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ON M-RINGS AND GENERAL ZPI-RINGS

Dedicated to Professor Kentaro Murata on his 60th birthday

TAKASABURO UKEGAWA

(Received January 7, 1981)

In the preceding paper [10], we have proved that a left Noetherian $M$-ring is a so called "general ZPI-ring" in the commutative case. Also we know that in an $M$-ring the multiplication of prime ideals is commutative [8]. In the present paper we define general ZPI-rings in section 1 and we study general properties of them, and as an important example of such rings we can give a left Noetherian semi-prime Asano left order. In section 2 we research the condition for a left Noetherian general ZPI-ring to be an $M$-ring, using minimal prime divisors of an ideal. The notation "<" means a proper inclusion as the preceding papers [8], [9], [10].

1. $M$-rings and general ZPI-rings

DEFINITION. If the multiplication of any two prime ideals of a ring $R$ is commutative, and any ideal of $R$ can be written as a produkt of powers of prime (considering $R$ as a prime ideal) ideals of $R$, then we call $R$ a general ZPI-ring. Therefore the multiplication of ideals is commutative.

In the commutative case a general ZPI-ring is necessarily Noetherian no matter whether the ring has an identity or not. But in our case the general ZPI-ring is not necessarily Noetherian as the example in [9] shows.

Proposition 1. Let $R$ be a left Noetherian general ZPI-ring, let $P$ be any prime ideal of $R$, and let $q$ be maximal in the set of prime ideals such that $q<P$. Then for any ideal $a$ with $q<a<P$, there is an ideal $b$ such that $a=Pb=bP$.

Proof. Let $a=p_1\ldots p_i<P$, since $R$ is a general ZPI-ring. Then $p_i\subseteq P$ for some $p_i$. Since $q\leq a\leq p_i$, $q<p_i\subseteq P$, so $p_i=P$. Therefore $a=pp_1\ldots p_{i-1}p_{i+1}\ldots p_i=bP$, where $b=p_1\ldots p_{i-1}p_{i+1}\ldots p_i$.

As in the commutative case we have

Proposition 2. Let $R$ be a left Noetherian general ZPI-ring, and let $P$ be a maximal ideal of $R$. Then there are no ideals between $P$ and $P^2$ (including the case that $P=P^2$), more generally for any positive integer $n$, the only ideals
between \( P \) and \( P^n \) are \( P, P^2, \ldots, P^n \) (including the case that \( P^i = P^{i+1} \) for some \( i, 1 \leq i < n \)).

**Remark.** Let \( R \) be as above. If every proper ideal \( \alpha \) of \( R \) can be written as a product of minimal prime divisors of \( \alpha \), then for any proper prime ideal \( \mathfrak{p} \) of \( R \) and for any positive integer \( n \), the only ideals between \( \mathfrak{p} \) and \( \mathfrak{p}^n \) are \( \mathfrak{p}^i \), \( i = 1, 2, \ldots, n \).

**Proposition 3.** Let \( R \) be a left Noetherian general \( \mathcal{ZPI} \)-ring, and let \( \min-\mathcal{O} = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_s \} \) be the set of minimal prime ideals of \( R \). Then for any subset \( \{ \mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_k} \} \) of \( \min-\mathcal{O} \), \( \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_k} = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_k} \). Especially for the prime radical \( N \) of \( R \), \( N = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = \mathfrak{p}_1 \cdots \mathfrak{p}_r \).

**Proof.** Since \( R \) is a general \( \mathcal{ZPI} \)-ring, \( \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = \mathfrak{p}_1 \cdots \mathfrak{p}_r \) for some prime ideals \( \mathfrak{p}_1, \ldots, \mathfrak{p}_r \) of \( R \). Then for any \( \mathfrak{p}_j \) \( 1 \leq j \leq r \) we have \( \mathfrak{p}_j = 0 \pmod{\mathfrak{p}_j} \) for some \( \mathfrak{p}_j \), and so \( \mathfrak{p}_j = \mathfrak{p}_j \), therefore \( \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = \mathfrak{p}_1 \cdots \mathfrak{p}_r \). Now \( \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = \mathfrak{p}_1 \cdots \mathfrak{p}_r \), hence \( \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = \mathfrak{p}_1 \cdots \mathfrak{p}_r \).

