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Author(s)	Takai, Shigemasu; Kumar, Ratnesh
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Non-Uniformly Bounded Inference Observability for Decentralized Supervisory Control of Discrete Event Systems

Shigemasa Takai¹ and Ratnesh Kumar²

Abstract—In the inference-based decentralized supervisory control framework for discrete event systems, multiple local supervisors perform multi-level inference against self-ambiguity and ambiguities of the others to jointly arrive at a correct control decision. For a chosen nonnegative integer N , the notion of N -inference observability together with controllability guarantees that the given specification language is enforced using N -level inference. The class of N -inference observable languages increases *strictly* monotonically as the parameter N is increased, i.e., there is no upper bound for the required level of inference. Informed by this fact, we introduce a notion of *non-uniformly bounded inference observability*, which does not involve an upper bound of inference level, and is weaker than N -inference observability for any nonnegative integer N . It together with controllability serves as a necessary and sufficient condition for the existence of a decentralized supervisor that uses inference with unbounded level.

I. INTRODUCTION

In decentralized decision-making paradigms for discrete event systems (DESSs) such as decentralized control [1]–[8], decentralized diagnosis [9]–[13], and decentralized prognosis [14], [15], multiple local agents make local decisions based on their own local observations, and these local decisions are fused to arrive at a global decision according to a certain decision fusion rule. In general, sensing capabilities of local agents are limited due to distributed nature of the system, causing uneven ambiguities in their decision-making. Frameworks for managing such ambiguities have been developed in the literature. In the context of decentralized supervisory control, a knowledge-based approach for assessing self-ambiguity was introduced in [16], and it was later extended to assess not only self-ambiguity but also ambiguities of the others in [17]. The process of utilizing the knowledge of self-ambiguity together with ambiguities of the others for the sake of decision-making was referred to as “inference” in [17]. The frameworks that utilize single-level inference were proposed for decentralized supervisory control [18] and decentralized diagnosis [19]. A general framework that supports multi-level inference was first introduced in [20] for decentralized control, in [21]–[24] for decentralized diagnosis, and in [25] for decentralized prognosis. Unlike the frameworks of [5], [18], the framework

of [20] does not require partitioning the controllable event set such that some controllable events are controlled under the disjunctive fusion rule and others are controlled under the conjunctive one. A method for synthesizing an inference-based decentralized supervisor that enforces a sublanguage (respectively, a superlanguage) of a specification language was presented in [26] (respectively, [27]). An arborescent architecture was proposed to realize the inference-based decentralized control in [28].

In the existing works on the inference-based approaches for control [20], [26], [27], diagnosis [21]–[24], and prognosis [25], the level of inference is bounded by a chosen nonnegative integer N . For example in the setting of control, the notion of N -inference observability of [20] together with controllability characterizes the class of languages enforceable using N -level inference. It was later shown in [29] that the class of N -inference observable languages increases *strictly* monotonically as the parameter N is increased, i.e., in general, there is no upper bound to the required level of inference. Informed by this, in this paper, we introduce a new notion of inference observability, named *non-uniformly bounded inference observability* (or “nUB-inference observability” for short), which does not involve an upper bound of inference level, and is weaker than N -inference observability for any nonnegative integer N . As the main result, we show that nUB-inference observability and controllability serve as necessary and sufficient conditions for the existence of a decentralized supervisor that enforces the specification using inference (with unbounded level).

