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COMPLETIONS AND MAXIMAL QUOTIENT RINGS OVER REGULAR RINGS

Dedicated to Professor Hiroyuki Tachikawa for his 60th birthday

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Let R be a (Von Neumann) regular ring with rank function N. Then there exists the N-metric completion $\bar{R}$ of R with respect to N. It is well known that $\bar{R}$ is a right and left self-injective regular ring. On the other hand, we have always the maximal right quotient ring $Q_r^{\max}(R)$ of R, which is a right self-injective regular ring.

In this paper, we are concerned with the connection between the N-metric completion of R and the maximal right quotient ring of R.

In [1, Theorem 23.17], Goodearl has shown that if for any essential right ideal I of R, $N(I)=1$, then $Q_r^{\max}(R)$ is embeded in $\bar{R}$ as a subring. Under this condition, we shall show that $\bar{R}$ is isomorphic to the maximal left quotient ring of $Q_r^{\max}(R)$.

Furthermore, we shall show the simillar result with the above result to the case that N is a pseudo-rank function.

§1. Preliminaries

Throughout of this paper, we assume that all rings are associative with identity element and all modules are unitary.

Let R be a reuglar ring. A pseudo-rank function on R is a map $N: R \to [0, 1]$ such that

(1) $N(1)=1$ and $N(0)=0$

(2) $N(xy) \leqq \mathrm{Max}\,\{N(x), N(y)\}$, for any x,y in R

(3) $N(e+f)=N(e)+N(f)$, for each orthogonal idempotent elements e, f of R.

A rank function on R is a pseudo-rank function with additional property

(4) $N(x)=0$ implies $x=0$.

If N is a pseudo-rank function on R, the the rule $\delta(x, y)=N(x-y)$ defines a pseudo-metric on R. Clearly, δ is metric if and only if N is a rank function. The completion $\bar{R}$ of R with respect to δ is a right and left self-injective regular ring which is complete with respect to the $\bar{N}$-metric, where $\bar{N}$ is the unique extension of N to $\bar{R}$.

If N is a pseudo-rank function, then we denote that $\ker N=\{x\in R \mid N(x)|=0\}$.

We set $P(R)$ is the set of pseudo-rank functions over R. We say that N of $P(R)$ is an extreme point of $P(R)$ provided that N can not be expressed as a positive convex combination of two distinct points of $P(R)$.

Let I be a right ideal of R. Then we denote that $N(I)=\operatorname{Sup}\{N(x) \mid x\in I\}$. Finally, if A and B are modules, then the notation $A\subseteq_e B$ means that A is an essential submodule of B.

§2. Completions over regular rings

In this section, we prove the following main theorem.

Theorem 1. *Let R be a regular ring with rank function N. If for any essential right ideal I of R, $N(I)=1$, then the completion $\bar{R}$ of R with respect to N-metric is isomorphic to the maximal left quotient ring of the maximal right quotient ring of R.*

Proof. First we shall show that there exists a ring monomorphism ψ from the maximal right quotient ring Q of R to the completion $\bar{R}$ of R. This is proved by the same idea of [1, Theorem 21.17], but we shall give a proof for completness. Since $\bar{R}$ is right and left-self-injective regular ring, $\bar{R}$ is a injective right R-module. Let x be any element of $\bar{R}$ and I is an essential right idel ideal of R such that $xI=0$. Then since $N(I)=1$, for any positive number ε, there exists an idempotent e of I such that $N(e)>1-\varepsilon$. Now since $x=x-xe$, $\bar{N}(x)=\bar{N}(x-xe)\leqq\bar{N}(1-e)=N(1-e)<\varepsilon$. Thus $\bar{N}(x)=0$. Since $\bar{N}$ is a rank function, $x=0$. Therefore $\bar{R}$ is a non-singular right R-module. Since R is essential right submodule of Q and $\bar{R}$ is an injective right R-module, the identity map on R extends to a right R-module monomorphism $\psi\colon Q\to\bar{R}$. We claim that ψ is a ring homomorphism. Given any elements x, y in Q, we have $yJ\subseteq R$ for some essential right ideal J of R. Then $\psi(xy)r=\psi(xyr)=\psi(x)yr=\psi(x)(y)r$ for all r in J. Whence $[\psi(x)\psi(y)-\psi(xy)]J=0$. Since $\bar{R}$ is non-singular right R-module, we obtain that $\psi(xy)=\psi(x)\psi(y)$, so that ψ is a ring homomorphism. Next we shall show that for any essential left ideal K of Q, $N_Q(K)=1$, where N_Q is an extension of N. Let K be an essential left ideal of Q. Then by [1, Lemma 9.7], there exist orthogonal idempotents $e_1, e_2, \cdots$, such that $\sum_{n=1}^{\infty}\oplus Qe_n\subseteq_e K\subseteq_e Q$. We set $J=\bigoplus_{n=1}^{\infty} e_nQ$. We claim that J is an essential right ideal of Q. If $J\cap eQ=0$, then $J\oplus eQ\subseteq_e Q$. On the other hand, since Q is a right self-injective regular ring, there exist orthogonal idempotents e'_n, e' such that $e'_nQ=e_nQ$, $eQ=e'Q$. We claim that $\bigoplus_{n=1}^{\infty} Qe_n\cap Qe'=0$. Let $re'=a_1e_{n_1}+\cdots+a_ke_{n_k}$ be an any element of $\bigoplus_{n=1}^{\infty} Qe_n\cap Qe'$. Since $e_n=e'_ne_n$, $e'_n=e_ne'_n$, we have that

