suppress the IC ratio, leverage, and credit spreads, preventing these variables from rising to risky levels.<sup>18</sup> From the lender's perspective, the role of covenants in mitigating default risk may lead to  $\phi$  that exceeds the level optimal for maximizing firm value. We also discuss debt maturity, transaction costs, costly refinancing, and debt overhang in Appendix F.

## 5. Conclusion

We study a debt financing model with a financial covenant that imposes a lower limit on a firm's IC ratio. When the ratio falls below the limit due to deteriorating earnings, shareholders are obligated to partially repay their debt to comply with the limit or default. In our model,

<sup>16</sup> We consider renegotiation by which shareholders can postpone repayment until the IC ratio reaches the renegotiated level. In typical renegotiation, as noted by Xiang (2024), debt holders allow firms to delay repayment and temporarily breach the limit, rather than permitting the issuance of new debt as modeled in Gamba and Mao (2020).

<sup>17</sup> Decreasing  $\phi$  is possible for any  $X(t)$ , although increasing  $\phi$  can cause an instant covenant violation and default for low  $X(t)$ .

<sup>18</sup> As discussed above, shareholders can default due to time lags and jumps, and a tighter covenant (i.e., higher  $\phi$ ) decreases default risk due to time lags and jumps.

equity, debt, firm values, debt repayment policy, and capital structure are derived explicitly. The main results are summarized below.

For very low levels of the limit, the covenant does not affect the firm, and the results are the same as in Leland (1994). For low levels of the limit, a covenant violation triggers default, and the covenant turns out to be inefficient (i.e., decreasing equity, debt, and firm values). For high levels of the limit, the following debt reduction outcome arises. Shareholders prefer to reduce their debt level and continue operation at each covenant violation because the IC ratio is sufficiently high. In other words, the covenant renders their default option worthless by accelerating their choice between default and repayment. It enhances firm value if the positive effect on debt value outweighs the negative effect on equity value. The covenant tends to improve firm value when bankruptcy costs and volatility are higher, although with risks due to time lags and jumps, the firm adopts a more conservative strategy (i.e., a tighter covenant and lower leverage), which diminishes its positive impact. It can also enhance firm value by easing the restriction on future debt financing, particularly in environments with higher growth and tax rates. With such an efficient covenant, the firm tends to start with high leverage to benefit from the decreased cost of debt. These results are consistent with empirical evidence and support the optimal contracting hypothesis.

## CRedit authorship contribution statement

**Michi Nishihara:** Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Takashi Shibata:** Writing – review & editing, Supervision, Methodology, Investigation, Funding acquisition. **Benoit Chevalier-Roignant:** Writing – review & editing, Methodology, Formal analysis.

## Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGpt in order to improve language and readability. After using ChatGpt, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

## Appendix A. Proof of Proposition 1

Shareholders are obligated to default or reduce debt when  $X(t)$  falls below  $\phi C(t)$ . The tax shield decreases as they reduce their debt level. Hence, shareholders have no reason to reduce their debt more (or earlier) than is required by the covenant. Shareholders also have an option to default (i.e., stop coupon payments) at any time.

First, assume  $\phi \leq \phi_1$  (i.e., Case (i)). We consider shareholders' optimal default timing problem with no covenant. As in Leland (1994), we can easily show that the no-covenant default threshold is equal to

$$x_N^d(c) = \frac{\gamma}{\gamma - 1} \frac{(r + \lambda - \mu)c}{r + \lambda}.$$

The no-covenant default policy is feasible even with the covenant, i.e.,  $x_N^d(c) \geq \phi c$ , if and only if  $\phi \leq \phi_1$ . Then, for  $\phi \leq \phi_1$ , shareholders suffer no loss from the covenant, and the equity, debt, and firm values agree with the following values of the no-covenant model (i.e., Leland (1994) and Goldstein et al. (2001)):