II. PRELIMINARIES

We consider a DES modeled by an automaton $G = (Q, \Sigma, \delta, q_0, Q_m)$, where Q is the set of states, Σ is the finite set of events, a partial function $\delta : Q \times \Sigma \rightarrow Q$ is the transition function, $q_0 \in Q$ is the initial state, and $Q_m \subseteq Q$ is the set of marked states. Let Σ^* be the set of all finite traces of elements of Σ , including the empty trace ε . The function δ can be generalized to $\delta : Q \times \Sigma^* \rightarrow Q$ in a usual way. The generated and marked languages of G , denoted by $L(G)$ and $L_m(G)$, respectively, are defined as $L(G) = \{s \in \Sigma^* \mid (\exists q \in Q)\delta(q_0, s) = q\}$ and $L_m(G) = \{s \in \Sigma^* \mid \delta(q_0, s) \in Q_m\}$. For a language $K \subseteq \Sigma^*$, the set of all prefixes of traces in it is denoted by $\overline{K}$, i.e., $\overline{K} = \{s \in \Sigma^* \mid (\exists t \in \Sigma^*)st \in K\}$. For the supervisory control purposes [30], the event set Σ is partitioned into two disjoint subsets Σ_c and Σ_{uc} of controllable and uncontrollable events, respectively. K is said to be controllable if $\overline{K}\Sigma_{uc} \cap L(G) \subseteq \overline{K}$ [30].

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¹Shigemasa Takai is with Division of Electrical, Electronic and Infocommunications Engineering, The University of Osaka, Suita, Osaka 565-0871, Japan takai@eei.eng.osaka-u.ac.jp

²Ratnesh Kumar is with the Department of Electrical and Computer Engineering, Iowa State University, Ames, IA 50011-3060 USA rkumar@iastate.edu

The set of control decisions possibly made by a supervisor is $C = \{0, 1, \phi\}$, where “0” represents a disablement decision, “1” represents an enablement decision, and “ ϕ ” represents an unsure (or pass) decision. Formally, a supervisor is defined as a map $S : L(G) \times \Sigma \rightarrow C$ such that $(\forall s \in L(G))(\forall \sigma \in \Sigma_{uc}) S(s, \sigma) = 1$. We inductively define the generated language $L(S/G)$ under the control action of S as follows:

- $\varepsilon \in L(S/G)$,
- $(\forall s \in L(S/G))(\forall \sigma \in \Sigma)$
 $s\sigma \in L(S/G) \Leftrightarrow [s\sigma \in L(G) \wedge S(s, \sigma) = 1]$.

S is said to be *valid* if

$$(\forall s \in L(S/G))(\forall \sigma \in \Sigma_c) s\sigma \in L(G) \Rightarrow S(s, \sigma) \neq \phi,$$

i.e., none of the control decisions for feasible controllable events are unsure.

III. INFERENCE-BASED DECENTRALIZED CONTROL FRAMEWORK

We present the inference-based decentralized control framework, originally introduced in [20]. In a decentralized control setting, there exist n local supervisors, whose local decisions are fused to make a global control decision so that a given (global) specification is enforced in the controlled plant. Let $\Sigma_{c,i}$ be the set of locally controllable events for the i th local supervisor S_i ($i \in I := \{1, 2, \dots, n\}$), in which case, $\Sigma_c := \bigcup_{i \in I} \Sigma_{c,i}$. For each controllable event $\sigma \in \Sigma_c$, the index set of local supervisors for which σ is controllable is denoted by $In(\sigma) = \{i \in I \mid \sigma \in \Sigma_{c,i}\}$. The limited sensing capability of the i th local supervisor S_i ($i \in I$) is represented by a local observation mask, $M_i : \Sigma \rightarrow \Delta_i \cup \{\varepsilon\}$, where Δ_i is the set of locally observed symbols. The observation mask M_i can be extended to $M_i : \Sigma^* \rightarrow \Delta_i^*$ in a usual way. Two traces $s, s' \in \Sigma^*$ with $M_i(s) = M_i(s')$ are said to be M_i -indistinguishable. For any languages $L \subseteq \Sigma^*$ and $L' \subseteq \Delta_i^*$, $M_i(L) \subseteq \Delta_i^*$ and $M_i^{-1}(L') \subseteq \Sigma^*$ are defined as $M_i(L) = \{M_i(s) \in \Delta_i^* \mid s \in L\}$ and $M_i^{-1}(L') = \{s \in \Sigma^* \mid M_i(s) \in L'\}$, respectively.

