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SIDE-EFFECTS OF MEASURE REPRESENTATIONS ON AXIOM (D)

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The two most important examples of axiomatic potential theory are the harmonic space $(\boldsymbol{R}^n, \mathcal{H}^*_\Delta)$ of the solutions of the Laplace equation on open subsets of $\boldsymbol{R}^n$ and the space $(\boldsymbol{R}^{n+1}, \mathcal{H}^*_\Omega)$ of solutions of the heat equation on open subsets of $\boldsymbol{R}^{n+1}$. The sheaves of solutions of these two partial differential equations—and of a large class of other elliptic and parabolic differential equations—behave similarly in many respects, which led to the development of a common theory—the theory of harmonic spaces. In this paper we want to focus on two properties with respect to which the Laplace operator Δ and the heat operator $\Omega=\Delta-\dfrac{\partial}{\partial t}$ differ:

1) The Laplace operator Δ satisfies the *bounded energy principle*

$$\text{(E)}\quad \int\int G_\Delta(x, y)\, \mu(dx)\, \mu(dy) \geq 0$$

for every signed measure μ such that the potential

$$x \mapsto \int G_\Delta(x, y)\, |\mu|(dy)$$

is bounded, where G_Δ denotes the Newtonian kernel; on the contrary the heat operator Ω and its kernel G_Ω do not satisfy (E).

2) The Laplace operator Δ is a *fine local operator,* in contrast with Ω which is not (for the definition of fine local operators and the proof of this statement see §1). It turns out that it is Axiom (D), introduced in axiomatic potential theory by M. Brelot, which is responsible for the "fine local" property as well as for the bounded energy principle. The purpose of this paper is to formulate and prove this statement within the framework of the theory of harmonic spaces in the sense of [2] or [5].

In the theory of harmonic spaces the starting-point is a sheaf of functions—called harmonic or hyperharmonic functions—without the intervention of a defining operator. In some situations however it is useful or even necessary to have such a defining operator at one's disposal—a substitute for the differential operators on $\boldsymbol{R}^n$. This led F.-Y. Maeda to introduce the concept of measure representations into the theory of harmonic spaces (see [11] and the bibliography cited there).

By definition a *measure representation* of a harmonic space $(X, \mathcal{H}^*)$ is a sheaf homomorphism $\sigma=(\sigma_U)$ of the sheaf $\mathcal{R}=(\mathcal{R}(U))$ of local differences of continuous superharmonic functions into the sheaf $\mathcal{M}=(\mathcal{M}(U))$ of signed Radon measures determining completely the harmonic structure:

$$\sigma_U(f)\geq 0 \Leftrightarrow f \text{ is superharmonic} \qquad (f\in\mathcal{R}(U),\ U \text{ open}).$$

Especially the kernel of σ consists exactly of the harmonic functions.
In [19] it was shown that every harmonic space $(X, \mathcal{H}^*)$ with a countable base admits a measure representation. We recall the construction of σ for the case that $(X, \mathcal{H}^*)$ is even a $\mathfrak{P}$-or strong harmonic space:

Let $\bar{\mu}$ be a (positive) Radon measure on X such that

$$0<\int p\,d\bar{\mu}<\infty$$

for every $p\in\mathcal{P}_0(X)$, the cone of continuous potentials having compact support, $p\neq 0$, and let g be a strictly positive locally bounded Borel measurable function. Then $\bar{\mu}$ and g determine σ as follows: Let $f\in\mathcal{R}(U)$, U open, and let V open and relatively compact, $\bar{V}\subset U$. By definition of $\mathcal{R}$ and the extension theorem there exist $u, v\in\mathcal{P}_0(X)$ such that

$$f=u-v \text{ on } V.$$

For every $\varphi\in\mathcal{C}_+(X)$ with $\mathrm{Supp}(\varphi)\subset V$ we set

$$\sigma(f)(\varphi):=\int((g\varphi)\odot u-(g\varphi)\odot v)\,d\bar{\mu}$$

(where $\odot$ denotes the specific multiplication). Then $\sigma(f)=\sigma_U(f)$ is a well-defined signed measure on U and

$$\sigma: f\mapsto\sigma(f)$$

is a measure representation for X.
It is possible and reasonable to allow for $\bar{\mu}$ not only Radon measures but also *strictly positive H-integrals* (see [4]), i.e. additive, positive-homogeneous, increasing functionals

$$\bar{\mu}: \mathcal{S}_+(X)\to\bar{\boldsymbol{R}}_+$$

defined on the cone $\mathcal{S}_+(X)$ of all positive superharmonic functions on X, which are continuous in order from below on $\mathcal{S}_+(X)$ and finite and strictly positive on $\mathcal{P}_0(X)\setminus\{0\}$. For $f=p-p'$, $p, p'\in\mathcal{S}_+(X)$, we use the symbols

$$\int f\, d\bar{\mu} \text{ and } \bar{\mu}(f) := \bar{\mu}(p) - \bar{\mu}(p')$$

interchangeably, provided that $\bar{\mu}(p') < \infty$.

At the Oberwolfach meeting on potential theory in 1984 I. Netuka suggested to characterize certain special properties of harmonic spaces by suitable additional properties of measure representations. In the first two sections of this paper relations between Axiom (D), the fine local property of σ and the bounded energy principle are discussed. Finally in §3 it is shown that Doob's convergence axiom implies strong continuity properties of σ. In particular the following result is proved for a harmonic space having a Green function G: If $p_n = G^{\mu_n}$ is a sequence of potentials converging pointwise outside a semipolar set to a potential $p = G^{\mu}$, then the corresponding sequence of measures (μ_n) converges vaguely to μ (A similar result has been proved by R.-M. Hervé for Brelot spaces).

The notations used are those of Constantinescu-Cornea's book [5]. Especially $\mathcal{S}_+(U)$ and $\mathcal{P}(U)$ denote the cones of positive superharmonic functions and potentials on the open set U respectively. In contrast with [5] however we denote by $\mathcal{P}_c(U)$, $\mathcal{P}_0(U)$, $\mathcal{P}_b(U)$, $\mathcal{P}_{00}(U)$ respectively the subcones of all continuous, continuous with compact support, bounded, respectively of all potentials with compact support. X will always stand for a locally compact space with a *countable base* of its topology; $C(X)$, $\mathfrak{K}(X)$ denote the spaces of all continuous, respectively continuous with compact support functions f on X.

1. Axiom (D) and the fine local property

Let $(X, \mathcal{H}^*)$ be a $\mathfrak{P}$-harmonic space such that $1 \in \mathcal{H}^*_+(X)$. Let $\bar{\mu}$ be a Radon measure on X such that

$$0 < \int p\, d\bar{\mu} < \infty \quad \text{for every} \quad p \in \mathcal{P}_0(X) \setminus \{0\},$$

(or, more generally, a strictly positive H-integral) and $g: X \to \boldsymbol{R}$ a locally bounded strictly positive Borel function. To every bounded potential p on X a Radon measure $\sigma(p)$ is assigned as follows:

$$\sigma(p)(\varphi) := \int (g\varphi) \odot p\, d\bar{\mu}, \quad \varphi \in \mathfrak{K}_+(X).$$

The aim of this section is to characterize Axiom (D) via a fine local property of σ.

