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THEOREMS OF RIESZ TYPE ON THE BOUNDARY BEHAVIOUR OF HARMONIC MAPS

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Introduction

In 1960–61, Constantinescu-Cornea [3] and Doob [6] investigated the Martin compactification of open Riemann surfaces and obtained the theorem of Riesz type on the boundary behaviour of analytic maps. Constantinescu-Cornea extended that theorem in an axiomatic system, where they considered a harmonic map of a harmonic space satisfying the axioms of Brelot, together with a resolutive compactification of the space[1]. In the case of the Martin space, it seems that the effort to obtain a result concerning fine filters by using their argument would meet inevitable difficulties. One of the reasons is as follows. They considered the trace filters consisting of neighbourhoods of boundary points. A fine filter of the Martin space is not a trace filter. In order to connect fine filters with trace filters, we have to consider a larger compactification, e.g., the Wiener's one. In the Wiener compactification, however, we know a case where any Wiener boundary point which lies over a Martin boundary point and possesses neighbourhoods, whose traces intersect every fine neighbourhood of the underlying Martin boundary point is not contained in the harmonic boundary.

The purpose of this paper is to get rid of this obstacle. We shall consider a compactification of a harmonic space X and filters $\mathcal{F}_x$ converging to boundary points x. For a harmonic map φ of X into another harmonic space X', we define the cluster set

$$\varphi^*(x) = \bigcap_{U \in \mathcal{F}_x} \overline{\varphi(U)},$$

where the closure is taken in an arbitrary compactification X'^* of X'. Our main theorem asserts that under some additional conditions, if the set $\varphi^*(x)$ falls in a polar set of X'^* for every point x of a boundary set A, then there exists a positive superharmonic function s on X such that

1) Cf. [5], Theorem 4. 10, p. 43.

$$\lim_{\mathcal{F}_x} s = +\infty \qquad \text{for every } x \in A.$$

From this theorem, we may derive some theorems of Riesz type including an extension due to Constantinescu-Cornea and Doob.

In § 1, we list up the hypothesis for harmonic spaces and the notations which will be used in this paper. § 2 is devoted to the main theorem and its proof. Applications of our main theorem yield theorems of Riesz type sharper than those of Constantinescu-Cornea [5], which are stated in § 3. An extension of the theorem of Constantinescu-Cornea and Doob concerning the Martin compactification and fine filters can be derived from our main theorem as well. To see this, we define first the Martin compactification of a harmonic space. The Martin space may be constructed in various manners[2]. Here we shall define it along M. Brelot [2] as we do not need Martin kernels in the following. The definition of the Martin space and some results are stated in § 4. In the last section, we shall give a theorem of Riesz type concerning fine filters, which gives another proof of the theorem due to Constantinescu-Cornea and Doob in the case of open Riemann surfaces.

1. Preliminaries and notations

Let X be a harmonic space in the sense of Brelot, i.e., X is a locally compact connected Hausdorff space satisfying the axioms 1,2 and 3 of Brelot [1]. In the sequel, we shall always assume the following:

1) X is non-compact,
2) X has a countable base for open sets,
3) $X \in \mathcal{P}$, i.e., there exists a positive potential on X,
4) $1 \in \mathcal{W}(X)$, i.e., the constant functions are Wiener functions [5].

Let X' be a second harmonic space in the sense of Brelot, which may be compact. The assumptions for X' are

1′) X' has a countable base for open sets,
2′) $X' \in \mathcal{P} \cup \mathcal{H}$, i.e., there exists a positive superharmonic function on X',
3′) when $X' \in \mathcal{H} - \mathcal{P}$, i.e., X' has no positive potentials, there exists a non-polar subset E' of X' each point of which is polar.

Let φ be a non-constant harmonic map [5] of X into X'. In an arbitrary compactification X'^* of X' we consider a polar set A'.