In inference-based decentralized control framework of [20], each inference-based local supervisor S_i is defined as a map $S_i : M_i(L(G)) \times \Sigma_{c,i} \rightarrow C \times (\mathbb{N} \cup \{\infty\})$, where $\mathbb{N}$ denotes the set of nonnegative integers, i.e., for any $s \in L(G)$ and any $\sigma \in \Sigma_{c,i}$, $S_i(M_i(s), \sigma)$ is of the form:

$$S_i(M_i(s), \sigma) = (c_i(M_i(s), \sigma), n_i(M_i(s), \sigma)).$$

Here, $c_i(M_i(s), \sigma) \in C$ denotes the control decision of S_i for a locally controllable event $\sigma \in \Sigma_{c,i}$ following an observation $M_i(s) \in M_i(L(G))$, and $n_i(M_i(s), \sigma) \in \mathbb{N} \cup \{\infty\}$ denotes the ambiguity level of the control decision $c_i(M_i(s), \sigma)$. Note that, unlike the original framework of [20], $n_i(M_i(s), \sigma)$ is allowed to be ∞ in the above definition. When S_i is certain about the own control decision, “0” is attached as its ambiguity level. The higher the ambiguity level is, the more ambiguous S_i is about its control decision. Let $n(s, \sigma)$ be the minimal ambiguity level of local decisions, i.e., it is defined as follows:

$$n(s, \sigma) := \min\{n_i(M_i(s), \sigma) \in \mathbb{N} \cup \{\infty\} \mid i \in In(\sigma)\}.$$

Here, $n(s, \sigma) = \infty$ implies that the ambiguity level of every local decision on σ is ∞ .

The decentralized supervisor $\{S_i\}_{i \in I}$ that consists of local supervisors S_i ($i \in I$) issues global decisions on controllable events, and is formally defined as a map $\{S_i\}_{i \in I} : L(G) \times \Sigma \rightarrow C$ so that for any $s \in L(G)$ and any $\sigma \in \Sigma$, the global control decision $\{S_i\}_{i \in I}(s, \sigma)$ is given as follows:

- If $\sigma \in \Sigma_c$,

$$\{S_i\}_{i \in I}(s, \sigma) := \begin{cases} 1, & \text{if } (\forall i \in In(\sigma)) n_i(M_i(s), \sigma) = n(s, \sigma) \\ & \Rightarrow c_i(M_i(s), \sigma) = 1 \\ 0, & \text{if } (\forall i \in In(\sigma)) n_i(M_i(s), \sigma) = n(s, \sigma) \\ & \Rightarrow c_i(M_i(s), \sigma) = 0 \\ \phi, & \text{otherwise.} \end{cases}$$

- If $\sigma \in \Sigma_{uc}$, $\{S_i\}_{i \in I}(s, \sigma) := 1$.

In other words, for a controllable event, the global control decision is taken to be the same as the minimal ambiguity level local consensus decision, which we call the “winning” decision.

Given a specification language $K \subseteq L(G)$ of the plant G , a pair of sublanguages of $\bar{K}$ is constructed for each controllable event $\sigma \in \Sigma_c$. The set $D_0(\sigma) \subseteq \bar{K}$ is the set of traces in $\bar{K}$ where σ must be disabled (i.e., $s \in D_0(\sigma) \Leftrightarrow s\sigma \in L(G) - \bar{K}$), whereas the set $E_0(\sigma) \subseteq \bar{K}$ is the set of traces where σ must be enabled (i.e., $s \in E_0(\sigma) \Leftrightarrow s\sigma \in \bar{K}$). Using these as the base step, we inductively define a monotonically decreasing sequence of language pairs $(D_k(\sigma), E_k(\sigma))$ as follows:

$$\begin{aligned} & \bullet \begin{cases} D_0(\sigma) &:= \{s \in \bar{K} \mid s\sigma \in L(G) - \bar{K}\}, \\ E_0(\sigma) &:= \{s \in \bar{K} \mid s\sigma \in \bar{K}\}. \end{cases} \\ & \bullet \begin{cases} D_{k+1}(\sigma) \\ &:= D_k(\sigma) \cap \left(\bigcap_{i \in In(\sigma)} M_i^{-1} M_i(E_k(\sigma)) \right), \\ E_{k+1}(\sigma) \\ &:= E_k(\sigma) \cap \left(\bigcap_{i \in In(\sigma)} M_i^{-1} M_i(D_k(\sigma)) \right). \end{cases} \end{aligned}$$