DEFINITION. σ is called a fine local operator iff the following condition holds: For any two potentials $p, p' \in \mathcal{P}_b(X)$ which coincide on a finely open set V the representation measures $\sigma(p)$ and $\sigma(p')$ coincide on V.

A typical example, where σ is *not* a fine local operator, is the following

EXAMPLE. Let $X=]-1,1[$, $\bar{\mu}$ the Lebesgue measure on X and $g=\mathbf{1}$. For every open interval $U\subset X$ let $\mathcal{H}(U)$ denote the vector space of all continuous functions $f\colon U\to\boldsymbol{R}$ such that

1) f is locally affine on $U\setminus\{0\}$

2) f is constant on $U\cap]-1,1[$, provided that $0\in U$

([5], Exercise 3.1.7). Then X is a harmonic space.

The potential $p:=\mathbf{1}_{]0,1[}(x)\,(\mathbf{1}-x)$ coincides on the finely open set $]-1,0[$ with $\mathbf{0}$, but the restriction of the measure

$$\sigma(p)=1/2\,\varepsilon_0$$

to $]-1,0]$ is not the zero measure.

The main result of this section is the following theorem:

Theorem. *σ is a fine local operator iff Axiom (D) is satisfied.*

The proof is divided into several lemmas. First of all the same argument as used in the example shows that X cannot have non-trivial absorbent sets provided that σ is a fine local operator. More generally in that case it will be shown that X is an elliptic space.

Lemma 1. *Let U be an open subset of X, A a non-trivial absorbent set of the harmonic space $(U,\mathcal{H}^*_{|U})$ and let V be an open, non-empty and relatively compact subset of $\partial A\cap U$. Then there exists a bounded potential $p\in\mathcal{P}(U)$ with compact non-empty support $S(p)$ contained in $\bar{V}$ and vanishing on A.*

Proof. Let f be continuous on $\partial(U\setminus A)$, strictly positive on V and vanishing outside V. Since A is an absorbent set the function

$$p':=\begin{cases}\mathbf{0} & \text{on } A\\ H_f^{U\setminus A} & \text{on } U\setminus A\end{cases}$$

is superharmonic on U. The points of $\partial A\cap U$ are regular boundary points of $U\setminus A$, hence p' is harmonic in small neighbourhoods of points $x\in\partial A\cap U\setminus\bar{V}$. Let p be the potential part of $p'\in\mathcal{S}(U)$. By construction p is bounded, $\phi\neq S(p)\subset\bar{V}$ and p vanishes on A.

Lemma 2. *Let U, A, V as in Lemma 1. Then there exists a bounded potential $\tilde{p}\in\mathcal{P}(X)$ such that $\phi\neq S(\tilde{p})\subset\bar{V}$ and $\tilde{p}=\hat{R}_{\tilde{p}}^{CU}$ on A.*

Proof. Let p be the potential defined in Lemma 1. According to an extension theorem of R.-M. Hervé ([9], Théorème 13.2, [8]) there exists $\tilde{p}\in\mathcal{P}(X)$

such that $\tilde{p}-\hat{R}_{\tilde{p}}^{CU}=p$ on U and $S(\tilde{p})=S(p)$. Since $\tilde{p}$ is bounded on a neighbourhood of $S(p)$, it is bounded on X; $p=\mathbf{0}$ on A implies $\tilde{p}=\hat{R}_{\tilde{p}}^{CU}$ on A.

Corollary. *If σ is a fine local operator, then $(X, \mathcal{H}^*)$ is elliptic.*

Proof. Assume that $(X, \mathcal{H}^*)$ is not elliptic. Then there exist an open subset $U\subset X$ and an absorbent set $A\subset U$, $A\neq\phi$, $A\neq U$. By Lemma 2 there is a bounded potential $\tilde{p}$ on X such that $S(\tilde{p})$ is compact, non-empty and contained in $\partial A\cap U$, and such that $\tilde{p}$ and $\hat{R}_{\tilde{p}}^{CU}$ coincide on the finely open set A. Hence the fine local property implies

$$\sigma(\tilde{p})_{|A}=\sigma(\hat{R}_{\tilde{p}}^{CU})_{|A}=\mathbf{0}\,,$$

since $\hat{R}_{\tilde{p}}^{CU}$ is harmonic on U.
But then $\sigma(\tilde{p})$ is the zero measure, since $\tilde{p}$ is harmonic outside a compact subset of A. This implies $f\odot\tilde{p}=0$ for every continuous function f, hence the harmonicity of $\tilde{p}$. This is a contradiction.

Lemma 3. *If σ is a fine local operator, then $(X, \mathcal{H}^*)$ satisfies Axiom (D).*

Proof. Since $\mathbf{1}\in\mathcal{H}_+^*(X)$ it suffices to prove $R_p^{S(p)}=p$ for every bounded (and not just locally bounded) potential p. Let p be a bounded potential and let $u\in\mathcal{H}_+^*(X)$, $u\geq p$ on $S(p)$. For every $\varepsilon>\mathbf{0}$ set $p_\varepsilon:=\inf(p, u+\varepsilon)$, hence $p_\varepsilon=p$ in a fine neighbourhood V_ε of $S(p)$. By assumption this implies

$$\sigma(p)_{|V_\varepsilon}=\sigma(p_\varepsilon)_{|V_\varepsilon}\,.$$

Since $p=\mathbf{1}_{S(p)}\odot p$ and $p_\varepsilon\leq p$ we have (recall that g is the Borel function appearing in the definition of σ)

$$\begin{aligned}\int\frac{1}{g}\cdot\mathbf{1}_{S(p)}\,d\sigma(p_\varepsilon)&=\int((g\cdot\frac{1}{g}\mathbf{1}_{S(p)})\odot p_\varepsilon)\,d\overline{\mu}\leq\\ &\leq\int\mathbf{1}_{S(p)}\odot p_\varepsilon\,d\overline{\mu}+\int\mathbf{1}_{X\setminus S(p)}\odot p_\varepsilon\,d\overline{\mu}=\int p_\varepsilon\,d\overline{\mu}\leq\int p\,d\overline{\mu}=\\ &=\int\mathbf{1}_{S(p)}\odot p\,d\overline{\mu}=\int\frac{1}{g}\cdot\mathbf{1}_{S(p)}\,d\sigma(p)=\int\frac{1}{g}\cdot\mathbf{1}_{S(p)}\,d\sigma(p_\varepsilon)\,,\end{aligned}$$

hence $\int\mathbf{1}_{X\setminus S(p)}\odot p_\varepsilon\,d\overline{\mu}=\mathbf{0}$, i.e. $S(p_\varepsilon)\subset S(p)$.

The next Lemma will show that the set $A:=\{x\in X\colon p(x)=p_\varepsilon(x)\}$ is an absorbent set of X. The Corollary then implies $A=X$, which finishes the proof of Lemma 3.