$$\begin{aligned} E(x, c) &= E_N(x, c) = (1 - \tau) \left( \frac{x}{r + \lambda - \mu} - \frac{c}{r + \lambda} - \left( \frac{x}{x_N^d(c)} \right)^\gamma \left( \frac{x_N^d(c)}{r + \lambda - \mu} - \frac{c}{r + \lambda} \right) \right), \\ D(x, c) &= D_N(x, c) = \frac{c}{r + \lambda} - \left( \frac{x}{x_N^d(c)} \right)^\gamma \left( \frac{c}{r + \lambda} - \frac{(1 - \alpha)(1 - \tau)x_N^d(c)}{r + \lambda - \mu} \right), \\ F(x, c) &= F_N(x, c) = \frac{(1 - \tau)x}{r + \lambda - \mu} + \frac{\tau c}{r + \lambda} - \left( \frac{x}{x_N^d(c)} \right)^\gamma \left( \frac{\alpha(1 - \tau)x_N^d(c)}{r + \lambda - \mu} + \frac{\tau c}{r + \lambda} \right) \quad (27) \end{aligned}$$

hold for  $x \geq x_N^d(c)$ , while  $E(x, c) = 0$  and  $D(x, c) = F(x, c) = (1 - \alpha)(1 - \tau)x/(r + \lambda - \mu)$  hold for  $x < x_N^d(c)$ . Shareholders will keep  $C(t) = c$  constant.

Next, assume  $\phi > \phi_1$  (i.e., Cases (iii) and (ii)). The no-covenant default timing is later than the first hitting time to the limit (i.e.,  $x_N^d(c) < \phi c$ ); hence, shareholders have no reason to default before EBIT  $X(t)$  falls below  $\phi c$ . Then, there is  $c^d \in [0, c]$  such that shareholders reduce their debt at covenant violations with  $C(t) > c^d$  and default when  $X(t)$  falls to  $\phi c^d$ . They choose  $c^d$  (or equivalently, default threshold  $x^d(c) = \phi c^d$ ) to maximize their value.

Shareholders' optimal default timing problem is given by

$$E(x, c) = \sup_{T^d} \mathbb{E} \left[ \int_0^{\min(S, T^d)} e^{-rt} (1 - \tau)(X(t) - C(t)) dt + \int_0^{\min(S, T^d)} \frac{e^{-rt}}{r + \lambda} dC(t) \right] \quad (28)$$

where  $S$  denotes the bankruptcy time caused by a Poisson jump, the default time  $T^d = \inf\{t \geq 0 \mid X(t) \leq x^d(c) = \phi c^d\}$  is optimized over all  $c^d \in [0, c]$ , and  $C(t) = \min\{c, \min_{0 \leq s \leq t} X(s)/\phi\}$ . The first and second integrals in (28) represent cash flows and repayment costs until default, respectively. Note that shareholders repay  $-dC(t)/(r + \lambda)$ , where  $dC(t) \leq 0$ . As in Chapter 11.1.A of Dixit and Pindyck (1994),  $E(x, c)$  satisfies

$$\mu x \frac{\partial E(x, c)}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 E(x, c)}{\partial x^2} + (1 - \tau)(x - c) - \lambda E(x, c) = rE(x, c) \quad (x > \phi c) \quad (29)$$

in the non-rebalancing region  $\{(x, c) \mid x > \phi c\}$  with the boundary conditions

$$\frac{\partial E(\phi c, c)}{\partial c} = -\frac{1}{r + \lambda} \quad (c > c^d) \quad (30)$$

on the rebalancing boundary  $\{(x, c) \mid x = \phi c, c > c^d\}$ ,

$$E(\phi c^d, c^d) = 0 \quad (31)$$

at the default point  $(x, c) = (\phi c^d, c^d)$ , and

$$\lim_{x \rightarrow \infty} \frac{E(x, c)}{x} < \infty. \quad (32)$$

Intuitively, condition (30) means that shareholders repay  $1/(r + \lambda)$  when reducing a unit of coupon, whereas condition (31) means that equity value becomes 0 at default threshold  $\phi c^d$ . Conditions (30) and (31) correspond to the well-known value matching conditions (e.g., Dixit and Pindyck (1994)). We will later derive the optimality condition for default threshold  $\phi c^d$ .<sup>19</sup> Condition (32) readily follows from the trivial inequality  $E(x, c) \leq (1 - \tau)x/(r + \lambda - \mu)$ .

