Title	Neyman factorization theorems in the case of nonequivalent dominating measures
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Citation	Osaka Journal of Mathematics. 1981, 18(3), p. 621-632
Version Type	VoR
URL	https://doi.org/10.18910/10701
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Yamada, S.
Osaka J. Math.
18 (1981), 621-632

NEYMAN FACTORIZATION THEOREMS IN THE CASE OF NON-EQUIVALENT DOMINATING MEASURES

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(Received Jaunary 21, 1980)

We extend the Neyman factorization theorem to the cases of localizable and locally localizable dominating measures and prove that a subfield has a Neyman factorization if and only if it is pairwise sufficient and contains carriers (PSCC for short). Thus the additional assumption, which was assumed by Ghosh, Morimoto and Yamada ([3]), that the dominating measure is equivalent to the family of probability measures considered was removed. It is proved that locally localizable measure can be extended in some sense to a localizable measure on an enlarged sigma-field, without the assumption of finite subset property of the measure. And this constitutes an important step of the proof of the above theorem. Furthermore an extended notion of carrier plays a remarkable role in the proof.

1. Introduction

We assume that each element in the family of probability measures considered has a density w.r.t. a common measure, called a dominating measure, which is not necessarily sigma-finite. The Neyman factorization theorem, in case that the dominating measure is sigma-finite, says that a particular factorization (see Definition 4 below, and hereafter we call this a Neyman factorization) of the densities of probability measures is equivalent to sufficiency of a subfield. Ghosh, Morimoto and Yamada ([3], which is hereinafter refered as the "previous paper") extended this Neyman factorization theorem to the cases where the dominating measures are localizable and locally localizable (we call them weak domination and local weak domination respectively). In those cases it was shown that the same factorization is equivalent to a property of a subfield called PSCC, which is somewhat weaker than sufficiency, under the additional assumption that the dominating measure is equivalent to the family of probability measures considered.

In the case of weak domination the general case could be "reduced" to the case of an equivalent dominating measure in the following way. Namely if there exists a localizable dominating measure then there exists an equivalent localizable dominating measure ([6] Lemma 2.9 (2)). And the previous paper ([3]) did not show the Neyman factorization theorem about the densities w.r.t. a general localizable dominating measure, but an equivalent localizable dominating measure. Moreover in the case of local weak domination it is not even known whether there exists an equivalent locally localizable dominating measure.

It is the purpose of this paper to establish the Neyman factorization theorem generally, removing these restrictions, for the cases of weak domination and local weak domination. Though weak domination is a part of local weak domination, the results are stated in separate Theorems (Theorem 1 and Theorem 2) for reasons that the measurability of a function appearing in the Neyman factorization differs slightly in those two cases, and that we first show the result for weakly dominated case then extend it to the locally weakly dominated case. The definition of "a carrier of a probability measure" appearing in the Theorem is given by Definition 3 below. In the case of equivalent dominating measure such as in the previous paper, Definition 3 is equivalent to $[p(x, P)>0]$, where $p(x, P)$ is a version of the density of a probability measure P ([3] Remark 1.1 (c)). However in the general case some carriers in the sense of Definition 3 are not the type of $[p(x, P)>0]$. If we ignore the fact, we cannot prove the Theorems (see Example below).

We prove in Lemma 3 an intersting fact that locally localizable measure can be extended in some sense to a localizable measure on the sigma-field of locally measurable sets. This fact was proved in the previous paper under the additional assumption that the measure has the finite subset property ([3] Lemma 2.4). Any equivalent dominating measure has the finite subset property, but general dominating measure may not have this property. So the result without this assumption is useful here and in fact it is an essential measure theoretical tool when we extend the theorem under localizable dominating measure to that under locally localizable dominating measure.

2. Definitions and notations

We give some definitions and notations which will be used in the sequel. Let X be a set and $\mathcal{A}$ be a sigma-field of subsets of X. Let $\mathcal{P}$ be a family of probability measures on $(X, \mathcal{A})$. We call the triplet $(X, \mathcal{A}, \mathcal{P})$ a statistical structure. If there exists a measure m on $(X, \mathcal{A})$ such that each P in $\mathcal{P}$ has a non-negative density w.r.t. m, we call this measure a dominating measure for $(X, \mathcal{A}, \mathcal{P})$. Then it is clear that all P in $\mathcal{P}$ is absolutely continuous w.r.t. m and this is denoted by $\mathcal{P} \ll m$. $dP/dm(x)$ denotes any fixed version of the density of P w.r.t. m. Every statistical structure considered in the sequel will be assumed to have a dominating measure. We denote by $N(m)$ the set of all m-null sets and by $N(\mathcal{P})$ the set of all A in $\mathcal{A}$ with $A \in N(P)$ for all P in $\mathcal{P}$. If a dominating measure m for $(X, \mathcal{A}, \mathcal{P})$ satisfies $N(m)=N(\mathcal{P})$ then m is called to be equivalent to $\mathcal{P}$ and it is written $\mathcal{P}\sim m$. For any measure m on $(X, \mathcal{A})$ we define $\mathcal{A}(m)=\{A\in\mathcal{A};\ m(A)<\infty\}$, and $\mathcal{A}_l(m)=\{A\subset X;\ A\cap E\in\mathcal{A}$ for all E in $\mathcal{A}(m)\}$. It is easy to see that $\mathcal{A}_l(m)$ is a sigma-field. Elements of $\mathcal{A}_l(m)$ are called locally $\mathcal{A}$-measurable w.r.t. m.

