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ON SQUARE-INTEGRABLE $\bar{\partial}$-COHOMOLOGY SPACES ATTACHED TO HERMITIAN SYMMETRIC SPACES

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0. Introduction

In the theory of unitary representations of Lie groups, it seems important to realize the representation spaces geometrically (cf. [9] (b)). We shall work on this problem for some types of real semi-simple Lie groups in this paper.

In §1, we shall give a method to construct Hilbert spaces for $\bar{\partial}$-cohomology spaces associated with a hermitian vector bundle over a hermitian manifold. This will be applied to the hermitian vector bundles constructed from the groups in §3. In §4 we shall discuss the problem when the spaces do not vanish. Theorem 4.3 is our main result in this paper, which is a non-vanishing theorem for a certain type of representation spaces. This can be considered as a first step to attack the Langlands' conjecture given in [10].

1. Generalities

Let M be a hermitian manifold of dimension n, and E a holomorphic vector bundle over M. We denote by $C^{p,q}(E)$ the space of all C^∞-differentiable E-valued forms of type (p,q) on M, and by $C_c^{p,q}(E)$ the subspace of $C^{p,q}(E)$ composed of forms with compact supports. The hermitian structure of M defines the real operator $*$ on forms on M as usual, and we extend this operator $*$ complex linearly to $C^{p,q}(E)$. Then $*$ maps $C^{p,q}(E)$ to $C^{n-q,n-p}(E)$ (cf. [1] 1.4 a) β) p. 86).

Now suppose that a hermitian metric is given on each fibre of E which depends differentiably on the base space M (cf. [1] 1.2 a)). The hermitian metric of M and the hermitian metric of the bundle E give a conjugate-linear isomorphism

$$\sharp : C^{p,q}(E) \to C^{q,p}(E^*)$$

where E^* is the complex dual bundle of E (cf. [1] 1.4 1) β)).

Now we define a pre-Hilbert metric on $C_c^{p,q}(E)$ by

$$(\varphi, \psi) = \int_M \varphi \wedge *\sharp\psi$$

for φ, ψ in $C_c^{p,q}(E)$.

The usual d''-operator maps $C^{p,q}(E)$ into $C^{p,q+1}(E)$. We define

$$\delta'' : C^{p,q}(E) \to C^{p,q-1}(E) \quad \text{by}$$
$$\delta''\varphi = -\#^{-1} {*} d'' {*} \# .$$

Then, we have

$$(d''\varphi, \psi) = (\varphi, \delta''\psi)$$

for $\varphi \in C_c^{p,q}(E)$, $\psi \in C_c^{p,q+1}(E)$. Let $L_2^{p,q}(E)$ be the completion of $C_c^{p,q}(E)$ with respect to the inner product (,). We denote by $\bar\partial_0$ the restriction of d'' to $C_c^{p,q}(E)$ and by ϑ_0 the restriction of δ'' to $C_c^{p,q}(E)$. Define

$$\bar\partial = (\vartheta_0)^* \quad \text{and} \quad \vartheta = (\bar\partial_0)^*$$

where ( )* denotes the adjoint operator of ( ) with respect to the inner product (,). Then $\bar\partial$ (resp. ϑ) is a closed, densely defined operator of $L_2^{p,q}(E)$ into $L_2^{p,q+1}(E)$ (resp. $L_2^{p,q-1}(E)$). Let $D_{\bar\partial}^{p,q}$ (resp. $D_{\vartheta}^{p,q}$) be the domain of the operator $\bar\partial$ (resp. ϑ) in $L_2^{p,q}(E)$. We put

$$Z_{\bar\partial}^{p,q}(E) = \{\varphi \in D_{\bar\partial}^{p,q};\, \bar\partial\varphi = 0\} ,$$
$$Z_{\vartheta}^{p,q}(E) = \{\varphi \in D_{\vartheta}^{p,q};\, \vartheta\varphi = 0\} .$$

Since $\bar\partial$ and ϑ are closed operators, $Z_{\bar\partial}^{p,q}(E)$ and $Z_{\vartheta}^{p,q}(E)$ are closed in $L_2^{p,q}(E)$. Let $B_{\bar\partial}^{p,q}(E)$ and $B_{\vartheta}^{p,q}(E)$ be the closure of $\bar\partial(D_{\bar\partial}^{p,q-1})$ and $\vartheta(D_{\vartheta}^{p,q+1})$, respectively. We define, finally, the square-integrable $\bar\partial$-cohomology spaces attached to the hermitian vector bundle E by

$$H_2^{p,q}(E) = Z_{\bar\partial}^{p,q}(E) \ominus B_{\bar\partial}^{p,q}(E)$$

where $\ominus$ denotes the orthogonal complement of $B_{\bar\partial}^{p,q}(E)$. It is easy to see that

$$H_2^{p,q}(E) = Z_{\bar\partial}^{p,q} \cap Z_{\vartheta}^{p,q} .$$

Since $Z_{\bar\partial}^{p,q}(E)$ and $Z_{\vartheta}^{p,q}(E)$ are closed in $L_2^{p,q}(E)$, $H_2^{p,q}(E)$ has canonically the structure of a Hilbert space.

Now we have the following orthogonal decomposition theorem.

Theorem 1.1. $L_2^{p,q}(E) = H_2^{p,q}(E) \oplus B_{\bar\partial}^{p,q}(E) \oplus B_{\vartheta}^{p,q}(E)$.

For a proof, see [8] (1.1.5), p. 92.

And we have the following Serre's duality theorem for these cohomology spaces.

Theorem 1.2. $H_2^{p,q}(E) = H_2^{n-p,n-q}(E^*)$ (*isomorphic as Hilbert spaces*).

In order to prove this theorem, we have only to notice that the following diagram is commutative.

$$\begin{array}{ccc} C_c^{p,q}(E) & \xrightarrow{\;*\#\;} & C_c^{n-p,n-q}(E^*) \\ \vartheta_0 \big\downarrow \big\uparrow \bar\partial_0 & & \bar\partial_0 \big\downarrow \big\uparrow \vartheta_0 \\ C_c^{p,q-1}(E) & \xrightarrow{\;\varepsilon*\#\;} & C_c^{n-p,n-q+1}(E^*) \end{array}$$

where $\varepsilon=(-1)^{p+q}$.