As in Chapter 11.1.A in Dixit and Pindyck (1994), the general solution to (29) with boundary condition (32) is expressed as

$$E(x, c) = \frac{(1 - \tau)x}{r + \lambda - \mu} - \frac{(1 - \tau)c}{r + \lambda} + f(c)x^\gamma, \quad (33)$$

where  $f(c)$  is an unknown function of  $c$ . In (33), the last term stands for the value from reducing debt until  $C(t) \geq c^d$  and default at  $X(t) = \phi c^d$ . It follows from (30) and (33) that

$$\frac{\tau}{r + \lambda} + (\phi c)^\gamma f'(c) = 0. \quad (34)$$

The general solution to (34) is

$$f(c) = k - \frac{\tau c^{1-\gamma}}{\phi^\gamma (r + \lambda)(1 - \gamma)}, \quad (35)$$

where  $k$  is an unknown constant. By substituting (35) into (33), we have

$$E(x, c) = \frac{(1 - \tau)x}{r + \lambda - \mu} - \frac{(1 - \tau)c}{r + \lambda} + \left( k - \frac{\tau c^{1-\gamma}}{\phi^\gamma (r + \lambda)(1 - \gamma)} \right) x^\gamma. \quad (36)$$

<sup>19</sup> No smooth pasting condition arises on the rebalancing boundary  $\{(x, c) \mid x = \phi c, c > c^d\}$  because the rebalancing boundary is not optimized by shareholders but determined by the covenant.

In (36),  $k$  is determined by boundary condition (31). Indeed, we obtain

$$E(x, c) = \frac{(1 - \tau)x}{r + \lambda - \mu} - \frac{(1 - \tau)c}{r + \lambda} - \left( \frac{x}{\phi c^d} \right)^\gamma \left( \frac{(1 - \tau)\phi c^d}{r + \lambda - \mu} - \frac{(1 - \tau)c^d}{r + \lambda} \right) - \left( \frac{x}{\phi c} \right)^\gamma \left( \frac{\tau c}{(r + \lambda)(1 - \gamma)} - \left( \frac{c}{c^d} \right)^\gamma \frac{\tau c^d}{(r + \lambda)(1 - \gamma)} \right) \quad (37)$$

$$= \frac{(1 - \tau)x}{r + \lambda - \mu} - \frac{(1 - \tau)c}{r + \lambda} - \left( \frac{x}{\phi c} \right)^\gamma \frac{\tau c}{(r + \lambda)(1 - \gamma)} + \left( \frac{x}{\phi c^d} \right)^\gamma \left( \frac{\gamma - \gamma\tau - 1}{\phi(r + \lambda)(\gamma - 1)} - \frac{1 - \tau}{r + \lambda - \mu} \right) \phi c^d. \quad (38)$$

In (37), the first, second, third, and fourth terms represent the values of receiving EBIT and paying coupon  $c$  constantly, loss at default threshold  $\phi c^d$ , and gradual loss of the tax shield due to debt reductions until default threshold  $\phi c^d$ , respectively. Shareholders optimize  $c^d \in [0, c]$  (or default threshold  $x^d(c) = \phi c^d$ ) based on the trade-off between the tax shield and default loss.

Now, we derive optimal  $c^d$ . The right-hand side of (38) monotonically increases with  $c^d$  when

$$\frac{\gamma - \gamma\tau - 1}{\phi(r + \lambda)(\gamma - 1)} - \frac{1 - \tau}{r + \lambda - \mu} > 0$$

(i.e.,  $\phi < \phi_2$ ). Then, we have a corner solution  $c^d = c$  (i.e., defaulting without reducing debt) for  $\phi \in (\phi_1, \phi_2)$ , which leads to (3)–(6) in Case (ii) of Proposition 1.<sup>20</sup> The right-hand side of (38) monotonically decreases with  $c^d$  when  $\phi > \phi_2$ . Then, we have the other corner solution  $c^d = 0$  for  $\phi > \phi_2$  (i.e., avoiding default until  $X(t)$  falls to 0 by a Poisson shock), which leads to (7)–(10) in Case (iii) of Proposition 1. For  $\phi = \phi_2$ , shareholders are indifferent to  $c^d$ , and for simplicity, we assume that shareholders reduce debt in this case.

