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# THE AVERAGE EDGE ORDER OF TRIANGULATIONS OF 3-MANIFOLDS

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## 1. Introduction

Let $K$ be a triangulation of compact 3-manifold $M$ with $V(K)$, $E(K)$, $F(K)$ and $T(K)$ the numbers of vertices, edges, faces, and tetrahedra in $K$, respectively. Note that we distinguish a triangulation from a cell decomposition into a union of 3-simplices, that is, such a cell decomposition is a triangulation when the intersection of any two simplices is actually a face of each of them. The order of an edge in $K$ is the number of triangles incident to that edge. The *average edge order* of $K$ is then $3F(K)/E(K)$, which we will denote $\mu(K)$. Feng Luo and Richard Stong showed in [2] that for a closed 3-manifold $M$, $\mu(K)$ being small implies that the topology of $M$ is fairly simple and restricts the triangulation $K$. This is the following theorem.

**Theorem 1** [2]. *Let $K$ be any triangulation of a closed connected 3-manifold $M$ without boundary. Then*

(a) $3 \leq \mu(K) < 6$, *equality holds if and only if $K$ is the triangulation of the boundary of a 4-simplex.*

(b) *For any $\varepsilon > 0$, there are triangulations $K_1$ and $K_2$ of $M$ such that $\mu(K_1) < 4.5 + \varepsilon$ and $\mu(K_2) > 6 - \varepsilon$.*

(c) *If $\mu(K) < 4.5$, then $K$ is a triangulation of $S^3$. There are an infinite number of distinct such triangulations, but for any constant $c < 4.5$ there are only finitely many triangulations $K$ with $\mu(K) \leq c$.*

(d) *If $\mu(K) = 4.5$, then $K$ is a triangulation of $S^3$, $S^2 \times S^1$, or $S^2 \tilde{\times} S^1$. Furthermore, in the last two cases, the triangulations can be described.*

The purpose of this note is to establish similar results for compact 3-manifolds with non-empty boundary. In fact we get the following theorem.

**Theorem 2.** *Let $K$ be any triangulation of a compact connected 3-manifold $M$ with non-empty boundary. Then*

(a) $2 \leq \mu(K) < 6$, *equality holds if and only if $K$ is the triangulation of one 3-simplex.*

(b) *For any rational number $r$ with $3<r<6$, there is a triangulation $K'$ of $M$ such that $\mu(K')=r$.*

(c) *If $\mu(K)<3$, then $K$ is a triangulation of $B^3$. There are an infinite number of distinct such triangulations, but for any constant $c<3$ there are only finitely many triangulations $K$ with $\mu(K)\leq c$.*

(d) *If $\mu(K)=3$, then $K$ is a triangulation of $B^3$, $D^2\times S^1$, or $D^2\tilde{\times}S^1$. Furthermore, in the last two cases, the triangulations can be described.*

We note that a result similar to Theorem 2(b) also holds for closed 3-manifolds.

**Proposition 3** (A refinement of Theorem 1(b)). *For any closed 3-manifold $M$ and for any rational number $r$ satisfying $4.5<r<6$, there is a triangulation $K'$ of $M$ such that $\mu(K')=r$.*

## 2. Proof of Theorem 2 (a)

This proof is similar to that of Theorem 1 (a) in [2]. Since each edge has order at least 2, we have $\mu(K)\geq 2$. Let $N=M\cup_{\partial M}M$ be the canonical double of $M$ and $K_N$ (resp. $K_\partial$) denote the cell decomposition of $N$ (resp. $\partial M$) induced by $K$. Though $K_N$ may not be a triangulation, the upper bound in Theorem 1 (a) is valid for any cell decomposition of a closed 3-manifold consisting of 3-simplices. Hence we have

$$\mu(K_N)=3\{2F(K)-F(K_\partial)\}/\{2E(K)-E(K_\partial)\}<6,$$

and therefore

$$3F(K)<6E(K)-3E(K_\partial)+(3/2)F(K_\partial).$$

As $\partial M$ is a closed 2-manifold, $3F(K_\partial)=2E(K_\partial)$. Thus

$$\mu(K)=3F(K)/E(K)<6-2E(K_\partial)/E(K)<6.$$

Finally, suppose $\mu(K)=2$. Then every edge of $K$ has order 2, and hence we see $K$ consists of only one 3-simplex. This completes the proof of Theorem 2 (a).

## 3. Proof of Theorem 2 (b)

Let $r=q/p$ be a rational number with $3<r<6$, where $p$ and $q$ are relatively prime integers. Choose an integer $\alpha$ such that $3\leq\alpha$ and $q/p<6\alpha/(\alpha+1)$. Then we can easily see that there is a triangulation $K$ of $M$ with three edges $e_1$, $e_2$, and $e_3$ which satisfy the following conditions:

(1) the order of $e_1$ is 2, thus $e_1$ lies on $\partial M$,
(2) the orders of $e_2$ and $e_3$ are $\alpha$ and $\alpha+1$, respectively, and
(3) $st(e_i)$ $(1\leq i\leq 3)$ have mutually disjoint interiors, where $st(e)$ is the star

neighborhood of $e$ in $K$.