**Lemma 4.** Let \( R \) be a left Noetherian semi-prime general \( \mathcal{ZPI} \)-ring, and let \( \min-\mathcal{O} = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_r \} \) be the set of minimal prime ideals of \( R \). Then for any \( 1 \leq i < r \) and any positive integers \( m_1, \ldots, m_r \), \( \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r} = 0 \).

**Theorem 1.** Let \( R \) be a left Noetherian semi-prime general \( \mathcal{ZPI} \)-ring, and let \( \min-\mathcal{O} = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_r \} \) be the set of minimal prime ideals of \( R \). If a proper ideal \( \alpha \) of \( R \) has the form \( \alpha = \mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_r^{i_r} \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_r} \) where \( \mathfrak{p}_i \in \min-\mathcal{O} \) for \( i = 1, \ldots, s \) and \( \mathfrak{p}_j \in \min-\mathcal{O} \) for \( j = 1, \ldots, t \), then \( \mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_r^{i_r} \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_r} \subseteq R \), i.e. essential as a left \( R \)-module, and the set \( \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_r \} \) is uniquely determined by \( \alpha \).

**Proof.** Let \( \mathfrak{p} \) be a prime ideal of \( R \). By proposition 2.11 [5] and Lemma 4, \( \mathfrak{p} \) is not essential as a left \( R \)-module if and only if \( \mathfrak{p} \in \min-\mathcal{O} \). Hence \( \mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_r^{i_r} \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_r} \subseteq R \) as a left \( R \)-module. Let \( \alpha = \mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_r^{i_r} \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_r} \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \) where \( \mathfrak{p}_j \in \min-\mathcal{O} \) for \( 1 \leq j \leq k, \mathfrak{Q}_j \in \min-\mathcal{O} \) for \( 1 \leq i \leq w \) be another form of \( \alpha \). Assume that two set \( \mathcal{M}_1 = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_s \}, \mathcal{M}_2 = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_d \} \) are distinct. If \( \mathcal{M}_1 > \mathcal{M}_2 \), then \( 0 = \alpha \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_t} \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \), \( \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \) contains some regular element, hence \( 0 = \mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_r^{i_r} \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_r} \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \), contradicting Lemma 4. Next we consider the case that \( \mathcal{M}_1 \nsubseteq \mathcal{M}_2 \) and also \( \mathcal{M}_1 \nsubseteq \mathcal{M}_2 \). We denote the product of minimal prime ideals belonging to the set \( \mathcal{M}_1 \) by \( [\mathcal{M}_1] \), for example. Then \( 0 = \alpha [\min-\mathcal{O} - \mathcal{M}_1] \) since \( \alpha = \mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_r^{i_r} \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_r} \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \). On the other hand, \( \min-\mathcal{O} - \mathcal{M}_1 \cap \min-\mathcal{O} - \mathcal{M}_2 \) and \( \alpha = \mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_r^{i_r} \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \), hence \( 0 = \alpha [\min-\mathcal{O} - \mathcal{M}_1] \) which is a contradiction. So we have \( \mathcal{M}_1 = \mathcal{M}_2 \).

As a result of Theorem 1 we have

**Proposition 5.** Let \( R \) be as above, and let \( \min-\mathcal{O} = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_r \} \). Then
(p_{i_1} \cdots p_{i_r}) is a regular\(^{1)}\) ideal of \(R\), where \(p_i, \ldots, p_r\) are distinct minimal prime ideals of \(R\), \(1 \leq i < j \leq r\) and \(\alpha_i, \ldots, \alpha_j\) are any positive integers.

