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Unitary representations of locally compact groups
—— Reproduction of Gelfand-Raikov's theorem ——

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1. INTRODUCTION. It is reported in Mathematical Reviews that I. Gelfand and D. Raikov proved, in 1943, that every locally compact group G admits sufficiently many irreducible continuous unitary representations on Hilbert spaces (that is, for any element g of G different from its identity element, there exists a representation of G which maps g to an operator different from the identity one) [3][1]. For the present author it is yet impossible to read their paper, but he had fortunately the opportunity to read the papers [6, 7] by I. E. Segal and was suggested the original direct proof by Gelfand and Raikov. The purpose of the present paper is to reproduce this direct proof.

Later the paper on the same subject by R. Godement [4] became available to the present author. The contents of the present paper are, in essential, involved in this paper.

2. SUMMARY. After some preliminary remarks in 3, we shall, in 4, establish the correspondence between measurable positive definite functions on G and so-called simple continuous unitary representations of G on Hilbert spaces. There we shall make use of the group algebra as [2], [6] and [7] do[2]. Every continuous unitary representation of G is obtained by the method used in 4, if, analogously to the finite dimensional case, we regard two representations as equivalent when they are mutually transformed by a unitary operator. In 5, we shall consider the correspondence between irreducible representations and so-called extreme positive definite functions and prove Gelfand-Raikov's theorem

1) Number in LITERATURE at the end of this paper.

2) We shall also utilize, as [7] does, some results in [1], which is reviewed in Math. Reviews but, to the present author, is yet unavailable.

on the existence of irreducible representations.

3. PRELIMINARIES. Throughout this paper, we shall consider an arbitrary but fixed locally compact group and denote it by G.

(1°) One of our principal subjects is *unitary representation* $\{U(g), H\}$ [3]) of G on a Hilbert space ($=$ generalized Euclidean space) H, which is continuous in the strong topology. $\{U(g), H\}$ is said to be *simple* [4]) if there exists an element ζ of H such that $\{U(g)\zeta \mid g \in G\}$ spans H, and to be *irreducible* if there is no non-trivial projection which commutes with every $U(g)$.

(2°) Let us denote by $m(\cdot)$ a fixed left-invariant Haar measure on G. The *group algebra* A over G shall be the ring arising from $L^1(G)$ [5]) in defining in it the multiplication by convolution:

$$x \cdot y(g) = \int x(h)\, y(h^{-1}g)\, dh, \text{ for } x, y \text{ of } A.$$

For every x of A, x^* is defined as follows: $x^*(g) = \overline{x(g^{-1})}\rho(g)$ [6]), where $\rho(g)$ is the density of a right-invariant Haar measure: $m(B^{-1}) = \int_B \rho(g)\, m(dg)$. Then $*$ satisfies the following conditions: $(x^*)^* = x$, $\|x^*\|_A = \|x\|_A$ [7]), $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda} x^*$, $(xy)^* = y^* x^*$. [8])

(3°) Let $\{V_\alpha\}$ be a system of (conditionally) compact neighbourhoods of e [9]), put $d_\alpha(g) = c_{V_\alpha}(g)/m(V_\alpha)$, where $c_B(\cdot)$ denotes the characteristic function of B. We shall call $\{e_\alpha \mid e_\alpha = d_\alpha^* d_\alpha\}$ the *approximate identity* of A. It is easy to see that $\|e_\alpha\|_A = 1$, $e_\alpha^* = e_\alpha$ and that, for every x of A, both $\{e_\alpha x\}$ and $\{x e_\alpha\}$ converge to x with α.

(4°) Let us call a complex-valued bounded linear functional $F(\cdot)$ on A *positive*, if $F(x^* x) \geqq 0$ for any x of A.

(5°) A complex-valued function $f(g)$ on G is called *positive definite*

3) We shall denote by $\{U(g), H\}$ the representation which maps g to unitary operator $U(g)$ on H.

4) It is called *normal* in [7].

5) We shall denote by $L^1(G)$ the Banach space of all complex-valued m-integrable functions on G.

6) $\bar{z}$ denotes the conjugate complex number of z.

7) $\|\cdot\|_E$ denotes the norm in Banach space E.

8) For these notions, see [6].

9) e denotes the identity element of G.