Lemma 4. *The set $A:=\{x\in X\colon p(x)=p_\varepsilon(x)\}$ is an absorbent set.*

Proof. By [3] A must be shown to be closed and finely open. Since $p=p_\varepsilon$ on $S(p)$ we have

$$X \backslash A = \{x \in X: p_\varepsilon(x) < p(x)\} \cap (X \backslash S(p)) .$$

On $X\backslash S(p)$ both p and p_ε are harmonic, hence A is closed. In order to show that A is finely open let $x \in A$. If $x \in S(p)$, then $x \in V_\varepsilon \subset A$. If $x \notin S(p)$, then x lies in one of the (open) components G of $X\backslash S(p)$. On G the function $p - p_\varepsilon$ is ≥ 0, harmonic and vanishes at x. According to the ellipticity of X we have $p - p_\varepsilon = 0$ on G, hence $x \in G \subset A$.
The Lemmas 1–4 show that the fine local property of σ implies Axiom (D). The proof of the converse implication relies on a decomposition lemma proved by A. Cornea (unpublished) and D. Feyel [6].

Lemma 5 (Decomposition lemma). *Let $E \subset X$.*
a) *Every $s \in \mathcal{S}_+(X)$ can be decomposed as $s = s_E + s'_E$ where*

$$s_E = \sup\{t < s: \hat{R}_t^E = t\} ,$$

b) *The superharmonic function s_E is the specific infimum of the sequence (u_n), defined by*

$$u_0 := s, \quad u_{n+1} := u_n - R(u_n - \hat{R}_{u_n}^E), \quad n \geq 0 ,$$

and

$$s'_E = \sum_{n=0}^{\infty} R(u_n - \hat{R}_{u_n}^E) .$$

c) $\hat{R}_{s_E}^E = s_E$ *and* $\hat{R}_{s'_E}^{X\backslash E} = s'_E$.

Corollary. *Let $s, \tilde{s} \in \mathcal{S}_+(X)$ such that*

$$s = \tilde{s} \text{ on } X\backslash b(E), \quad \hat{R}_s^E = \hat{R}_{\tilde{s}}^E \text{ on } X\backslash b(E) ,$$

Then $s'_E = \tilde{s}'_E$.

Proof. Let (u_n) and $(\tilde{u}_n)$ be the defining sequences for the decomposition. Then by induction

(*) $R(u_n - \hat{R}_{u_n}^E) = R(\tilde{u}_n - \hat{R}_{\tilde{u}_n}^E)$, $n \geq 0$;

hence $s'_E = \tilde{s}'_E$.
In fact by assumption

$$u_0 - \hat{R}_{u_0}^E = s - \hat{R}_s^E = \tilde{s} - \hat{R}_{\tilde{s}}^E = \tilde{u}_0 - \hat{R}_{\tilde{u}_0}^E \text{ on } X\backslash b(E) \text{ and} = 0 \text{ on } b(E) .$$

This implies (*) for $n=0$. If (*) is true for all $k \leq n$, then

$$r_n := \sum_{k=0}^{n} R(u_k - \hat{R}_{u_k}^E) = \sum_{k=0}^{n} R(\tilde{u}_k - \hat{R}_{\tilde{u}_k}^E) =: \tilde{r}_n ,$$

and

$$s = r_n + u_{n+1}, \quad \tilde{s} = \tilde{r}_n + \tilde{u}_{n+1} ;$$

$$(**) \quad \hat{R}_s^E = \hat{R}_{r_n}^E + \hat{R}_{u_{n+1}}^E, \quad \hat{R}_{\tilde{s}}^E = \hat{R}_{\tilde{r}_n}^E + \hat{R}_{\tilde{u}_{n+1}}^E .$$

The assumption $s = \tilde{s}$ on $X \setminus b(E)$ and $r_n = \tilde{r}_n$ imply $u_{n+1} = \tilde{u}_{n+1}$ on $X \setminus b(E)$; $\hat{R}_s^E = \hat{R}_{\tilde{s}}^E$ on $X \setminus b(E)$ and (**) imply $\hat{R}_{u_{n+1}}^E = \hat{R}_{\tilde{u}_{n+1}}^E$ on $X \setminus b(E)$, hence $u_{n+1} - \hat{R}_{u_{n+1}}^E = \tilde{u}_{n+1} - \hat{R}_{\tilde{u}_{n+1}}^E$ on $X \setminus b(E)$, and$=\mathbf{0}$ on $b(E)$.
Consequently (*) is true for $n+1$.

Lemma 6. *Let V be a finely open set and s, $\tilde{s}$ locally bounded potentials on X such that $s = \tilde{s}$ on V. Then Axiom (D) implies*

a) $\hat{R}_s^{b(X \setminus V)} = \hat{R}_{\tilde{s}}^{b(X \setminus V)}$ *on* $X \setminus b(X \setminus V)$;

b) $f \odot s = f \odot \tilde{s}$ *for every bounded Borel function f vanishing on $X \setminus V$.*

Proof. a) follows from the fact (see [9], §28, [7], 4.7) that the measures $\varepsilon_x^{b(X \setminus V)}$ for $x \notin b(X \setminus V)$ are carried by the fine boundary of $b(X \setminus V)$; this fine boundary is contained in the fine closure of V, where s and $\tilde{s}$ coincide. According to a) and the assumption the Corollary of the Decomposition Lemma can be applied to $E := b(X \setminus V)$, since s and $\tilde{s}$ coincide on the fine closure of V, especially on $X \setminus b(X \setminus V)$. Hence $s'_E = \tilde{s}'_E$ in the decomposition $s = s_E + s'_E$; $\tilde{s} = \tilde{s}_E + \tilde{s}'_E$. But $s_E = \hat{R}_{s_E}^{b(X \setminus V)}$ implies $f \odot s_E = \mathbf{0}$ (and analogously $f \odot \tilde{s}_E = \mathbf{0}$) for every bounded Borel function f vanishing on $X \setminus V$ ([7], 11.13, 11.21).

Corollary. *If Axiom (D) is satisfied, then σ is a fine local operator.*

The assertion follows immediately from the definition of σ.

REMARK (Existence of *fine measure representations*). If Axiom (D) is satisfied, then σ can be extended to a defining operator of the sheaf $\hat{\mathfrak{R}}$ of fine-local differences of finite finely hyperharmonic functions:
For every finely open set V and f: $V \to \mathbf{R}$

$f \in \hat{\mathfrak{R}}(V) \underset{\text{def}}{\Leftrightarrow}$ for every point $x \in V$ there exist a fine neighbourhood W and two finite finely hyperharmonic functions u, v on W such that $f = u - v$ on W.

$\Leftrightarrow$ for every point $x \in V$ there exist a fine neighbourhood W and $u, v \in \mathcal{P}_b(X)$ such that $f = u - v$ on W (due to the "local extension property", [7], 9.9).

To $f \in \hat{\mathfrak{R}}(V)$ a representing measure is assigned in the usual way:
If $f = u - v$ on a finely open set W, $u, v \in \mathcal{P}_b(X)$, then define

$$\hat{\sigma}(f)_{|W} := \sigma(u)_{|W} - \sigma(v)_{|W} .$$

According to the fine local property, $\hat{\sigma}$ is well-defined and has the "measure representation property"

$$\hat{\sigma}(f)_{|V} \geq 0 \Leftrightarrow f \text{ is finely hyperharmonic on } V.$$

REMARK. If the assumption $1 \in \mathcal{H}_+^*(X)$ does not hold, then the statement of the theorem is still true provided that the cone of all bounded potentials is replaced by the cone of locally bounded potentials.