Put $a=E(K)$ and $b=F(K)$. Let $K'$ be the steller subdivision of $K$ obtained by adding $l$, $m$, and $n$ vertices in the interior of $e_1$, $e_2$, and $e_3$, respectively. Then we have the following (see Figure 1).

$$\mu(K')=3\{b+3l+2\alpha m+2(\alpha+1)n\}/\{a+3l+(\alpha+1)m+(\alpha+2)n\}.$$

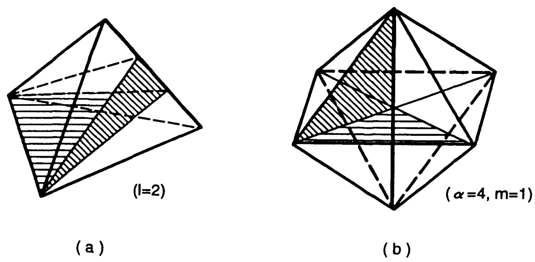


Fig. 1.

The proof is completed if we show the following lemma.

**Lemma 4.** *There exists a non-negative integers $l$, $m$, and $n$ satisfying $\mu(K')=q/p$, that is,*

(1) $3(3p-q)l+\{6\alpha p-(\alpha+1)q\}m+\{6(\alpha+1)p-(\alpha+2)q\}n=aq-3bp.$

Proof. Let $\Delta$ be the greatest common divisor $\Delta$ of $3(3p-q)$, $6\alpha p-(\alpha+1)q$. Since $p$ and $q$ are relatively prime, we have

$$\Delta=(3(3p-q),\ 6\alpha p-(\alpha+1)q,\ 6(\alpha+1)p-(\alpha+2)q)=(3p,q)=(3,q).$$

This shows that $\Delta$ divides $aq-3bp$, because $aq-3bp$ is a multiple of 3 when $(3,q)=3$. Hence we have an integral solution $[l_1,m_1,n_1]$ of Equation (1). Put $l_2=\{6\alpha p-(\alpha+1)q\}+\{6(\alpha+1)p-(\alpha+2)q\}$ and $m_2=n_2=-3(3p-q)$. By the definition of $p$, $q$, and $\alpha$, all of $l_2$, $m_2$, and $n_2$ are positive. Therefore, for a sufficiently large positive integer $k$, the triple $[l_1+kl_2, m_1+km_2, n_1+kn_2]$ is a solution of (1) consisting of non-negative integers. □

REMARK 5. Proposition 3 can be proved similarly by taking three edges $e_1$, $e_2$, and $e_3$ with the orders 3, $\alpha$, and $\alpha+1$, respectively.

## 4. Proof of Theorem 2 (c) and (d)

Let $K$ be a triangulation of a compact 3-manifold $M$ with non-empty boundary and $\mathscr{S}$ the 1-skeleton of the dual cell complex of $K$. For each vertex $v$ (resp.

edge $e$) of $\mathscr{S}$, let $\Delta(v)$ (resp. $f(e)$) be the 3-simplex (resp. face) of $K$ dual to $v$ (resp. $e$). For each subcomplex $\mathscr{C}$ of $\mathscr{S}$, let $D(\mathscr{C})$ be the cell complex obtained from the disjoint union of the 3-simplices $\{\Delta(v)\}_{v\in\mathscr{V}(\mathscr{C})}$ by identifying their faces along $\{f(e)\}_{e\in\mathscr{E}(\mathscr{C})}$. Here $\mathscr{V}(\mathscr{C})$ and $\mathscr{E}(\mathscr{C})$ denote the vertex set and the edge set of $\mathscr{C}$, respectively. Then we see $K=D(\mathscr{S})$. Though $D(\mathscr{C})$ may not be a triangulation, we can define the average edge order of $D(\mathscr{C})$ by $\mu(D(\mathscr{C}))=3F(D(\mathscr{C}))/E(D(\mathscr{C}))$. Put $\xi(D(\mathscr{C}))=F(D(\mathscr{C}))-E(D(\mathscr{C}))$. Then the following is obvious:

REMARK 6. $\mu(D(\mathscr{C}))$ is greater than (resp. equal to) 3, if and only if $\xi(D(\mathscr{C}))$ is positive (resp. 0).

In the following we choose an increasing sequence of the subcomplexes of $\mathscr{S}$,

$$\mathscr{C}_0\subset\mathscr{C}_1\subset\cdots\subset\mathscr{C}_n=\mathscr{S},$$

and estimate $\xi(D(\mathscr{C}_i))$ successively. To do this we need the following notation. Let $\mathscr{C}$ and $\mathscr{C}'$ be subcomplexes of $\mathscr{S}$ such that $\mathscr{E}(\mathscr{C})\cap\mathscr{E}(\mathscr{C}')=\emptyset$ and $\mathscr{V}(\mathscr{C})=\mathscr{V}(\mathscr{C}')$. Then $D(\mathscr{C}\cup\mathscr{C}')$ is obtained from $D(\mathscr{C})$ by identifying their faces along $\{f(e)\}_{e\in\mathscr{E}(\mathscr{C}')}$. So we say that $D(\mathscr{C}\cup\mathscr{C}')$ is obtained from $D(\mathscr{C})$ by gluing operations corresponding to the edges of $\mathscr{C}'$, and we denote $D(\mathscr{C})\xrightarrow{\mathscr{C}'}D(\mathscr{C}\cup\mathscr{C}')$. Suppose further that $\mathscr{E}(\mathscr{C}')=\{e\}$ where $e=[v_1,v_2]$ is an edge of $\mathscr{S}$. For $i=1,2$, let $f(e,v_i)$ be the face of the 3-simplex $\Delta(v_i)$ of $D(\mathscr{C})$ which projects to the face $f(e)$ of $D(\mathscr{C}\cup\mathscr{C}')$. Let $d$ be the number of the common edges of $f(e,v_1)$ and $f(e,v_2)$ in $D(\mathscr{C})$. Then $d=0,1,2$, or 3, and we say that the edge $e$ is of *type* $d$ with respect to $\mathscr{C}$. It should be noted that the intersection of $f(e_1,v_1)$ and $f(e_2,v_2)$ in $D(\mathscr{C})$ may contain isolated vertices beside $d$ edges when $d=0$ or 1.