## Appendix B. Proof of Proposition 2

First, consider Case (i). The solution is the same as that of the no-covenant model (see Shibata and Nishihara (2012) and Sundaresan et al. (2015)):

$$\begin{aligned} c(x) &= c_N(x) = \frac{(\gamma - 1)(r + \lambda)x}{\gamma(r + \lambda - \mu)h}, \quad x^d(x) = x_N^d(x) = x/h, \\ E(x) &= E_N(x) = \left( 1 - h^{\gamma-1} - \frac{(\gamma - 1)(1 - h^\gamma)}{\gamma h} \right) \frac{(1 - \tau)x}{r + \lambda - \mu}, \\ D(x) &= D_N(x) = \left( \frac{(\gamma - 1)(1 - h^\gamma)}{\gamma} + (1 - \alpha)(1 - \tau)h^\gamma \right) \frac{x}{(r + \lambda - \mu)h}, \\ F(x) &= F_N(x) = \left( 1 - \tau + \frac{\tau}{h} \right) \frac{x}{r + \lambda - \mu}, \end{aligned} \quad (39)$$

where

$$h = \left[ 1 - \gamma \left( 1 - \alpha + \frac{\alpha}{\tau} \right) \right]^{-1/\gamma} (> 1). \quad (40)$$

In Case (ii), we derive  $c(x) = \arg \max_{c \geq 0} F(x, c)$  by the first-order condition. By using  $c(x)$ , we also derive  $E(x) = E(x, c(x))$ ,  $D(x) = D(x, c(x))$ , and  $F(x) = F(x, c(x))$ . The details are omitted because the derivation is standard.

Last, consider Case (iii). In (10),  $\partial F(x, c)/\partial c > 0$  holds for  $c \in (0, x/\phi)$ . Then, we have  $c(x) = \arg \max_{c \in [0, x/\phi]} F(x, c) = x/\phi$ . We also obtain (18)–(20) by substituting  $c = x/\phi$  into (8)–(10).

## Appendix C. Proof of Proposition 3

First, we derive the equity, debt, and firm values by supposing that shareholders continuously adjust coupon  $C_{AD}(t) = X(t)/\phi$  ( $t \geq 0$ ). In

<sup>20</sup> We cannot derive  $c^d$  (or default threshold  $x^d(c) = \phi c^d$ ) from the corresponding smooth pasting condition. Indeed, the smooth pasting condition coincides with the first-order condition (i.e., the derivative of the right-hand side of (38) with respect to  $c^d$  equals 0), and there is no solution.

this case, debt value is the minimal-risk debt value (i.e.,  $D_{AD}(x) = x/(\phi(r + \lambda))$ ). The firm receives flows of the tax benefits,  $\tau X(t)/\phi$ , and, hence, firm value  $F_{AD}(x)$  is derived as

$$F_{AD}(x) = \mathbb{E} \left[ \int_0^S e^{-rt} \left( 1 - \tau + \frac{\tau}{\phi} \right) X(t) dt \right] = \left( 1 - \tau + \frac{\tau}{\phi} \right) \frac{x}{r + \lambda - \mu},$$

where  $S$  denotes the bankruptcy time caused by a Poisson jump. We also obtain equity value  $E_{AD}(x)$  in (23) by calculating  $E_{AD}(x) = F_{AD}(x) - D_{AD}(x)$ .

Next, we confirm that shareholders will not default until  $X(t)$  falls to 0 by a Poisson shock under the assumption of Proposition 3. Indeed,  $E_{AD}(x)$  in (23) is nonnegative if and only if  $\phi \geq \phi_{2AD}$  holds. We can also show that  $\phi_{2AD} < \phi_2$  (i.e.,  $E_{AD}(x) > E(x)$ ) as follows. For  $\mu \geq 0$ , the inequality immediately follows from (2) and (22). For  $\mu < 0$ ,  $\gamma$  satisfies  $(r + \lambda)/\mu < \gamma < 0$ . Then, we have

$$\phi_2 = \frac{\gamma - 1/(1 - \tau)}{\gamma - 1} \frac{r + \lambda - \mu}{r + \lambda} > \frac{(r + \lambda)/\mu - 1/(1 - \tau)}{(r + \lambda)/\mu - 1} \frac{r + \lambda - \mu}{r + \lambda} = \phi_{2AD}.$$