2. Axiom (D) and the energy principle

In classical potential theory the energy of a potential

$$p(\cdot) = G^{\mu}(\cdot) = \int_{R^3} \frac{\mu(dy)}{||\cdot - y||}$$

respectively its measure μ is defined by

$$E(p) := \int G^{\mu} \, d\mu .$$

More generally, for a harmonic space with a Green function G the energy of a signed measure μ could be defined as the quantity $\int G^{\mu} \, d\mu = \iint G(x, y) \, \mu(dx) \, \mu(dy)$. But unfortunately this real number is not always positive—as it is true in classical potential theory for the usual Newtonian kernel as a Green function. As we shall see, the deeper reasons for this misbehavior are:
1) Positivity can only be expected if G is "normalized in a suitable way".
2) Positivity for normalized Green functions is equivalent to Axiom (D).
For $\mathfrak{P}$-harmonic spaces X having no Green function a natural definition of energy would be

$$E(p) := \int p d\sigma(p) ,$$

where σ is a suitable chosen measure representation of X.

HISTORICAL REMARKS. 1) The fact that Axiom (D) implies the bounded energy principle

(*E*) $E(p) := \int p d\mu_p \geq 0$ for all differences p of bounded potentials, where μ_p is the Revuz measure,

has been established by P.A. Meyer [12] in the framework of Hunt processes with an absolutely continuous proper potential kernel; see also [14] and [15] (The Axiom (D) of the theory of harmonic spaces is just the bounded maximum principle in the language of Hunt processes).
2) In [16], M. Rao showed the converse for a class of Lévy processes. The proof carries over to more general situations provided that $E(\cdot)$ is definite

$$E(p) = 0 \Rightarrow p = 0 .$$

In our proof, we use Rao's idea and add some considerations ensuring the definiteness. Unfortunately the proof as it stands uses tools from the theory of harmonic spaces and does not work immeditaely for general Hunt processes with absolutely continuous resolvents or for standard H-cones.
The following example will show that the validity of (E) can only be expected for suitably normalized Green functions or—more generally—for measure representations σ of "*standard form*":

$$\sigma(p)(\varphi) = \bar{\mu}(\varphi \odot p)$$

and *not* for those defined by

$$\sigma(p)(\varphi) = \bar{\mu}(g\varphi \odot p) \qquad (p \in \mathcal{P}_0(X), \varphi \in \mathfrak{K}_+(X))$$

where g is an arbitrary strictly positive continuous function, as considered before.

EXAMPLE. Let X be the harmonic space of the solutions of the one-dimensional Laplace equation on $]-1, 1[$. A Green function for X is given by

$$G(x, y) = \min((1+x)(1-y), (1-x)(1+y)) \quad \text{for } x, y \in X .$$

With respect to this Green function

$$\iint G(x, y)\, \mu(dx)\, \mu(dy) \geq 0 \quad \text{for all signed measures } \mu.$$

Let now g: $]-1, 1[\to \boldsymbol{R}$ be a strictly positive bounded continuous function such that $g\left(\frac{1}{2}\right)=1, g\left(-\frac{1}{2}\right)=\alpha$. By

$$G_g(x, y) := \frac{1}{g(y)} G(x, y),\ x, y \in X ,$$

another Green function G_g is given. For $p \in \mathcal{P}(X)$ we have

$$p(\cdot) = \int G_g(\cdot, y)\, \mu_g(dy) = \int \frac{1}{g(y)} G(\cdot, y)\, \mu_g(dy) = \int G(\cdot, y)\, \mu(dy)$$

i.e. the measures $\mu = \sigma(p)$ and $\mu_g = \sigma_g(p)$ corresponding to G and G_g are related by

$$\mu = \frac{1}{g} \cdot \mu_g ,$$

the corresponding measure representations σ and σ_g by

$$\sigma = \frac{1}{g} \cdot \sigma_g .$$

If $q:=4\,G\left(\cdot, \frac{1}{2}\right)-G\left(\cdot, -\frac{1}{2}\right)$, then an easy computation shows that for $\mu_g=\sigma_g(q)$

$$\int q d\sigma_g(q) = \iint G_g(x, y)\, \mu_g(dx)\, \mu_g(dy) = \iint G(x, y)\, g(y)\, \sigma(q)\,(dx)\, \sigma(q)\,(dy)$$
$$= 11-\frac{\alpha}{4},$$

and this quantity is <0 for $\alpha>44$. The measure representation σ which assigns to each Greenian potential

$$p = G^{\mu}: = \int G(\cdot, y)\, \mu(dy)$$

the corresponding measure μ can be represented in standard form

$$\sigma(p)\,(\varphi) = \bar{\mu}(\varphi \odot p),\ \varphi \in \mathfrak{K}_+(X),$$

where $\bar{\mu}$ is the normalizing H-integral for G:

$$\bar{\mu}(p): = \frac{1}{2}\,(p'_+(-1)-p'_-(1))$$

(right and left derivatives at $-1, +1$); ($\bar{\mu}$ is determined by the fact, that

$$\mathbf{1} = \bar{\mu}(G(\cdot, y)) \quad \text{for every} \quad y \in X).$$

Consequently, if $p=G^{\mu}$, then

$$\sigma(p)\,(\varphi) = \bar{\mu}\left(\int G(\cdot, y)\, \varphi(y)\, \mu(dy)\right) = \int \varphi d\mu,\ \varphi \in \mathfrak{K}_+(X),$$
$$\text{i.e. } \sigma(p) = \mu\,.$$

The measure representation σ_g corresponding to the Green function G_g is not representable in standard form; this will follow from the next Theorem.
The follownig remarks will show that:

1) Measure representations σ corresponding to a pair of resolvents $(U_\alpha)_{\geq \alpha 0}$ and $(V_\alpha)_{\alpha \geq 0}$ in duality with respect to an excessive measure ξ are representable in standard form (see remark 2)). For a potential $p=U_0 f$ we have

$$\sigma(p) = f \cdot \xi$$

and
$$E(p) = \int (U_0 f)\,(y) f(y)\, \xi(dy)\,.$$

This situation has been studied by P.A. Meyer for general Hunt processes.

2) (special case of 1)) Measure representations corresponding to Green functions defining a duality between a pair of harmonic spaces (for the exact for-

mulation see remark 1)), especially
3) measure representations corresponding to *symmetric* Green functions are representable in standard form.
As in the example the functionals $\bar{\mu}$ normalizing G

$$\bar{\mu}(G(\cdot, y)) = \mathbf{1} \quad (\text{for all } y \in X)$$

are in general no longer Radon measures on X but strictly positive H-integrals.