**Proposition 6.** Let \(R\) be a left Noetherian general ZPI-ring, and let \(P\) be a maximal ideal of \(R\) such that \(P^i > P^{i+1}\) for any positive integer \(i\). Then \(\bigcap_{n=1}^{\infty} P^n\) is a prime ideal of \(R\).

Proof. Set \(\bigcap_{n=1}^{\infty} P^n = \mathfrak{a}\). Let \(A, B\) be ideals of \(R\) such that \(AB \equiv 0\) (mod \(\mathfrak{a}\)) \(A \equiv 0\) and \(B \equiv 0\) (mod \(\mathfrak{a}\)). Therefore there is a maximal \(i \geq 0\) such that \(A \subseteq P^i\) and so \(A \subseteq P^{i+1}\). Similarly there is a maximal \(j \geq 0\) such that \(B \subseteq P^j\) and so \(B \subseteq P^{j+1}\). Then \(P^{i+1} < (A, P^{i+1}) \subseteq P^i\), therefore \((A, P^{i+1}) = P^i\) by Proposition 2, and similarly \((B, P^{j+1}) = P^j\). Hence \(P^{i+j} = (A, P^{i+1}) (B, P^{j+1}) \subseteq P^{i+j+1}\), thus \(P^{i+j} = P^{i+j+1}\) contradicting the assumptions.

**Remark.** Let \(R\) be as above. Let \(\mathfrak{p}\) be any proper prime ideal such that for any positive integer \(i\) \(\mathfrak{p}^i > \mathfrak{p}^{i+1}\). If every proper ideal of \(R\) can be written as a product of minimal prime divisors, then \(\bigcap_{n=1}^{\infty} \mathfrak{p}^n\) is a prime ideal of \(R\).

**Theorem 2.** Let \(R\) be a Noetherian (left and right) prime ring with an identity. If \(R\) satisfies the following

1) \(R\) is a general ZPI-ring;
2) every non-zero proper prime ideal of \(R\) is maximal;
3) every ideal of \(R\) is injective both as a left and as a right \(R\)-module,

the \(R\) is an M-ring.

Proof. We shall prove the existence of an ideal \(C\) with \(A = BC = CB\) for ideals \(A, B\) such that \(0 < A < B < R\). Let \(A = P_1^{e_1} \cdots P_r^{e_r} < B = Q_1^{f_1} \cdots Q_m^{f_m}\) where \(P_1, \ldots, P_r, Q_1, \ldots, Q_m\) are prime ideals of \(R\) and \(e_k > 0\) for \(k = 1, \ldots, \alpha, f_j > 0\) for \(j = 1, \ldots, \beta\), so for every \(Q_k\) there is some \(P_k = Q_k\) for \(k = 1, \ldots, \beta\). Hence \(A = Q_1^{j_1} \cdots Q_m^{j_m} P_1^{e_1} \cdots P_r^{e_r} < B = Q_1^{j_1} \cdots Q_m^{j_m}\). Now by Proposition 2.2 [3], each maximal ideal of \(R\) is either idempotent or invertible. Let \(Q_1, \ldots, Q_j\) be the set of idempotent maximal ideals in the set of maximal ideals \(Q_1, \ldots, Q_j, \ldots, Q_{j+1}\) (including the case that \(\{Q_1, \ldots, Q_j\}\) is empty). Then \(A = Q_1^{j_1} \cdots Q_j^{j_j} \cdots Q_{j+1}^{j_{j+1}} \cdots Q_m^{j_m} P_1^{e_1} \cdots P_r^{e_r} < B = Q_1^{j_1} \cdots Q_j^{j_j} \cdots Q_{j+1}^{j_{j+1}} \cdots Q_m^{j_m}\), where \(Q_{j+1}, \ldots, Q_{j+1}\) are invertible ideals of \(R\). If \(e_{j+1} < j_{j+1}\) for example, multiplying \((Q_{j+1})^{j_{j+1}}\) on each side, we have \(Q_1 \cdots Q_j Q_{j+1}^{j_{j+1} - j_{j+1}} \cdots Q_m^{j_m} \equiv 0\) (mod \(Q_{j+1}\)), which is a contradiction. Therefore \(e_{j+1} \geq j_{j+1}, \ldots, e_{j+1} \geq f_j\). Thus \(A = B Q_1^{j_1} \cdots Q_{j+1}^{j_{j+1} - j_{j+1}} \cdots Q_m^{j_m} P_1^{e_1} \cdots P_r^{e_r}\), hence \(R\) is an M-ring.