We denote by X^* a compactification of X and $\Delta = X^* - X$. For each point $x \in A \subset \Delta$, let $\mathcal{F}_x$ be a filter on X converging to x. We define

2) Cf. [2] and [10].

$$\varphi^*(x) = \bigcap_{U \in \mathcal{F}_x} \overline{\varphi(U)}$$

where the closure is taken in X'^*.

Let Y be a subset of X, s be a positive superharmonic function on Y and E be a subset of Y. We define

$$(\mathrm{R}_s^E)_Y = \inf \{v;\ \text{positive superharmonic on } Y \text{ dominating } s \text{ on } E\}$$

and

$$(\hat{\mathrm{R}}_s^E)_Y = \text{the lower semi-continuous regularization of } (\mathrm{R}_s^E)_Y,$$

i.e., $(\hat{\mathrm{R}}_s^E)_Y(x) = \liminf_{y \to x} (\mathrm{R}_s^E)_Y(y)$. When $Y = X$, we write simply $\hat{\mathrm{R}}_s^E$ instead of $(\hat{\mathrm{R}}_s^E)_X$.

2. The main theorem

2.1.

Theorem 1. *Let X' have no positive potentials and A' be a polar set in an arbitrary compactification X'^* of X'.*

Hypothesis: 1) *there exists a relatively compact domain D' of X' such that $D' \cap A' = \phi$,*

2) *there exist a point z_0' of a set E' (described in* 3′) *of* § 1), *a non-polar compact set $F' \subset D'$, a positive superharmonic function v on X and a positive number α satisfying*

a) $U' \cap F'$ is non-polar for every neighbourhood U' of z_0',

b) v is continuous in a neighbourhood of $\varphi^{-1}(F')$,

c) for each $x \in A$ there exists $V_x \in \mathcal{F}_x$ such that $\inf_{V_x} v \geq \alpha$,

d) $\hat{\mathrm{R}}_v^{\varphi^{-1}(F')}$ is a potential,

3) *$\varphi^*(x) \subset A'$ for every $x \in A$.*

Conclusion: *there exists a positive superharmonic function s on X such that*

$$\lim_{\mathcal{F}_x} s = +\infty \qquad \textit{for every } x \in A.$$

To prove the theorem we shall prepare some lemmas.

Lemma 1

Under the hypothesis of Theorem 1, *we have a decreasing sequence of compact non-polar sets $\{F_n'\}$ such that $F_1' \subset F'$ and*

$$\mathrm{R}_v^{\varphi^{-1}(F_n')}(y_0) < 1/4^n \qquad (n = 1, 2, \cdots),$$

where y_0 is a point of $X - \varphi^{-1}(\{z_0'\})$.

Proof. It is known that $\varphi^{-1}(\{z_0'\})$ is polar[3]. Let $\tilde{X}=X-\overline{\varphi^{-1}(\{z_0'\})}$ and let $\{\tilde{X}_k\}$ be an exhaustion of $\tilde{X}$, i.e., $\tilde{X}_k$ are relatively compact, $\overline{\tilde{X}_k}\subset\tilde{X}_{k+1}$ and $\bigcup_{k=1}^{\infty}\tilde{X}_k=\tilde{X}$. By the hypothesis 2), a) of Theorem 1, we may take a sequence of regular neighbourhoods $\{D_\nu'\}$ of z_0' such that

$$
(2.1)\qquad \begin{cases} D_1' = D' , \\ D_\nu' \supset \overline{D'}_{\nu+1} , \\ \bigcap_{\nu=1}^{\infty} D_\nu' = \{z_0'\} , \\ F' \cap D_\nu' = F_\nu'' \text{ is compact and non-polar.} \end{cases}
$$

We can find a sequence of integers $\{\nu(k)\}$ so that

$$
\varphi(\overline{\tilde{X}_k}) \cap \overline{D'_{\nu(k)}} = \phi \qquad (k=1, 2, \cdots).
$$