Let us define a function $\bar{m}$ on $\mathcal{A}_l(m)$ by $\bar{m}(A)=\sup\{m(A\cap E);\ E\in\mathcal{A}(m)\}$, $A\in\mathcal{A}_l(m)$. That $\bar{m}$ is a measure on $(X, \mathcal{A}_l(m))$ is noted by Diepenbrock ([2] Section 1) (see also Berberian ([1] p. 32, Theorem 1)).

DEFINITION 1. ([3]) A measure m on $(X, \mathcal{A})$ is said to be *localizable* (resp. *locally localizable*) if for any subfamily $\mathcal{F}$ of $\mathcal{A}(m)$ there exists an "ess-sup $\mathcal{F}$" w.r.t. m in $\mathcal{A}$ (resp. $\mathcal{A}_l(m)$) such that

(a) $m(F-\text{ess-sup}\,\mathcal{F})=0$ for all F in $\mathcal{F}$, and

(b) For any A in $\mathcal{A}$ (resp. $\mathcal{A}_l(m)$) such that $m(F-A)=0$ for all F in $\mathcal{F}$, it follows that $m(\text{ess-sup}\,\mathcal{F}-A)=0$ (resp. $\bar{m}(\text{ess-sup}\,\mathcal{F}-A)=0$).

It is easy to see that a localizable measure is locally localizable.

DEFINITION 2. A statistical structure $(X, \mathcal{A}, \mathcal{P})$ is said to be *weakly dominated* (resp. *locally weakly dominated*) if there exists a localizable (resp. locally localizable) dominating measure for $(X, \mathcal{A}, \mathcal{P})$.

REMARK 1. Assuming that there exists an equivalent locally localizable dominating measure, the previous paper showed a Neyman factorization theorem for such a dominating measure. However whether the following two conditions are equivalent is not known: 1) There exists a locally localizable dominating measure for $(X, \mathcal{A}, \mathcal{P})$. 2) There exists an equivalent locally localizable dominating measure for $(X, \mathcal{A}, \mathcal{P})$. That we can prove now is the following: Let m be a locally localizable dominating measure for $(X, \mathcal{A}, \mathcal{P})$ which is not necessarily equivalent to $\mathcal{P}$. Then there exists an equivalent dominating measure n for $(X, \mathcal{A}, \mathcal{P})$ such that the completion of n is locally localizable on the completed sigma-field w.r.t. n.

A subfield (i.e., sub-sigma-field) $\mathcal{B}$ of $\mathcal{A}$ is said to be sufficient for $(X, \mathcal{A}, \mathcal{P})$ if for any A in $\mathcal{A}$ there exists a $\mathcal{B}$-measurable function $g(x)$ such that $P(A\cap B)=\int_B g(x)dP(x)$, for all B in $\mathcal{B}$ and P in $\mathcal{P}$. A subfield $\mathcal{B}$ is said to be pairwise sufficient for $(X, \mathcal{A}, \mathcal{P})$ if it is sufficient for $(X, \mathcal{A}, \{P_1, P_2\})$ for each pair $\{P_1, P_2\}$ from $\mathcal{P}$. Let m be a measure on $(X, \mathcal{A})$ and $\Pi(x)$ be any propositional function of x. Then $[\Pi(x)]$ denotes the set of all points x in X which satisfy $\Pi(x)$. And $\Pi(x)\ [m]$ means that there exists a set N in $N(m)$ such that $X-[\Pi(x)]\subset N$.

DEFINITION 3. ([3]) Let C be a set in $\mathcal{A}$ and let $P\in\mathcal{P}$. If the following two conditions are satisfied then we say that $C=C(P)$ is a *carrier* of P (w.r.t. $\mathcal{P}$):

(a) $P(C)=1$, and

(b) $A\in\mathcal{A}$, $A\subset C$ and $P(A)=0$ imply $A\in N(\mathcal{P})$.

If a subfield $\mathcal{B}$ contains a carrier $C(P)$ for each P in $\mathcal{P}$, then we shall say that $\mathcal{B}$ *contains carriers* of $\mathcal{P}$.

A subfield $\mathcal{B}$ is said to be PSCC if it is pairwise sufficient for $(X, \mathcal{A}, \mathcal{P})$ and contains carriers of $\mathcal{P}$.

DEFINITION 4. ([3]) A subfield $\mathcal{B}$ is said to have a *Neyman factorization* when for each P in $\mathcal{P}$ $dP/dm(x)$ is factored as

$$dP/dm(x) = g(x, P)h(x) \qquad \text{a.e.},$$

where $g(x, P)$ is a non-negative $\mathcal{B}$-measurable function and $h(x)$ is a non-negative function which is independent of P.