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1) We call an \(R\)-ideal a regular ideal.
REMARK. If $R$ is a Noetherian semi-prime ring with an identity, then we may replace the condition 2) by the following:

2') the proper prime ideals of $R$ are either comaximal minimal prime ideals or maximal prime ideals of $R$.

The theorem is valid also in this case, because $R=R_1 \oplus \cdots \oplus R_i \oplus \cdots \oplus R_n$ where $R_i \cong R/p_i$ for every $i$ and $\{p_1, \ldots, p_n\} = \text{min}(\mathfrak{D})$, so every $R_i$ is a Noetherian general $ZPI$-ring satisfying the condition 2).

**Theorem 3.** Let $R$ be a left Noetherian semi-prime Asano left order. Then $R$ is a general $ZPI$-ring and also an $M$-ring, and the proper prime ideals of $R$ are either comaximal idempotent minimal prime ideals or maximal prime ideals of $R$. Every proper ideal $\alpha$ of $R$ has the form $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_i \oplus \cdots \oplus \alpha_n$ where $\alpha_i \in \text{min}(\mathfrak{D})$ for $1 \leq k \leq i$ and $\alpha_i, \ldots, \alpha_n$ are maximal prime ideals of $R$ which are regular.

**Proof.** Let $Q = Q_1 \oplus \cdots \oplus Q_i \oplus \cdots \oplus Q_n$ be the left quotient ring of $R$ which is semisimple Artinian, where $Q_1, \ldots, Q_n$ are simple Artinian rings. Now we can deduce that $R=R_1 \oplus \cdots \oplus R_i \oplus \cdots \oplus R_n$ where $R_i$ is a left Noetherian Asano left order of $Q_i$ for $1 \leq i \leq n$. Each proper prime ideal of $R$ has either the form $\mathfrak{p}_i = R_1 \oplus \cdots \oplus R_i-1 \oplus R_{i+1} \oplus \cdots \oplus R_n$ or the form $\mathfrak{p}_i = R_1 \oplus \cdots \oplus R_i-1 \oplus \mathfrak{p}_{(i)} \oplus R_{i+1} \oplus \cdots \oplus R_n$ where $\mathfrak{p}_{(i)}$ is a maximal prime ideal of $R_i$ for $1 \leq i \leq n$. Every proper ideal $\alpha$ of $R$ has the form $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_i \oplus \cdots \oplus \alpha_n$ where $\alpha_i$ is an ideal of $R_i$ for $1 \leq i \leq n$. In order to make the proof concise we assume that $\alpha_i = \cdots = \alpha_{i-1} = 0$ (including the case that $\{\alpha_1, \ldots, \alpha_{i-1}\}$ is empty) and $\alpha_i = \mathfrak{p}_{(i)}^{e_{(i)}} \cdots \mathfrak{p}_{(i)}^{e_{(i)}}$, $\alpha_n = \mathfrak{p}_{(n)}^{e_{(n)}} \cdots \mathfrak{p}_{(n)}^{e_{(n)}}$. Then $\alpha = \mathfrak{p}_{(1)} \cdots \mathfrak{p}_{(i-1)} \mathfrak{p}_{(i)}^{e_{(i)}} \cdots \mathfrak{p}_{(n)}^{e_{(n)}}$ where $\mathfrak{p}_{(i)} = R_i \oplus \cdots \oplus R_{i-1} \oplus \mathfrak{p}_{(i)} \oplus R_{i+1} \oplus \cdots \oplus R_n$, thus $R$ is a general $ZPI$-ring. Then it is easy to see that $R$ is an $M$-ring.