#### Appendix D. Proof of Proposition 5

It follows from (10), (20), and (25) in Case (iii) that

$$F(X(t), C(t)) \leq F(X(t), X(t)/\phi) = F(X(t)) \leq F_{AD}(X(t))$$

for any  $X(t)$ . In the no-covenant model,  $F_N(X(t), c_N(x))$  in (27) approaches the post-bankruptcy firm value  $(1 - \alpha)(1 - \tau)x_N^d(x)/(r + \lambda - \mu)$  (i.e.,  $(1 - \alpha)$  times the unlevered firm value) when EBIT  $X(t)$  approaches the no-covenant default threshold  $x_N^d(x)$ . However,  $F(X(t), C(t))$  in (10) and  $F_{AD}(X(t))$  in (25) remain higher than the unlevered firm value  $(1 - \tau)x_N^d(x)/(r + \lambda - \mu)$  even if  $X(t)$  reaches  $x_N^d(x)$ . Hence,  $F(X(t), C(t))$  and  $F_{AD}(X(t))$  are higher than  $F_N(X(t), c_N(x))$  when  $X(t)$  is sufficiently close to  $x_N^d(x)$ . We also have

$$F_N(X(t), c_N(x)) < \frac{(1 - \tau)X(t)}{r + \lambda - \mu} + \frac{\tau c_N(x)}{r + \lambda}. \quad (41)$$

$F_{AD}(X(t))$  in (25) is higher than the right-hand side of (41) when  $X(t)$  is sufficiently high (i.e.,  $X(t) > \phi(r + \lambda - \mu)c_N(x)/(r + \lambda)$ ).

#### Appendix E. Discrete-time model

As discussed in Section 4, our baseline continuous-time model allowing infinitesimal reductions of debt upon each covenant violation is not realistic. To study the robustness of our results, we compute several discretized versions of the baseline model, including tree models and fine-grained lattice models that truncate the tails beyond  $\pm 3\sigma$  and  $\pm 4\sigma$  of the normal distribution. We solve the dynamic programming equation numerically using the value iteration method to compute default timing and equity, debt, and firm values. Fig. 6 shows the results with optimal capital structure for  $\Delta t = 0.25$  and 0.5 years in the lattice model with truncation at  $4\sigma$ .<sup>21</sup> For comparison, Fig. 6 also shows the baseline and no-covenant results in the continuous-time model.

For  $\Delta t = 0.25$  and 0.5, the optimal covenant parameter  $\phi$  (i.e., the maximizer of the firm value) is equal to 1.23 and 1.36 (see the center-right panel of Fig. 6), respectively, and the threshold for Case (iii) (i.e., the minimal-risk debt region),  $\phi_{2D}$ , is equal to 1.35 and 1.6, respectively. Neither  $\phi_{2D}$  nor the optimal  $\phi$  has a closed-form expression, and therefore both are computed numerically. The discrete-time model's threshold  $\phi_{2D}$  is approximately equal to  $\phi_2/L$ , where  $L = e^{(\mu - 0.5\sigma^2)\Delta t - 4\sigma\sqrt{\Delta t}}$  is determined by the left endpoint of the truncation.<sup>22</sup>

<sup>21</sup> Fig. 6 does not depict the default thresholds because the default policy depends on  $(X(t), C(t))$  in the discrete-time model.

<sup>22</sup> For example,  $L = e^{-\sigma\sqrt{\Delta t}}$  in the binomial tree model, and  $L = e^{(\mu - 0.5\sigma^2)\Delta t - 3\sigma\sqrt{\Delta t}}$  in the model with truncation at  $3\sigma$  of the normal distribution. If  $L$  approaches 0,  $\phi_{2D}$  becomes infinitely large, which implies that the minimal-risk region (i.e., Case (iii)) disappears. If  $L$  is high, the optimal  $\phi$  can be equal