The measure to which "a.e." refers and the measurability of $h(x)$ vary from context to context.

3. Weakly dominated case

Let $(X, \mathcal{A}, \mathcal{P})$ be a statistical structure and T be a set in $\mathcal{A}$. Then we define $T\cap\mathcal{A}=\{T\cap A;\, A\in\mathcal{A}\}$ and $\mathcal{P}|_T=\{P|_T;\, P\in\mathcal{P}\}$, where $P|_T$ is a measure on $(T, T\cap\mathcal{A})$ which is defined by $P|_T(T\cap A)=P(T\cap A)$, $A\in\mathcal{A}$. For any measure m on $(X, \mathcal{A})$, $m|_T$ is defined in the same manner. If $f(x)$ is a function on X we denote $f|_T(x)$ as the restriction of $f(x)$ on T.

The following two Theorems in [3] are quoted here as Lemmas.

Lemma 1 (Ghosh, Morimoto and Yamada, [3] Theorem 1). *Let a statistical structure $(X, \mathcal{A}, \mathcal{P})$ have a dominating measure m such that $\mathcal{P}\sim m$. Suppose that a subfield $\mathcal{B}$ has a Neyman factorization*

$$dP/dm(x) = g(x, P)h(x) \qquad [m],$$

where $g(x, P)$ is a non-negative $\mathcal{B}$-measurable function for each P in $\mathcal{P}$ and $h(x)$ is a non-negative functions such that $h(x)>0$ $[m]$.

Then $\mathcal{B}$ is PSCC.

Lemma 2 (Ghosh, Morimoto and Yamada, [3] Theorem 4 and Remark 2.3). *Let $(X, \mathcal{A}, \mathcal{P})$ be weakly dominated (resp. locally weakly dominated by a localizable (resp. locally localizable) dominating measure m such that $\mathcal{P}\sim m$. Suppose that a subfield $\mathcal{B}$ is PSCC. Then it has a Neyman factorization*:

$$dP/dm(x) = g(x, P)h(x) \qquad [m] \quad (resp.\ [\overline{m}]),$$

where $g(x, P)$ is a non-negative $\mathcal{B}$-measurable function for each P in $\mathcal{P}$ and $h(x)$ is *a non-negative $\mathcal{A}$-measurable (resp. $\mathcal{A}_l(m)$-measurable) function.*

Theorem 1. *Let a statistical structure $(X, \mathcal{A}, \mathcal{P})$ have a localizable dominating measure m. Then a subfield $\mathcal{B}$ is PSCC if and only if it has a Neyman factorization:*

$$dP/dm(x) = g(x, P)h(x) \qquad [m], \qquad \cdots\cdots(1)$$

where $g(x, P)$ is a non-negative $\mathcal{B}$-measurable function for each P in $\mathcal{P}$ and $h(x)$ is a non-negative $\mathcal{A}$-measurable function.

Proof. "Only if" part. For any P in $\mathcal{P}$ we denote $T_P=[dP/dm(x)>0]$. Since m is localizable and each T_P is a sigma-finite set w.r.t m it is easy to show the family $\{T_P;\ P\in\mathcal{P}\}$ has an essential supremum T w.r.t. m belonging to $\mathcal{A}$. It follows that, for each P in $\mathcal{P}$,

$$\begin{aligned} P(X-T) &= P((X-T)\cap T_P)+P((X-T)\cap(X-T_P)) \\ &\leqq P(T_P-T)+P(X-T_P) = 0\,, \end{aligned}$$

by $m(T_P-T)=0$ and $\mathcal{P}\ll m$. Therefore $P|_T$ is a probability measure on $(T, T\cap\mathcal{A})$, and hence $(T, T\cap\mathcal{A}, \mathcal{P}|_T)$ is a statistical structure. Clearly we have $\mathcal{P}|_T\ll m|_T$. Conversely let us assume that $P|_T(T\cap A)=0$ for all P in $\mathcal{P}$. Then $P(A)=0$, and $m(T_P\cap A)=0$ for each P in $\mathcal{P}$ by the definition of T_P. Since T is an essential supremum of $\{T_P; P\in\mathcal{P}\}$ w.r.t. m, we have $m(T\cap A)= m(T-(X-A))=0$. Hence we have $\mathcal{P}|_T\sim m|_T$.