2. A consequence of completeness of the hermitian metric on M

Put

$$N_{d''}^{p,q}(E) = \{\varphi\in C^{p,q}(E);\ d''\varphi = 0\}$$
$$N_{\delta''}^{p,q}(E) = \{\varphi\in C^{p,q}(E);\ \delta''\varphi = 0\}\ .$$

Proposition 2.1. *If the hermitian metric on M is complete, then we have*

$$N_{d''}^{p,q}(E)\cap L_2^{p,q}(E)\subset Z_{\bar{\partial}}^{p,q}(E)$$
$$N_{\delta''}^{p,q}(E)\cap L_2^{p,q}(E)\subset Z_{\vartheta}^{p,q}(E)\ .$$

In order to prove the proposition, we need some results due to Andreotti-Vesentini.

We assume that the hermitian metric on M is complete. We take a C^∞-differentiable function μ on R satisfying

1) $0\leqslant\mu\leqslant 1$ on R
2) $\mu(t)=1$ for $t\leqslant 1$
3) $\mu(t)=0$ for $t\geqslant 2$.

We fix a point o in M, and for each point p in M, we denote by $d(p)$ the distance from o to p in M, and set

$$w_k(p) = \mu(d(p)/k) \qquad \text{for} \quad k=1, 2, 3, \cdots .$$

We simply write $|\varphi|$ instead of $\sqrt{*(\varphi\wedge*\sharp\varphi)}$.

Lemma 2.1. (*Andreotti-Vesentini*) *Under the above notations, there exists a positive number c, depending only on μ, such that*

1) $|d''w_k\wedge\varphi| \leqslant \dfrac{c}{k}|\varphi|$,

2) $|d'w_k\wedge*\varphi| \leqslant \dfrac{c}{k}|\varphi|$

for all $\varphi\in C^{p,q}(E)$.

For a proof, see [1] 2.6 (13) and (14), p. 91.

Since the function w_k has a compact support, we remark that $w_k\varphi\in D_{\bar{\partial}}^{p,q}\cap D_{\vartheta}^{p,q}$ for every $\varphi\in C^{p,q}(E)$ and that

$$d''(w_k\varphi) = \bar{\partial}(w_k\varphi)$$
$$\delta''(w_k\varphi) = \vartheta(w_k\varphi)\ .$$

Now we come to the proof of Proposition 2.1. Let φ be in $N_{d''}^{p,q}\cap L_2^{p,q}$. By the above remarks

$$\begin{aligned}\bar{\partial}(w_k\varphi) &= d''(w_k\varphi)\\ &= d''w_k\wedge\varphi+w_kd''\varphi\\ &= d''w_k\wedge\varphi\ .\end{aligned}$$

Hence, by Lemma 2.1, we have

$$|\bar{\partial}(w_k\varphi)| \leqslant \frac{c}{k}|\varphi| .$$

Putting $\varphi_k = w_k\varphi$, we know that $\bar{\partial}\varphi_k$ converges strongly to 0. On the other hand, φ_k converges strongly to φ by the choice of w_k. Since $\bar{\partial}$ is a closed operator, we see that φ is in $D_{\bar{\partial}}^{p,q}$ and $\bar{\partial}\varphi = 0$. This shows $\varphi \in Z_{\bar{\partial}}^{p,q}(E)$. In the same way, we have $N_{\delta''}^{p,q}(E) \cap L_2^{p,q}(E) \subset Z_{\vartheta}^{p,q}(E)$. This completes the proof of Proposition 2.1.

Now we need also the following result due to Andreotti-Vesentini.

Lemma 2.2. (*Andreotti-Vesentini*) *Let the metric on M be complete. If* $\varphi \in L_2^{p,q}(E) \cap C^{p,q}(E)$ *is such that* $\square\varphi = 0$, *then* $\varphi \in N_{d''}^{p,q}(E) \cap N_{\delta''}^{p,q}(E)$ *where* $\square = d''\delta'' + \delta''d''$.

For a proof, see [1] Proposition 7, p. 93.

From Proposition 2.1 and Lemma 2.2 we have the following proposition.

Proposition 2.2. *Let the metric on M be complete. If* $\varphi \in L_2^{p,q}(E) \cap C^{p,q}(E)$ *is such that* $\square\varphi = 0$, *then* $\varphi \in H_2^{p,q}(E)$.

3. Definition of $H_2^{p,q}(E_\Lambda)$

Let G be a connected semisimple Lie group which has a faithful representation and K a maximal compact subgroup of G. We put $M = G/K$. In this paper we shall be concerned with the case where $M = G/K$ is a bounded symmetric domain. We denote by $\mathfrak{g}$ the Lie algebra of left invariant vector fields on G, and by $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to the subgroup K. Let $\mathfrak{g}^C$ be the complexification of $\mathfrak{g}$. We put

$$\mathfrak{p} = \{Y \in \mathfrak{g};\ B(X, Y) = 0 \quad \text{for all } X \in \mathfrak{k}\}$$

where B denotes the Killing form of the Lie algebra $\mathfrak{g}^C$. Then we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \quad \mathfrak{k} \cap \mathfrak{p} = (0), [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} .$$

We denote by π the canonical projection mapping of G onto $M = G/K$. Put $\pi(e) = p_0$ where e is the identity element of G. We may identify $\mathfrak{p}$ with the tangent vector space T_{p_0} of M at the point p_0 by the mapping $Y \to d\pi_e Y_e (Y \in \mathfrak{p})$. Let T_p^C be the complexification of the tangent space T_p $(p \in M)$. For any subset $\mathfrak{m}$ of $\mathfrak{g}^C$, we denote by $\mathfrak{m}^C$ the complex subspace of $\mathfrak{g}^C$ spanned by $\mathfrak{m}$. We denote by T_p^+ (resp. T_p^-) the set of all holomorphic (resp. anti-holomorphic) tangent vectors in T_p^C. Put

$$\mathfrak{p}_- = \{Y \in \mathfrak{p}^C;\ d\pi_e Y_e \in T_{p_0}^+\} ,$$
$$\mathfrak{p}_+ = \{Y \in \mathfrak{p}^C;\ d\pi_e Y_e \in T_{p_0}^-\} .$$

Then since we assumed that the complex structure on M is G-invariant, we have

$d\pi_g Y_g \in T^+_{\pi(g)}$ (resp. $d\pi_g Y_g \in T^-_{\pi(g)}$) if $Y \in \mathfrak{p}_-$ (resp. $Y \in \mathfrak{p}_+$). Moreover, it is well known (see [6]) that

$$(2,1) \qquad \begin{aligned} &[\mathfrak{p}_+, \mathfrak{p}_+] = (0), \quad [\mathfrak{p}_-, \mathfrak{p}_-] = (0). \\ &[\mathfrak{k}^C, \mathfrak{p}_+] \subset \mathfrak{p}_+, \quad [\mathfrak{k}^C, \mathfrak{p}_-] \subset \mathfrak{p}_-. \end{aligned}$$