This is intuitive because in the discrete-time model, the IC ratio can fall to  $\phi L$ , at which point shareholders decide whether to default or reduce their debt. Fig. 7 shows the optimal  $\phi$  and threshold  $\phi_{2D}$  for  $\Delta t = 1/12, 0.25, 0.5, 0.75$ , and 1 years, together with  $\phi_2/L$  for truncation at  $1\sigma, 2\sigma, 3\sigma$ , and  $4\sigma$ . In the figure, the optimal  $\phi$  is lower than  $\phi_{2D}$  and is rather close to  $\phi_2/L$  determined by truncation at  $3\sigma$ . In other words, the firm sets a sufficiently large  $\phi$  to induce shareholders to reduce debt for negative shocks within  $3\sigma$ , but shareholders may default when a large negative shock beyond  $3\sigma$  occurs, albeit with very low probability. This logic resembles that of the model with default risks due to time lags, incomplete adjustment, and jumps in Section 4; indeed, the optimal covenant is not in the minimal-risk debt region (i.e., Case (iii)) but allows default caused by large negative shocks with very low probability.

Fig. 6 also shows that with higher  $\Delta t$ , the firm adopts a tighter covenant and lower leverage and extracts less value from the covenant than in the continuous-time model. The efficient covenant region shrinks,<sup>23</sup> and the optimal  $\phi$  lies within the commonly observed range. Our main result – that the covenant can reduce shareholders' incentive to default and, through this channel, increase firm value – remains qualitatively unchanged in the discrete-time models.

#### Appendix F. Debt maturity, transaction costs, costly refinancing, and debt overhang

Although for simplicity, this paper assumes debt with infinite maturity,<sup>24</sup> we discuss the effects of debt maturity on the main results. Below, we focus on the debt reduction outcome (i.e., Case (iii)). First, we consider one-time debt issuance with no rollover. Following the standard literature (e.g., Leland and Toft (1996), Demarzo and He (2021), and Dangi and Zechner (2021)), suppose that debt has an exponential amortization rate. The equity value of reducing debt and continuing operation decreases with shorter maturity because the tax benefits of debt decrease over time even when the IC ratio does not reach the limit of the covenant. Then, the debt reduction region (i.e., Case (iii)) and firm value decrease with shorter maturity. In other words, shorter maturity decreases the covenant value shown in this paper. Under the covenant that ensures debt reductions, the optimal maturity becomes infinite. The optimal maturity remains infinite even if we consider one-time debt issuance with costly rollover. In the additional debt model with no debt issuance cost, shareholders are indifferent to maturity choices, and all the results remain unchanged in the debt reduction region.

In the real world, there are transaction costs of debt rollover and issuance (e.g., Dangi and Zechner (2021) and Benzoni et al. (2022)). Dangi and Zechner (2021) show that with debt rollover costs, the debt level can decrease through expiring debt in low-earning states. In other words, shareholders can credibly commit to leverage reductions by choosing a short maturity structure. Then, the optimal maturity structure exists based on the trade-off between the commitment effect and transaction costs. Benzoni et al. (2022) show that fixed debt issuance costs play a role in mitigating new debt issuance. Then, a certain degree of debt issuance costs can suppress the leverage ratchet effect (i.e., the ex post agency problem) and increase firm value. In contrast

to  $\phi_{2D}$  even in the discrete-time models. In this sense, the existence of Case (iii) or the optimality of  $\phi_{2D}$  is a technical issue related to truncation, or to the choice of distribution used in the discrete-time models.

<sup>23</sup> According to our computations, the left endpoint of the efficient covenant region increases with  $\Delta t$ , while the right endpoint does not change significantly with  $\Delta t$ .

<sup>24</sup> In the baseline model where repaid debt cannot be reissued, covenant parameter  $\phi$  effectively measures debt maturity; higher  $\phi$  means shorter expected maturity.