Next we shall prove that $m|_T$ is localizable. Take any subfamily $\mathcal{F}$ of $T\cap\mathcal{A}(m|_T)$, which is the family of sets A in $T\cap\mathcal{A}$ such that $m|_T(A)<\infty$. Then there exists an ess-sup $\mathcal{F}\in\mathcal{A}$ w.r.t. m because $\mathcal{F}\subset\mathcal{A}(m)$ and m is localizable. Denote $S=T\cap(\text{ess-sup }\mathcal{F})$. Then S belongs to $T\cap\mathcal{A}$, and we have for each F in $\mathcal{F}$,

$$m|_T(F-S) = m(F-S) = m(F-\text{ess-sup }\mathcal{F}) = 0\,. \qquad \cdots\cdots(2)$$

Take any A in $T\cap\mathcal{A}$ such that $m|_T(F-A)=0$ for all F in $\mathcal{F}$. Then it follows that

$$m|_T(S-A) = m(S-A)\leqq m\,(\text{ess-sup }\mathcal{F}-A) = 0\,. \qquad \cdots\cdots(3)$$

From (2) and (3) S is an essential supremum of $\mathcal{F}$ w.r.t. $m|_T$. Hence $m|_T$ is localizable.

Since it is clear that each $P|_T$ has a density w.r.t. $m|_T$, $(T, T\cap\mathcal{A}, \mathcal{P}|_T)$ is weakly dominated by an equivalent localizable dominating measure $m|_T$.

Let $\mathcal{B}$ be a subfield of $\mathcal{A}$ which is PSCC. Then $T\cap\mathcal{B}=\{T\cap B; B\in\mathcal{B}\}$ is a subfield of $T\cap\mathcal{A}$. We show that $T\cap\mathcal{B}$ contains carriers of $\mathcal{P}|_T$. Take any P in $\mathcal{P}$ and let $C(P)$ be a carrier of P which belongs to $\mathcal{B}$. Then we have $C'(P)\equiv T\cap C(P)\in T\cap\mathcal{B}$ and $P|_T(C'(P))=P(C(P)\cap T)=P(C(P))=1$. Let A

be any set such that $A \subset C'(P)$, $A \in T \cap \mathcal{A}$ and $P|_T(A)=0$. Then it follows that $Q(A)=0$ for all Q in $\mathcal{P}$ because it holds that $A \subset C(P)$ and $P(A)=0$. Therefore $Q|_T(A)=0$ for all Q in $\mathcal{P}$. Thus $C'(P)$ is a carrier of $P|_T$ which belongs to $T \cap \mathcal{B}$.

We can easily show that $T \cap \mathcal{B}$ is pairwise sufficient for $(T, T \cap \mathcal{A}, \mathcal{P}|_T)$. Hence $T \cap \mathcal{B}$ is PSCC for $(T, T \cap \mathcal{A}. \mathcal{P}|_T)$. So, from Lemma 2, we have a Neyman factorization for $T \cap \mathcal{B}$: for each P in $\mathcal{P}$ it follows that

$$dP|_T/dm|_T(x) = g'(x, P|_T)h'(x) \qquad [m|_T], \qquad \cdots\cdots(4)$$

where $g'(x, P|_T)$ is a non-negative $T \cap \mathcal{B}$-measurable function on T and $h'(x)$ is a non-negative $T \cap \mathcal{A}$-measurable function on T. Define a function $h(x)$ on X by

$$\begin{aligned} h(x) &= h'(x) \qquad \text{if } x \in T, \\ &= 0 \qquad \text{if } x \in X-T. \end{aligned}$$

Then $h(x)$ is a non-negative $\mathcal{A}$-measurable function on X. K.P.S.B. Rao and B.V. Rao ([7], p. 5) showed if $(X, \mathcal{A})$ is a measurable space, $Y \subset X$, f is a real measurable function on $(Y, Y \cap \mathcal{A})$ then there is a real measurable function f' on $(X, \mathcal{A})$ such that $f=f'$ on Y. We can take a finite valued function as $g'(x, P|_T)$, because in the previous paper ([3] Theorem 4) we showed that any $\mathcal{B}$-measurable version of the density of $P|_T$ w.r.t. a "pivotal measure" served as the $g'(x, P|_T)$. Hence there exists a non-negative $\mathcal{B}$-measurable function $g(x, P)$ on X which is an extension of $g'(x, P|_T)$ to X. Since $dP/dm|_T(x)$, which is the restriction of $dP/dm(x)$ on T, is a version of the density of $P|_T$ w.r.t. $m|_T$, we have from (4)

$$dP/dm|_T(x) = g'(x, P|_T)h'(x) = g(x, P)h(x) \qquad [m|_T]. \qquad \cdots\cdots(5)$$

Since $m(T_P \cap (X-T))=0$ and $h(x)=0$ on $X-T$, we have (1) on $X-T$. But on T we have the relation (5). So (1) holds true on T. Therefore (1) holds true on X.