By Proposition 6 we have

**Corollary 4.** Let $R$ be a left Noetherian semi-prime Asano left order and let $P$ be a regular prime ideal of $R$, then $\bigcap_{n=1}^\infty P^n = \mathfrak{p}$ is a minimal prime ideal of $R$.

2. Minimal prime divisors of ideals

Let $\alpha$ be a proper ideal of $R$. A minimal prime divisor of $\alpha$ is a prime ideal $\mathfrak{p}$ with $\alpha \subseteq \mathfrak{p}$ such that there are no prime ideals $\mathfrak{p}'$ with $\alpha \subseteq \mathfrak{p}' \subset \mathfrak{p}$. We denote the set of minimal prime divisors of $\alpha$ by $\text{min}(\mathfrak{D}_\alpha)$. The set $\text{min}(\mathfrak{D})$ of minimal prime ideals of $R$ is $\text{min}(\mathfrak{D})$. As a consequence of Theorem 3 [10] and Proposition 1 [8], we have

**Proposition 7.** Let $R$ be a left Noetherian general $ZPI$-ring. Moreover if $R$ is an $M$-ring, then
For any prime ideal $p, q$ with $p < q, p = q = q$. 

(ii) Let $a$ be any proper ideal of $R$, and let $\text{min-}\mathfrak{a} = \{p_1, \ldots, p_s\}$. Then $a = a_1 \cdots a_s$ for some positive integers $f_1, \ldots, f_s$.

Remark. Let $R$ be a left Noetherian general ZPI-ring. Then i) of the above condition (*) is equivalent to the following:

i') For any prime ideal $p$ and any ideal $b$ properly containing $p$, $p = b = p$.

Next we consider the converse of this apparent proposition.

Proposition 8. Let $R$ be a left Noetherian general ZPI-ring which satisfies the condition (*) in Proposition 7 and let $a$ be a proper ideal of $R$. Then for any minimal prime divisor $p$ of $a$, either $p = p^{i+1}$ for some positive integer $i$ or else there is some positive integer $j$ such that $p^j \not\supseteq a$.

Proof. We assume that for any positive integer $i$, $p^i > p^{i+1}$, and we shall show that $p^j \supseteq a$ for some positive integer $j$. If $p^i > p^{i+1}$ for any positive integer $i$ and moreover $p^k \supseteq a$ for any positive integer $k$, then $a = \prod_{i=1}^n p^i = n < p$ where $n$ is a prime ideal by the remark of Proposition 6, a contradiction.

Proposition 9. Let $R$ be a left Noetherian general ZPI-ring which satisfies the condition (*), let $a$ be a proper ideal of $R$, and let $\text{min-}\mathfrak{a} = \{p_1, \ldots, p_s\}$. Then for any $i \neq j$ and any positive integer $e_i, e_j, (\psi_{i}, \psi_{j})$ is an idempotent ideal of $R$.