**Lemma 7.** *Let $\mathscr{C}$ be a subcomplex of $\mathscr{S}$ and $e$ an edge of $\mathscr{S}$ such that $\mathscr{C}\cap e=\partial e$.*

(1) *If $e$ is of type $d$ with respect to $\mathscr{C}$, then*

$$\xi(D(\mathscr{C}\cup\{e\}))=\xi(D(\mathscr{C}))+2-d.$$

*In particular, $\xi(D(\mathscr{C}\cup\{e\}))<\xi(D(\mathscr{C}))$ if and only if $e$ is of type* 3.

(2) *If $\mathscr{C}$ is a tree, then $\xi(D(\mathscr{C}))=-2$.*

Proof. (1) This follows from the facts that each gluing operation decreases the number of faces by 1 and that the type $d$ gluing operation decreases the number of edges by $3-d$.

(2) Put $T=T(D(\mathscr{C}))$. Then $D(\mathscr{C})$ is obtained from $\amalg\ {}^{T}\Delta^3$ by gluing operations of type 0 corresponding to the edges of $\mathscr{C}$. Since $\mathscr{C}$ has $(T-1)$ edges, $\xi(D(\mathscr{C}))=-2T+2(T-1)=-2$ by using (1) of this lemma. $\square$

For an edge $[ab]$ in $D(\mathscr{C})$, let $N([ab], D(\mathscr{C}))$ (or simply $N[ab]$) be the star neighborhood of $[ab]$ in $D(\mathscr{C})$, and let $L([ab], \mathscr{C})$ (or simply $L[ab]$) be the 1-skeleton of the dual cell complex of $N[ab]$, *i.e.*, $L[ab]$ is the subcomplex of $\mathscr{C}$ consisting of those cells whose duals contain $[ab]$. By the construction of $\mathscr{C}$ and the fact that $K$ is a triangulation of a 3-manifold, we see $L[ab]$ is either a simple cycle or a simple edge path in $\mathscr{S}$. In the former case, we call $L[ab]$ an *edge cycle* (see Figure 2).

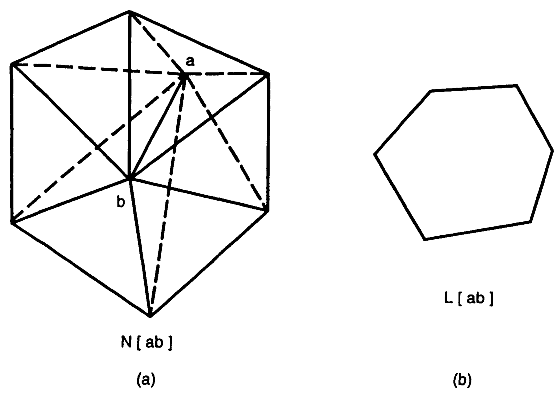


Fig. 2.

**Lemma 8.** *For two distinct edges* $[ab]$ *and* $[a'b']$ *of* $D(\mathscr{C})$, $L[ab] \cap L[a'b']$ *is an empty set, a vertex, or an edge.*

Proof. Suppose $L[ab] \cap L[a'b']$ is neither an empty set nor a vertex. Then it contains two vertices, say $v_1$ and $v_2$, of $\mathscr{S}$ and both $\Delta(v_1)$ and $\Delta(v_2)$ contain both edges $[ab]$ and $[a'b']$. If $[ab] \cap [a'b'] = \phi$, then we must have $\Delta(v_1) = [aba'b'] = \Delta(v_2)$ since $K$ is a triangulation, a contradiction. So we may assume $a' = a$ and $b' \neq b$. Then $\Delta(v_1) \cap \Delta(v_2)$ is the 2 simplex $[abb']$, and hence $v_1$ and $v_2$ span an edge $[v_1 v_2]$ in $L[ab] \cap L[a'b']$. Suppose $L[ab] \cap L[a'b']$ has another vertex $v_3$, then the corresponding 3-simplex $\Delta(v_3)$ also has $[abb']$ as a face. This contradicts the fact that $K$ is a triangulation of a 3-manifold. Thus $L[ab] \cap L[a'b']$ is an edge $[v_1 v_2]$. □

Let $e$ be an edge of $\mathscr{C}$, and put $[abc] = f(e)$. Then $\mathscr{L}(e, \mathscr{C})$ (or simply $\mathscr{L}(e)$) denotes the subcomplex $L[ab] \cup L[bc] \cup L[ca]$ of $\mathscr{C}$ (see Figure 3). Note that the intersection of any two of the three summands of $\mathscr{L}(e)$ is equal to $e$ by Lemma 8.