We put $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\,\mathfrak{p}$. Then $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}^C$. We denote by $*$ the multiple of -1 of the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}_u$. We define the inner product (,) in $\mathfrak{g}^C$ by

$$(X, Y) = B(X, Y^*) \quad (X, Y \in \mathfrak{g}^C).$$

Now we know that $\mathfrak{k}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\sum$ be the system of all non-zero roots of $\mathfrak{g}^C$ with respect to the Cartan subalgebra $\mathfrak{h}^C$. It is easy to see that for each $\alpha \in \sum$ we can choose an eigenvector X_α of the root α such that $(X_\alpha, X_\alpha) = 1$. Then we have $\bar{X}_\alpha = a_\alpha X_{-\alpha}$ for some $a_\alpha \in \boldsymbol{C}$ where $-$ denotes the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Since $[\mathfrak{h}^C, \mathfrak{k}^C] \subset \mathfrak{k}^C$ and $[\mathfrak{h}^C, \mathfrak{p}^C] \subset \mathfrak{p}^C$, it is clear that either $X_\alpha \in \mathfrak{k}^C$ or $X_\alpha \in \mathfrak{p}^C$ $(\alpha \in \sum)$. A root $\alpha \in \sum$ is called compact or non-compact according to $X_\alpha \in \mathfrak{k}^C$ or $X_\alpha \in \mathfrak{p}^C$. Since $\mathfrak{h}^C \subset \mathfrak{k}^C$, it follows from (2,1) that there exists a subset $P_n \subset \sum$ such that $\mathfrak{p}_+ = \sum_{\alpha \in P_n} CX_\alpha$. Moreover, it is obvious that we can introduce a linear order on $\sum$ such that the set P of all positive roots contains P_n. Since $\mathfrak{p}_+ = \bar{\mathfrak{p}}_-$ and $\bar{X}_\alpha = a_\alpha X_{-\alpha}$ $(\alpha \in \sum)$, we see that $\mathfrak{p}_- = \sum_{\alpha \in P_n} CX_{-\alpha}$. Since G is assumed to have a faithful representation, there exists a complex form G^C of G, that is, a complex Lie group with the Lie algebra $\mathfrak{g}^C$ which has G as a real analytic subgroup corresponding to $\mathfrak{g} \subset \mathfrak{g}^C$. We denote by K^C (resp. P_+, P_-) the complex analytic subgroup of G^C corresponding to $\mathfrak{k}^C$ (resp. $\mathfrak{p}_+$, $\mathfrak{p}_-$). Put $U = K^C P_+$. Then U is a complex analytic group of G^C with the Lie algebra $\mathfrak{k}^C + \mathfrak{p}_+$ and P_+ is a normal subgroup of U. Consider the complex homogeneous space G^C/U. Then we know (see [3]) that the G-orbit of U is open in G^C/U and can be identified with $M = G/K$, for $U \cap G = K$. Moreover, we can show that the identification is compatible with both complex structures of G^C/U and G/K; i.e. the above complex structure of G/K coincides with the one as the open submanifold of G^C/U.

For a linear form λ on $\mathfrak{h}^C$, we shall denote by H_λ the element of $\mathfrak{h}^C$ such that $B(H_\lambda, H) = \lambda(H)$ for all $H \in \mathfrak{h}^C$; the inner product (λ, μ) of two linear forms λ, μ means the value (H_λ, H_μ). Let $\mathfrak{F}$ denote the set of all integral forms on $\mathfrak{h}^C$, i.e. $\mathfrak{F}$ is the set of all linear forms λ on $\mathfrak{h}^C$ such that $\frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$ are integers for all $\alpha \in \sum$. We put

$$\mathfrak{F}' = \{\lambda \in \mathfrak{F}\,;\, (\lambda + \rho, \alpha) \neq 0 \quad \text{for all } \alpha \in \textstyle\sum\},$$
$$\mathfrak{F}'_0 = \{\lambda \in \mathfrak{F}'\,;\, (\lambda + \rho, \alpha) > 0 \quad \text{for all } \alpha \in P_k\}$$

where $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha$ and P_k is the set of all compact positive roots. For any $\Lambda\in\mathfrak{F}_0'$ we denote by τ_Λ the irreducible unitary representation of K with the highest weight Λ on the representation space V_Λ. Then one knows that τ_Λ is uniquely extended to a holomorphic representation of K^C. Since P_+ is normal in U, we can extend τ_Λ uniquely to a holomorphic representation of U such that $\tau_\Lambda(g)=1$ for all $g\in P_+$. In the following, we denote by the same notation τ_Λ this extended representation. Let τ_Λ^* denote the contragradient representation to τ_Λ on the dual space V_Λ^* of V_Λ. G^C is the principal fibre bundle over the base space G^C/U with the structure group U. Let $\tilde{E}_\Lambda$ denote the associated holomorphic vector bundle with the holomorphic representation τ_Λ^*. We denote by E_Λ the portion of $\tilde{E}_\Lambda$ over the open submanifold G/K; i.e. E_Λ is the induced bundle by the injection mapping of G/K into G^C/U. Since we started from the unitary representation of K, we have the canonical reduction of the structure group of the vector bundle E_Λ to the unitary subgroup $\tau_\Lambda^*(K)$. So we have the canonical hermitian metrics on the fibers of E_Λ. Making use of the linear isomorphism of $\mathfrak{p}$ onto $T_{\pi(g)}$ induced by the projection mapping π, we can introduce an inner product (,) in $T_{\pi(g)}$ by

$$(d\pi_g X_g, d\pi_g Y_g) = (X, Y) \qquad X, Y\in\mathfrak{p}\,.$$

Then it is easy to see that the inner products thus introduced on the tangent space T_p $(p\in M)$ are invariant by the actions by G and that it defines the Kähler metric on $M=G/K$. Thus we have constructed a hermitian vector bundle $E_\Lambda$on M for each $\Lambda\in\mathfrak{F}_0'$. Finally we denote by $H_2^{0,q}(E_\Lambda)$ the square-integrable $\bar{\partial}$-cohomology spaces attached to E_Λ.

