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Let $R$ be a (Von Neumann) regular ring with rank function $N$. Then there exists the $N$-metric completion $\overline{R}$ of $R$ with respect to $N$. It is well known that $\overline{R}$ is a right and left self-injective regular ring. On the other hand, we have always the maximal right quotient ring $Q^\text{max}_r(R)$ of $R$, which is a right self-injective regular ring.

In this paper, we are concerned with the connection between the $N$-metric completion of $R$ and the maximal right quotient ring of $R$.

In [1, Theorem 23.17], Goodearl has shown that if for any essential right ideal $I$ of $R$, $N(I)=1$, then $Q^\text{max}_r(R)$ is embedded in $\overline{R}$ as a subring. Under this condition, we shall show that $\overline{R}$ is isomorphic to the maximal left quotient ring of $Q^\text{max}_r(R)$.

Furthermore, we shall show the simillar result with the above result to the case that $N$ is a pseudo-rank function.

§ 1. Preliminaries

Throughout of this paper, we assume that all rings are associative with identity element and all modules are unitary.

Let $R$ be a regular ring. A pseudo-rank function on $R$ is a map $N: R \to [0, 1]$ such that

1. $N(1)=1$ and $N(0)=0$
2. $N(xy) \leq \text{Max} \{N(x), N(y)\}$, for any $x, y$ in $R$
3. $N(e+f)=N(e)+N(f)$, for each orthogonal idempotent elements $e, f$ of $R$.

A rank function on $R$ is a pseudo-rank function with additional property

4. $N(x)=0$ implies $x=0$.

If $N$ is a pseudo-rank function on $R$, the the rule $\delta(x, y)=N(x-y)$ defines a pseudo-metric on $R$. Clearly, $\delta$ is metric if and only if $N$ is a rank function.

The completion $\overline{R}$ of $R$ with respect to $\delta$ is a right and left self-injective regular ring which is complete with respect to the $\overline{N}$-metric, where $\overline{N}$ is the unique extension of $N$ to $\overline{R}$.
If $N$ is a pseudo-rank function, then we denote that \( \ker N = \{ x \in R \mid N(x) = 0 \} \).

We set \( P(R) \) is the set of pseudo-rank functions over \( R \). We say that \( N \) of \( P(R) \) is an extreme point of \( P(R) \) provided that \( N \) can not be expressed as a positive convex combination of two distinct points of \( P(R) \).

Let \( I \) be a right ideal of \( R \). Then we denote that \( N(I) = \text{Sup} \{ N(x) \mid x \in I \} \).

Finally, if \( A \) and \( B \) are modules, then the notation \( A \subseteq B \) means that \( A \) is an essential submodule of \( B \).

\section*{§2. Completions over regular rings}

In this section, we prove the following main theorem.

**Theorem 1.** Let \( R \) be a regular ring with rank function \( N \). If for any essential right ideal \( I \) of \( R \), \( N(I) = 1 \), then the completion \( \bar{R} \) of \( R \) with respect to \( N \)-metric is isomorphic to the maximal left quotient ring of the maximal right quotient ring of \( R \).