REMARKS. 1) J.C. Taylor showed in [21]; Let X be a harmonic space with a Green function G' such that the constant function $\mathbf{1}$ is hyperharmonic. Then there exists a strictly positive continuous function g and a Radon measure ξ such that the kernels

$$U_0(\cdot, dy) := G(\cdot, y)\,\xi(dy),\ V_0(\cdot, dx) := G(x, \cdot)\,\xi(dx)$$

(where $G(x, y) = \dfrac{1}{g(y)}\, G'(x, y),\ x, y \in X$) are the potential kernels of two submarkovian resolvents $(U_\alpha)_{\alpha \geq 0}$ and $(V_\alpha)_{\alpha \geq 0}$ in duality with respect to ξ, which define two Hunt processes on X in duality.
2) Let X be a $\mathfrak{P}$-harmonic space and let $(U_\alpha)_{\alpha \geq 0}$ be a submarkovian resolvent with proper potential kernel U_0 such that the excessive functions coincide with the positive hyperharmonic functions. For every excessive reference measure ξ it is possible to choose a density u such that

$$U_0(\cdot, dy) = u(\cdot, y)\,\xi(dy)$$

and such that

$$V_0(\cdot, dx) = u(x, \cdot)\,\xi(dx)$$

is the potential kernel of a dual resolvent (V_α). Then there exists a strictly positive H-integral $\bar{\mu}$ such that the corresponding measure representation σ defined by

$$\sigma(p)\,(\varphi) := \bar{\mu}(\varphi \odot p),\ (\varphi \in \mathfrak{K}_+(X))$$

assigns to a potential $p = U_\mu = \int u(\cdot, y)\,\mu(dy)$ the measure μ: In fact, the functional $\bar{\mu} = L$, defined in [12], p. 435, satisfies all required properties. It is the unique H-integral extension of the functional

$$U_0 f \mapsto \int f d\xi\,.$$

Some easy considerations or [4] (Proposition 4.2.3 and p. 98) show that $\bar{\mu}$ is finite on $\mathcal{P}_0(X)$.

3) P.A. Meyer showed the following [12]: Let $(U_\alpha)_{\alpha\geq 0}$ be the resolvent of a Hunt process on X, such that U_0 is proper and absolutely continuous with respect to an excessive measure ξ.
Let u be a density function on $X\times X$ such that $U_0(\cdot, dy)=u(\cdot, y)\,\xi(dy)$ and such that the kernel

$$V_0(\cdot, dx) = u(x, \cdot)\,\xi(dx)$$

is the potential kernel of a dual resolvent $(V_\alpha)_{\alpha\geq 0}$. If f is a Borel function such that $U_0 f$ is finite and the integral $\int U_0 f\cdot f\, d\xi$ exists (more generally: if μ is a signed measure such that both $U_0\mu_+$ and $U_0\mu_-$ are regular potentials and such that the integral $\int U_0\mu\, d\mu$ exists) then this integral is positive [12, p. 441].
As we have seen by the example and the remarks it is reasonable to concentrate on measure representations σ in standard form

$$\sigma(p)\,(\varphi) = \bar{\mu}(\varphi \odot p),\ (p\in\mathcal{P}(X),\ \varphi\in\mathfrak{K}_+(X))$$

where $\bar{\mu}$ is a strictly positive H-integral. Again σ can be extended in a natural way to the cone $\mathcal{P}_b(X)-\mathcal{P}_b(X)$ of differences of bounded potentials.
Now we can state the main result:

Theorem. *Let $(X, \mathcal{H}^*)$ be a $\mathfrak{P}$-harmonic space with a countable base such that $\mathbf{1}\in\mathcal{H}^*_+(X)$. Then Axiom (D) is valid if and only if*
(E) $\int p\, d\sigma(p)\geq 0$ for every $p\in\mathcal{P}_b(X)-\mathcal{P}_b(X)$ holds.

Lemma 1. *Axiom (D) is equivalent to the following condition: (D^*). Let $p\in\mathcal{P}_b(X)$ such that the restriction $p_{|S(p)}$ to its superharmonic support $S(p)$ is continuous and let $u\in\mathcal{P}_b(X)$ such that $u\leq p$, $u=p$ in a fine neighbourhood of $S(p)$ and u continuous on $X\backslash S(p)$. Then $u=p$.*

Proof. Axiom (D) is equivalent to the condition

$$(*)\quad R_p^{S(p)} = p$$

for every locally bounded potential p. Hence Axiom (D) clearly implies the above condition (D^*).
Conversely, since every locally bounded potential is the countable sum of a sequence (p_n) of bounded potentials whose restrictions to their supports are continuous (according to [5], Theorem 8.2.2, and taking into account that $\mathbf{1}\in\mathcal{H}^*_+(X)$) it suffices to prove $(*)$ for all $p\in\mathcal{P}_b(X)$ such that $p_{|S(p)}$ is continuous. Let $v\in\mathcal{H}^*_+(X)$ such that $v\geq p$ on $S(p)$. Since $p\cdot\mathbf{1}_{S(p)}$ is upper semicontinuous and $\inf(v, p)$ is lower semicontinuous there exists a continuous function φ such that

$$p\cdot\mathbf{1}_{S(p)}\leq\varphi\leq\inf(v,p)\,.$$

But then $v':=R\varphi$ is a continuous potential, $v'=p$ on $S(p)$ and $v'\leq\inf(v,p)$. For every $\varepsilon>0$

$$u_\varepsilon:=\inf(v'+\varepsilon,p)$$

satisfies the assumptions of (D^*): $u_\varepsilon=p$ on the fine open set $[p<v'+\varepsilon]$. Hence $u_\varepsilon=p$. Since this is true for every $\varepsilon>0$ we conclude

$$\inf(v',p)=p,\ v'=p$$

and finally

$$v\geq p\,.$$

Proof of the implication $(E)\Rightarrow(D^*)$. Let $p,u\in\mathcal{P}_b(X)$ such that $u\leq p$, $u=p$ in a fine neighbourhood V of $S(p)$, and such that u is continuous on $X\backslash S(p)$. The condition (E) for $p-u\in\mathcal{P}_b(X)-\mathcal{P}_b(X)$ implies

$$0\leq\int(p-u)\,(d\sigma(p)-d\sigma(u))=-\int(p-u)\,d\sigma(u)\leq 0\,,$$

since $p=u$ on Supp $\sigma(p)$ and $p-u\geq 0$ on X. (This is M. Rao's trick. If $\int fd\sigma(f)=0$ would imply $f=0$ for $f\in\mathcal{P}_b(X)-\mathcal{P}_b(X)$, then the proof would be finished at this point: we could conclude $f=p-u=0$, $u=p$. The following considerations show $u=p$ without this assumption). Using the continuity of u and p on $X\backslash S(p)$ and the fact that $u=p$ on $S(p)$,

$$\int(p-u)\,d\sigma(u)=0$$

implies

$$u=p\text{ on Supp }\sigma(u)\,,$$

i.e. both u and p are harmonic on the open set

$$U=\{x\in X: u(x)<p(x)\}=\{x\in X\backslash S(p): u(x)<p(x)\}\,.$$

The next Lemma 2 will show that all boundary points of U are regular, i.e. $F:=X\backslash U\subset b(F)$.

Let now $f:=p-u\in\mathcal{P}_b(X)-\mathcal{P}_b(X)$, then $f\geq 0$, Supp $\sigma(f)\subset F$ and $\int fd\sigma(f)=0$. If $u\neq p$, then $U\neq\phi$; and it is possible to choose a continuous bounded potential q such that

$$\phi\neq S(q)=\text{Supp }\sigma(q)\subset U\,.$$

Let $g:=q-\hat{R}_q^F\in\mathcal{P}_b(X)-\mathcal{P}_b(X)$, then $g\geq 0$, $g=0$ on F; hence $\int g\, d\sigma(f)=0$ (since Supp $\sigma(f)\subset F$)
and

$$\int f\, d\sigma(g)=\int f\, d\sigma(q)>0$$

(since $f=0$ on $F\supset$ Supp $\sigma(\hat{R}_q^F)$ and $f>0$ on the non-empty set Supp $\sigma(q)$).
But then (E) applied to $\alpha f-g$ is violated for sufficiently large $\alpha>0$

$$\begin{aligned}\int(\alpha f-g)\,(d\sigma(\alpha f)-d\sigma(g)) &= \alpha^2\cdot 0-\alpha\int fd\sigma(q)-\alpha\cdot 0+\int gd\sigma(g)\\ &=\int gd\sigma(g)-\alpha\int fd\sigma(q)\,.\end{aligned}$$

Hence $U=\phi$, $u=p$.