4. Main theorem

In this section we shall discuss "the non-vanishing theorem" of the cohomology spaces $H_2^{0,q}(E_\Lambda)$ defined in the previous section. Fix a $\Lambda\in\mathfrak{F}_0'$, once for all. Define τ_Λ, V_Λ, τ_Λ^* as in Section 3. Let $C^0(G, V_\Lambda^*)$ denote the set of all V_Λ^*-valued functions on G. For any $f\in C^0(G, V_\Lambda^*)$ and $v\in V_\Lambda$, we denote by $f_v(x)$ $(x\in G)$ the value of $f(x)$ at v. Then $x\rightarrow f_v(x)$ $(x\in G)$ defines a complex valued function on G. We put

$$C^\infty(G, V_\Lambda^*) = \{f\in C^0(G, V_\Lambda^*)\ ;\ f_v\in C^\infty(G)\}\,.$$

Fix an orthonormal base $(v_1, \cdots, v_r)$ of V_Λ where $r=\dim V_\Lambda$. Let $(v_1^*, \cdots, v_r^*)$ be its dual base of V_Λ^*. In the following we shall use the notation defined in Section 3 without further comment. Let P_k be the set of all compact roots with respect to the linear order that is introduced in Section 3. Fix an orthonormal base $(H_1, \cdots, H_l)$ of $\mathfrak{h}^C$ where $l=\dim_C\mathfrak{h}^C=\text{rank } G$. Let $P_n=\{\alpha_1, \cdots, \alpha_m\}$ and $P_k=\{\alpha_{m+1}, \cdots, \alpha_{m+k}\}$. To simplify the notation, we put henceforth

$$z_i \begin{cases} = X_{\alpha i} & (i=1, \cdots, m+k)\,, \\ = H_i & (i=m+k+1, \cdots, m+k+l)\,, \\ = z^*_{n-i+1} & (i=m+k+l+1, \cdots, n)\,. \end{cases}$$

Then we have

$$z_i \begin{cases} \in \mathfrak{p}_+ & (i=1, \cdots, m) \\ \in \mathfrak{k}^C & (i=m+1, \cdots, n-m) \\ \in \mathfrak{h}^C & (i=m+k+1, \cdots, m+k+l) \\ \in \mathfrak{p}_- & (i=n-m+1, \cdots, n)\,. \end{cases}$$

Now we define an injection mapping

$$\eta : C^{0,q}(E_\Lambda) \longrightarrow \Lambda^q \mathfrak{p}_- \otimes C^\infty(G) \otimes V^*_\Lambda$$

by putting

$$\eta(\varphi) = \sum_{1 \leqslant i_1 < < i_q \leqslant m} \sum_{j=1}^{r} \Lambda^q_{s=1} z^*_{is} \otimes \varphi(z^1_{il}, \cdots, z^1_{iq})_{vj} \otimes v^*_j\,.$$

We denote by Ad^q (resp. ad^q) the representation of K (resp. $\mathfrak{k}^C$) in $\Lambda^q \mathfrak{p}_-$ induced by the adjoint action of K (resp. $\mathfrak{k}^C$) in $\mathfrak{g}^C$. We define a representation R of G on $C^\infty(G)$ by

$$(R(g)f)(x) = f(xg) \qquad (x \in G)$$

for each $g \in G$. We denote by $R|K$ the restriction of R to K. Let $\sigma^q = \mathrm{Ad}^q \otimes R|K$ denote the representation obtained by the tensor product of Ad^q and $R|K$. We put

$$\begin{aligned} C^q(G)_\Lambda &= \Lambda^q \mathfrak{p}_- \otimes C^\infty(G) \otimes V^*_\Lambda \\ C^q_c(G)_\Lambda &= \Lambda^q \mathfrak{p}_- \otimes C^\infty_c(G) \otimes V^*_\Lambda \quad \text{and} \\ L^q_2(G)_\Lambda &= \Lambda^q \mathfrak{p}_- \otimes L_2(G) \otimes V^*_\Lambda\,. \end{aligned}$$

Define

$$\begin{aligned} C^q(G)^0_\Lambda &= \{u \in C^q(G)_\Lambda;\ (\sigma^q \otimes \tau^*_\Lambda)(k) u = u \ \text{ for all } k \in K\}\,, \\ C^q_c(G)^0_\Lambda &= \{u \in C^q_c(G)_\Lambda;\ (\sigma^q \otimes \tau^*_\Lambda)(k) u = u \ \text{ for all } k \in K\} \ \text{ and} \\ L^q_2(G)^0_\Lambda &= \{u \in L^q_2(G)_\Lambda;\ (\sigma^q \otimes \tau^*_\Lambda)(k) u = u \ \text{ for all } k \in K\}\,. \end{aligned}$$

Then it is easy to see that η maps $C^{0,q}(E_\Lambda)$ isomorphically onto $C^q(G)^0_\Lambda$. Moreover, we see that the mapping η induces the isometry of $L^{0,q}_2(E_\Lambda)$ onto $L^q_2(G)^0_\Lambda$. There we define the metric in $L^q_2(G)_\Lambda$ by

$$(u, u') = \sum_{1 \leqslant i_1 < \cdots < i_q \leqslant m} \sum_{j=1}^{r} \int_G f_{i_1 \cdots i_q, j}(g) \overline{f'_{i_1 \cdots i_q, j}(g)}\, dg$$

where

$$u = \sum_{1 \leqslant i_1 < \cdots < i_q \leqslant m} \sum_{j=1}^{r} z^*_{i_1} \wedge \cdots \wedge z^*_{i_q} \otimes f_{i_1 \cdots i_q, j} \otimes v^*_j\,,$$

$$u' = \sum_{1 \leqslant i_1 < \ldots < i_q \leqslant m} \sum_{j=1}^{r} z^*_{i_1} \wedge \cdots \wedge z^*_{i_q} \otimes f'_{i_1 \cdots i_q, j} \otimes v^*_j\,.$$

For any $X \in \mathfrak{g}$ we define $\nu(X)$ by

$$\nu(X)f = Xf \qquad (f \in C^\infty(G))$$

where

$$(Xf)(g) = \left[\frac{d}{dt} f(g \exp tX)\right]_{t=0} \qquad (g \in G).$$

Then $X \to \nu(X)$ $(X \in \mathfrak{g})$ is a representation of $\mathfrak{g}$ on $C^\infty(G)$. This representation ν is complex linearly extended to a representation of $\mathfrak{g}^C$ on $C^\infty(G)$. Let θ denote the representation of $\mathfrak{g}^C$ in $\Lambda\mathfrak{g}^C$ induced by the adjoint representation of $\mathfrak{g}^C$. We put

$$\theta_\nu(z) = \theta(z) \otimes 1 + 1 \otimes \nu(z) \qquad (z \in \mathfrak{g}^C).$$

Then θ_ν is a representation of $\mathfrak{g}^C$ on $\Lambda\mathfrak{g}^C \otimes C^\infty(G)$.