(we shall, abbreviate this as "p. d."), if it is measurable, essentially bounded and satisfies the inequality:

$$\int x^* x(g) f(g)\, dg = \int\int f(h^{-1} g)\, \overline{x(h)}\, x(g)\, dh dg \geq 0$$

for every x of $L(G)$. When $f(g)$ is moreover continuous, this inequality is equivalent with

$$\sum_{k,j} \bar{\xi}_j \xi_k f(g_j^{-1} g_k) \geq 0$$

for complex ξ_k $(1 \leq k \leq n)$.[10] Every p. d. function satisfies $\overline{x(g^{-1})} = x(g)$ for almost all g.

If we put

$$F_f(x) = \int x(g) f(g)\, dg$$

for x of A, then F_f is positive on A and satisfies $F_f(x^*) = \overline{F_f(x)}$; conversely, every positive functional on A is obtained by this manner from some p. d. function.

(6°) The important examples of p. d. functions are, among others, the following two: If $x(g)$ is bounded and vanishes outside a compact set in G, then

$$\int x(gh)\, \overline{x(h)}\, dh$$

is a continuous p. d. function in g. Another example is given by

THEOREM 1. *If $\{U(g), H\}$ is a continuous unitary representation of G and $\varphi \in H$, then $(U(g)\varphi, \varphi)_H$[11] is a continuous p. d. function in g.*

4. POSITIVE DEFINITE FUNCTIONS AND UNITARY REPRESENTATION. As for the further relation of p. d. functions to unitary representations, it is easy to prove

THEOREM 2. *If $\{U(g), H\}$ and $\{U'(g), H'\}$ are unitary representations, and $\{U(g)\zeta \mid g \in G\}$ and $\{U'(g)\zeta' \mid g \in G\}$ span H and H', respectively, and if $(U(g)\zeta, \zeta)_H = (U'(g)\zeta', \zeta')_{H'}$ on G, then $\{U(g), H\}$*

10) Cf. [8].

11) $(\cdot, \cdot)_H$ denotes the inner product in H.

and $\{U'(g), H'\}$ *are unitary equivalent, i.e., there exists a unitary transformation* T *from* H *onto* H' *such that* $U(g)=T^{-1}U'(g)T$.

In this paragraph, we shall prove that, for every (measurable) p.d. function, there exists a simple continuous unitary representation in such a manner that this correspondence is reciproque to that one stated in THEOREM 1.[12]

THEOREM 3. *For every measurable p.d. function* $f(g)$ *there exists a continuous unitary representation* $\{U(g), H\}$ *and an element* ζ *of* H *such that* $\{U(g)\zeta \mid g \in G\}$ *spans* H *and*

(1) $\qquad (U(g)\zeta, \zeta)_H = f(g)$ *almost everywhere in* G.

PROOF. We shall divide our proof into four steps:

(1°) *Construction of* H. Put $L_f=\{x \mid F_f(yx)=0$ for all y of $A\}$; then L_f is a closed left-ideal in A. Hence we can make the factor space (as a Banach space) A/L_f; we shall denote by $[x]$ the class containing x. Define

(2) $$([x], [y])_f = F_f(y^* x),$$

for x, y of A; it is easy to verify that this definition is possible, and, by means of Schwarz' inequality, that (2) is a (positive) inner product in A/L_f. At the same time it is proved that

(3) $$L_f=\{x \mid F_f(x^* x)=0\}.$$

Let $H\equiv H_f$ be the completion of A/L_f by the norm $\|[x]\|_f^2=([x], [x])_f$. Then H is a Hilbert space, and the mapping from x of A to $[x]$ of H is continuous.

(2°) *Construction of* $U(g)$. Define $U(g)$, for every g of G, as the mapping on A/L_f which transforms $[x]$ to $[x_g]$, where $x_g(h)=x(g^{-1}h)$; this definition is possible since every left-ideal in A, and consequently L_f in particular, contains x_g together with x for all g. Then $U(g)$ leaves the inner product (2) invariant, since $(y_g)^*(x_g)=y^*x$. Hence $U(g)$ can be extended to a unitary mapping on H. That this extended $U(g)$ is strongly continuous follows from the continuity of

12) This is easily proved for continuous p.d. functions, but it is necessary in the following to prove it for all measurable, p.d. functions, and in order to do that we shall have to make use of the group algebra over G.

$U(g)[x]$ in H for every x of A, and this, in turn, follows from the facts that the mapping $[x]$ is continuous from A into H and that x_a is continuous from G into $L^1(G)$.

