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COMPLETIONS AND MAXIMAL QUOTIENT RINGS OVER REGULAR RINGS

Dedicated to Professor Hiroyuki Tachikawa for his 60th birthday

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Let R be a (Von Neumann) regular ring with rank function N . Then there exists the N -metric completion \bar{R} of R with respect to N . It is well known that \bar{R} is a right and left self-injective regular ring. On the other hand, we have always the maximal right quotient ring $Q_r^{\max}(R)$ of R , which is a right self-injective regular ring.

In this paper, we are concerned with the connection between the N -metric completion of R and the maximal right quotient ring of R .

In [1, Theorem 23.17], Goodearl has shown that if for any essential right ideal I of R , $N(I)=1$, then $Q_r^{\max}(R)$ is embeded in \bar{R} as a subring. Under this condition, we shall show that \bar{R} is isomorphic to the maximal left quotient ring of $Q_r^{\max}(R)$.

Furthermore, we shall show the simillar result with the above result to the case that N is a pseudo-rank function.

§1. Preliminaries

Throughout of this paper, we assume that all rings are associative with identity element and all modules are unitary.

Let R be a reuglar ring. A pseudo-rank function on R is a map $N: R \rightarrow [0, 1]$ such that

- (1) $N(1)=1$ and $N(0)=0$
- (2) $N(xy) \leq \text{Max} \{N(x), N(y)\}$, for any x, y in R
- (3) $N(e+f)=N(e)+N(f)$, for each orthogonal idempotent elements e, f of R .

A rank function on R is a pseudo-rank function with additional property

- (4) $N(x)=0$ implies $x=0$.

If N is a pseudo-rank function on R , the the rule $\delta(x, y)=N(x-y)$ defines a pseudo-metric on R . Clearly, δ is metric if and only if N is a rank function. The completion \bar{R} of R with respect to δ is a right and left self-injective regular ring which is complete with respect to the \bar{N} -metric, where \bar{N} is the unique extension of N to \bar{R} .

If N is a pseudo-rank function, then we denote that $\ker N = \{x \in R \mid N(x) = 0\}$.

We set $P(R)$ is the set of pseudo-rank functions over R . We say that N of $P(R)$ is an extreme point of $P(R)$ provided that N can not be expressed as a positive convex combination of two distinct points of $P(R)$.

Let I be a right ideal of R . Then we denote that $N(I) = \text{Sup}\{N(x) \mid x \in I\}$. Finally, if A and B are modules, then the notation $A \subseteq_e B$ means that A is an essential submodule of B .

§2. Completions over regular rings

In this section, we prove the following main theorem.

Theorem 1. *Let R be a regular ring with rank function N . If for any essential right ideal I of R , $N(I) = 1$, then the completion \bar{R} of R with respect to N -metric is isomorphic to the maximal left quotient ring of the maximal right quotient ring of R .*