Theorem 4.1. *Under the above defined notations, we have*

$$\eta(\Box\varphi) = \frac{1}{2}\{(\Lambda+2\rho, \Lambda) - 1 \otimes R^\nu \otimes 1\}\eta(\varphi) \qquad (\varphi \in C^{0,q}(E_\Lambda))$$

where $R^\nu = \sum_{i=1}^{n} \nu(z_i^*)\nu(z_i)$.

A proof of this theorem will be given in the next section.

Let r be the right regular representation of G on $L_2(G)$; i.e.

$$(r(g)f)(x) = f(xg) \qquad (x \in G)$$

for any $g \in G$ and $f \in L_2(G)$.

Owing to the profound result of Harish-Chandra ([5] (c), Theorem 16), for any $\lambda \in \mathfrak{F}'$, we can find a closed subspace $\mathfrak{H}_\lambda$ of $L_2(G)$ invariant by r such that the restriction π_λ of r to $\mathfrak{H}_\lambda$ is irreducible and $\chi_\lambda(\Omega) = (\lambda+2\rho, \lambda)$ where χ_λ is the infinitesimal character of the irreducible unitary representation π_λ and Ω denotes the Casimir operator of $\mathfrak{g}$.

Now we consider the irreducible unitary representation π_Λ of G on $\mathfrak{H}_\Lambda$ defined above. We denote by $\pi_\Lambda | K$ the restriction of π_Λ to the subgroup K. Put

$$\sigma_\Lambda^q = \mathrm{Ad}^q \otimes \pi_\Lambda | K.$$

Then we have the following "non-vanishing theorem" for $H_2^{0,q}(E_\Lambda)$.

Theorem 4.2. *Let* $\Lambda \in \mathfrak{F}_0'$. *If* σ_Λ^q *contains the irreducible representation of* K *with the highest weight* Λ, *then* $H_2^{0,q}(E_\Lambda) \neq (0)$.

Proof. Let ξ_Λ denote the character of the representation τ_Λ. Define a projection operator e_Λ by

$$e_\Lambda = r\int_K \overline{\xi_\Lambda(k)}\,\sigma_\Lambda^q(k)\,dk$$

where dk denotes the normalized Haar-measure of K. We assume that σ_Λ^q contains the irreducible representation of K with the highest weight Λ.
Then we have

$$e_\Lambda(\wedge^q\mathfrak{p}_-\otimes\mathfrak{H}_\Lambda)\neq(0).$$

For each $\lambda\in\mathfrak{F}_0'$ we define a projection operator $e_\lambda(\pi_\Lambda)$ by

$$e_\lambda(\pi_\Lambda)=r\int_K\overline{\xi_\lambda(k)}\pi_\Lambda(k)\,dk.$$

Then it is clear that there exists a finite numbers of element, $\lambda_1\cdots,\lambda_s$ of $\mathfrak{F}_0'$ such that $e_\Lambda(\wedge^q\mathfrak{p}_-\otimes\mathfrak{H}_\Lambda)\subset\wedge^q\mathfrak{p}_-\otimes\sum_{i=1}^s e\lambda_i(\pi_\Lambda)\mathfrak{H}_\Lambda$. It follows from the result of Harish-Chandra (see [5] (b) Lemma 33, p. 108) that

$$e_\Lambda(\wedge^q\mathfrak{p}_-\otimes\mathfrak{H}_\Lambda)\subset\wedge^q\mathfrak{p}_-\otimes C^\infty(G).$$

It is easy to see that

$$R^\nu f=\chi_\Lambda(\Omega)f=(\Lambda+2\rho,\ \Lambda)f$$

for all $f\in C^\infty(G)\cap\mathfrak{H}_\Lambda$.
It follows from Theorem 4.1 that

$$\Box\eta^{-1}u=0\quad\text{for all}\quad u\in(e_\Lambda(\wedge^q\mathfrak{p}_-\otimes\mathfrak{H}_\Lambda)\otimes V_\Lambda^*)^0$$

where $(e_\Lambda(\wedge^q\mathfrak{p}_-\otimes\mathfrak{H}_\Lambda)\otimes V_\Lambda^*)^0=(e_\Lambda(\wedge^q\mathfrak{p}_-\otimes\mathfrak{H}_\Lambda)\otimes V_\Lambda^*)\cap C^q(G)_\Lambda^0$.
We notice that $\eta^{-1}(e_\Lambda(\wedge^q\mathfrak{p}_-\otimes\mathfrak{H}_\Lambda)\otimes V_\Lambda^*)^0\subset C^{0,q}(E_\Lambda)\cap L_2^{0,q}(E_\Lambda)$. It follows from Proposition 2.2 that

$$\eta^{-1}(e_\Lambda(\wedge^q\mathfrak{p}_-\otimes\mathfrak{H}_\Lambda)\otimes V_\Lambda^*)^0\subset H_2^{0,q}(E_\Lambda).$$

Since $(e_\Lambda(\wedge^q\mathfrak{p}_-\otimes\mathfrak{H}_\Lambda)\otimes V_\Lambda^*)^0\neq(0)$, we see that $H_2^{0,q}(E_\Lambda)\neq(0)$.
This proves the theorem.

We put

$$Q_\Lambda=\{\alpha\in P_n;\ (\Lambda+\rho,\ \alpha)>0\}.$$

And we denote by q_Λ the number of elements of the set Q_Λ. Let

$$\Phi_\Lambda=\wedge_{\gamma\in Q_\Lambda}X_{-\gamma}.$$

Proposition 4.1. *Φ_Λ is a weight vector belonging to the weight which is one of the lowest weight with respect to the representation ad^{q_Λ} of $\mathfrak{k}^C$.*

Proof. Since $[\mathfrak{k}^C,\mathfrak{p}_+]\subset\mathfrak{p}_+$, for any $\alpha\in P_n$ and $\beta\in P_k$ we have $\alpha+\beta\in P_n$ if $\alpha+\beta$ is a root. On the other hand, for any $\gamma\in Q_\Lambda$ and $\alpha\in P_k$ we have

$$(\Lambda+\rho,\ \gamma+\alpha)=(\Lambda+\rho,\ \gamma)+(\Lambda+\rho,\ \alpha)>0.$$

This shows that

$$\{\gamma+\alpha\,;\,\gamma\in Q_\Lambda,\ \alpha\in P_k\}\subset Q_\Lambda\,.$$

It follows immediately that

$$ad^{q_\Lambda}(X_{-\alpha})\Phi_\Lambda=0 \qquad \text{for all}\ \ \alpha\in P_k\,.$$

This proves the proposition.