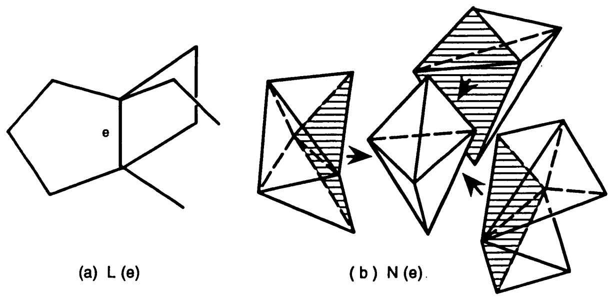


Fig. 3.

**Lemma 9.** *If $e\in\mathscr{C}$ is an edge of type $d$ with respect to $\mathscr{C}-e$, then*

$$\xi(D(\mathscr{L}(e,\mathscr{C})))=d-2.$$

Proof. Let $\mathscr{M}$ be a maximal tree of $D(\mathscr{L}(e,\mathscr{C}))$. Since $D(\mathscr{L}(e))$ consists of $d$ edge cycles and $3-d$ edge paths, $\mathscr{L}(e)/\mathscr{M}$ is the set of $d$ edges of type 1 with respect to $\mathscr{M}$ (see Figure 4). Therefore by Lemma 7, we have

$$\xi(D(\mathscr{L}(e)))=\xi(D(\mathscr{M}))+d=d-2.$$

□

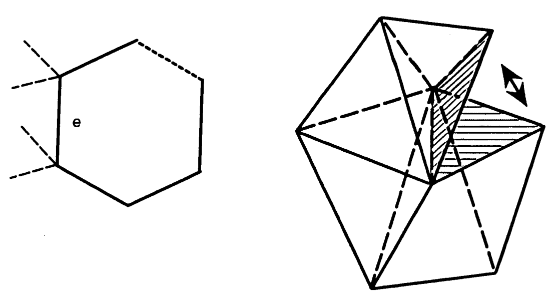


Fig. 4.

Now we choose an ordering $e_1,\cdots,e_n$ of the edges of $\mathscr{S}$, and let $\mathscr{C}_0,\cdots,\mathscr{C}_n$ be the increasing sequence of the subcomplexes of $\mathscr{S}$ defined by $\mathscr{C}_0=\mathscr{V}(\mathscr{S})$, and $\mathscr{C}_i=\mathscr{C}_{i-1}\cup\{e_i\}$ $(1\leq i\leq n)$. Then we obtain the following sequence of cell complexes:

$$K_0=D(\mathscr{C}_0)\xrightarrow{e_1}D(\mathscr{C}_1)\xrightarrow{e_2}D(\mathscr{C}_2)\xrightarrow{e_3}\cdots\xrightarrow{e_n}D(\mathscr{C}_n)=K.$$

Here $K_0$ is the disjoint union of the 3-simplices of $K$. The edge $e_i$ is said to be of type $d$ in this sequence if $e_i$ is of type $d$ with respect to $\mathscr{C}_{i-1}$.

The following is the key observation for the proof of Theorem 2 (c) and (d).

CLAIM 10. (a) *If* $\mu(K)\leq 3$, *then there are no edges of type* 3.
(b) *If* $\mu(K)<3$, *then there are no edges of type* 2 *nor those of type* 3.

We will postpone the proof of Claim 10 until the end of this section.

Now we prove Theorem 2 (c) by assuming Clain 10. Suppose $\mu(K)<3$, and put $T=T(K)$. Let $\mathscr{M}$ be a maximal tree of $\mathscr{S}$. Then $K$ is obtained as the final term of the sequence

$$K_0=D(\mathscr{C}_0)\xrightarrow{\mathscr{M}} D(\mathscr{M})\xrightarrow{\mathscr{S}\backslash\mathscr{M}} D(\mathscr{S})=K.$$

By Lemma 7, $\xi(D(\mathscr{M}))=-2$. By Clain 10, any edge of $\mathscr{S}\backslash\mathscr{M}$ is of type 0 or 1. Since we have $\xi(K)<0$ by Remark 6, we see, by Lemma 7, either $\mathscr{S}\backslash\mathscr{M}$ is empty or consists of only one edge of type 1. If $\mathscr{S}\backslash\mathscr{M}$ is empty, then $K=D(\mathscr{M})$, *i.e.*, it is a boundary connected sum $\natural^T\Delta^3$ of $T$ 3-simplices along their faces; in particular, $M=B^3$. If $\mathscr{S}\backslash\mathscr{M}$ consists of only one edge of type 1, then $K$ is obtained from $D(\mathscr{M})=\natural^T\Delta^3$ by applying a type 1 gluing operation exactly once. Since $D(\mathscr{M})$ is a triangulation of $B^3$, two fases in $D(\mathscr{M})$ which will be identified by the next type 1 gluing operation hold only one edge in common: so we see $D(\mathscr{S})$ is a triangulation of $B^3$. Thus we have obtained the first half of Theorem 2 (c). To see the latter half, note that $\mu(K)=3\{3T+1-j\}/\{3T+3-2j\}$, where $j$ is the number of the edges in $\mathscr{S}\backslash\mathscr{M}$. Since $j=0$ or 1, we see $\mu(K)<3$ for any $T$. Hence there are infinitely many $K$ with $\mu(K)<3$. Let $c$ be a constant less than 3 and suppose $\mu(K)=3\{3T+1-j\}/\{3T+3-2j\}<c$. Then we get $T<\{3(c-1)+(3-2c)j\}/\{3(3-c)\}$. Thus the number of $K$ with $\mu(K)<c$ is finite. This completes the proof of Theorem 2 (c).