$e'e_n=e'(e'_ne_n)=0$. Now $re'e_{n_i}=a_ie_{n_i}$, so $re'=0$, as claimed. Thus $e'=0$, so that $e=0$. This shows that $\bigoplus_{n=1}^{\infty} e_nQ$ is an essential right ideal in Q. Therefore by assumption, $N_Q(J)=1$, which is concluded to that $N_Q(K)=1$. In this case, the maximal left quotient ring of Q is embeded in $\bar{R}$ as a subring. Now in order to prove this Theorem, it suffices to show that $\bar{R}$ is a left essential extension of Q. Let x be any element of $\bar{R}$. Since $\bar{R}$ is complete, for any positive number ε, there exists an element x' of Q such that $\bar{N}(x-x')<\varepsilon/2$. We put $x-x'=y$, where y is an element of $\bar{R}$. With respect to y, there exists an element x_1 of Q such that $\bar{N}(y-x_1)<\varepsilon/2\cdot 3$. Put $y-x_1=y_1$. In general, thre there exist elements x_n of Q such that $\bar{N}(y_{n-1}-x_n)<\varepsilon/2\cdot 3^n$. We put $y_{n-1}-x_n=y_n$. Now we have that $\bar{N}(y-\sum_{i=1}^{n} x_i)<\varepsilon/2\cdot 3^n$. Thus $\sum x_i$ is a Cauchy sequence. Since $\bar{R}$ is complete, $\sum_{i=1}^{\infty} x_i=y$. Furthermore, $\sum_{i=1}^{\infty} \bar{N}(x_i)<\bar{N}(y)+2\{\bar{N}({}_1y)+\bar{N}(y_2)+\cdots\}<\varepsilon/2+\sum_{n=1}^{\infty}\varepsilon/3^n=\varepsilon/2+\varepsilon/2=\varepsilon$. Let I be a right ideal generated by x', x_1, $\cdots$. Since $\sum \bar{N}(x_i)<\varepsilon$, $\bar{N}(I)<\varepsilon$. On the other hand, since $\sum_{i=1}^{\infty} x_i=y\in\bar{R}$, y is in $\bar{I}$, where $\bar{I}$ is the N_Q-closure of I. Let e be an idempotent element of Q such that $I\subseteq_e(1-e)Q$, then $eI=0$. And since $I\oplus eQ\subseteq_e Q$, $N_Q(I\oplus eQ)=N_Q(I)+N_Q(e)=1$. Hence $N_Q(e)>1-.\varepsilon$. Furthermore since $l_{\bar{R}}(I)=l_{\bar{R}}(\bar{I})$, $e\bar{I}=0$, where $l_{\bar{R}}(\ )$ means the left annhilator ideal. In particular, $ey=0$. Now since $y=x-x'$, $ex=ex'$. Finally, we shall show that $ex'\neq 0$. Since $ex=x-(1-e)x$, $\bar{N}(ex)>\bar{N}(x)-\bar{N}((1-e)x)>\bar{N}(x)-\bar{N}(1-e)>\bar{N}(x)-\varepsilon$. If we set ε as $\bar{N}(x)>\varepsilon$, then $N(ex)\neq 0$, so $ex=ex'\neq 0$. Therefore $\bar{R}x\cap Q\neq 0$, hence $\bar{R}$ $\bar{R}$ is a left essential extention of Q. This completes the proof.