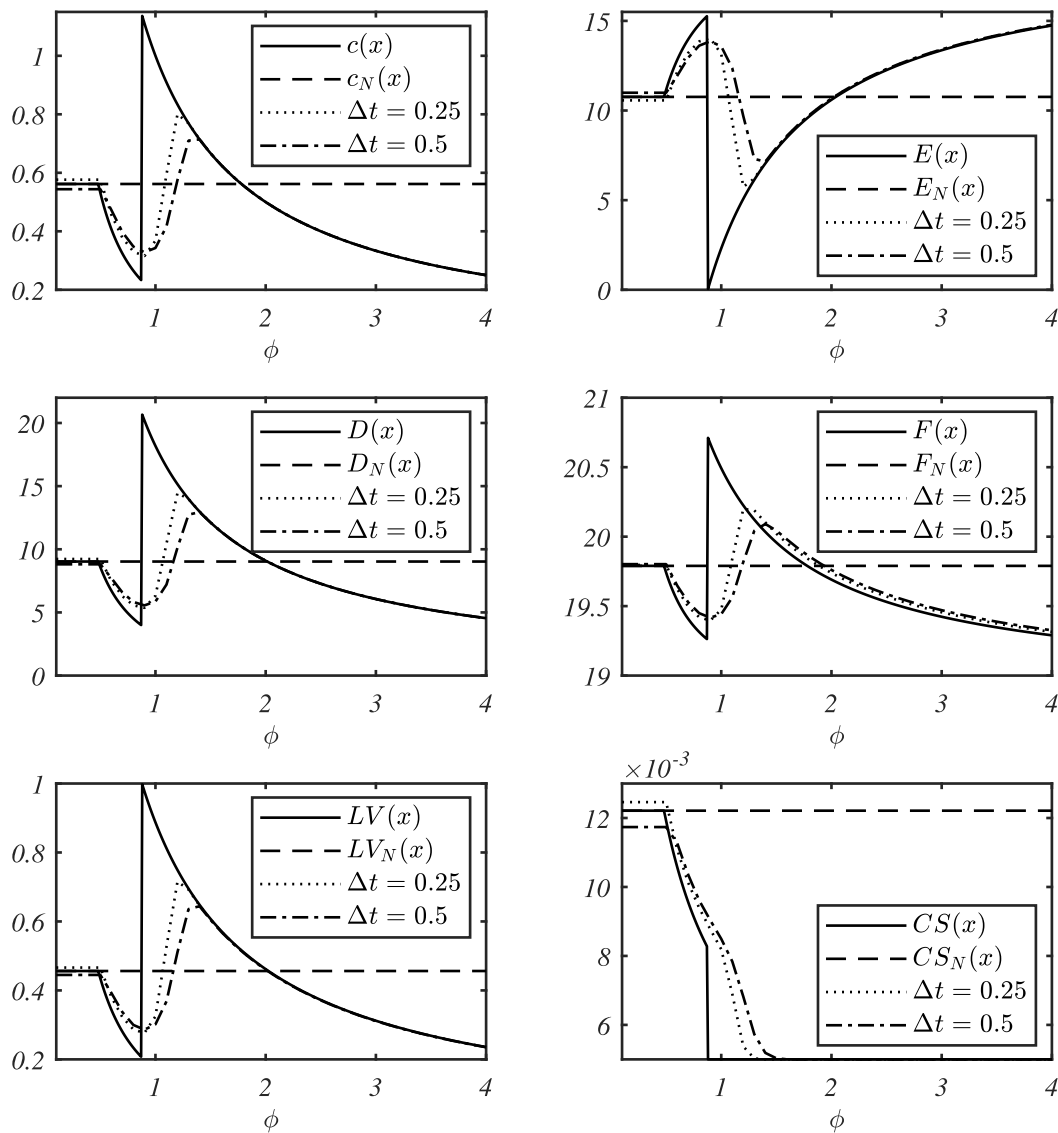


**Fig. 6.** Results with optimal coupon  $c(x)$  for covenant parameter  $\phi \in [0.1, 4]$  in the discrete-time model. The figure depicts coupon  $c(x)$ , equity value  $E(x)$ , debt value  $D(x)$ , firm value  $F(x)$ , leverage  $LV(x) = D(x)/F(x)$ , and credit spreads  $CS(x) = c/D(x) - r$  in the continuous-time baseline and no-covenant models with solid and dashed lines, respectively. Note that  $\phi_1 = 0.48$  and  $\phi_2 = 0.88$  in the baseline model. The figure depicts the values in the discrete-time model with  $\Delta t = 0.25$  and  $0.5$  years, with dotted and chain lines. The other parameter values are set as in Table 1.

to these studies, our paper focuses on the IC ratio covenant as a commitment device for leverage reductions. With the covenant that ensures debt reductions, short maturity and debt issuance costs decrease firm value.