Next we prove Theorem 2(d) by assuming Claim 10. Suppose $\mu(K)=3$, and put $T=T(K)$.

**Case 1.** None of $e_i$ $(1\leq i\leq n)$ is of type 2. Consider the sequence in the proof of Theorem 2(c);

$$K_0=D(\mathscr{C}_0)\xrightarrow{\mathscr{M}} D(\mathscr{M})\xrightarrow{\mathscr{S}\backslash\mathscr{M}} D(\mathscr{S})=K.$$

Then by Lemma 7, $\xi(D(\mathscr{M}))=-2$. By Lemma 7 and the assumption, $\mathscr{S}\backslash\mathscr{M}$ consists of either two edges of type 1 or only one edge of type 0.

In the former case, $K$ can be obtained from $\natural^T\Delta^3$ by applying type 1 gluing operations twice. Let $\{e_1,e_2\}$ be the set of two edge of type 1 in $\mathscr{S}\backslash\mathscr{M}$ and put $[v_{11},v_{12}]=e_1$ and $[v_{21},v_{22}]=e_2$. As we mentioned before, since $D(\mathscr{M})$ is a triangulation of $B^3,f(e_1,v_{11})\cap f(e_1,v_{12})$ consists of only one edge. Thus $D(\mathscr{M}\cup\{e_1\})$

is a cell-decomposition of $B^3$. If $f(e_2,v_{21})\cap f(e_2,v_{22})$ consists of an edge and an isolated vertex $v$, then the point $v$ in $D(\mathscr{M}\cup\{e_1,e_2\})$ cannot have an Euclidean neighborhood, a contradiction. Therefore $f(e_2,v_{21})\cap f(e_2,v_{22})$ consists of only one edge. Hence $D(\mathscr{S})=D(\mathscr{M}\cup D\{e_1,e_2\})$ is a triangulation of $B^3$.

In the latter case, $K$ can be obtained from $\natural^T\Delta^3$ by applying a type 0 gluing operation once. Since $M=D(\mathscr{S})$ is a 3-manifold, the faces in $\partial D(\mathscr{M})$ to be identified by the last type 0 gluing operation are disjoint. Thus $M$ is $D^2\times S^1$ or $D^2\tilde{\times}S^1$ according to whether the operation is compatible with the orientation or not.

**Case 2.** Some $e_i$ is of type 2. Then we reorder the edge of $\mathscr{S}$ so that $\{e_1,\cdots,e_l\}=\mathscr{L}(e_i)$. We call a subcomplex $\mathscr{X}$ of $\mathscr{S}$ an *extended maximal tree containing* $\mathscr{C}$ if $\mathscr{X}$ is the inverse image in $\mathscr{S}$ of a maximal tree of the graph $\mathscr{S}/\mathscr{C}$, where $\mathscr{S}/\mathscr{C}$ denotes the graph obtained from $\mathscr{S}$ by collapsing each connected component of $\mathscr{C}$ to a point. Let $\mathscr{X}$ be an extended maximal tree containing $\mathscr{L}(e_i)$, and consider the following sequence;

$$K_0=D(\mathscr{C}_0)\xrightarrow{\mathscr{L}(e_i)}D(\mathscr{C}_l)\xrightarrow{\mathscr{X}\backslash\mathscr{L}(e_i)}D(\mathscr{X})\xrightarrow{\mathscr{S}\backslash\mathscr{X}}D(\mathscr{S})=K,$$

where $\mathscr{C}_l=\mathscr{L}(e_i)\cup\mathscr{C}_0$. Put $t=T(K)-T(\mathscr{L}(e_i))$, then

$$D(\mathscr{C}_l)=D(\mathscr{L}(e_i))\amalg(\coprod^t\Delta^3).$$

By Lemma 9, we have $\xi(D(\mathscr{L}(e_i)))=0$. Therefore

$$\xi(D(\mathscr{C}_l))=\xi(D(\mathscr{L}(e_i)))+t\xi(\Delta^3)=-2t.$$

Since $\mathscr{X}\backslash\mathscr{L}(e_i)$ consists of $t$ edges of type 0, we have

$$\xi(D(\mathscr{X}))=\xi(D(\mathscr{L}(e_i)))+2t=0.$$

Since $\xi(D(\mathscr{S}))=0$ by the assumption and Remark 6, we see, by Lemma 7 and Claim 10, that $\mathscr{S}\backslash\mathscr{X}$ is empty or consists of only edges of type 2. Let $m$ be the number of edges of $\mathscr{S}\backslash\mathscr{X}$.

If $m=0$, *i.e.* $\mathscr{S}\backslash\mathscr{X}=\emptyset$, then $K=D(\mathscr{X})=D(\mathscr{L}(e_i))\natural(\natural\Delta^3)$ and $M=B^3$.

Suppose $m=1$, choose an edge $e'$ of $\mathscr{S}\backslash\mathscr{X}$, and put $[abc]=f(e_i)$ and $[a'b'c']=f(e')$. Since $e_i$ is of type 2, we may assume $L[ab]$ and $L[ac]$ are edge cycles and $L[bc]$ is an edge path. Similarly, we may assume $L[a'b']$ and $L[a'c']$ are edge cycles and $L[b'c']$ is an edge path. It should be noted that $L[ab]$ and $L[ac]$ are considered in $\mathscr{X}$, whereas $L[a'b']$ and $L[a'c']$ are considered in $\mathscr{X}\cup\{e'\}$. Let $\mathscr{L}_0$ be the union of these four edge cycles. Then we have the following sublemma which is proved later.