In fact, if we have $\varphi(\overline{\tilde{X}_{k_0}}) \cap \overline{D_\nu'} \neq \phi$ for some k_0 and for all ν, then we would have by (2.1)

$$
\phi \neq \bigcap_{\nu=1}^{\infty} \{\varphi(\overline{\tilde{X}_{k_0}}) \cap \overline{D_\nu'}\} = \varphi(\overline{\tilde{X}_{k_0}}) \cap \{z_0'\} ,
$$

which would imply

$$
z_0' \in \varphi(\overline{\tilde{X}_{k_0}}) \subset \varphi(\tilde{X}_{k_0+1}) ,
$$

i.e.,

$$
\varphi^{-1}(\{z_0'\}) \cap \tilde{X}_{k_0+1} \neq \phi ,
$$

which is absurd.

Next, we shall show that the restriction q of $\hat{R}_v^{\varphi^{-1}(F')}$ to $\tilde{X}$ is a potential on $\tilde{X}$. To see this, let u be the greatest harmonic minorant of q on $\tilde{X}$. The u is bounded on $K \cap \tilde{X}$, where K is a compact set of X. Since $\varphi^{-1}(\{z_0'\})$ is polar, u has a harmonic extension on X. The extension is the greatest harmonic minorant of a potential $\hat{R}_v^{\varphi^{-1}(F')}$. Thus $u\equiv 0$.

Finally, since $\hat{R}_v^{\varphi^{-1}(F_\nu''(k))}$ is bounded and harmonic on $\tilde{X}_k$, we have

$$
\hat{R}_v^{\varphi^{-1}(F''_{\nu(k)})} = H^{\tilde{X}_k}_{\hat{R}_v^{\varphi^{-1}(F_\nu''(k))}} \qquad \text{on } \tilde{X}_k .
$$

From the evident facts

$$
H^{\tilde{X}_k}_{\hat{R}_v^{\varphi^{-1}(F_\nu''(k))}} \leq H^{\tilde{X}_k}_q
$$

and

3) Cf. [5], Theorem 3.2, p. 21.

$$\lim_{k\to\infty} \mathrm{H}_q^{\tilde{X}_k} \doteq 0$$

it follows that there exists an increasing sequence $\{\nu(k_n)\}$ such that

$$\mathrm{R}_v^{\varphi^{-1}(F_\nu''(k_n))}(y_0) < 1/4^n \qquad (n=1, 2, \cdots).$$

Thus, $\{F_n'\}$ with $F_n'=F_\nu''{}_{(k_n)}$ is the desired one, q.e.d.

Lemma 2.

Under the hypothesis of Theorem 1, *we have a decreasing sequence of closed sets* $\{Q_n\}$ *such that*

$$(2.2) \qquad \begin{cases} Q_n \supset \varphi^{-1}(F_n'), \text{ where } F_n' \text{ is the set constructed in Lemma 1,} \\ p_n = \hat{\mathrm{R}}_v^{Q_n} \text{ is continuous,} \\ p_n(y_0) < 1/4^n. \end{cases}$$

Proof. Since v is continuous in a neighbourhood of $\varphi^{-1}(F_n')$, we have

$$\mathrm{R}_v^{\varphi^{-1}(F_n')}(y_0) = \inf\{\mathrm{R}_v^{\omega}(y_0);\ \omega \text{ is open and } \omega \supset \varphi^{-1}(F_n')\}^{4)}.$$

Thus, by Lemma 1, we have a decreasing sequence $\{G_n\}$ of open sets satisfying

$$(2.3) \qquad \begin{cases} \varphi^{-1}(F_n') \subset G_n, \\ \mathrm{R}_v^{G_n}(y_0) < 1/4^n. \end{cases}$$