"If" part. The first half of the proof of this part is the same as the proof of Lemma 1 ([3]). In fact, for the proof of pairwise sufficiency, the assumptions $\mathcal{P} \sim m$ and $h(x) > 0$ $[m]$ were not used. We sketch here the proof for the sake of self-containedness. For any two measures P_1 and P_2 in $\mathcal{P}$ define a $\mathcal{B}$-measurable function $k(x)$ as follows:

$$\begin{aligned} k(x) &= \frac{g(x, P_1)}{g(x, P_1)+g(x, P_2)} \qquad \text{if the denominator is positive,} \\ &= 0. \qquad \text{otherwise.} \end{aligned}$$

Take any set A in $\mathcal{A}$ and let B be a subset of A defined by

$$B = A \cap [dP_1/dm(x)+dP_2/dm(x)>0] \cap [g(x, P_1)+g(x, P_2)>0]$$
$$\cap [dP_i/dm(x) = g(x, P_i)h(x), i = 1, 2] .$$

Then it can be proved that $B \in \mathcal{A}$ and $(P_1+P_2)(A-B)=0$. Further $h(x)$ is positive on B. Hence

$$k(x) = \frac{dP_1/dm(x)}{dP_1/dm(x)+dP_2/dm(x)}$$

on B, so that

$$\int_A k(x)d(P_1+P_2) = \int_B \frac{dP_1/dm(x)}{dP_1/dm(x)+dP_2/dm(x)} d(P_1+P_2)$$
$$= P_1(B) = P_1(A) .$$

Hence $\mathcal{B}$ is sufficient for $(X, \mathcal{A}, \{P_1, P_2\}$.

To prove that $\mathcal{B}$ contains carriers of $\mathcal{P}$ we use $\mathcal{A}$-measurability of $h(x)$ instead of $h(x)>0$ $[m]$ assumed in [3]. Take any P in $\mathcal{P}$, and define

$$\begin{aligned} p(x, P) &= dP/dm(x) && \text{if } dP/dm(x)>0 , \quad g(x, P)>0 \\ &= 1 && \text{if } dP/dm(x)=0 , \quad g(x, P)>0 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Note that $[p(x, P)>0]=[g(x, P)>0]$ belongs to $\mathcal{B}$. As we have assumed that any density is non-negative, it follows that

$$[dP/dm(x) \neq p(x, P)] = [dP/dm(x)>0, g(x, P)=0]$$
$$\cup [dP/dm(x)=0, g(x, P)>0] \subset N_P \cup [h(x)=0]$$
$$\subset N_P \cup N ,$$

where N_P is a set in $N(m)$ such that $N_P \supset [dP/dm(x) \neq g(x, P)h(x)]$, and N is a set in $N(m)$ such that $[h(x)=0]=X-[h(x)>0] \subset N$. Therefore we have $m([dP/dm(x) \neq p(x, P)])=0$. Hence $p(x, P)$ is a version of the density of P w.r.t. m such that $[p(x, P)>0] \in \mathcal{B}$.

We will show $[g(x, P)>0]$ itself is a carrier of P which belongs to $\mathcal{B}$ using $\mathcal{A}$-measurability of $h(x)$. Take any P in $\mathcal{P}$. It is clear from (1) that $P([g(x, P)>0])=1$. Let A be any set in $\mathcal{A}$ such that $A \subset [g(x, P)>0]$ and $P(A)=0$. Then it follows that

$$0 = P(A) = \int_{A \cap [dP/dm(x)>0]} g(x, P)h(x)dm(x)$$
$$= \int_{A \cap [h(x)>0]} g(x, P)h(x)dm(x) ,$$

because, if we write $N_P=[dP/dm(x) \neq g(x, P)h(x)]$,

$$(X-N_P) \cap A \cap [dP/dm(x)>0] = (X-N_P) \cap A \cap [h(x)>0]$$

holds true by $A \subset [g(x, P) > 0]$. So we have $m(A \cap [h(x) > 0]) = 0$. In these few lines we used $\mathcal{A}$-measurability of $h(x)$. For any Q in $\mathcal{P}$ it is clear that $Q([h(x) = 0]) = 0$, so it follows that $Q(A \cap [h(x) = 0]) = 0$ and hence $Q(A) = 0$. This completes the proof.

To show that $\mathcal{B}$ contains carrier of $\mathcal{P}$ in Lemma 1, Ghosh, Morimoto and Yamada proved a carrier of a special type of $[p(x, P) > 0]$ belongs to $\mathcal{B}$. In fact, under the assumption that $\mathcal{P} \sim m$, $\mathcal{B}$ contains carriers of $\mathcal{P}$ in the sense of Definition 3 if and only if for each P in $\mathcal{P}$ there exists a version $p(x, P)$ of the density of P w.r.t. m such that $[p(x, P) > 0] \in \mathcal{B}$ ([3] Remark 1.1 (c)). However if we drop the assumption that $\mathcal{P} \sim m$, it may happen that $\mathcal{B}$ contains carriers of $\mathcal{P}$ in the sense of Definition 3 but there does not exist a version $p(x, P)$ such that $[p(x, P) > 0] \in \mathcal{B}$ for some P in $\mathcal{P}$. Therefore to prove the "if" part of Theorem 1, it is essential to use such a broader concept of carrier as in Definition 3. The following example illustrating this point is due to B.V. Rao.