Proof. First we prove that $\psi_{i}(\psi_{i}, \psi_{j}) = \psi_{i}$, and similarly $\psi_{j}(\psi_{i}, \psi_{j}) = \psi_{j}$. Since $\psi_{i} \equiv 0 \pmod{p_{i}}$, $\psi_{j} \not\equiv 0 \pmod{p_{i}}$, $\psi_{j} \equiv P_{1} \cdots P_{s}$ where $P_1, \ldots, P_s$ are minimal prime divisors of $(\psi_{i}, \psi_{j})$. Now we know that $p_{i} \equiv 0 \pmod{P_{k}}$ for every $P_{k}$, $1 \leq k \leq s$. If $p_{i} = P_{k}$ for some $P_{k}$, then $(\psi_{i}, \psi_{j}) \equiv P_{1} \cdots P_{k-1} \psi_{i} P_{k+1} \cdots P_{s} \equiv 0 \pmod{p_{i}}$, hence $p_{j} \equiv 0 \pmod{p_{i}}$, a contradiction. Therefore $p_{j} \not\equiv 0 \pmod{P_{k}}$, $1 \leq k \leq s$, hence $p_{j} \equiv 0 \pmod{P_{k}}$, $1 \leq k \leq s$. Then $(\psi_{i}, \psi_{j}) = (\psi_{i}(\psi_{i}, \psi_{j}), \psi_{j}(\psi_{i}, \psi_{j})) = (\psi_{i}, \psi_{j})$.

Lemma. Under the same assumptions as above, for any $i \neq j$ and any positive integer $e_i, e_j, \psi_{i} \cap \psi_{j} = \psi_{i} \psi_{j}$.

Proof. First we prove that $\psi_{i} \cap \psi_{j} = (\psi_{i} \cap \psi_{j}) (\psi_{i}, \psi_{j})$. For some positive integer $\rho$, $\alpha \leq \psi_{i} \cap \psi_{j} = P_{1} \cdots P_{s} \equiv 0 \pmod{p_{i}}$, where $P_1, \ldots, P_s$ are minimal prime divisors of $\psi_{i} \cap \psi_{j}$. Therefore $\alpha \leq P_{1} \equiv 0 \pmod{p_{i}}$ for some $P_{1}$, so $P_{1} = p_{i}$. Similarly for some $P_{s}$, $\alpha \leq P_{s} \equiv 0 \pmod{p_{i}}$, so $P_{s} = p_{i}$, and $\psi_{i} \cap \psi_{j} = \psi_{i} \psi_{j} P_{1} \cdots P_{s}$. Let $(\psi_{i}, \psi_{j}) = Q_{1} \cdots Q_{t}$, where $Q_1, \ldots, Q_t$ are minimal prime divisors of $(\psi_{i}, \psi_{j})$. For every $Q_{s}$, $p_{i} \equiv 0 \pmod{Q_{s}}$ and $p_{j} \equiv 0 \pmod{Q_{s}}$, hence $p_{i} < Q_{s}$ and $p_{j} < Q_{s}$ for every $Q_{s}, 1 \leq k \leq t$. From the above arguments $(\psi_{i} \cap \psi_{j}) (\psi_{i}, \psi_{j}) = \psi_{i} \psi_{j}$.
Theorem 5. Let \( R \) be a left Noetherian general \( \mathbb{Z} \)-PI-ring which satisfies the condition (*)\), let \( a \) be a proper ideal of \( R \), and let \( \text{min-} \mathcal{O}_a = \{ P_1, \ldots, P_r \} \) and \( x_i > 0 \) for \( 1 \leq i \leq r \). Let \( \{ P_1, \ldots, P_r \} \) be the subset of \( \text{min-} \mathcal{O}_a \) every \( P_i \) of which has a maximal index \( \alpha_i \) such that \( P_i \supseteq a \) and so \( P_i \supseteq \mathcal{O}_a \), and assume that for \( P_{k+1}, \ldots, P_r \), there are no maximal \( \beta_i \) among indices \( \beta_i \) such that \( P_i \supseteq a \) (including the case that one of the sets \( \{ 1, \ldots, k \} \), \( \{ k+1, \ldots, r \} \) is empty). Then \( a \) has the form \( a = P_1^{\beta_1} \cdots P_r^{\beta_r} \), hence \( a \) is primary for any positive integer such with \( x_i \leq \beta_i \leq \alpha_i \) for \( 1 \leq i \leq r \). Let \( p_i = p_i^{\beta_i} \cdots P_r^{\beta_r} \), where \( \beta_i \) is any positive integer such that \( x_i \leq \beta_i \leq \alpha_i \) for \( 1 \leq i \leq r \) and \( y_j \) is any positive integer for \( k < j \leq r \).