We may assume that v is continuous in G_1. Let $\{X_k\}$ be an exhaustion of X. We shall fix n and put

$$(2.4) \qquad \begin{cases} K_1 = \varphi^{-1}(F_n') \cap \overline{X_2}, \\ K_k = \varphi^{-1}(F_n') \cap (\overline{X_{k+1}} - X_{k-1}) \qquad \text{for } k \geq 2 \end{cases}$$

and

$$(2.5) \qquad \begin{cases} \omega_1 = X_3 \cap G_n, \\ \omega_2 = X_4 \cap G_n, \\ \omega_k = (X_{k+2} - \overline{X_{k-2}}) \cap G_n \qquad \text{for } k \geq 3. \end{cases}$$

Obviously, K_k are compact, ω_k are open, $\omega_k \supset K_k$ $(k=1, 2, \cdots)$ and

$$\bigcup_{k=1}^{\infty} K_k = \varphi^{-1}(F_n').$$

In a sufficiently small neighbourhood V_y of each point y of K_k, there exists a compact set U_y containing y in its interior and each boundary point of U_y is

4) Cf. [1], Theorem 23, p. 122.

regular with respect to the Dirichlet problem on $V_y - U_y$[5]. A finite union W_k of these U_y covers K_k. $Q_n = \bigcup_{k=1}^{\infty} W_k$ is closed. From $\varphi^{-1}(F_n') \subset Q_n \subset G_n$ we have

$$\mathrm{R}_v^{Q_n}(y_0) < 1/4^n.$$

We may construct successively $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$. It is easily seen

$$\mathrm{R}_v^{Q_n} = \mathrm{H}_f^{X_k - Q_n} \qquad \text{in } X_k - Q_n,$$

where

$$f = \begin{cases} v & \text{on } X_k \cap \partial Q_n, \\ \mathrm{R}_v^{Q_n} & \text{on } \partial X_k \cap (X - Q_n). \end{cases}$$

Since each boundary point of Q_n is regular with respect to the Dirichlet problem[6] on $X_k - Q_n$,

$$\lim_{\substack{z \to y \\ z \in X - Q_n}} \mathrm{R}_v^{Q_n}(z) = \lim_{\substack{z \to y \\ z \in X_k - Q_n}} \mathrm{H}_f^{X_k - Q_n}(z) = f(y) = v(y)$$

at each $y \in \partial Q_n$.

This means that $p_n = \hat{\mathrm{R}}_v^{Q_n} = \mathrm{R}_v^{Q_n}$ and p_n is continuous, q.e.d..

2.2.

Now, we shall proceed to the proof of our main theorem. We shall fix a point y_0 and assume $\varphi(y_0) \notin A' \cup F'$ and $v(y_0) = 1$. Let $\{v_n'\}$ be a sequence of positive superharmonic functions such that

$$(2.6) \qquad \begin{cases} v_n' \text{ is defined on } X' - F_n', \\ \lim_{y' \to x'} v_n'(y') = +\infty \text{ for every } x' \in A', \\ v_n'[\varphi(y_0)] < 1/2^n, \end{cases}$$

and let

$$V_0^{(n)} = \{y \in X;\, p_n(y) > (3/4) \cdot v(y)\}$$

and

$$V_1^{(n)} = \{y \in X;\, p_n(y) > (1/2) \cdot v(y)\},$$

where F_n' is the closed set in Lemma 1 and p_n is the function in Lemma 2. We have

$$V_1^{(n)} \supset \overline{V_0^{(n)}} \cap X \supset V_0^{(n)} \supset Q_n.$$

Since $p_n \geq p_{n+1}$, it is readily seen

5) Cf. [9]. Lemma 7.1, p. 439.
6) Cf. [1], Theorem 22, p. 118.

$$V_i^{(n)} \supset V_i^{(n+1)} \qquad (i=0,\ 1).$$

We define

$$s_n = \begin{cases} v & \text{on } \bar{V}_0^{(n)} \cap X, \\ \hat{\mathrm{R}}_v^{\bar{V}_0^{(n)}} + (\hat{\mathrm{R}}^{X-V_1^{(n)}}_{\min(v_n' \circ \varphi,\,(1/4)\cdot v)})_{X-\bar{V}_0^{(n)}} & \text{on } X-\bar{V}_0^{(n)}. \end{cases}$$

In the following it will be shown that

$$s = \sum_{n=1}^{\infty} (s_n + p_n)$$

fulfills the requirement of Theorem 1.