EXAMPLE. Let X be the real line and $\mathcal{A}$ be the power set of X. Let $\mathcal{P} = \{P_x; x \in X, x \neq 0\}$ be the family of unit probability measures P_x on $x \neq 0$. Then the counting measure m on $(X, \mathcal{A})$ is a localizable dominating measure for $(X, \mathcal{A}, \mathcal{P})$. And m is not equivalent to $\mathcal{P}$. Let $\mathcal{B}$ be the subfield of $\mathcal{A}$ which is defined by $\mathcal{B} = \{A \in \mathcal{A}; 0, 1 \in A \text{ or } 0, 1 \notin A\}$. Then, as is easily shown, the set $C(P_x)$ defined by

$$\begin{aligned} C(P_x) &= \{0, 1\} \qquad \text{if } x = 1\,, \\ &= \{x\} \qquad\quad\ \text{if } x \neq 0, 1\,, \end{aligned}$$

is a carrier of P_x which belongs to $\mathcal{B}$ and $\mathcal{B}$ is pairwise sufficient. However the unique density of P_1 w.r.t. m is the following:

$$\begin{aligned} dP_1/dm(x) &= 1 \qquad \text{if } x = 1\,, \\ &= 0 \qquad \text{if } x \neq 1\,. \end{aligned}$$

And $[dP_1/dm(x) > 0] = \{1\} \notin \mathcal{B}$.

REMARK 2. The above Example also gives a counter example for the following result of Mussmann ([5] Lemma 2.2) in case that the dominating measure is not necessarily equivalent to $\mathcal{P}$: Let a statistical structure $(X, \mathcal{A}, \mathcal{P})$ have an equivalent dominating measure m. Let $\mathcal{B}$ be sufficient for $(X, \mathcal{A}, \mathcal{P})$. Then for each P in $\mathcal{P}$ there exists a version $p(x, P)$ of the density of P w.r.t. m such that $[p(x, P) > 0] \in \mathcal{B}$. In fact the subfield $\mathcal{B}$ in the above Example is sufficient.

REMARK 3. Halmos and Savage ([4] Corollary 1) showed, in the result in the case of domination which corresponds to the "only if" part of Theorem 1, that we can take $h(x)$ satisfying

$$m(A\cap[h(x)>0])=0 \qquad \cdots\cdots(6)$$

for all A in $N(\mathcal{P})$. Here domination of a statistical structure means that it has a sigma-finite dominating measure.

Even if the dominating measure m is not sigma-finite, the relation (6) holds true if m is equivalent to $\mathcal{P}$. In either case m has the *finite subset property* (Mussamnn, [6] Lemma 2.9 (1)); that is, for any A in $\mathcal{A}$ such that $m(A)>0$ there exists a set B in $\mathcal{A}$ such that $B\subset A$ and $0<m(B)<\infty$. Actually, as the following proposition shows, the relation (6) can be proved generally under the assumption that m has the finite subset property.

Proposition. *Let $(X, \mathcal{A}, \mathcal{P})$ be weakly dominated by a localizable measure m and let m have the finite subset property. Then a subfield $\mathcal{B}$ is PSCC if and only if it has the following Neyman factorization:*

$$dP/dm(x)=g(x, P)h(x) \qquad [m],$$

where $g(x, P)$ is a non-negative $\mathcal{B}$-measurable function for each P in $\mathcal{P}$ and $h(x)$ is a non-negative $\mathcal{A}$-measurable function which satisfies

$$m(A\cap[h(x)>0])=0$$

for all A in $N(\mathcal{P})$.

Proof. "If" part has been proved in Theorem 1. We only need to prove "only if" part. Let n be an equivalent localizable dominating measure for $(X, \mathcal{A}, \mathcal{P})$ such that there exists a $\mathcal{B}$-measurable version $q(x, P)$ of the density of P w.r.t. n for each P in $\mathcal{P}$. The existence of such a measure, called a pivotal measure, is guaranteed by Lemma 2.3 a) and Theorem 2 in the previous paper ([3]). For any set A which is sigma-finite w.r.t. m let $k(x, A)$ be a non-negative version of the density on A of n w.r.t. m. Define a function $h(x, A)$ on X by

$$\begin{aligned} h(x, A) &= k(x, A) \qquad \text{if} \quad x\in A\,, \\ &= 0 \qquad\qquad\quad \text{if} \quad x\in X-A\,. \end{aligned}$$

Then it follows that $\{h(x, A);\ A \text{ is sigma-finite w.r.t. } m\}$ is an m-cross section; that is, for any A_1 and A_2 which are sigma-finite w.r.t. m, it follows that $I(x, A_1\cap A_2)h(x, A_1)=I(x, A_1\cap A_2)h(x, A_2)$ $[m]$, where $I(x, A)$ is the indicator function of A. Then, since m is localizable and has the finite subset property, there exists a non-negative $\mathcal{A}$-measurable function $h(x)$ such that $h(x)I(x, A)=h(x, A)$ $[m]$ for all A which is sigma-finite w.r.t. m (the previous paper [3] Lemma 2.2).