**Sublemma 11.** *$\mathscr{L}_0$ is as illustrated in Figure* 5.

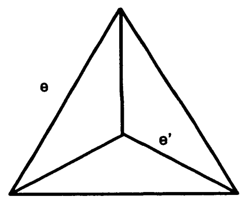


Fig. 5.

By this sublemma, we have $K=D(\mathscr{L}_0)\natural(\natural\Delta^3)$ and $M$ is $B^3$.

Suppose $m\geq 2$, *i.e.*, $\mathscr{S}\backslash\mathscr{X}$ has another edge $e''$ of type 2. Then $e''$ has its endpoints in $\mathscr{L}_0$ and $e''\notin\mathscr{L}_0$. But this contradicts Sublemma 11 and the fact that $K$ is a triangulation. Thus $m\leq 1$ and we have proved the part (d) of Theorem 2. □

Proof of Sublemma 11. Put $J=(L[a'b']\cup L[a'c'])\backslash\{e'\}$. Then $J$ is a simple cycle in $\mathscr{X}$ by Lemma 8. By the construction of $\mathscr{X}$, the cycle $J$ is contained in $\mathscr{L}(e_i)$. So $J$ is contained in $L[ab]\cup L[ac]$. Hence the endpoints of $e'$ lie in $L[ab]\cup L[ac]$. Suppose $\partial e'$ lies in $L[ab]$. Let $\overline{\partial e'}$ be the sub-edge path in $L[ab]\backslash\{e\}$ cut off by $\partial e'$ as in Figure 6.

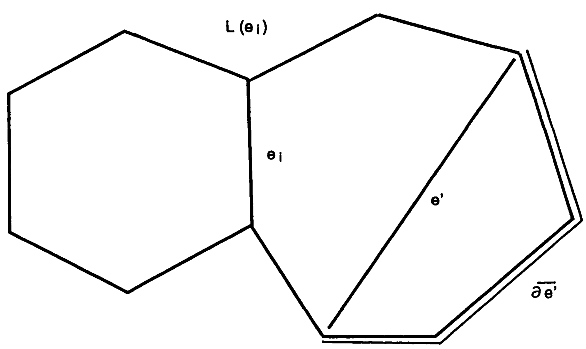


Fig. 6.

Then one of $L[a'b']$ and $L[a'c']$, say $L[a'b']$, is $\overline{\partial e'}$. Since $\overline{\partial e'}=L[ab]\cap L[a'b']$, $\overline{\partial e'}$ consists of one edge by Lemma 8. So it follows that $e'=\overline{\partial e'}$ since $K$ is a triangulation; this contradicts the fact that $e'$ is not contained in $\mathscr{X}$. Hence $\partial e'$ does not lie in $L[ab]$. Similarly $\partial e'$ does not lie in $L[ac]$. Hence $e'$ has its endpoints in each of $L[ab]$ and $L[ac]$. By Lemma 8, any two of the edge cycles $L[ab]$, $L[ac]$, $L[a'b']$, and $L[a'c']$ intersects in an edge. Therefore, we see that the union $\mathscr{L}_0$ of these four cycles is as illustrated in Figure 5. □

Proof of Claim 10 (a). Suppose $\mu(K)\leq 3$ and some edge $e_i$ is of type 3. Put

$\mathscr{A}=\mathscr{L}(e_i,\mathscr{C}_{i-1})$. Then we reorder the edge of $\mathscr{S}$ so that $\{e_1,\cdots,e_l\}=\mathscr{A}$, and consider the following sequence;

$$K_0=D(\mathscr{C}_0)\xrightarrow{\mathscr{A}}D(\mathscr{C}_l)\xrightarrow{e_{l+1}}D(\mathscr{C}_{l+1})\to\cdots\xrightarrow{e_n}D(\mathscr{C}_n)=K.$$

We need the following sublemma which is proved later.

**Sublemma 12.** *By further reordering* $\mathscr{S}\backslash\mathscr{A}=\{e_{l+1},\cdots,e_n\}$, *we may assume no* $e_i(l+1\leq i\leq n)$ *is of type* 3.

Put $t=T(K)-T(D(\mathscr{A}))$. Then $D(\mathscr{C}_l)=D(\mathscr{A})\amalg(\amalg^t\Delta^3)$. By Lemma 9,

$$\xi(D(\mathscr{C}_l))=\xi(D(\mathscr{A}))+t\xi(\Delta^3)=1-2t.$$

By Lemma 7 and Sublemma 12, the sequence $\{\xi(D(\mathscr{C}_i))\}_{l\leq i\leq n}$ is not decreasing. Since $D(\mathscr{C}_l)$ has $(t+1)$ components, there are $t$ edges of type 0 among $\{e_{l+1},\cdots,e_n\}$. Hence, by using Lemma 7, we see