(3°) *Representation of* A. Define S_x, for every x of A, as follows:

(4) $$S_x\varphi = \int x(g)U(g)\varphi dg \text{ [13]}, \text{ for } \varphi \text{ of } H.$$

Then $\|S_x\varphi\|_H \leq \|x\|_A \cdot \|\varphi\|_H$; hence S_x is a uniformly continuous representation of A by operators on H. From (4) and the unitarity of $U(g)$ it follows that $S_{x^*} = S_x{}^*$; and a simple calculation together with the fact that $\{[z] \mid z \in A\}$ is dense in H implies that

(5) $$S_x[y] = [x \cdot y] \qquad \text{for } y \text{ of } A.$$

Now, let $\{e_\alpha\}$ be the approximate identity of A. Then $\{[e_\alpha]\}$ converges weakly in H [14], since $([x], [e_\alpha])_H = F_f(e_\alpha x)$ converges to $F_f(x)$ for every x of A, and since on the other hand $\{[x] \mid x \in A\}$ is dense and $\{[e_\alpha]\}$ is bounded in H. Let ζ be the limit of $\{[e_\alpha]\}$. Then

(6) $$([x], \zeta)_H = F_f(x).$$

From (5) and (6) follows that

(7) $$S_x\zeta = [x],$$

because, for every y of A,

$$([y], S_x\zeta) = (S_x{}^*[y], \zeta) = ([x^*y], \zeta) = F_f(x^*y) = ([y], [x]).$$

By (6) and (7) it holds that

(8) $$(S_x\zeta, \zeta)_H = F_f(x).$$

(4°) *Proof of* (1). From (4), (7) and the definition of the integral, it is proved that $\{U(g)\zeta \mid g \in G\}$ spans H: that is, if $(U(g)\zeta, \varphi) = 0$ for all g, then φ must be the zero element.

From (8) it follows that

$$\int x(g)(U(g)\zeta, \zeta)dg = (S_x\zeta, \zeta) = F_f(x):$$

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13) This integration shall be understood as in the sense of Pettis, i.e.,
$$\left(\int x(g)U(g)\varphi dg, \psi\right)_H = \int (x(g)U(g)\varphi, \psi)_H dg,$$
for every ψ of H.

on the other hand $F_f(x) = \int x(g) f(g)\, dg$. Therefore $f(g) = (U(g)\zeta, \zeta)_H$ almost everywhere in G, q. e. d.

COROLLARY. *Every measurable p. d. function coincides with a continuous one almost everywhere in G.*

For the later application we shall prove here

LEMMA. *For every non-zero x of A, there exists a positive functional $F(\cdot)$ such that $F(x^* x) \neq 0$.*

PROOF. (1°) When $x(\cdot)$ belongs not only to $L^1(G)$ but also to $L^2(G)$, $x^* x(\cdot)$ is bounded and continuous on G and $x^* x(e) \neq 0$. Hence if we put

$$f(g) = \int c_V(gh) c_V(h)\, dh \quad [15],$$

where V is sufficiently small neighbourhood of e, then F_f satisfies the required condition.

(2°) In the general case, let $y(\cdot)$ belong both to $L^1(G)$ and to $L^2(G)$ and yx be $\neq 0$. Then, from (1°), there exists an F such that $F((yx)^*(yx)) \neq 0$, since $yx(\cdot) \in L^2(G)$. Then, from (3), $F(x^* x) \neq 0$, too, q. e. d.

5. IRREDUCIBLE REPRESENTATION. Let Ω be the family of all p. d. functions, each of which is bounded by 1 in absolute value almost everywhere. Let us consider Ω as a subset of the conjugate space of A. Then, by COROLLARY in 4, we can assume that every element $f(\cdot)$ of Ω is continuous and $|f(g)| \leq f(e) \leq 1$. Ω is bounded, convex and weakly compact. Hence, according to the theorem by M. Krein and D. Milman [5] every point of Ω is weakly approximated by convex combinations of *extreme* ones, where we mean under an extreme point of Ω such a point belonging to Ω that is not an inner point of the segment combining any pair of points of Ω. It is easy to see that every extreme f is of norm one, except the zero element.

We shall establish in THEOREMS 4 and 5 the correspondence be-

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14) Moreover it can be proved that this convergence is the strong one.

15) $c_V(\cdot)$ denotes, as in 3 (3°), the characteristic function of V.

tween irreducible representations and extreme p. d. functions. [16])

THEOREM 4. *If* $\{U(g), H\}$ *is irreducible, then* $g(g) \equiv (U(g)\zeta, \zeta)_H$ *is an extreme p.d. function for every* ζ *of norm one.*

PROOF. Suppose that

$$f(g) = f_1(g) + f_2(g),$$

where both f_1 and f_2 belong to Ω. It is sufficient to show that $f_1 = \lambda f$ for some λ.