Take any version $dP/dm(x)$ of the density of P w.r.t. m. The restriction of m on $[dP/dm(x)>0]$ is sigma-finite. Hence we have $dP/dm(x)=dP/dn(x) \times dn/dm(x)=q(x,P)h(x)$ $[m]$ on $[dP/dm(x)>0]$.

We will denote

$$M_P = \{X-[dP/dm(x)>0]\} \cap [q(x, P)>0] \cap [h(x)>0]$$

and show that $m(M_P)=0$. For, suppose that $m(M_P)>0$. Then there exists a set B in $\mathcal{A}$ such that $B \subset M_P$ and $0<m(B)<\infty$ because m has the finite subset property. Hence we have $h(x)I(x, B)=h(x, B)$ $[m]$. Because of this relation and by $0=k(x, B)=h(x, B)$ if $x \in B \cap [k(x, B)=0]$ we have

$$m(B \cap [k(x, B) = 0]) = 0\,. \qquad \cdots\cdots(7)$$

On the other hand

$$0 = P(M_P) = \int_{M_P} q(x, P)dn(x)$$

implies that $n(M_P)=0$ and hence $n(B)=0$. So we have

$$m(B \cap [k(x, B)>0]) = 0\,. \qquad \cdots\cdots(8)$$

From (7) and (8) it follows that $m(B)=0$. This is a contradiction. Hence we have $m(M_P)=0$.

Therefore $\mathcal{B}$ has a Neyman factorization on X: $dP/dm(x)=q(x, P)h(x)$ $[m]$.

Take any set A in $N(\mathcal{P})$. If $m(A \cap [h(x)>0])>0$ then there exists a set B in $\mathcal{A}$ such that $B \subset A \cap [h(x) > 0]$ and $0<m(B)<\infty$ because m has the finite subset property. Denote $N_B=B \cap [h(x)I(x, B) \neq h(x, B)]$. Then we have $m(B \cap [h(x, B)=0]) \leqq m(N_B)=0$. So it follows that $m(B \cap [h(x, B)>0])>0$ because $m(B)>0$. On the other hand $B \subset A$ and $\mathcal{P} \sim n$ imply that

$$0 = n(B) = \int_B k(x, B)dm(x) = \int_B h(x, B)dm(x)\,.$$

Hence $m(B \cap [h(x, B)>0])=0$. This is a contradiction. Thus it follows that $m(A \cap [h(x)>0])=0$. This completes the proof.

4. Locally weakly dominated case

The next lemma has been proved in the previous paper ([3] Lemma 2.4) under the additional assumption that m has the finite subset property.

Lemma 3. *If m is a locally localizable measure on $(X, \mathcal{A})$ then $\overline{m}$ is localizable on $(X, \mathcal{A}_l(m))$.*

Proof. Let $\overline{\mathcal{F}}$ be any family of sets in $\mathcal{A}_l(m)$ which have finite $\overline{m}$-measure. For any F in $\overline{\mathcal{F}}$ there exists a set E_F in $\mathcal{A}$, which is sigma-finite w.r.t. m, such that

$$\overline{m}(F) = m(F \cap E_F)\,, \qquad \cdots\cdots(9)$$

by the definition of $\overline{m}$. It is clear that $\overline{m}(E)=m(E)$ for all E in $\mathcal{A}(m)$. Hence it follows that

$$m(F \cap E_F) = \overline{m}(F \cap E_F)\,. \qquad \cdots\cdots(10)$$

Denote $\mathcal{F} = \{F \cap E_F;\, F \in \overline{\mathcal{F}}\}$. Since $\mathcal{F}$ is a subfamily of $\mathcal{A}(m)$ and m is locally localizable, there exists an essent'al supremum S of $\mathcal{F}$ in $\mathcal{A}_l(m)$ w.r.t. m. We prove that S is also an essential supremum of $\overline{\mathcal{F}}$ w.r.t. $\overline{m}$. For any set F in $\overline{\mathcal{F}}$ we have

$$\begin{aligned}\overline{m}(F-S) &\leqq \overline{m}((F-F \cap E_F) \cup (F \cap E_F - S)) \\ &\leqq \overline{m}(F-F \cap F_F)+\overline{m}(F \cap E_F - S) = 0\,,\end{aligned}$$

by (9), (10) and the definition of S. Let A be any set in $\mathcal{A}_l(m)$ such that $\overline{m}(F-A)=0$ for all F in $\overline{\mathcal{F}}$. Then we have

$$m(F \cap E_F - A) = \overline{m}(F \cap E_F - A) \leqq \overline{m}(F-A) = 0$$

for all $F \cap E_F$ in $\mathcal{F}$. Hence it follows that $\overline{m}(S-A)=0$ by the definition of S. This proves that $\overline{m}$ is localizable on $\mathcal{A}_l(m)$.

Theorem 2. *Let $(X, \mathcal{A}, \mathcal{P})$ be locally weakly dominated by a locally localizable measure m. Let $\mathcal{B}$ be a subfield of $\mathcal{A}$. Then $\mathcal{B}$ is PSCC if and only if it has a Neyman factorization*:

$$dP/dm(x) = g(x, P)h(x) \qquad [\overline{m}]\,,$$

where $g(x, P)$ is a non-negative $\mathcal{B}$-measurable function for each P in $\mathcal{P}$ and $h(x)$ is a non-negative $\mathcal{A}_l(m)$-measurable function.