**Proof.** First we shall show that there exists a ring monomorphism \( \psi \) from the maximal right quotient ring \( Q \) of \( R \) to the completion \( \bar{R} \) of \( R \). This is proved by the same idea of [1, Theorem 21.17], but we shall give a proof for completeness. Since \( \bar{R} \) is right and left-self-injective regular ring, \( \bar{R} \) is a injective right \( R \)-module. Let \( x \) be any element of \( \bar{R} \) and \( I \) is an essential right idel ideal of \( R \) such that \( xI = 0 \). Then since \( N(I) = 1 \), for any positive number \( \varepsilon \), there exists an idempotent \( e \) of \( I \) such that \( N(e) > 1 - \varepsilon \). Now since \( x = x - xe \), \( \bar{N}(x) = \bar{N}(x - xe) \leq \bar{N}(1 - e) = N(1 - e) = 1 - \varepsilon \). Thus \( \bar{N}(x) = 0 \). Since \( \bar{N} \) is a rank function, \( x = 0 \). Therefore \( \bar{R} \) is a non-singular right \( R \)-module. Since \( R \) is essential right submodule of \( Q \) and \( \bar{R} \) is an injective right \( R \)-module, the identity map on \( R \) extends to a right \( R \)-module monomorphism \( \psi : Q \to \bar{R} \). We claim that \( \psi \) is a ring homomorphism. Given any elements \( x, y \) in \( Q \), we have \( yf \subseteq R \) for some essential right ideal \( J \) of \( R \). Then \( \psi(xy)r = \psi(xy)r = \psi(x)y\psi(y)r = \psi(x)(y)r \) for all \( r \) in \( J \). Whence \( [\psi(x)(y) - \psi(xy)]J = 0 \). Since \( \bar{R} \) is non-singular right \( R \)-module, we obtain that \( \psi(xy) = \psi(x)\psi(y) \), so that \( \psi \) is a ring homomorphism. Next we shall show that for any essential left ideal \( K \) of \( Q \), \( N_Q(K) = 1 \), where \( N_Q \) is an extension of \( N \). Let \( K \) be an essential left ideal of \( Q \). Then by [1, Lemma 9.7], there exist orthogonal idempotents \( e_1, e_2, \ldots \), such that \( \sum_{n=1}^{\infty} Qe_n \subseteq K \subseteq Q \). We set \( J = \bigoplus_{n=1}^{\infty} e_nQ \). We claim that \( J \) is an essential right ideal of \( Q \). If \( J \cap eQ = 0 \), then \( J \oplus eQ \subseteq eQ \). On the other hand, since \( Q \) is a right self-injective regular ring, there exist orthogonal idempotents \( e', e' \) such that \( e'Q = eQ, eQ = e'Q \). We claim that \( \bigoplus_{n=1}^{\infty} Qe_n \cap Qe' = 0 \). Let \( re' = a_1e_{a_1} + \cdots + a_k e_{a_k} \) be an any element of \( \bigoplus_{n=1}^{\infty} Qe_n \cap Qe' \). Since \( e_n = e_{a_1}e_n, e_n = e_n e' \), we have that
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\[ e'e_n = e'(e'e_n) = 0. \] Now \( re'e_n = a_re_n \), so \( re' = 0 \), as claimed. Thus \( e' = 0 \), so that \( e = 0 \). This shows that \( \bigoplus_{n=1}^{\infty} e_nQ \) is an essential right ideal in \( Q \). Therefore by assumption, \( N_Q(J) = 1 \), which is concluded to that \( N_Q(K) = 1 \). In this case, the maximal left quotient ring of \( Q \) is embeded in \( R \) as a subring. Now in order to prove this Theorem, it suffices to show that \( R \) is a left essential extension of \( Q \). Let \( x \) be any element of \( R \). Since \( R \) is complete, for any positive number \( \varepsilon \), there exists an element \( x' \) of \( Q \) such that \( \overline{N}(x - x') < \varepsilon /2 \). We put \( x - x' = y \), where \( y \) is an element of \( R \). With respect to \( y \), there exists an element \( x_i \) of \( Q \) such that \( \overline{N}(y - x_i) < \varepsilon /2 \cdot 3^n \). Put \( y = x_i = y_1 \). In general, there exist elements \( x_i \) of \( Q \) such that \( \overline{N}(y_n - x_n) < \varepsilon /2 \cdot 3^n \). We put \( y_n - x_n = y_n \). Now we have that \( \overline{N}(y - \sum_{i=1}^{n} x_i) < \varepsilon /2 \cdot 3^n \). Thus \( \sum_{i=1}^{n} x_i \) is a Cauchy sequence. Since \( R \) is complete, \( \sum_{i=1}^{\infty} x_i = y \). Furthermore, \( \sum_{i=1}^{\infty} \overline{N}(x_i) < \overline{N}(y) + 2 \{ \overline{N}(y_1) + \overline{N}(y_2) + \cdots \} < \varepsilon /2 + \sum_{i=1}^{\infty} \varepsilon /3^i = \varepsilon /2 + \varepsilon /2 = \varepsilon \). Let \( I \) be a right ideal generated by \( x', x_1, \ldots \). Since \( \sum \overline{N}(x_i) < \varepsilon \), \( \overline{N}(I) < \varepsilon \). On the other hand, since \( \sum_{i=1}^{n} x_i = y \in R \), \( y \) is in \( \overline{I} \), where \( \overline{I} \) is the \( N_Q \)-closure of \( I \). Let \( e \) be an idempotent element of \( Q \) such that \( I \subseteq (1 - e)Q \), then \( eI = 0 \). And since \( I \oplus eQ \subseteq Q \), \( N_Q(I \oplus eQ) = N_Q(I) + N_Q(e) = 1 \). Hence \( N_Q(e) = 1 - \varepsilon \). Furthermore since \( \overline{l}(I) = \overline{l}(I) \), \( e \overline{I} = 0 \), where \( \overline{l}(I) \) means the left annihilator ideal. In particular, \( ey = 0 \). Now since \( y = x - x' \), \( ex = ex' \). Finally, we shall show that \( ex' \neq 0 \). Since \( ex = x - (1 - e)x \), \( \overline{N}(ex) > \overline{N}(x) - \overline{N}((1 - e)x) > \overline{N}(x) - \overline{N}(1 - e) > \overline{N}(x) - \varepsilon \). If we set \( \varepsilon = \overline{N}(x) > \varepsilon \), then \( N(ex) \neq 0 \), so \( ex = ex' = 0 \). Therefore \( \overline{R}x \cap Q = 0 \), hence \( \overline{R} \) is a left essential extention of \( Q \). This completes the proof.