Note that $D_{k+1}(\sigma)$ is a sublanguage of $D_k(\sigma)$ consisting of traces for each of which there exists an M_i -indistinguishable trace in $E_k(\sigma)$ for each $i \in In(\sigma)$. As a result, all local supervisors that can control σ become ambiguous about their control decisions on σ following the execution of a trace in $D_{k+1}(\sigma)$. The sublanguage $E_{k+1}(\sigma)$ of $E_k(\sigma)$ can be understood in a similar fashion. Accordingly, the notion of N -inference observability for a nonnegative integer $N \in \mathbb{N}$ was introduced in [20] as follows:

Definition 1: [20] Given a nonnegative integer $N \in \mathbb{N}$, a language $K \subseteq L(G)$ is said to be N -inference observable if, for any $\sigma \in \Sigma_c$, $D_{N+1}(\sigma) = \emptyset$ or $E_{N+1}(\sigma) = \emptyset$.

Remark 1: It was shown in [20] that C&PVD&A coobservability [5] and conditional C&PVD&A coobservability [18] are the same as 0-inference observability and 1-inference observability, respectively, and furthermore, the notion of N -inference observability becomes weaker as the parameter $N \in \mathbb{N}$ is increased. It was later shown in [29] that the generalization resulting from increasing N is strictly monotonic.

N -inference observability together with controllability characterizes the existence of a certain class of valid decentralized supervisors, called N -inferring decentralized supervisors, that enforce the specification language $K \subseteq L(G)$. A decentralized supervisor is said to be N -inferring if, for each controllable event, all winning enablement or all winning disablement decisions have ambiguity levels below N . Formally, it is defined as follows:

Definition 2: [20] Given a nonnegative integer $N \in \mathbb{N}$, a valid decentralized supervisor $\{S_i\}_{i \in I} : L(G) \times \Sigma \rightarrow C$ is said to be N -inferring if, for any $\sigma \in \Sigma_c$, either

$$(\forall s \in L(\{S_i\}_{i \in I}/G)) \\ [s\sigma \in L(G) \wedge \{S_i\}_{i \in I}(s, \sigma) = 0] \Rightarrow n(s, \sigma) \leq N,$$

or

$$(\forall s \in L(\{S_i\}_{i \in I}/G)) \\ [s\sigma \in L(G) \wedge \{S_i\}_{i \in I}(s, \sigma) = 1] \Rightarrow n(s, \sigma) \leq N.$$

The following result was originally derived in [20]:

Theorem 1: [20] Given a nonempty language $K \subseteq L(G)$ and a nonnegative integer $N \in \mathbb{N}$, there exists a valid N -inferring decentralized supervisor $\{S_i\}_{i \in I} : L(G) \times \Sigma \rightarrow C$ such that $L(\{S_i\}_{i \in I}/G) = \overline{K}$ if and only if K is controllable and N -inference observable.

IV. NON-UNIFORMLY BOUNDED INFERENCE OBSERVABILITY

From Theorem 1, N -inference observability together with controllability of the specification language characterizes the existence of a valid N -inferring decentralized supervisor that enforces it. As shown in [29], the class of N -inference observable languages increases strictly monotonically as the value of N increases, i.e., in general, there is no upper bound to the number of required level of inference. Informed by this fact, we introduce a new notion of inference observability, named non-uniformly bounded inference observability, which does not impose an apriori upper bound to the inference level.