(1) $\qquad v \geq \hat{\mathrm{R}}_v^{\bar{V}_0^{(n)}} + (\hat{\mathrm{R}}^{X-V_1^{(n)}}_{\min(v_n' \circ \varphi,\,(1/4)\cdot v)})_{X-\bar{V}_0^{(n)}} \qquad$ *on* $X-\bar{V}_0^{(n)}$

In fact, $(1/2)\cdot v \geq p_n$ on $X-V_1^{(n)}$ implies $v-(1/4)v \geq p_n/(3/4)$ on $X-V_1^{(n)}$. From $p_n/(3/4) \geq \mathrm{R}_v^{\bar{V}_0^{(n)}}$ on X we deduce $v-(1/4)v \geq \hat{\mathrm{R}}_v^{\bar{V}_0^{(n)}}$ on $X-V_1^{(n)}$, that is, $v-\hat{\mathrm{R}}_v^{\bar{V}_0^{(n)}} \geq (1/4)v \geq \min(v_n' \circ \varphi,\ (1/4)v)$ on $X-V_1^{(n)}$. Since $v-\hat{\mathrm{R}}_v^{\bar{V}_0^{(n)}}$ is non-negative and superharmonic on $X-\bar{V}_0^{(n)}$, we have

$$v-\hat{\mathrm{R}}_v^{\bar{V}_0^{(n)}} \geq (\hat{\mathrm{R}}^{X-V_1^{(n)}}_{\min(v_n' \circ \varphi,\,(1/4)v)})_{X-\bar{V}_0^{(n)}}\,.$$

(2) s_n *is lower semi-continuous.*

The only point in question is the boundary of $V_0^{(n)}$. Let $y \in \partial V_0^{(n)}$, ω_1 be a regular neighbourhood of y and $\omega = \omega_1 - \bar{V}_0^{(n)}$. The function

$$f = \begin{cases} v & \text{on } \partial V_0^{(n)} \cap \omega_1 \\ \mathrm{R}_v^{\bar{V}_0^{(n)}} & \text{on } \partial\omega_1 \cap (X-V_0^{(n)}) \end{cases}$$

is resolutive for ω with respect to the Dirichlet problem. Clearly we have

$$\mathrm{R}_v^{\bar{V}_0^{(n)}} \geq \overline{\mathrm{H}}_f^{\omega} = \underline{\mathrm{H}}_f^{\omega} \qquad \text{in } \omega.$$

Since y is a regular boundary point with respect to the Dirichlet problem,

$$\liminf_{\substack{x \to y \\ x \in \omega}} \underline{\mathrm{H}}_f^{\omega}(x) \geq \liminf_{\substack{z \to y \\ z \in \partial\omega}} f(z) \geq v(y)\,.$$

Thus, we have

$$\liminf_{\substack{x \to y \\ x \in X-\bar{V}_0^{(n)}}} \mathrm{R}_v^{\bar{V}_0^{(n)}}(x) \geq v(y)\,,$$

which means

$$\liminf_{\substack{x \to y \\ x \in X-\bar{V}_0^{(n)}}} s_n(x) \geq s_n(y)$$

at each $y \in \partial V_0^{(n)}$.

From (1) and (2) we know that s_n is superharmonic on X.

(3) $s(y_0) < +\infty$.

From $p_n(y_0) < 1/4^n < 1/2 = (1/2) \cdot v(y_0)$ we see $y_0 \notin \bar{V}_1^{(n)}$. Hence,

$$\begin{aligned} s_n(y_0) &= \hat{R}_v^{\bar{V}_0^{(n)}}(y_0) + \min(v_n'[\varphi(y_0)], (1/4)v(y_0)) \\ &\leq (4/3)p_n(y_0) + v_n'[\varphi(y_0)] \\ &\leq (4/3)(1/4^n) + 1/2^n \end{aligned}$$

and

$$s(y_0) \leq (7/3)\sum_{n=1}^{\infty}(1/4^n) + \sum_{n=1}^{\infty} 1/2^n < +\infty.$$

Finally, we shall show

(4) $\lim_{\mathcal{F}_x} s = +\infty$ *for every* $x \in A$.