Lemma 2. *$F:=X\setminus U$ is not thin at any of its points.*

Proof. Let $z\in\partial F$. If $z\in S(p)$, then z belongs to the fine neighbourhood V of $S(p)$, where $u=p$. By definition of F

$$z\in V\subset F\,.$$

If $z\notin S(p)$, then $p-u$ is continuous at z:

$$0=\lim_{\substack{x\to z\\ x\in U}} p(x)-u(x);$$

i.e. $p-u$ is a barrier function for z and U; in other words: z is a regular boundary point of U, hence $z\in b(F)$.

Proof of the implication $(D)\Rightarrow(E)$. This implication has been proved by Meyer [12], since by (D) all bounded potentials are regular (see remark 3). The proof uses the machinery of processes and additive functionals and is valid in the more general context of Hunt processes with absolutely continuous potential kernel.

A second proof relies on a result of F.-Y. Maeda [11, p. 41]. He proved—under the assumption $\mathbf{1}\in\mathcal{H}_+^*(X)$—that for every $f\in\mathcal{P}_c(X)-\mathcal{P}_c(X)$

$$2f\sigma(f)-\sigma(f^2)\geq 0\,,$$

hence $2f\odot f-f^2$ is superharmonic, and—as it is easily seen—even a (continuous) potential.
Let now $f=p-p'$, where p and p' are bounded and hence rgeular potentials. Then p and p' are countable sums of continuous potentials

$$p=\sum_{k=1}^{\infty}p_k,\quad p'=\sum_{k=1}^{\infty}p'_k\,.$$

Maeda's result applied to

$$\bar{P}_n := \sum_{k=1}^{n} (p_k - p'_k),\ n \in \boldsymbol{N},$$

yields

$$2\,\bar{P}_n \odot \bar{P}_n - \bar{P}_n^2 \in \mathcal{P}_c(X).$$

It turns out that $2f \odot f - f^2$ is the pointwise limit of the sequence $(2\bar{P}_n \odot \bar{P}_n - \bar{P}_n^2)_{n\in N}$ of these potentials (since $|f \odot f - \bar{P}_n \odot \bar{P}_n| = |f \odot \bar{Q}_n + \bar{Q}_n \odot \bar{P}_n|$, where $\bar{Q}_n = Q_n - Q'_n$, $Q_n = p - P_n$, $Q'_n = p' - P'_n$ and $|f \odot \bar{Q}_n| \le (\|p\|_\infty + \|p'\|_\infty)(Q_n + Q'_n)$, $|\bar{Q}_n \odot \bar{P}_n| \le (Q_n + Q'_n) \odot (p + p')$).

Hence by Lebesgue's theorem

$$2\int f \odot f\, d\bar{\mu} - \int f^2 d\bar{\mu} = \lim_{n\to\infty} \int (2\bar{P}_n \odot \bar{P}_n - \bar{P}_n^2)\, d\bar{\mu}$$

$$= \lim_{n\to\infty} \int 1\, d\sigma(2\bar{P}_n \odot \bar{P}_n - \bar{P}_n^2) \ge 0,$$

i.e.

$$\int f d\sigma(f) = \int f \odot f\, d\bar{\mu} \ge \frac{1}{2} \int f^2\, d\bar{\mu} \ge 0.$$

REMARK. The quantity $\bar{\mu}(f^2)$ can be zero even if f is not identically zero. For instance in the example of the one-dimensional Laplace equation on $]-1, 1[$ $\bar{\mu}$ is the H-integral

$$\bar{\mu}(p) = \frac{1}{2}(p'_+(-1) - p'_-(1)), \quad p \in \mathcal{P}_0(X).$$

Hence $\bar{\mu}(f^2) = 0$ provided that f vanishes near $+1$, -1.

3. Doob's convergence axiom and continuity of measure representations

Let $(X, \mathcal{H}^*)$ be a $\mathfrak{P}$-harmonic space with a countable base, let $\bar{\mu}$ be a Radon measure on X such that

$$0 < \int p d\bar{\mu} < \infty \quad \text{for all} \quad p \in \mathcal{P}_0(X)$$

(or: a strictly positive H-integral), and let g be a strictly positive continuous function on X. The given data $\bar{\mu}$ and g determine a measure representation σ on X, as explained in the introduction.

For every open set U the map $\sigma = \sigma_U$ can be extended in a natural way to the cone $\mathcal{S}_+(U)$ of all superharmonic functions ≥ 0 on U, but—unless *all* potentials with compact support are integrable with respect to $\bar{\mu}$—$\sigma(q)$ is in general not a Radon measure for a discontinuous superharmonic function q.

In [19] it was shown that σ is a continuous map from the cone $\mathcal{S}_c(U)$ of all continuous superharmonic functions with the topology of locally uniform convergence to the set $\mathcal{M}_+(U)$ of all (positive) Radon measures endowed with the vague topology.

The cone $\mathcal{S}_+(X)$ of *all* positive superharmonic functions on X is usually endowed with much coarser topologies:

1) The topology of *graph convergence* introduced by Mokobodzki in order to prove the existence of integral representations (see [13] and [1]).
2) The topology τ introduced in Constantinescu-Cornea's book [5], p. 288, defined by the semi-norms

$$p_{K,f}\colon s \mapsto \sup_K |f \odot s| ,$$

where f is a continuous and positive function on the Alexandroff compactification X_ω of X and K is a compact subset of $X\backslash \operatorname{Supp} f$.
3) The *natural topology* τ_{nat} on standard-H-cones (see [4]).

The topologies 1) and 3) coincide on cones of positive superharmonic functions. A sequence $(s_n)_{n\in N}$ in $\mathcal{S}_+(X)$ is τ_{nat}-convergent to $s\in\mathcal{S}_+(X)$ iff for every subsequence $(s_{n_k})_{k\in N}$

$$s = \widehat{\liminf_{k\to\infty} s_{n_k}} .$$

Especially sequences converging pointwise on the complement of a semipolar set converge with respect to the natural topology. In general the topology τ is finer than τ_{nat}; the two tpologies coincide, provided that the harmonic space satisfies Doob's convergence axiom; in that case the specific multiplication map

$$s \mapsto f \odot s,\ f\in\mathcal{C}_+(X_\omega) ,$$

is τ_{nat}-continuous (see [18] and [20]).

The aim of this final section is to prove the continuity of measure representations with respect to $\tau=\tau_{\mathrm{nat}}$ for harmonic spaces satisfying Doob's convergence axiom.

Theorem. *Assume that Doob's convergence axiom and the following condition (A) hold*

(A). *Every point $x\in X$ has an open neighbourhood V_x such that the smallest absorbing set A_{CV_x} containing CV_x is the whole space X.*

Then for every open set U

$$\sigma=\sigma_U\colon \mathcal{S}_+(U) \to \mathcal{M}_+(U)$$

is continuous with respect to the topology τ on $\mathcal{S}_+(U)$ and the vague topology on $\mathcal{M}_+(U)$ for a suitably chosen measure $\bar{\mu}$.