Definition 3: A language $K \subseteq L(G)$ is said to be *non-uniformly bounded inference observable* (or *nUB-inference observable* for short) if, for any $\sigma \in \Sigma_c$,

$$(\forall s \in D_0(\sigma))(\exists N \in \mathbb{N})s \notin D_N(\sigma) \quad (1)$$

and

$$(\forall s \in E_0(\sigma))(\exists N \in \mathbb{N})s \notin E_N(\sigma). \quad (2)$$

Note according to Definition 3, the inference level required to resolve the ambiguity about an event following a trace needing a disablement or enablement decision is allowed to be trace dependent and not necessarily uniformly bounded across the traces. It turns out that this type of generalization indeed results in a *strictly weaker* notion of inference observability as shown in the following result.

Theorem 2: For any nonnegative integer $N \in \mathbb{N}$, if a language $K \subseteq L(G)$ is N -inference observable, then it is nUB-inference observable. The converse does not hold in general.

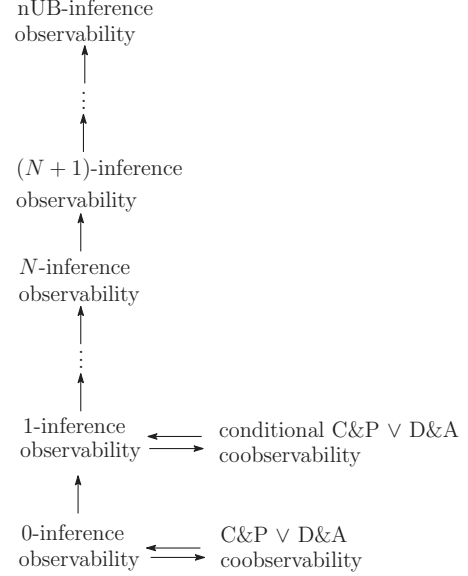


Fig. 1. Relation among notions of observability for decentralized control.

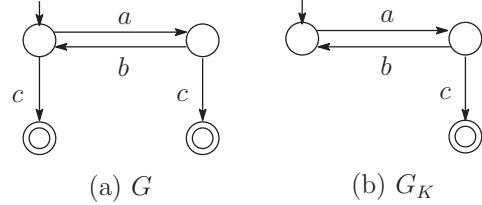


Fig. 2. Automata G and G_K .

A relation among notions of observability for decentralized control is shown in Fig. 1. The following example shows that nUB-inference observability is strictly weaker than N -inference observability for any $N \in \mathbb{N}$:

Example 1: Consider the finite automaton G shown in Fig. 2(a). Let $n = 2$, $\Sigma_c = \Sigma_{c,1} = \Sigma_{c,2} = \{c\}$, $\Delta_1 = \{a, c\}$, $\Delta_2 = \{b, c\}$, and, for each $i \in I$,

$$M_i(\sigma) = \begin{cases} \sigma, & \text{if } \sigma \in \Delta_i \\ \varepsilon, & \text{otherwise.} \end{cases}$$

Also, let $K \subseteq L(G)$ be a language marked by the finite automaton G_K shown in Fig. 2(b). These automata G and G_K are taken from [29].

We have $D_0(c) = (ab)^*$ and $E_0(c) = a(ba)^*$. It was shown in [29] that

$$D_{2k}(c) = D_{2k-1}(c) = (ab)^k(ab)^* \neq \emptyset$$

for any $k \in \mathbb{N}$ with $k \geq 1$ and

$$E_{2k}(c) = E_{2k+1}(c) = a(ba)^k(ba)^* \neq \emptyset$$

for any $k \in \mathbb{N}$. It follows that K is not N -inference observable for any $N \in \mathbb{N}$. On the other hand, K is nUB-inference observable. Any $s \in D_0(c) = (ab)^*$ is of the form $s = (ab)^k$ for some $k \in \mathbb{N}$. We have $s = (ab)^k \notin$

$D_{2(k+1)}(c) = (ab)^{k+1}(ab)^*$ for $2(k+1) \in \mathbb{N}$. Also, any $s \in E_0(c) = a(ba)^*$ is of the form $s = a(ba)^k$ for some $k \in \mathbb{N}$. It follows that $s = a(ba)^k \notin E_{2(k+1)}(c) = a(ba)^{k+1}(ba)^*$ for $2(k+1) \in \mathbb{N}$. Thus, K is nUB-inference observable.