Theorem 4.3. *Let* $\Lambda\in\mathfrak{F}_0'$. *If* $\pi_\Lambda|K$ *contains the irreducible representation of* K *with the highest weight* $\Lambda+\sum_{\alpha\in Q_\Lambda}\alpha$, *then* $H_2^{0,q_\Lambda}(E_\Lambda)\neq(0)$.

Proof. Assume that $\pi_\Lambda|K$ contains the irreducible representation of K with the highest weight $\Lambda+\sum_{\alpha\in Q_\Lambda}\alpha$. Put $\lambda_0=\sum_{\alpha\in Q_\Lambda}\alpha$. Then we have $e_{\Lambda+\lambda_0}(\pi_\Lambda)\mathfrak{H}_\Lambda\neq(0)$. On the other hand, we know from Proposition 4.1 that $-\lambda_0$ is the lowest weight of the representation ad^{q_Λ}. Then it is clear that we can choose an irreducible component of $\wedge^{q_\Lambda}\mathfrak{p}_-\otimes V_\Lambda^*$ with the lowest weight $-\Lambda-\lambda_0$ which is the contragredient representation to the irreducible representation with the highest weight $\Lambda+\lambda_0$. It follows immediately that

$$(\wedge^{q_\Lambda}\mathfrak{p}_-\otimes e_{\Lambda+\lambda_0}(\pi_\Lambda)\otimes V_\Lambda^*)\cap L_2^{q_\Lambda}(G)\neq(0)\,.$$

This means that $\wedge^{q_\Lambda}\mathfrak{p}_-\otimes e_{\Lambda+\lambda_0}(\pi_\Lambda)$ contains an irreducible component with the highest weight Λ. Therefore, from Theorem 4.2 we have $H_2^{0,q_\Lambda}(E_\Lambda)\neq(0)$. This completes the proof of the theorem.

REMARK. $G=SU(m,1)$ (for the notation, see [6] p. 340) satisfies all the conditions in §3. It is not difficult to verify that the assumption in Theorem 4.3 is always satisfied for $SU(m,1)$, using a result of Gel'fand-Graev [4]. We conjecture that it holds in general.

5. Proof of Theorem 4.1

In this section we shall give a proof of Theorem 4.1. We just follow the Kostant's method given in [9] (a). We shall use the notations given in Section 3 and 4 without further comment.

First we shall summarize some notions and the known results about the cohomology theory of Lie algebras.

For any $X\in\mathfrak{g}^C$ we denote by $\varepsilon(X)$ the operator of the exterior multiplication by X and by $i(X)$ the operator on $\Lambda\mathfrak{g}^C$ defined as usual by the formula

$$i(X)(X_1\wedge\cdots\wedge X_q)=\sum_{j=1}^{q}(-1)^{j-1}B(X,X_j)X_1\wedge\cdots\wedge\hat{X}_j\wedge\cdots\wedge X_q$$

for $X_1,\cdots,X_q\in\mathfrak{g}^C$. Then it is easy to check that

$$i(X)^*=\varepsilon(X^*) \qquad \text{for all}\ \ X\in\mathfrak{g}^C$$

where $i(X)^*$ denotes the adjoint operator of $i(X)$ with respect to the inner product in $\mathfrak{g}^C$ defined in the previous section.

Now we define the inner product (,) in $\wedge\mathfrak{g}^C\otimes C_c^\infty(G)$ by

$$(u, u') = \sum_{q=0}^{n} \sum_{1\leqslant i_1<\cdots<i_q\leqslant n} \int_G f_{i_1\cdots i_q}(g)\overline{f'_{i_1\cdots i_q}(g)}\,dg$$

where $u=\sum_{q=0}^{n} \sum_{1\leqslant i_1<\ldots<i_q\leqslant n} z_{i_1}\wedge\cdots\wedge z_{i_q}\otimes f_{i1\cdots iq}$ and $u'=\sum_{q=0}^{n} \sum_{1\leqslant i_1<\cdots<i_q\leqslant n} z_{i_1}\wedge\cdots\wedge z_{i_q}\otimes f'_{i_1\cdots i_q}$. Let $X\to\bar{X}$ ($X\in\mathfrak{g}^C$) denote the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Consider the representation ν of $\mathfrak{g}^C$ on $C^\infty(G)$ defined in the previous section. For any $X\in\mathfrak{g}^C$ we define the "formal adjoint" operator $\nu(X)^*$ of $\nu(X)$ by

$$\nu(X)^* = -\nu(\bar{X})\,.$$

Then we have

$$(\nu(X)f_1, f_2) = (f_1, \nu(X)^*f_2) \qquad (X\in\mathfrak{g}^C)$$

for all f_1, $f_2\in C_c^\infty(G)$ where the inner product (,) is that of $L_2(G)$. We define operators d_1 and d_2 on $\wedge\mathfrak{g}^C\otimes C^\infty(G)$ by the formulas

$$d_1 = \sum_{i=1}^{n} \varepsilon(z_i^*)\otimes\nu(z_i)$$
$$d_2 = \sum_{i=1}^{m} \varepsilon(z_i^*)\otimes\nu(z_i)\,.$$

We put

$$d_3 = d_1 - d_2 = \sum_{i=m+1}^{n} \varepsilon(z_i^*)\otimes\nu(z_i)\,.$$

And define the "formal adjoint" operators d_1^*, d_2^* and d_3^* by the formulas

$$d_1^* = \sum_{i=1}^{n} i(z_i)\otimes\nu(z_i)^*\,,$$
$$d_2^* = \sum_{i=1}^{m} i(z_i)\otimes\nu(z_i)^* \quad \text{and}$$
$$d_3^* = \sum_{i=m+1}^{n} i(z_i)\otimes\nu(z_i)^*\,.$$

Then it is easy to see that

$$(d_s u, u') = (u, d_s^* u') \qquad (s=1, 2, 3)$$

for all $u, u'\in\wedge\mathfrak{g}^C\otimes C_c^\infty(G)$.

We notice that the operators $*$ and $-$ commute. For any $X\in\mathfrak{g}^C$ we define

$$\sigma(X) = -\bar{X}^*\,.$$

Then it is easy to see that σ is an involutive automorphism of $\mathfrak{g}^C$. Obviously we have

$$\nu(X)^* = \nu(\sigma X^*) \quad \text{for all } X\in\mathfrak{g}^C\,.$$

Now we shall prove some lemmas which will be used in the sequel.