Especially if (s_n) *is a sequence in* $\mathcal{S}_+(U)$ *converging pointwise outsidde a semipolar set to some* $s \in \mathcal{S}_+(U)$, *then the sequence of Radon measures* $(\sigma(s_n))_{n \in N}$ *converges vaguely to the Radon measure* $\sigma(s)$.

REMARKS. 1) Condition (A) was introduced by K. Janßen in [10] in the study of harmonic spaces with Green functions.
2) In the presence of Doob's convergence axiom condition (A) is even necessary for the convergence property of σ (see Lemma 4).
3) It is clear that the result cannot be valid for an *arbitrary* measure $\bar{\mu}$; for if (s_n) is a decreasing sequence in $\mathcal{S}_+(X)$ such that the semipolar set

$$A := \{x: \widehat{\inf s_n(x)} < \inf s_n(x)\}$$

is non-empty and $\bar{\mu}$ charges A, then $(\sigma(s_n))_{n \in N}$ does not converge vaguely to $\sigma(s)$.
4) It is open whether the convergence property of σ conversely implies Doob's convergence axiom (or at least nuclearity).
Doob's convergence axiom is needed on three occasions: first in proving continuity properties of a "localizing process" (Lemma 2), then for the construction of $\bar{\mu}$ (Lemma 4) and finally it is responsible for the crucial coincidence of the different topologies on the cone $\mathcal{S}_+(X)$.
For every open set $U \subset X$ let

$$\mathcal{P}_{00}(U) := \{p \in \mathcal{P}(U) : S(p) \text{ compact} \subset U\}$$

and, for $L \subset U$,

$$\mathcal{P}_L(U) := \{p \in \mathcal{P}(U) : S(p) \text{ compact} \subset L\} \quad (\subset \mathcal{P}_{00}(U)).$$

By [9], Théorème 13.2 and [8] the restriction map

$$\begin{aligned} Q\colon \mathcal{P}_U(X) &\to \mathcal{P}_{00}(U) \\ \tilde{p} &\mapsto \text{potential part of } \tilde{p}_{|U} = \tilde{p}_{|U} - R^{CU}_{\tilde{p}|U} \end{aligned}$$

is bijective, additive and positively homogeneous, the extension map $T = Q^{-1}$

$$T\colon \mathcal{P}_{00}(U) \to \mathcal{P}_U(X)$$

is also bijective, increasing and respects suprema of increasing sequences.

The following lemma follows immediately from these properties and from the definition of the specific multiplication.

Lemma 1. *Let* $f \in C_+(X)$ *such that* Supp (f) *is a compact subset of* U *and let* $\tilde{p} \in \mathcal{P}_U(X)$. *Then* $Q(f \odot \tilde{p}) = (f_{|U}) \odot Q(\tilde{p})$.

Lemma 2. *If Doob's convergence axiom is satisfied and* ∂U *is compact, then*

a) *the restriction of Q to $\mathcal{P}_L(X)$ is continuous for every compact subset L of U,*
b) *The restriction of T to $\mathcal{P}_L(U)$ is continuous for every compact subset L of U.*

Proof. *Continuity of Q on $\mathcal{P}_L(X)$.* Let us first remark that the trace of the topology τ on $\mathcal{P}_L(U)$ is defined by the seminorms $p_{K,f}$, where $f \in \mathcal{C}_+(X)$ with compact support contained *in* U and K a compact subset of $U \setminus \operatorname{Supp} f$. Let $(\tilde{p}_n)_{n \in N}$ be a sequence in $\mathcal{P}_L(X)$ converging to some $\tilde{p} \in \mathcal{P}_L(X)$ and let $f \in \mathcal{C}_+(X)$ such that $\operatorname{Supp}(f)$ is compact $\subset U$. By definition of τ we must show that $(f_{|U} \odot Q(\tilde{p}_n))_{n \in N}$ converges locally uniformly to $f_{|U} \odot Q(\tilde{p})$ on $U \setminus \operatorname{Supp} f$.
For $n \in N$ let $\tilde{p}'_n := f \odot \tilde{p}_n$, $\tilde{p}' := f \odot \tilde{p}$. Then

$$\tilde{p} = \tau - \lim_{n \to \infty} \tilde{p}_n$$

implies the local uniform convergence of $(\tilde{p}'_n)$ to $\tilde{p}'$ on $X \setminus \operatorname{Supp} f$; especially this sequence converges uniformly on the compact set ∂U. But then

$$\lim_{n \to \infty} R^{CU}_{\tilde{p}'_n}(x) = \lim_{n \to \infty} \int \tilde{p}'_n \, d\varepsilon^{CU}_x = R^{CU}_{\tilde{p}'}(x) \quad \text{for every} \quad x \in U\,;$$

i.e. the potentials

$$f_{|U} \odot Q(\tilde{p}_n) = Q(\tilde{p}'_n) = \tilde{p}'_{n_{|U}} - R^{CU}_{\tilde{p}'_{n_{|U}}}$$

converge pointwise to $f_{|U} \odot Q(\tilde{p})$ on $U \setminus \operatorname{Supp} f$.
According to [10], (2.1) (which uses Doob's convergence axiom), this convergence is even locally uniform. This completes the first part of the proof.

Continuity of T on $\mathcal{P}_L(U)$. Let (p_n) be a sequence in $\mathcal{P}_L(U)$ converging to $p \in \mathcal{P}_L(U)$, and let $\tilde{p}_n := T(p_n)$; $\tilde{p} := T(p)$. If $(\tilde{p}_n)$ does not converge to $\tilde{p}$, there exists a subsequence $(\tilde{p}'_n)$ converging to some $\tilde{p}' \neq \tilde{p}$.
From the definition of τ-convergence follows $S(\tilde{p}') \subset L$, hence $\tilde{p}' \in \mathcal{P}_L(X)$. But then by the continuity and bijectivity of Q

$$\lim_{n \to \infty} Q(\tilde{p}'_n) = Q(\tilde{p}') \neq Q(\tilde{p}) = p\,,$$

which is a contradiction.

Lemma 3. *Let K' be a compact subset of X, ν a measure on X with support disjoint from K' and such that all $p \in \mathcal{P}_0(X)$ are ν-integrable, and let (s_n) be sequence in $\mathcal{P}_{K'}(X)$ converging pointwise on $X \setminus K'$. Then*

$$\lim_{n \to \infty} \int_{X \setminus K'} s_n d\nu = \int_{X \setminus K'} \lim_{n \to \infty} s_n d\nu < \infty\,.$$

The assertion follows immediately from [10], (proof of) Theorem (2.4).

Lemma 4. *For a $\mathfrak{P}$-harmonic space X satisfying Doob's convergence axiom condition (A) is equivalent to the following condition*

(R) *There exists a measure $\bar{\mu}$ on X such that*

$$(*) \quad 0<\int p d\bar{\mu}<\infty \text{ for all } p\in\mathcal{P}_0(X)\setminus\{0\}$$

which is continuous on the sets $\mathcal{P}_K(X)$, K compact; (i.e. the map $p\mapsto\int p\,d\bar{\mu}$ is continuous with respect to τ and real-valued).