Given an nUB-inference observable language $K \subseteq L(G)$, each inference-based local supervisor computes its control decision and the associated ambiguity level as described next. For any $s \in L(G)$ and any $\sigma \in \Sigma_{c,i}$, the i th local supervisor $S_i : M_i(L(G)) \times \Sigma_{c,i} \rightarrow C \times (\mathbb{N} \cup \{\infty\})$ ($i \in I$) computes $n_i^d(M_i(s), \sigma) \in \mathbb{N} \cup \{\infty\}$ and $n_i^e(M_i(s), \sigma) \in \mathbb{N} \cup \{\infty\}$ as follows:

$$n_i^d(M_i(s), \sigma) = \begin{cases} \min\{k \in \mathbb{N} \mid M_i(s) \notin M_i(E_k(\sigma))\}, & \text{if } (\exists k \in \mathbb{N}) M_i(s) \notin M_i(E_k(\sigma)) \\ \infty, & \text{otherwise} \end{cases} \quad (3)$$

and

$$n_i^e(M_i(s), \sigma) = \begin{cases} \min\{k \in \mathbb{N} \mid M_i(s) \notin M_i(D_k(\sigma))\}, & \text{if } (\exists k \in \mathbb{N}) M_i(s) \notin M_i(D_k(\sigma)) \\ \infty, & \text{otherwise.} \end{cases} \quad (4)$$

When $n_i^d(M_i(s), \sigma)$ takes a finite value in $\mathbb{N}$, it denotes the minimal index $k \in \mathbb{N}$ such that the observation $M_i(s)$ does not match with the observation of any of the traces in $E_k(\sigma)$. In addition, $n_i^d(M_i(s), \sigma) = \infty$ means that there exists an M_i -indistinguishable trace in $E_k(\sigma)$ for any $k \in \mathbb{N}$. $n_i^d(M_i(s), \sigma)$ thus represents the ambiguity level of a disablement decision “contemplated” by S_i for the event σ following the observation $M_i(s)$. Similarly, $n_i^e(M_i(s), \sigma)$ represents the ambiguity level of an enablement decision “contemplated” by S_i for the event σ following the observation $M_i(s)$. Comparing $n_i^d(M_i(s), \sigma)$ and $n_i^e(M_i(s), \sigma)$, the control decision of S_i and the associated ambiguity level, $S_i(M_i(s), \sigma) = (c_i(M_i(s), \sigma), n_i(M_i(s), \sigma))$, is determined as follows [20]:

$$c_i(M_i(s), \sigma) = \begin{cases} 1, & \text{if } n_i^e(M_i(s), \sigma) < n_i^d(M_i(s), \sigma) \\ 0, & \text{if } n_i^d(M_i(s), \sigma) < n_i^e(M_i(s), \sigma) \\ \phi, & \text{if } n_i^e(M_i(s), \sigma) = n_i^d(M_i(s), \sigma), \end{cases} \quad (5)$$

and

$$n_i(M_i(s), \sigma) = \min\{n_i^d(M_i(s), \sigma), n_i^e(M_i(s), \sigma)\}. \quad (6)$$

Since nUB-inference observability does not impose an upper bound of inference level, the decentralized supervisor $\{S_i\}_{i \in I} : L(G) \times \Sigma \rightarrow C$ that consists of the local supervisors $S_i : M_i(L(G)) \times \Sigma_{c,i} \rightarrow C \times (\mathbb{N} \cup \{\infty\})$ ($i \in I$) given by (3)–(6) need not be N -inferring for any fixed $N \in \mathbb{N}$, rather the inference level can differ for different traces with no apriori bound. A valid decentralized supervisor for which all winning enablement and disablement decisions have finite ambiguity levels is said to be non-uniformly bounded inferring, as defined below:

Definition 4: A valid decentralized supervisor $\{S_i\}_{i \in I} : L(G) \times \Sigma \rightarrow C$ is said to be *non-uniformly bounded inferring* (or *nUB-inferring* for short) if, for any $\sigma \in \Sigma_c$,

$$(\forall s \in L(\{S_i\}_{i \in I}/G)) s\sigma \in L(G) \Rightarrow n(s, \sigma) \in \mathbb{N}.$$

The main result of the paper that we show here is that nUB-unbounded inference observability together with controllability of the specification language is necessary and sufficient for the existence of a valid nUB-inferring decentralized supervisor that enforces it. We establish this result through the help of a few lemmas. The first lemma shows that the nUB-inference observability is sufficient for the decentralized supervisor, consisting of the local supervisors given by (3)–(6), to be valid, nUB-inferring, and furthermore, specification-enforcing in the presence of controllability.

Lemma 1: For a nonempty nUB-inference observable and controllable language $K \subseteq L(G)$, consider the decentralized supervisor $\{S_i\}_{i \in I} : L(G) \times \Sigma \rightarrow C$ that consists of the local supervisors $S_i : M_i(L(G)) \times \Sigma_{c,i} \rightarrow C \times (\mathbb{N} \cup \{\infty\})$ ($i \in I$) given by (3)–(6). Then, $\{S_i\}_{i \in I}$ is valid, nUB-inferring, and satisfies $L(\{S_i\}_{i \in I}/G) = \overline{K}$.

Next, the following lemma states that nUB-inference observability is a necessary condition for the existence of a valid nUB-inferring decentralized supervisor that enforces the given specification.

Lemma 2: For a nonempty language $K \subseteq L(G)$, if there exists a valid nUB-inferring decentralized supervisor $\{S_i\}_{i \in I} : L(G) \times \Sigma \rightarrow C$ such that $L(\{S_i\}_{i \in I}/G) = \overline{K}$, then K is nUB-inference observable.

Using Lemmas 1 and 2, the following theorem is obtained as the main result of the paper, which presents necessity and sufficiency conditions for the existence of an nUB-inferring decentralized supervisor that enforces the given specification.

Theorem 3: For a nonempty language $K \subseteq L_m(G)$, there exists a valid nUB-inferring decentralized supervisor $\{S_i\}_{i \in I} : L(G) \times \Sigma \rightarrow C$ such that $L(\{S_i\}_{i \in I}/G) = \overline{K}$ if and only if K is controllable and nUB-inference observable.

Example 2: We consider the setting of Example 1, where the plant and the specification are modeled by the automata G and G_K shown in Fig. 2(a) and (b), respectively.

Obviously, $K (= L_m(G_K))$ is controllable. For any $N \in \mathbb{N}$, since K is not N -inference observable as shown in Example 1, there does not exist a valid N -inferring decentralized supervisor that enforces K . However, since K is nUB-inference observable, by Theorem 3, there exists a valid nUB-inferring decentralized supervisor for enforcing it. For the controllable event $c \in \Sigma_c$, the local control decisions of the local supervisors S_1 and S_2 computed by (3)–(6) are given in Table I. For example, for $a^k \in M_1(L(G))$ with $k \geq 1$, $S_1(a^k, c) = (c_1(a^k, c), n_1(a^k, c))$ is computed as follows. Since $a^k \in M_1(E_{2k-1}(c)) - M_1(E_{2k}(c))$ and $a^k \in M_1(D_{2k}(c)) - M_1(D_{2k+1}(c))$, we have $n_1^d(a^k, c) = 2k$ and $n_1^e(a^k, c) = 2k + 1$ by (3) and (4), respectively. Then, since $n_1^d(a^k, c) < n_1^e(a^k, c)$, it follows from (5) and (6) that $c_1(a^k, c) = 0$ and $n_1(a^k, c) = 2k$, which implies that S_1 issues the disablement decision whose ambiguity level is $2k$. Based on the local control decisions of S_1