REMARK. Recently, H. Kambara [2] constructed the counter example of Roos conjecture (=Is every directly finite regular right self-injective ring necessary left self-injective?). He constructed the simple regular ring which is directly finite right self-injective and satisfies the assumption of above Theorem 1, but not left self-injective. By virtue of this example, Theorem 1 is not abstruct non-sensce.

If N is a pseudo-rank function, then we have the following theorem.

Theorem 2. *Let R be a regular ring with pseudo-rank function N. If for any essential right ideal I of R, $N(I)=1$ and $ker(N)$ is a prime ideal, then N is an extreme point and is extended to the maximal right quotient ring Q. In this case, $Q/kerN_Q$ is the maximal right quotient ring of $R/ker\,N$, where N_Q is the extension N. Furthermore, the completion of R is isomorphic to the left maximal quotient ring of the right maximal quotient ring of $R/ker\,(N)$.*

Proof. Let $\bar{R}$ be the N-completion of R. Then since $\bar{R}$ is a right and left self-injective regular ring, $\bar{R}$ is injective as a right R-module. Now there exists a R-module homomorphism f from Q to $\bar{R}$ such that the following diagram commute,

$$\begin{tikzcd} Q \arrow[r, dotted, "f"] & R \\ R \arrow[u] \arrow[r] & R/\ker(N) \arrow[u] \end{tikzcd}$$

In this case, by using the same proof of Theorem 1, we can see that $\bar{R}$ is a non-singular right R-module and f is a ring homomorphism. We extend N to Q as follows, for any element x of Q, we define that $N_Q(x)=N(f(x))$. Note that $\ker(f)=\ker N$ and $\ker N_Q \cap R=\ker N$. Clearly $R/\ker N$ has a rank function $\tilde{N}$ which is induced by N. We note that $\tilde{N}(K)=1$ for any essential right ideal K of $R/\ker N$. Therefore we apply Theorem 1 to $R/\ker N$, that is $\bar{R}$ is the maximal lef quotient ring of the maximal right quotient ring of $R/\ker N$. Next we claim that $Q/\ker N_Q$ is a right essential extension of $R/\ker N$. Given non-zero element $\bar{x}$ of $Q/\ker N_Q$, since x is in Q, there exists essential right ideal J of R such that $xJ\subseteq R$. Assume that $xJ\subseteq \ker N$, then $\bar{x}\bar{J}=\bar{0}$. In this case, we have that $\bar{x}=0$, which is a contradiction. Therefore $xJ \nsubseteq \ker N$, that is for some non-zero element t of J, $\bar{0}\neq\bar{x}\bar{t}\in R/\ker N$. So $Q/\ker N$ is essential right extention of $R/\ker N$. Note that $\ker N_Q$ is a prime ideal of Q. Since Q is a self-injective regular ring, there exists a central idempotent e of Q such that $\ker N_Q \subseteq_e eQ$. Thus $\ker N_Q \oplus (1-eQ)$ is an essential ideal of Q, so $N_Q(\ker N_Q \oplus (1-e)Q)=1$. This shows that $N_Q((1-e)Q)=1$. Now $N_Q(e)=0$, hence $\ker N_Q=eQ$. Therefore $Q/\ker N_Q= Q/eQ$ is a regular right self-injective ring. Consequently, $Q/\ker N_Q$ is a maximal right quotient ring of $R/\ker N$. Thus $Q/\ker N_Q$ is a prime regular self-injective ring with rank function. In this case, [1, Proposition 8.6] shows that $Q/\ker N_Q$ is a simple ring. Therefore $R/\ker N$ is also a simple ring. Now [1, Theorem 19.14] implies that N is an extreme point of $P(R)$. Thus the proof is complete.

References

[1] K.R. Goodearl: Von Neumann regular rings, Pitman, London, 1979.

[2] H. Kambara: *Example of a D.F right self-injective regular ring which is not left self-injective,* to appear.

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