Our baseline model corresponds to the case with infinite issuance costs, whereas our additional debt model corresponds to the case with no issuance costs. As shown in Fig. 5, coupon  $C(t)$  gradually decreases in the former, while  $C(t)$  is continuously adjusted to satisfy  $C(t) = X(t)/\phi$  in the latter. Compared to the additional debt model, this irreversibility in the baseline model decreases the incentive to reduce debt and, hence, reduces the covenant value and the efficient covenant region (see Proposition 4 and Fig. 4). In a more realistic model that allows debt issuance with a fixed cost, as in Benzon et al. (2022), a debt issuance threshold  $\psi (> \phi)$  arises, where shareholders issue debt when the IC ratio  $X(t)/C(t)$  exceeds  $\psi$ . The IC ratio then lies between the lower limit  $\phi$  and the upper limit  $\psi$ , which can explain the frequently

observed leverage target policy (e.g., Graham (2022)). In the model with costly debt issuance, the covenant value and the efficient covenant region are expected to be intermediate between those in our baseline and additional debt models.

Our baseline model can also be adjusted to include a growth option (e.g., Nishihara et al. (2019) and Bensoussan et al. (2021)). It is well known as underinvestment (or debt overhang) that a firm with debt decreases or delays growth investment due to the agency conflict, where shareholders pay the investment cost, but debt holders benefit from it without bearing any costs. Although the underinvestment problem creates the agency cost of debt without a covenant, a tight covenant that ensures debt reductions removes the agency cost. Indeed, growth investment does not affect debt value when the covenant eliminates shareholders' incentive to default. As discussed in Section 3.3, the covenant may also enable the firm to issue new debt in the growth stage and, through this channel, improve firm value.

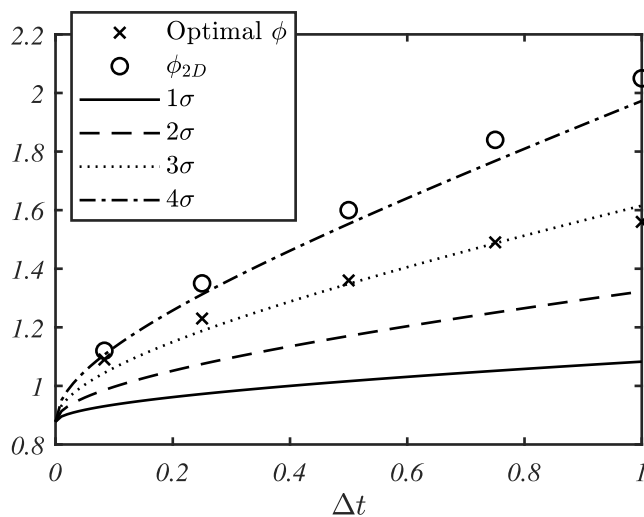


**Fig. 7.** The threshold for Case (iii),  $\phi_{2D}$ , and the optimal  $\phi$  in the discrete-time model with  $\Delta t = 1/12, 0.25, 0.5, 0.75$ , and 1 years. The figure also depicts the approximated threshold  $\phi_2/L$ , where  $L = e^{(\mu-0.5\sigma^2)\Delta t - i\sigma\sqrt{\Delta t}}$  by truncation at  $i\sigma$  ( $i = 1, 2, 3, 4$ ). The parameter values are set as in Table 1.

Convertible and callable debt rely on different state-contingent payoff mechanisms and play a positive role in mitigating the agency problem (e.g., Lyandres and Zhdanov (2014) and Flor et al. (2023)); however their effects differ from those of covenants. For convertible and callable debt, debt and equity holders, respectively, exercise their option when  $X(t)$  increases sufficiently. This mitigates underinvestment for high  $X(t)$  because existing debt holders are effectively eliminated at the option exercise time. However, convertible and callable debt do little to address the agency problem related to default for low  $X(t)$ , where shareholders choose the default time without considering debt holders' interests. In contrast, tight covenants have the advantage of addressing this default-related agency problem, thereby mitigating the underinvestment problem. Their drawback, however, is that they may force firms to adopt more conservative leverage (and thus reduce tax benefits) when  $X(t)$  stays at low, but not bankruptcy, levels.

## Data availability

No data was used for the research described in the article.

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