Lemma 5.1. *For any* $z \in \mathfrak{g}^C$ *we have*

$$[1 \otimes \nu(z), d_1^*] = \sum_{i=1}^{n} i(\sigma[z_i^*, z]) \otimes \nu(z_i) .$$

Proof. Since $d_1^* = \sum_{i=1}^{n} i(z_i) \otimes \nu(\sigma z_i^*)$, we have

$$[1 \otimes \nu(z), d_1^*] = \sum_{i=1}^{n} i(z_i) \otimes \nu([z, \sigma z_i^*]) .$$

We notice that $\sum_{i=1}^{n} z_i \otimes z_i^* \in \mathfrak{g}^C \otimes \mathfrak{g}^C$ is invariant under the adjoint representation of $\mathfrak{g}^C$ on $\mathfrak{g}^C \otimes \mathfrak{g}^C$, i.e.

$$\sum_{i=1}^{n} \{[z, z_i] \otimes z_i^* + z_i \otimes [z, z_i^*]\} = 0$$

for any $z \in \mathfrak{g}^C$. By operating $1 \otimes \sigma$ on both sides, we have

$$\sum_{i=1}^{n} z_i \otimes [z, \sigma z_i^*] = \sum_{i=1}^{n} [z_i, \sigma z] \otimes \sigma z_i^*$$

after replacing z by σz. Since $\sigma z_i = z_i$ or $-z_i$ $(i=1, \cdots, n)$, it follows that

$$\begin{aligned} [1 \otimes \nu(z), d_1^*] &= \sum_{i=1}^{n} i([z_i, \sigma z]) \otimes \nu(\sigma z_i^*) \\ &= \sum_{i=1}^{n} i(\sigma[z_i, z]) \otimes \nu(z_i^*) . \end{aligned}$$

This proves the lemma.

Lemma 5.2. *For any* $u \in \Lambda \mathfrak{p}_-$ *we have*

$$\sum_{i=1}^{m} \varepsilon(z_i^*) i(\sigma[z, z_i]) u = \begin{cases} 0 & \text{if } z \in \mathfrak{p}^C \\ \theta(z) u & \text{if } z \in \mathfrak{k}^C . \end{cases}$$

Proof. Since both sides of the equation are obviously derivations of degree 0, it suffices to verify the equality for $u \in \mathfrak{p}_-$. First assume that $z \in \mathfrak{p}^C$. Then for any $u \in \mathfrak{p}_-$ we have $[z, \sigma u] \in \mathfrak{k}^C$. Since $(\mathfrak{p}^C, \mathfrak{k}^C) = (0)$, it follows that

$$\begin{aligned} \sum_{i=1}^{m} \varepsilon(z_i^*) i(\sigma[z, z_i]) u &= \sum_{i=1}^{m} B(\sigma[z, z_i], u) z_i^* \\ &= -\sum_{i=1}^{m} (z_i^*, [z, \sigma u]) z_i^* \\ &= 0 . \end{aligned}$$

We now assume that $z \in \mathfrak{k}^C$. In the same way we can show that

$$\sum_{i=1}^{m} \varepsilon(z_i^*) i(\sigma[z, z_i]) u = \theta(z) u .$$

for all $u \in \mathfrak{p}_-$.

Lemma 5.3. *For any $z \in \mathfrak{g}^C$ we have*

$$d_1^*(\varepsilon(z^*)\otimes 1)+(\varepsilon(z^*)\otimes 1)d_1^* = 1\otimes\nu(z)^* .$$

Proof. For any $z\in\mathfrak{g}^C$ we have

$$\begin{aligned}&(i(z)\otimes 1)d_1+d_1(i(z)\otimes 1)\\ &\quad=\sum_{i=1}^{n}(i(z)\varepsilon(z_i^*)+\varepsilon(z_i^*)i(z))\otimes\nu(z_i) .\end{aligned}$$

On the other hand, one has the following equality;

$$i(x)\varepsilon(y)+\varepsilon(y)i(x) = B(x, y) = (x, y^*)$$

for any $x, y\in\mathfrak{g}^C$. It follows that

$$\begin{aligned}&(i(z)\otimes 1)d_1+d_1(i(z)\otimes 1)\\ &\quad=\sum_{i=1}^{n}(z, z_i)\otimes\nu(z_i)\\ &\quad= 1\otimes\nu(z) .\end{aligned}$$

Taking the adjoint operator of both sides, we obtain the equality in Lemma 5.3.

Lemma 5.4. *For any $u\in\wedge\mathfrak{p}_-$ we have*

$$\Big(\sum_{i=m+1}^{n-m}\theta(z_i^*)\theta(z_i)+\sum_{i=1}^{m}\theta([z_i, z_i^*])\Big)u = 0 .$$

For a proof, see [11] Lemma 4.1.

Lemma 5.5. *For any $u\in\wedge\mathfrak{p}_-\otimes C^\infty(G)$ we have $d_3^*u=0$.*

Proof. By the definition of d_3 we have

$$d_3^* = \sum_{i=m+1}^{n} i(z_i)\otimes\nu(z_i)^* .$$

We notice that z_i $(i=m+1, \cdots, n)$ is orthogonal to $\mathfrak{p}_+$. For any $y\in\mathfrak{p}_-$ we have $y^*\in\mathfrak{p}_+$. Hence, $i(z_i)y=(z_i, y^*)=0$ $(i=m+1, \cdots, n)$. This proves the lemma.

Proposition 5.1. *For any $u\in\wedge\mathfrak{p}_-\otimes C^\infty(G)$ we have*

$$(d_2 d_2^*+d_2^* d_2)u = \frac{1}{2}\{R_{\mathfrak{k}}^{\theta_\nu}+2\theta_\nu(H_\rho)-2\theta_\nu(H_{\rho_k})-1\otimes R^\nu\}u$$

where $R_{\mathfrak{k}}^{\theta_\nu}$ denotes the Casimir operator of the restriction of θ_ν to $\mathfrak{k}^C$ and $\rho_k=\frac{1}{2}\sum_{\alpha\in P_k}\alpha$.

Proof. First we notice that d_2 and d_2^* both map $\wedge\mathfrak{p}_-\otimes C^\infty(G)$ into itself. This is an immediate consequence of the definition of d_2, and d_2^*. It follows from Lemma 5.5 that

$$(d_2d_2^*+d_2^*d_2)u=(d_2d_1^*+d_1^*d_2)u$$

for all $u\in\Lambda\mathfrak{z}_-\otimes C^\infty(G)$.