$$\xi(K)\geq\xi(D_l)+2t=1>0.$$

Then $\mu(K)>3$ by Remark 6, contradiction. Thus Clain 10 (a) is proved. □

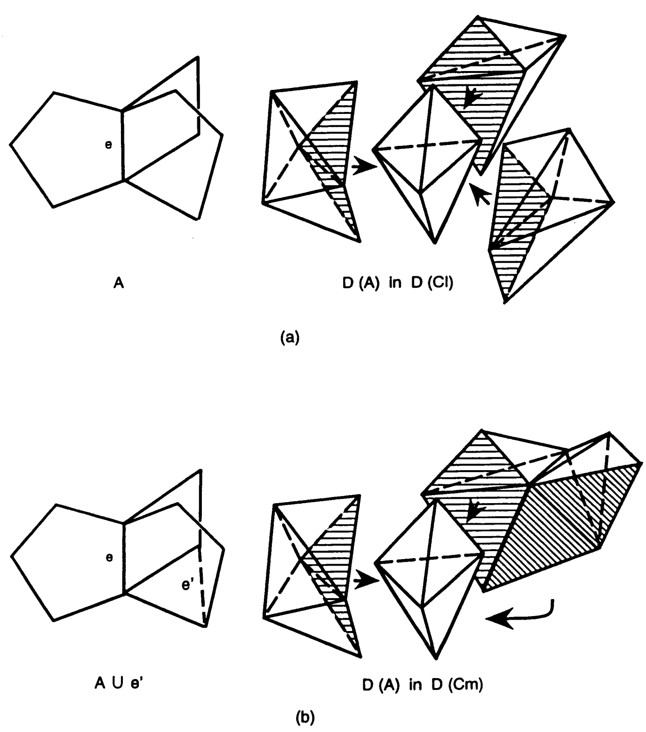


Fig. 7.

Proof of Sublemma 12. Suppose there are edge of type 3, say $e_m$ coming first, in the subsequences $\mathscr{S}\backslash\mathscr{A}$ and put $\mathscr{B}=\{e_{l+1},\cdots,e_m\}$. Here we regard $\mathscr{B}$ as an ordered set. In the following, we show that we can reorder $\mathscr{B}$ so that there are no edges of type 3 in the new ordered set $\mathscr{B}$.

Let $intD(\mathscr{A})$ be $D(\mathscr{A})\backslash\partial D(\mathscr{A})$ in $D(\mathscr{C}_l)$, and we can regard this subcomplex in $D(\mathscr{C}_l)$ as that of in $D(\mathscr{C}_m)$. In $D(\mathscr{C}_m)$, we define $\partial D(\mathscr{A})$ by $D(\mathscr{A})\backslash intD(\mathscr{A})$. Note that $D(\mathscr{A})$ in $D(\mathscr{C}_m)$ may be different from $D(\mathscr{A})$ in $D(\mathscr{C}_l)$ (see Figure 7). Thus $\partial D(\mathscr{A})$ in $D(\mathscr{C}_m)$ may be different from $\partial D(\mathscr{A})$ in $D(\mathscr{C}_l)$, but it satisfies the following property:

(*) for any two faces $f$ and $f'$, there is a sequence of faces $f=f_0,f_1,\cdots,f_t=f'$ of the complex such that $f_i$ and $f_{i+1}$ are adjacent.

Let $D$ be the connected component of $D(\mathscr{C}_m)$ containing $f(e_m)$. Let $f$ and $f'$ be any faces in $D\backslash intD(\mathscr{A})$, there is sequence of faces $f=f_0,f_1,\cdots,f_t=f'$ of $D$ such that $f_i$ and $f_{i+1}$ are adjacent. Suppose there is a subsequence $f_i,f_{i+1},\cdots,f_j$ whose elements are in $intD(\mathscr{A})$. Since $\partial D(\mathscr{A})$ satisfies the property (*), there is a

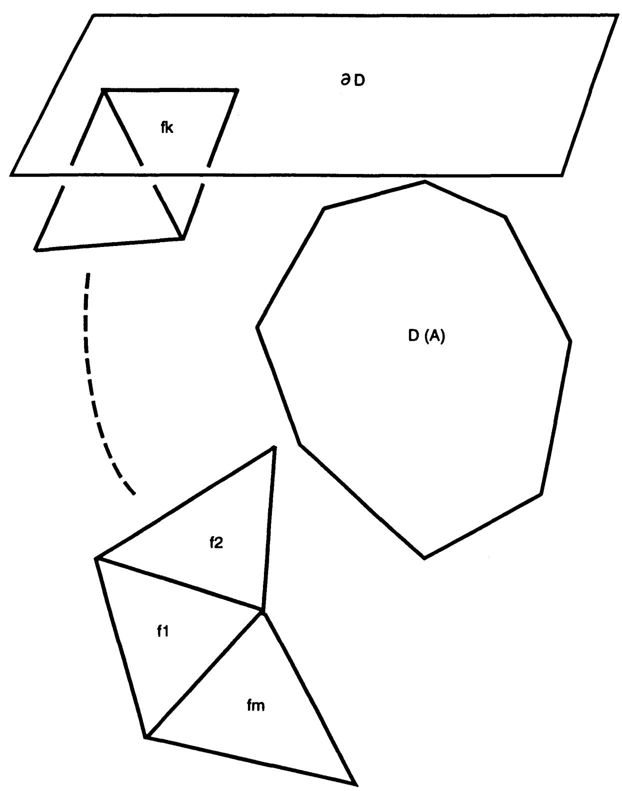


Fig. 8.

subsequence $f'_i, f'_{i+1}, \cdots, f'_r$ in $\partial D(\mathscr{A})$ such that any of $(f_{i-1}, f'_i)$, $(f'_s, f'_{s+1})$ $(i \leq s \leq r-1)$, and $(f'_k, f_{j+1})$ is a pair of adjacent faces. Thus $D \backslash intD(\mathscr{A})$ satisfies property $(*)$. So we can take a sequence of faces $f_0 = f(e_m), f_1, \cdots, f_t$ in $D \backslash intD(\mathscr{A})$ satisfying the following conditions:

(1) $f_i$ and $f_{i+1}$ are adjacent, and
(2) $f_t$ lies on $\partial D$.