Proof. First we shall show that there exists a ring monomorphism ψ from the maximal right quotient ring Q of R to the completion \bar{R} of R . This is proved by the same idea of [1, Theorem 21.17], but we shall give a proof for completeness. Since \bar{R} is right and left-self-injective regular ring, \bar{R} is a injective right R -module. Let x be any element of \bar{R} and I is an essential right ideal of R such that $xI = 0$. Then since $N(I) = 1$, for any positive number ε , there exists an idempotent e of I such that $N(e) > 1 - \varepsilon$. Now since $x = x - xe$, $\bar{N}(x) = \bar{N}(x - xe) \leq \bar{N}(1 - e) = N(1 - e) < \varepsilon$. Thus $\bar{N}(x) = 0$. Since \bar{N} is a rank function, $x = 0$. Therefore \bar{R} is a non-singular right R -module. Since R is essential right submodule of Q and \bar{R} is an injective right R -module, the identity map on R extends to a right R -module monomorphism $\psi: Q \rightarrow \bar{R}$. We claim that ψ is a ring homomorphism. Given any elements x, y in Q , we have $yJ \subseteq R$ for some essential right ideal J of R . Then $\psi(xy)r = \psi(xyr) = \psi(x)yr = \psi(x)(y)r$ for all r in J . Whence $[\psi(x)\psi(y) - \psi(xy)]J = 0$. Since \bar{R} is non-singular right R -module, we obtain that $\psi(xy) = \psi(x)\psi(y)$, so that ψ is a ring homomorphism. Next we shall show that for any essential left ideal K of Q , $N_Q(K) = 1$, where N_Q is an extension of N . Let K be an essential left ideal of Q . Then by [1, Lemma 9.7], there exist orthogonal idempotents e_1, e_2, \dots , such that $\sum_{n=1}^{\infty} \oplus Qe_n \subseteq_e K \subseteq_e Q$. We set $J = \bigoplus_{n=1}^{\infty} e_n Q$. We claim that J is an essential right ideal of Q . If $J \cap eQ = 0$, then $J \oplus eQ \subseteq_e Q$. On the other hand, since Q is a right self-injective regular ring, there exist orthogonal idempotents e'_n, e' such that $e'_n Q = e_n Q, eQ = e'Q$. We claim that $\bigoplus_{n=1}^{\infty} Qe_n \cap Qe' = 0$. Let $re' = a_1 e_{n_1} + \dots + a_k e_{n_k}$ be an any element of $\bigoplus_{n=1}^{\infty} Qe_n \cap Qe'$. Since $e_n = e'_n e_n, e'_n = e_n e'_n$, we have that

$e'e_n = e'(e'_n e_n) = 0$. Now $re'e_{n_i} = a_i e_{n_i}$, so $re' = 0$, as claimed. Thus $e' = 0$, so that $e = 0$. This shows that $\bigoplus_{n=1}^{\infty} e_n Q$ is an essential right ideal in Q . Therefore by assumption, $N_Q(J) = 1$, which is concluded to that $N_Q(K) = 1$. In this case, the maximal left quotient ring of Q is embedded in \bar{R} as a subring. Now in order to prove this Theorem, it suffices to show that \bar{R} is a left essential extension of Q . Let x be any element of \bar{R} . Since \bar{R} is complete, for any positive number ε , there exists an element x' of Q such that $\bar{N}(x - x') < \varepsilon/2$. We put $x - x' = y$, where y is an element of \bar{R} . With respect to y , there exists an element x_1 of Q such that $\bar{N}(y - x_1) < \varepsilon/2 \cdot 3$. Put $y - x_1 = y_1$. In general, there exist elements x_n of Q such that $\bar{N}(y_{n-1} - x_n) < \varepsilon/2 \cdot 3^n$. We put $y_{n-1} - x_n = y_n$. Now we have that $\bar{N}(y - \sum_{i=1}^n x_i) < \varepsilon/2 \cdot 3^n$. Thus $\sum x_i$ is a Cauchy sequence. Since \bar{R} is complete, $\sum_{i=1}^{\infty} x_i = y$. Furthermore, $\sum_{i=1}^{\infty} \bar{N}(x_i) < \bar{N}(y) + 2\{\bar{N}(y_1) + \bar{N}(y_2) + \dots\} < \varepsilon/2 + \sum_{n=1}^{\infty} \varepsilon/3^n = \varepsilon/2 + \varepsilon/2 = \varepsilon$. Let I be a right ideal generated by x', x_1, \dots . Since $\sum \bar{N}(x_i) < \varepsilon$, $\bar{N}(I) < \varepsilon$. On the other hand, since $\sum_{i=1}^{\infty} x_i = y \in \bar{R}$, y is in \bar{I} , where \bar{I} is the N_Q -closure of I . Let e be an idempotent element of Q such that $I \subseteq_e (1-e)Q$, then $eI = 0$. And since $I \oplus eQ \subseteq_e Q$, $N_Q(I \oplus eQ) = N_Q(I) + N_Q(e) = 1$. Hence $N_Q(e) > 1 - \varepsilon$. Furthermore since $l_{\bar{R}}(I) = l_{\bar{R}}(\bar{I})$, $e\bar{I} = 0$, where $l_{\bar{R}}(\)$ means the left annihilator ideal. In particular, $ey = 0$. Now since $y = x - x'$, $ex = ex'$. Finally, we shall show that $ex' \neq 0$. Since $ex = x - (1-e)x$, $\bar{N}(ex) > \bar{N}(x) - \bar{N}((1-e)x) > \bar{N}(x) - \bar{N}(1-e) > \bar{N}(x) - \varepsilon$. If we set ε as $\bar{N}(x) > \varepsilon$, then $N(ex) \neq 0$, so $ex = ex' \neq 0$. Therefore $\bar{R}x \cap Q \neq 0$, hence \bar{R} is a left essential extension of Q . This completes the proof.