Proof. By Theorem 5 \( a = P_1^{\beta_1} \cdots P_r^{\beta_r} \supseteq P_1^{\alpha_1} \cdots P_r^{\alpha_r} \cap \cdots \cap P_r^{\alpha_r} \cap \cdots \cap P_r^{\alpha_r} \), since \( x_i \leq \beta_i \leq \alpha_i \) for \( 1 \leq i \leq k \) and \( y_j \leq y_j \) for \( k < j \leq r \). Conversely \( a \subseteq P_i^{\beta_i} \) for \( 1 \leq i \leq k \) since \( \beta_i \leq \alpha_i \), and also \( a \subseteq P_1^{\alpha_1} \cdots P_r^{\alpha_r} \) for any \( y_j \geq x_j \), \( k < j \leq r \); hence \( a \subseteq P_1^{\alpha_1} \cdots P_r^{\alpha_r} \cap \cdots \cap P_r^{\alpha_r} \cap \cdots \cap P_r^{\alpha_r} \). Thus \( a = P_1^{\beta_1} \cap \cdots \cap P_r^{\beta_r} \cap \cdots \cap P_r^{\beta_r} \cap \cdots \cap P_r^{\beta_r} \). The following definition of primary ideal is defined in [2]. Let \( a \) be an ideal of \( R \). If for ideals \( A, B \) \( A B \equiv 0 \) (mod \( a \)) implies \( A \equiv 0 \) (mod \( a \)) or \( B \equiv 0 \) (mod \( a \)) for some positive integer \( \rho \), then \( a \) is called \( r \)-primary. And an \( l \)-primary ideal is defined similarly. A \( 1 \)- and \( r \)-primary ideal is called a primary ideal.

Theorem 7. Let \( R \) be as above. Then for every proper prime ideal \( \mathfrak{p} \) of \( R \) \( \mathfrak{p} \) is a primary ideal for any positive integer \( e \).

Proof. Let \( A B \equiv 0 \) (mod \( p^e \)) for ideals \( A, B \). We may assume that \( A \equiv a \) and \( B \equiv a \) where we set \( p^e = a \). We set anew \( A_1 = (A, a), B_1 = (B, a) \). Then \( A_1, B_1 \equiv 0 \) (mod \( p^e \)); and \( A \equiv 0 \) (mod \( p^e \)) if and only if \( A_1 \equiv 0 \) (mod \( p^e \)), etc.

minimal prime divisor of \( \alpha \); so \( A = \mathfrak{p}^{\beta_{1}} \mathfrak{p}^{\beta_{2}} \cdots \mathfrak{p}^{\beta_{s}} \), i.e. \( \mathfrak{p} \) is a minimal prime divisor of \( A \). Let \( \text{min-}\mathfrak{p}_{\mathfrak{j}} = \{ \mathfrak{q}_{1}, \cdots, \mathfrak{q}_{\nu} \} \). Since \( \alpha = \mathfrak{p}^{\nu_{1}}B = \mathfrak{q}_{1}^{\nu_{1}} \cdots \mathfrak{q}_{\nu}^{\nu_{\nu}} \) for some positive integers \( \nu_{1}, \cdots, \nu_{\nu} \), \( \nu < \mathfrak{q}_{i} \) for every \( \mathfrak{q}_{i} \) and since \( \mathfrak{p} \) is a factor of \( A \), \( AB = A \) by the condition (*), i.e. \( A \equiv 0 \pmod{\mathfrak{p}} \).

**Theorem 8.** Let \( R \) be a left Noetherian general ZPI-ring which satisfies the condition (*). Then \( R \) is an \( M \)-ring.