REMARK. K. Janßen showed in [10] the existence of measures $\bar{\mu}$ operating continuously on a cone $\mathcal{P}_K(X)$ for a given compact set K. The important point here is the existence of a measure $\bar{\mu}$ which works for all compact sets K simultaneously. Obviously condition (R) is necessary for the continuity property of σ stated in the Theorem.

Proof. $(A)\Rightarrow(R)$. Let $(V_x)_{x\in X}$ be a covering of X of relatively compact open sets such that $x\in V_x$ and

$$A_{CV_x}=X \quad \text{for every} \quad x\in X,$$

let (V_n) be a countable subfamily still covering X and let (φ_n) be a subordinate locally finite partition of the constant $\mathbf{1}$. For every $n\in\boldsymbol{N}$ there exists a measure μ_n satisfying $(*)$ with support $\operatorname{Supp}(\mu_n)$ disjoint from $\operatorname{Supp}\varphi_n$ (for example $\mu_n=\sum_{k=1}^{\infty}\lambda_k\varepsilon_{x_k}$, where $\{x_k\colon k\in\boldsymbol{N}\}$ is dense in $X\setminus V_n$ and λ_k are suitable strictly positive real numbers).

For $p\in\mathcal{P}_0(X)$ set

$$\bar{\mu}(p):=\sum_{n=1}^{\infty}\alpha_n\int_0^1\int_X R_p^{\{\varphi_n>\alpha\}}\,d\mu_n\,d\alpha,$$

where the sequence (α_n) of strictly positive numbers is chosen such that $\bar{\mu}(p)<\infty$ for all $p\in\mathcal{P}_0(X)$. Then by [5], p. 160, $\bar{\mu}$ can be extended to a unique Radon measure on X, also denoted by $\bar{\mu}$.

Let now K be a compact subset of X. In order to prove the continuity and finiteness of $\bar{\mu}$ on $\mathcal{P}_K(X)$ let (p_k) be a sequence in $\mathcal{P}_K(X)$ converging to p_0. Then by [1] and [13] the potentials

$$R_{p_k}^{\varphi}:=\int_0^1 R_{p_k}^{\{\varphi>\alpha\}}\,d\alpha$$

converge to

$$R_{p_0}^{\varphi}:=\int_0^1 R_{p_0}^{\{\varphi>\alpha\}}\,d\alpha$$

pointwise on $X\setminus\operatorname{Supp}\varphi$ for every $\varphi\in\mathfrak{K}(X)$, $0\le\varphi\le\mathbf{1}$.

Since the partition (φ_n) is locally finite eventually all φ_n vanish in a compact neighbourhood L of K: there exists $n_0 \in N$ such that Supp $\varphi_n \cap L = \phi$ for all $n > n_0$. We apply now Lemma 3 to

1) $K' = \mathrm{Supp}\,(\varphi_n)$, $\nu = \mu_n$ (for $n \leq n_0$) and the sequence $(R_{p_k}^{\varphi_n})_{k \in N}$ and get

$$\lim_{k\to\infty} \int R_{p_k}^{\varphi_n}\, d\mu_n = \int R_{p_0}^{\varphi_n}\, d\mu_n < \infty$$

and then to

2) $K' = K$, $\nu = \mu_0$, defined by

$$\mu_0(s) := \sum_{n > n_0} \alpha_n\, \mu_n(R_s^{\varphi_n}),\ s \in \mathcal{P}_0(X)\,,$$

and the sequence $(p_k)_{k \in N}$ and get

$$\lim_{k\to\infty} \mu_0(p_k) = \mu_0(p_0) < \infty$$

1) and 2) together imply the continuity and finiteness of $\bar{\mu}$ on $\mathcal{P}_K(X)$.

$(R) \Rightarrow (A)$. If (A) is violated, then there exist $x \in X$ and a sequence (V_n) of relatively compact open neighbourhoods of x decreasing to x such that $X \neq A_{CV_n}$ for every $n \in N$. But then we can find bounded potentials p_n vanishing on A_{CV_n} such that $\int p_n\, d\bar{\mu} = 1$. The sequence (p_n) converges to $\mathbf{0}$ with respect to τ, which contradicts the continuity of $\bar{\mu}$ on $\mathcal{P}_{\bar{V}_1}(X)$.

Proof of the Theorem. Let (s_n) be a sequence in $\mathcal{S}_+(U)$ converging to $s_0 \in \mathcal{S}_+(U)$ with respect to the topology τ, and let $f \in \mathcal{C}_+(X)$ such that $K := \mathrm{Supp}\,(f)$ is a compact subset of the open set U. Without loss of generality we can assume that U is relatively compact and each s_n is a potential which is harmonic outside some fixed compact subset L of U, $L \supset K$ (if not, replace U by a relatively compact open neighbourhood V of K and s_n by $\varphi \odot s_n$, where φ is a continuous function on V with compact support in V, $0 \leq \varphi \leq 1$ and $\varphi = 1$ on K).
Then

$$\tau\text{—}\lim_{n\to\infty} (fg)_{|U} \odot s_n = (fg)_{|U} \odot s$$

(g is the continuous function appearing in the definition of σ) and by Lemma 1 and 2

$$\begin{aligned}\tau\text{—}\lim_{n\to\infty} (fg) \odot T(s_n) &= \tau\text{—}\lim_{n\to\infty} T((fg)_{|U} \odot s_n) = T((fg)_{|U} \odot s) \\ &= (fg) \odot T(s)\,.\end{aligned}$$

Let $\bar{\mu}$ be a measure on X satisfying the condition (R) of Lemma 4. Then the measure representation σ determined by g and $\bar{\mu}$ has the required properties:

$$\lim_{n\to\infty} \sigma_U(s_n)(f) = \lim_{n\to\infty} \int (fg) \odot T(s_n)\, d\bar{\mu} = \int (fg) \odot T(s)\, d\bar{\mu} = \sigma_U(s)(f)\,.$$

Corollary. *Let $(X, \mathcal{H}^*)$ be a harmonic space with a Green function G. Then for every sequence (p_n) of potentials $p_n = G^{\mu_n} = \int G(\cdot, y)\, \mu_n(dy)$ converging pointwise outside a semipolar set to a potential $p = G^{\mu}$, the sequence of measures (μ_n) is vaguely convergent to the measure μ.*

Proof. According to [17] every harmonic space admitting a Green function satisfies Doob's convergence axiom; by [10] condition (A) is fulfilled.

Let $\bar{\mu}$ be a measure on X satisfying the properties of Lemma 4 and let

$$\begin{aligned} g\colon X &\to \boldsymbol{R}_+ \setminus \{0\} \\ y &\mapsto \int G(\cdot, y)\, d\bar{\mu}\,. \end{aligned}$$

The continuity of $\bar{\mu}$ on the cones $\mathcal{P}_K(X)$, K compact, implies the continuity of g. The measure representation σ on X determined by $\bar{\mu}$ and $\frac{1}{g}$ corresponds to the given Green function: for $p \in \mathcal{P}(X)$, $p = G^{\mu}$, and $\varphi \in \mathfrak{K}_+(X)$ we have

$$\sigma(p)(\varphi) = \iint \varphi(y) \frac{1}{g(y)} G(x, y)\, \mu(dy)\, \bar{\mu}(dx) = \mu(\varphi)\,,$$

i.e. $\sigma(p) = \mu$.

The assertion follows now immediately from the Theorem.

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