Making use of Lemma 5.1∼5.3 we have

$$\begin{aligned}(d_2d_1^*+d_1^*d_2)u &= \sum_{i=1}^{m}(\varepsilon(z_i^*)\otimes 1)\,[1\otimes\nu(z_i),\ d_1^*]u \\ &\quad+\sum_{i=1}^{m}(d_1^*(\varepsilon(z_i^*)\otimes 1)+(\varepsilon(z_i^*)\otimes 1)\,d_1^*)(1\otimes\nu(z_i))u \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m}\varepsilon(z_i^*)\,i(\sigma[z_j^*,\ z_i])\otimes\nu(z_j)u \\ &\quad+\sum_{i=1}^{m}1\otimes\nu(z_i)^*\nu(z_i)u \\ &= \sum_{i=m+1}^{n-m}\theta(z_i^*)\otimes\nu(z_i)u \\ &\quad+\sum_{i=1}^{m}1\otimes\nu(z_i)^*\nu(z_i)u\,.\end{aligned}$$

We consider the first term. Owing to the choice of the base $(z,\cdots,z_n)$, we have

$$\text{(A)}\qquad \begin{aligned}2\sum_{i=m+1}^{n-m}\theta(z_i^*)\otimes\nu(z_i) &= \sum_{i=m+1}^{n-m}\theta_\nu(z_i^*)\theta_\nu(z_i) \\ &\quad-\sum_{i=m+1}^{n-m}\theta(z_i^*)\theta(z_i)\otimes 1 \\ &\quad-\sum_{i=m+1}^{n-m}1\otimes\nu(z_i^*)\nu(z_i) \\ &= R_{\mathfrak{k}}^{\theta_\nu}-R_{\mathfrak{k}}^{\theta}\otimes 1-1\otimes R_{\mathfrak{k}}^{\nu}\,,\end{aligned}$$

where $R_{\mathfrak{k}}^{\theta_\nu}$ (resp. $R_{\mathfrak{k}}^{\theta}$, $R_{\mathfrak{k}}^{\nu}$) denotes the Casimir operator of the restriction of θ_ν (resp. θ, ν) to $\mathfrak{k}^C$.

Now we consider the second term.

$$\begin{aligned}&2\sum_{i=1}^{m}1\otimes\nu(z_i)^*\nu(z_i) \\ &\quad= -2\sum_{i=1}^{m}1\otimes\nu(z_i^*)\,\nu(z_i) \\ &\quad= -\sum_{i=1}^{n}1\otimes\nu(z_i^*)\,\nu(z_i)+\sum_{i=m+1}^{n-m}1\otimes\nu(z_i^*)\,\nu(z_i) \\ &\qquad+\sum_{i=1}^{m}1\otimes\nu([z_i,\ z_i^*])\,. \\ &\quad= -1\otimes R^\nu+1\otimes R_{\mathfrak{k}}^{\nu}+2\otimes\nu(H_\rho)-2\otimes\nu(H_{\rho_k})\,.\end{aligned}$$

It follows from (A) and Lemma 5.4 that

$$2(d_2d_1^*+d_1^*d_2)u=(R_{\mathfrak{k}}^{\theta_\nu}-1\otimes R^\nu\oplus 2\theta_\nu(H_\rho)-2\theta_\nu(H_{\rho_k}))u\,.$$

This completes the proof of the proposition.

Now we come to the proof of Theorem 4.1. First we observe that the diagram

$$\begin{array}{ccc} C^{0,q}(E_\Lambda) & \xrightarrow{\eta} & \wedge^q\mathfrak{p}_-\otimes C^\infty(G)\otimes V_\Lambda^* \\ d''\Big\downarrow\Big\uparrow\delta'' & & d_2\otimes 1\Big\downarrow\Big\uparrow d_2^*\otimes 1 \\ C^{0,q+1}(E_\Lambda) & \xrightarrow{\eta} & \wedge^{q+1}\mathfrak{p}_-\otimes C^\infty(G)\otimes V_\Lambda^* \end{array}$$

is commutative. It follows from Proposition 5.1 that

$$\eta(\Box\varphi) = \frac{1}{2}\{R_{\mathfrak{k}}^{0\nu}+2\theta_\nu(H_\rho)-2\theta_\nu(H_{\rho_k})-1\otimes R^\nu\}\otimes 1\eta(\varphi)\,.$$

We know that $\eta C^{0,q}(E_\Lambda)=C^q(G)_\Lambda^0$.

Consider the the projection operator e_Λ^q defined by

$$e_\Lambda^q = r\int_K \overline{\xi_\Lambda(k)}\sigma^q\cdot(k)\,dk\,.$$

Then it is easy to see that

$$C^q(G)_\Lambda^0\subset e_\Lambda^q(\wedge^q\mathfrak{p}_-\otimes C^\infty(G))\otimes V_\Lambda^*\,.$$

Therefore, since $\eta(\varphi)\in C^q(G)_\Lambda^0$, we have

$$R_{\mathfrak{k}}^{0\nu}\otimes 1\eta(\varphi) = (\Lambda+2\rho_{\mathfrak{k}},\ \Lambda)\eta(\varphi)\,.$$

Since $H_{2\rho-2\rho_k}$ is contained in the center of $\mathfrak{k}^C$ (see [9] (a) Lemma 5.5), $\theta_\nu(H_{2\rho-2\rho_k})$ reduces to a scalar operator on $e_\Lambda^q(\wedge^q\mathfrak{p}_-\otimes C^\infty(G))$. To determine the scalar it suffices to compute $\theta_\nu(H_{2\rho-2\rho_k})$ on a highest weight vector $u_\Lambda\in e_\Lambda^q(\wedge^q\mathfrak{p}_-\otimes C^\infty(G))$. Clearly we have

$$\theta_\nu(H_{2\rho-2\rho_k})u_\Lambda = (\Lambda,\ 2\rho-2\rho_k)u_\Lambda\,.$$

Hence

$$(\theta_\nu(H_{2\rho-2\rho_k})\otimes 1)\eta(\varphi) = (\Lambda,\ 2\rho-2\rho_k)\eta(\varphi)\,.$$

It follows that

$$\eta(\Box\varphi) = \frac{1}{2}\{(\Lambda+2\rho,\ \Lambda)-1\otimes R^\nu\}\otimes 1\eta(\varphi)\,.$$

This completes the proof of Theorem 4.1.

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