Proof. Let \( 0 < A < B < R \) be ideals of \( R \), let \( \text{min-}\mathfrak{p}_{\mathfrak{j}} = \{ \mathfrak{p}_{1}, \cdots, \mathfrak{p}_{\alpha} \} \), and let \( A = \mathfrak{p}_{1}^{\beta_{1}} \cdots \mathfrak{p}_{\alpha}^{\beta_{\alpha}} \), \( B = \mathfrak{q}_{1}^{\lambda_{1}} \cdots \mathfrak{q}_{\nu}^{\lambda_{\nu}} \). By Theorem 6 we can choose them as large as possible. Then for every \( \mathfrak{q}_{i} \), there is some \( \mathfrak{p}_{i} \) such that \( \mathfrak{p}_{i} \subseteq \mathfrak{q}_{i} \). If \( \mathfrak{p}_{i} < \mathfrak{q}_{i} \) for every \( \mathfrak{q}_{i} \), then \( A = AB = BA \), so there is nothing to prove. If there are some \( \mathfrak{q}_{i} \) such that \( \mathfrak{p}_{i} = \mathfrak{q}_{i} \) and \( \mathfrak{p}_{i} < \mathfrak{q}_{i} \) for every \( \mathfrak{q}_{i} \), there are some \( \mathfrak{p}_{k} \) with \( \mathfrak{p}_{k} < \mathfrak{q}_{k} \). Furthermore, as to \( \mathfrak{p}_{1}, \cdots, \mathfrak{p}_{\alpha} \), let \( \mathfrak{p}_{1}, \cdots, \mathfrak{p}_{s} \) be minimal prime divisors of \( A \) which have maximal indices such that \( \mathfrak{p}_{i} \subseteq \mathfrak{q}_{i} \) for \( 1 \leq i \leq s \) and let \( \mathfrak{p}_{s+1}, \cdots, \mathfrak{p}_{m} \) be those which do not have such indices as above. On prime ideals \( \mathfrak{p}_{j}, 1 \leq j \leq s, A \subseteq \mathfrak{p}_{j} \), and \( A < B \subseteq \mathfrak{q}_{j} \), \( R \) is an \( M \)-ring if, and only if, for \( 1 \leq j \leq s \), by Theorem 6. On prime ideals \( \mathfrak{p}_{s+1}, \cdots, \mathfrak{p}_{m} \), we may assume that \( \beta_{i} \leq \alpha_{i} \) for \( s < i \leq m \), by Theorem 6.

Therefore \( A = \mathfrak{p}_{1}^{\beta_{1}} \cdots \mathfrak{p}_{s}^{\beta_{s}} \cdots \mathfrak{p}_{m}^{\beta_{m}} \cdots \mathfrak{p}_{m+1}^{\beta_{m+1}} \cdots \mathfrak{p}_{s}^{\beta_{s}} = \mathfrak{p}_{1}^{\beta_{1}} \cdots \mathfrak{p}_{m}^{\beta_{m}} \cdots \mathfrak{p}_{s}^{\beta_{s}} \cdots \mathfrak{p}_{m}^{\beta_{m}} \cdots \mathfrak{p}_{m+1}^{\beta_{m+1}} \cdots \mathfrak{p}_{s}^{\beta_{s}} = B C \), say. Hence \( R \) is an \( M \)-ring.

We summarize

**Theorem 9.** Let \( R \) be a left Noetherian general ZPI-ring. Then \( R \) is an \( M \)-ring if, and only if,

1) For any prime ideals \( \mathfrak{p}, \mathfrak{q} \) of \( R \) such that \( \mathfrak{p} \subset \mathfrak{q}, \mathfrak{p} = \mathfrak{q} \), and

2) Any non-zero proper ideal \( \alpha \) of \( R \) can be written as a product of powers of minimal prime divisors of \( \alpha \).

**References**


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