**Remark.** Recently, H. Kambara [2] constructed the counter example of Roos conjecture (= Is every directly finite right self-injective ring necessary left self-injective?). He constructed the simple regular ring which is directly finite right self-injective and satisfies the assumption of above Theorem 1, but not left self-injective. By virtue of this example, Theorem 1 is not abstract non-sence.

If \( N \) is a pseudo-rank function, then we have the following theorem.

**Theorem 2.** Let \( R \) be a regular ring with pseudo-rank function \( N \). If for any essential right ideal \( I \) of \( R \), \( N(I) = 1 \) and \( ker(N) \) is a prime ideal, then \( N \) is an extreme point and is extended to the maximal right quotient ring \( Q \). In this case, \( Q/ker N_Q \) is the maximal right quotient ring of \( R/ker N \), where \( N_Q \) is the extension \( N \). Furthermore, the completion of \( R \) is isomorphic to the left maximal quotient ring of the right maximal quotient ring of \( R/ker(N) \).

Proof. Let \( \overline{R} \) be the \( N \)-completion of \( R \). Then since \( \overline{R} \) is a right and
left self-injective regular ring, $\mathcal{R}$ is injective as a right $R$-module. Now there exists a $R$-module homomorphism $f$ from $Q$ to $\mathcal{R}$ such that the following diagram commute,

$$
\begin{array}{c}
Q \xrightarrow{f} \mathcal{R} \\
\uparrow \\
R \xrightarrow{} R/\ker(N)
\end{array}
$$

In this case, by using the same proof of Theorem 1, we can see that $\mathcal{R}$ is a non-singular right $R$-module and $f$ is a ring homomorphism. We extend $N$ to $Q$ as follows, for any element $x$ of $Q$, we define that $N_Q(x)=N(f(x))$. Note that $\ker(f)=\ker N$ and $\ker N_Q \cap R = \ker N$. Clearly $R/\ker N$ has a rank function $\mathcal{N}$ which is induced by $N$. We note that $\mathcal{N}(K)=1$ for any essential right ideal $K$ of $R/\ker N$. Therefore we apply Theorem 1 to $R/\ker N$, that is $\mathcal{R}$ is the maximal left quotient ring of the maximal right quotient ring of $R/\ker N$. Next we claim that $Q/\ker N_Q$ is a right essential extension of $R/\ker N$. Given non-zero element $x$ of $Q/\ker N_Q$, since $x$ is in $Q$, there exists essential right ideal $I$ of $R$ such that $xI \subseteq R$. Assume that $xI \nsubseteq \ker N$, then $xI=0$. In this case, we have that $x=0$, which is a contradiction. Therefore $xI \nsubseteq \ker N$, that is for some non-zero element $t$ of $I$, $0 \neq xI \subseteq R/\ker N$. So $Q/\ker N$ is essential right extension of $R/\ker N$. Note that $\ker N_Q$ is a prime ideal of $Q$. Since $Q$ is a self-injective regular ring, there exists a central idempotent $e$ of $Q$ such that $\ker N_Q \subseteq eQ$. Thus $\ker N_Q \oplus (1-eQ)$ is an essential ideal of $Q$, so $N_Q(\ker N_Q \oplus (1-e)Q)=1$. This shows that $N_Q((1-e)Q)=1$. Now $N_Q(e)=0$, hence $\ker N_Q = eQ$. Therefore $Q/\ker N_Q = Q/eQ$ is a regular right self-injective ring. Consequently, $Q/\ker N_Q$ is a maximal right quotient ring of $R/\ker N$. Thus $Q/\ker N_Q$ is a prime regular self-injective ring with rank function. In this case, [1, Proposition 8.6] shows that $Q/\ker N_Q$ is a simple ring. Therefore $R/\ker N$ is also a simple ring. Now [1, Theorem 19.14] implies that $N$ is an extreme point of $P(R)$. Thus the proof is complete.

References


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