TABLE I

LOCAL CONTROL DECISIONS OF S_1 AND S_2 FOR $c \in \Sigma_c$

$t \in M_1(L(G))$	$n_1^d(t, c)$	$n_1^d(t, c)$	$c_1(t, c)$	$n_1(t, c)$
$a^k (k \in \mathbb{N})$	$2k$	$2k + 1$	0	$2k$
$a^k c (k \in \mathbb{N})$	0	0	ϕ	0
$t \in M_2(L(G))$	$n_2^d(t, c)$	$n_2^d(t, c)$	$c_2(t, c)$	$n_2(t, c)$
$b^k (k \in \mathbb{N})$	$2k + 2$	$2k + 1$	1	$2k + 1$
$b^k c (k \in \mathbb{N})$	0	0	ϕ	0

TABLE II

GLOBAL CONTROL DECISIONS OF $\{S_i\}_{i \in I}$ FOR $c \in \Sigma_c$

$s \in L(G)$	$n(s, c)$	$\{S_i\}_{i \in I}(s, c)$
$(ab)^k (k \in \mathbb{N})$	$2k$	0
$a(ba)^k (k \in \mathbb{N})$	$2k + 1$	1
$(ab)^k c (k \in \mathbb{N})$	0	ϕ
$a(ba)^k c (k \in \mathbb{N})$	0	ϕ

and S_2 , the global control decisions of the decentralized supervisor $\{S_i\}_{i \in I}$ are obtained as shown in Table II. For example, for $(ab)^k \in \bar{K}$, where $k \in \mathbb{N}$, since $(ab)^k c \in L(G) - \bar{K}$, c needs to be disabled. Since $2k = n_1(a^k, c) = n_1(M_1((ab)^k), c) < n_2(M_2((ab)^k), c) = n_2(b^k, c) = 2k + 1$ and $c_1(M_1((ab)^k), c) = c_1(a^k, c) = 0$, we have $\{S_i\}_{i \in I}((ab)^k, c) = 0$ with $n((ab)^k, c) = 2k$, i.e., c is disabled by $\{S_i\}_{i \in I}$, as expected. Also, for $a(ba)^k \in \bar{K}$, where $k \in \mathbb{N}$, since $a(ba)^k c \in \bar{K}$, c needs to be enabled. Since $2k + 1 = n_2(b^k, c) = n_2(M_2(a(ba)^k), c) < n_1(M_1(a(ba)^k), c) = n_1(a^{k+1}, c) = 2k + 2$ and $c_2(M_2(a(ba)^k), c) = c_2(b^k, c) = 1$, we have $\{S_i\}_{i \in I}(a(ba)^k, c) = 1$ with $n(a(ba)^k, c) = 2k + 1$, i.e., c is enabled by $\{S_i\}_{i \in I}$, as expected. By Table II, we can conclude that the decentralized supervisor $\{S_i\}_{i \in I}$ is valid and nUB-inferring and it also satisfies $L(\{S_i\}_{i \in I}/G) = \bar{K}$.

V. CONCLUSIONS

In the setting of decentralized supervisory control of DESs, this paper introduced a new notion of inference observability, named non-uniformly bounded inference observability (or nUB-inference observability for short). We showed that nUB-inference observability is strictly weaker than N -inference observability for any $N \in \mathbb{N}$. In addition, we proved that it together with controllability serves as a necessary and sufficient condition for the existence of a decentralized supervisor that uses inference with trace-dependent finite levels that are not necessarily uniformly bounded across all traces. Examining an algorithmic verification of the proposed nUB-inference observability and realization of an nUB-inferring decentralized supervisor will be our future work.

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