(1°) Let

$$\varphi = \sum_k \alpha_k U(g_k)\zeta, \quad \psi = \sum_l \beta_l U(h_l)\zeta, \tag{9}$$

where α_k, β_l are complex numbers and g_k, $h_l \in G$, $(1 \leqq k \leqq n,\ 1 \leqq l \leqq m)$; and define

$$[\varphi, \psi] = \sum_{k,l} \alpha_k \overline{\beta}_l f_1(h_l^{-1} g_k). \tag{10}$$

Then $[\varphi, \psi]$ is positive, bilinear and bounded for these φ's and ψ's:

$$\|\varphi\|_H^2 = \sum_{k,j} \overline{\alpha}_k \alpha_j f(g_k^{-1} g_j) \geqq \sum_{k,j} \overline{\alpha}_k \alpha_j f_1(g_k^{-1} g_j) \geqq 0.$$

Therefore, since these φ's are dense in H, there exists a positive-definite operator P such that

$$[\varphi, \psi] = (\varphi, P\psi)_H. \tag{11}$$

From (10) and (11) follows

$$f_1(g) = [U(g)\zeta, \zeta] = (U(g)\zeta, P\zeta). \tag{12}$$

(2°) Let φ, ψ be as in (9); then, from (10) and (11),

$$\begin{aligned}(PU(g)\varphi, \psi) &= (U(g)\varphi, P\psi) = [U(g)\varphi, \psi] \\ &= \sum \alpha_k \overline{\beta_l} f_2(h_l^{-1} g g_k) = \sum \alpha_k \overline{\beta_l} f_1((g^{-1}h_l)^{-1} g_k) \\ &= (P\varphi, U(g^{-1})\psi) = (U(g)P\varphi, \psi).\end{aligned}$$

Hence $PU(g) = U(g)P$.

(3°) Therefore necessarily $P = \lambda I$. where $\lambda \geqq 0$.

Hence it follows from (12) that $f_1(g) = \lambda f(g)$, q. e. d.

THEOREM 5. *If* $\{U(g), H\}$ *is a continuous unitary representation such that* $\{U(g)\zeta \mid g \in G\}$ *spans H, and if* $f(g) \equiv (U(g)\zeta, \zeta)_H$ *is an extreme*

16) In proving these theorems we do not refer the results obtained in 4.

p.d. function in g, then $\{U(g), H\}$ is irreducible.

PROOF. If there exists a projection which commutes with every $U(g)$, then

$$f(g)=(U(g)\zeta, \zeta)_H=(U(g)P\zeta, P\zeta)_H+(U(g)(I-P)\zeta, (I-P)\zeta)_H \text{ [17]}.$$

Since $f(g)$ is extreme and both $(U(g)P\zeta, P\zeta)/(P\zeta, P\zeta)$ and $(U(g)(I-P)\zeta, (I-P)\zeta/((I-P)\zeta, (I-P)\zeta)$ belong to Ω, it follows that

$$(U(g)\zeta, P\zeta)=\lambda(U(g)\zeta, \zeta)$$

and

$$(U(g)\zeta, (I-P)\zeta)=(I-\lambda)(U(g)\zeta, \zeta)$$

for some λ. Hence, for $\varphi=\sum_k \alpha_k U(g_k)\zeta$,

$$\lambda(\varphi, \varphi)=\lambda\sum \alpha_k\overline{\alpha}_j(U(g_j^{-1}g_k)\zeta, \zeta)=\sum \alpha_k\overline{\alpha}_j(U(g_j^{-1}g_k)\zeta, P\zeta)$$
$$=(P\varphi, P\varphi),$$

and consequently, for arbitrary φ of H,

$$\lambda(\varphi, \varphi)=(P\varphi, P\varphi);$$

analogously

$$(1-\lambda)(\varphi, \varphi)=((I-P)\varphi, (I-P)\varphi).$$

From these two identities it follows that $P=O$ [18] or $P=I$, q.e.d.

Now, refering the results obtained in 3, we can prove

THEOREM. 6. *If a is an element of G different from e, then there exists an irreducible unitary representation $\{U(g), H\}$ such that $U(a) \neq I$.*

PROOF. There exists an x of A such that $x_a \neq x$. It follows that, by means of LEMMA, $F_f((x_a-x)^*(x_a-x))\neq 0$ for some f of Ω, and consequently, for some extreme f. Hence $U(a)[x]\neq[x]$ on H_f, in the proof of THEOREM 3, q.e.d.

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17) I denotes the identity operator.
18) O denotes the zero operator.

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