Let $f_k$ be the first face in this sequence which has edges in $\partial \mathscr{D}$. These conditions are illustrated in Figure 8.

Now we define $\mathscr{B}^{(i)} = \{e^{(i)}_{l+1}, \cdots, e^{(i)}_m\}$ $(0 \leq i \leq k-1)$ inductively as follows. We regard $\mathscr{B}$ as $\mathscr{B}^{(0)} = \{e^{(0)}_{l+1}, \cdots, e^{(0)}_m\}$ and suppose we have constructed $\mathscr{B}^{(i)}$. Let $e^{(i)}_{m_{i+1}}$ be the dual of $f_{i+1}$, *i.e.* $f(e^{(i)}_{m_{i+1}}) = f_{i+1}$. Note that $e^{(i)}_{m_{i+1}} \in \mathscr{B}$ because $f(m_{i+1}) \in D \backslash intD(\mathscr{A})$. To make a new sequence $\mathscr{B}^{(i+1)}$, we shift $e^{(i)}_{m_{i+1}}$ to the last of $\mathscr{B}^{(i)}$, that is, if

$$\mathscr{B}^{(i)} = \{e^{(i)}_{l+1}, \cdots, e^{(i)}_{m_{i+1}}, \cdots, e^{(i)}_{m_i}\}$$

then we set

$$\mathscr{B}^{(i+1)} = \{e^{(i)}_{l+1}, \cdots, \check{e}^{(i)}_{m_{i+1}}, \cdots, e^{(i)}_{m_i}, e^{(i)}_{m_{i+1}}\}.$$

Strictly speaking, we deifne $e^{(i+1)}_j$ as following:

$$e^{(i+1)}_j = \begin{cases} e^{(i)}_j & (l+1 \leq j \leq m_{i+1} - 1) \\ e^{(i)}_{j+1} & (m_{i+1} \leq j \leq m-1) \\ e^{(i)}_{m_{i+1}} & (j = m). \end{cases}$$

Now we prove the following by induction of $i$: $e^{(i)}_j$ $(l+1 \leq j \leq m-1)$ is not of type 3 $(1 \leq i \leq k)$. Since $e^{(i)}_j$ $(l+1 \leq j \leq m-1)$ is not of type 3 in $\mathscr{A} \cup \mathscr{B}^{(i)}$, $e^{(i+1)}_j$ $(l+1 \leq j \leq m-2)$ is not of type 3 in $\mathscr{A} \cup \mathscr{B}^{(i+1)}$. By the condition (1), $e^{(i)}_m$ and $e^{(i)}_{m_{i+1}}$ belong to the same edge cycle. Thus we see $e^{(i+1)}_{m-1}$ is not of type 3 in $\mathscr{A} \cup \mathscr{B}^{(i+1)}$. By the condition (2), $\mathscr{L}(e^{(k)}_m, \mathscr{A} \cup \mathscr{B})$ does not consist of three edge cycles containing $e_m$. Thus $e^{(k)}_m$ is not of type 3 in $\mathscr{A} \cup \mathscr{B}^{(k)}$. Therefore we get a subsequence $\mathscr{A} \cup \mathscr{B}^{(k)}$ without edges of type 3 in $\mathscr{B}^{(k)}$.

If there are edges of type 3 in $e_{m+1}, \cdots, e_n$, we proceed in a similar fashion, changing the orders again. After making finitely many changes, we can get a sequence without edges of type 3 in $e_{l+1}, \cdots, e_n$. □

Proof of Claim 10 (b). Suppose $\mu(K) < 3$ and some edge $e_i$ is of type 2. As we observed before in the proof of Theorem 2(d),

$$\xi(D(\mathscr{X})) = \xi(D(\mathscr{L}(e_i))) + 2t = 0.$$

By Claim 10 (a) and Lemma 7, we have

$$\xi(K) = \xi(D(\mathscr{S})) \geq \xi(D(\mathscr{X})) = 0.$$

This contradicts $\mu(K)<3$. □

## 5. Concluding Remark

Prof. S. Kojima kindly suggested to the author that the more natural generalization of the average edge order of closed manifolds to that of manifolds with boundary is to count simplices on the boundary with weight $1/2$, *i.e.* $E(K)=E_i(K)+E_\partial(K)/2$ and $F(K)=F_i(K)+F_\partial(K)/2$, where $E_i$ (resp. $F_i$) is the number of edges (resp. faces) in $K\backslash\partial K$, and $E_\partial$ (resp. $F_\partial$) is the number of edges (resp. faces) on $\partial K$. This suggestion is based upon the fact that the average edge order is a geometric interpretation in terms of a global average of the curvature. The author conjectures a result similar to Theorem 2 also holds in this case.

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