Proof. "Only if" part. Let us take any P in $\mathcal{P}$. We define a probability measure $\bar{P}$ on $(X, \mathcal{A}_l(m))$ by $\bar{P}(A)=P(A \cap T_P)$, where $T_P=[dP/dm(x)>0]$. Since $A \cap T_P \in \mathcal{A}$ and $\overline{m}|_{A \cap T_P}=m|_{A \cap T_P}$ on $(A \cap T_P, (A \cap T_P) \cap \mathcal{A})$ for all A in $\mathcal{A}_l(m)$, where $\overline{m}|_{A \cap T_P}$ is the measure on $(A \cap T_P, (A \cap T_P) \cap \mathcal{A})$ defined by $\overline{m}|_{A \cap T_P}(B)=\overline{m}(B)$ for all $B \in (A \cap T_P) \cap \mathcal{A}$, $\bar{P}$ is an extension of P to $\mathcal{A}_l(m)$ and $dP/dm(x)$ is also a version of the density of $\bar{P}$ w.r.t. $\overline{m}$. By above Lemma 3, therefore, $\overline{m}$ is a localizable dominating measure for $(X, \mathcal{A}_l(m), \overline{\mathcal{P}})$, where $\overline{\mathcal{P}}=\{\bar{P};\, P \in \mathcal{P}\}$.

Next we shall prove that $\mathcal{B}$ is PSCC for $(X, \mathcal{A}_l(m), \overline{\mathcal{P}})$. The proof that $\mathcal{B}$ is pairwise sufficient for $(X, \mathcal{A}_l(m), \overline{\mathcal{P}})$ is the same as the first half of the proof of Lemma 2.7 in the previous paper ([3]). However we need a change on carrier. So we sketch a proof for the sake of completeness. Take any two measures $\bar{P}_1$ and $\bar{P}_2$ in $\overline{\mathcal{P}}$ and a $\mathcal{B}$-measurable version $f(x)$ of the density of P_1 w.r.t. P_1+P_2. T_{P_1} and T_{P_2} are sigma-finite w.r.t. m, and so $A \cap (T_{P_1} \cup T_{P_2}) \in \mathcal{A}$ for all A in $\mathcal{A}_l(m)$. Hence it follows that $f(x)$ is a version of the density of

$\bar{P}_1$ w.r.t. $\bar{P}_1+\bar{P}_2$. So $\mathcal{B}$ is pairwise sufficient for $(X, \mathcal{A}_l(m), \bar{\mathcal{P}})$.

Let $C(P)$ be a carrier of P w.r.t. $\mathcal{P}$ which belongs to $\mathcal{B}$. Then $\bar{P}(C(P))=P(C(P))=1$. Let A be any set in $\mathcal{A}_l(m)$ such that $A\subset C(P)$ and $\bar{P}(A)=0$ hold. Then it follows that $P(A\cap T_P)=0$ and $A\cap T_P\subset C(P)$, which imply that $A\cap T_P\in N(\mathcal{P})$ by the definition of a carrier. So for any Q in $\mathcal{P}$ we have

$$Q(A\cap T_P\cap T_Q)=0\,. \qquad \cdots\cdots(11)$$

On the other hand we have $P(A\cap(X-T_P)\cap T_Q)=0$ and $A\cap(X-T_P)\cap T_Q\subset C(P)$. Therefore $A\cap(X-T_P)\cap T_Q\in N(\mathcal{P})$, and hence

$$Q(A\cap(X-T_P)\cap T_Q)=0\,. \qquad \cdots\cdots(12)$$

From (11) and (12), it follows that $\bar{Q}(A)=Q(A\cap T_Q)=0$. Therefore $C(P)$ is a carrier of $\bar{P}$ w.r.t. $\bar{\mathcal{P}}$. By Theorem 1, we have the desired Neyman factorization.

"If" part. Since $(X, \mathcal{A}_l(m), \bar{\mathcal{P}})$ is weakly dominated by $\bar{m}$ and each $dP/dm(x)$ is a version of the density of $\bar{P}$ w.r.t. $\bar{m}$, from Theorem 1 again, $\mathcal{B}$ is PSCC for $(X, \mathcal{A}_l(m), \bar{\mathcal{P}})$. Then, because each $\bar{P}$ is an extension of P to $\mathcal{A}_l(m)$, it is clear that $\mathcal{B}$ is PSCC for $(X, \mathcal{A}, \mathcal{P})$. This completes the proof.

Acknowledgement. This paper is a continuation of the study carried out in the paper by Ghosh, Morimoto and Yamada ([3]). The author wishes to thank Professor J.K. Ghosh and Professor H. Morimoto for their collaboration. He is also grateful to Professor B.V. Rao for his useful advice.

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