REMARK. Recently, H. Kambara [2] constructed the counter example of Roos conjecture (=Is every directly finite regular right self-injective ring necessary left self-injective?). He constructed the simple regular ring which is directly finite right self-injective and satisfies the assumption of above Theorem 1, but not left self-injective. By virtue of this example, Theorem 1 is not abstract non-sense.

If N is a pseudo-rank function, then we have the following theorem.

Theorem 2. *Let R be a regular ring with pseudo-rank function N . If for any essential right ideal I of R , $N(I) = 1$ and $\ker(N)$ is a prime ideal, then N is an extreme point and is extended to the maximal right quotient ring Q . In this case, $Q/\ker N_Q$ is the maximal right quotient ring of $R/\ker N$, where N_Q is the extension N . Furthermore, the completion of R is isomorphic to the left maximal quotient ring of the right maximal quotient ring of $R/\ker(N)$.*

Proof. Let \bar{R} be the N -completion of R . Then since \bar{R} is a right and

left self-injective regular ring, \bar{R} is injective as a right R -module. Now there exists a R -module homomorphism f from Q to \bar{R} such that the following diagram commutes,

$$\begin{array}{ccc} Q & \xrightarrow{\dots \dots f \dots} & R \\ \uparrow & & \uparrow \\ R & \longrightarrow & R/\ker(N) \end{array}$$

In this case, by using the same proof of Theorem 1, we can see that \bar{R} is a non-singular right R -module and f is a ring homomorphism. We extend N to Q as follows, for any element x of Q , we define that $N_Q(x) = N(f(x))$. Note that $\ker(f) = \ker N$ and $\ker N_Q \cap R = \ker N$. Clearly $R/\ker N$ has a rank function \bar{N} which is induced by N . We note that $\bar{N}(K) = 1$ for any essential right ideal K of $R/\ker N$. Therefore we apply Theorem 1 to $R/\ker N$, that is \bar{R} is the maximal left quotient ring of the maximal right quotient ring of $R/\ker N$. Next we claim that $Q/\ker N_Q$ is a right essential extension of $R/\ker N$. Given non-zero element x of $Q/\ker N_Q$, since x is in Q , there exists essential right ideal J of R such that $xJ \subseteq R$. Assume that $xJ \subseteq \ker N$, then $x\bar{J} = \bar{0}$. In this case, we have that $x = 0$, which is a contradiction. Therefore $xJ \not\subseteq \ker N$, that is for some non-zero element t of J , $\bar{0} \neq x\bar{t} \in R/\ker N$. So $Q/\ker N$ is essential right extension of $R/\ker N$. Note that $\ker N_Q$ is a prime ideal of Q . Since Q is a self-injective regular ring, there exists a central idempotent e of Q such that $\ker N_Q \subseteq_e eQ$. Thus $\ker N_Q \oplus (1-e)Q$ is an essential ideal of Q , so $N_Q(\ker N_Q \oplus (1-e)Q) = 1$. This shows that $N_Q((1-e)Q) = 1$. Now $N_Q(e) = 0$, hence $\ker N_Q = eQ$. Therefore $Q/\ker N_Q = Q/eQ$ is a regular right self-injective ring. Consequently, $Q/\ker N_Q$ is a maximal right quotient ring of $R/\ker N$. Thus $Q/\ker N_Q$ is a prime regular self-injective ring with rank function. In this case, [1, Proposition 8.6] shows that $Q/\ker N_Q$ is a simple ring. Therefore $R/\ker N$ is also a simple ring. Now [1, Theorem 19.14] implies that N is an extreme point of $P(R)$. Thus the proof is complete.

References

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