Let $B = \bigcap_{n=1}^{\infty}(A \cap \bar{V}_1^{(n)})$ and β be an arbitrary positive number. First, we consider the case: $x \in A - B$. Then, there exists an integer m so that

$$x \in A - \bar{V}_1^{(m)} \tag{2.7}$$

(and $x \in A - \bar{V}_1^{(l)}$ for all $l \geq m$). Let N be a positive integer so large that $N > (4/\alpha) \cdot \beta$. From the hypothesis of the theorem, there exists $V_x \in \mathcal{F}_x$ such that

$$v \geq \alpha \qquad \text{on } V_x. \tag{2.8}$$

We can find $U_x \in \mathcal{F}_x$ so that

$$v_n' \circ \varphi \geq \alpha/4 \qquad \text{on } U_x \text{ for } n = 1, 2, \cdots, N+m. \tag{2.9}$$

In fact, since by (2.6)

$$\lim_{y' \to x'} v_n'(y') = +\infty \qquad \text{for every } x' \in A',$$

there exists a neighbourhood P' of A' such that

$$v_n' > \alpha/4 \qquad \text{on } P' \cap X' \text{ for } n = 1, 2, \cdots, N+m.$$

On the other hand, $\varphi^*(x) \subset A'$ implies the existence of $U_x \in \mathcal{F}_x$ satisfying

$$\overline{\varphi(U_x)} \subset P'.$$

From (2.7), we have a neighbourhood W of x such that

$$W \cap \bar{V}_1^{(m)} = \phi$$

and naturally

$$W \cap \bar{V}_1^{(l)} = \phi \qquad \text{for } l \geq m.$$

Therefore we have

$$s_n \geq \min(v_n' \circ \varphi, (1/4)v) \qquad \text{on } W \text{ for } n \geq m.$$

Consequently, in view of (2.8) and (2.9) it follows

$$s_n \geq \alpha/4 \qquad \text{on } V_x \cap U_x \cap W \text{ for } n = m+1, \cdots, m+N.$$

If we take $W_x \in \mathcal{F}_x$ so that $W_x \subset V_x \cap U_x \cap W$, then

$$s \geq (\alpha/4)N \geq \beta \qquad \text{on } W_x.$$

Next, let $x \in B$, then there exist V_x and U_x in $\mathcal{F}_x$ such that

$$\begin{cases} v \geq \alpha & \text{on } V_x \\ v_n' \circ \varphi \geq \alpha/4 & \text{on } U_x \text{ for } n = 1, 2, \cdots, 2N. \end{cases} \tag{2.10}$$

If we prove

$$s \geq \beta \text{ on any } W_x \in \mathcal{F}_x \text{ such that } W_x \subset V_x \cap U_x, \tag{2.11}$$

the proof will be completed.

To prove (2.11), we consider first $y \in W_x - \bar{V}_1^{(N)}$. Since $y \in W_x - \bar{V}_1^{(l)}$ for $l \geq N$,

$$s_n(y) \geq \min(v_n'[\varphi(y)], (1/4)v(y)) \qquad \text{for } n \geq N.$$

By virtue of (2.10) we have

$$s_n(y) \geq \alpha/4 \qquad \text{for } n = N+1, \cdots, 2N,$$

then, we conclude

$$s(y) \geq \beta \qquad \text{on } W_x - \bar{V}_1^{(N)}.$$

If $y \in W_x \cap \bar{V}_1^{(N)}$, then $y \in W_x \cap \bar{V}_1^{(n)}$ for all $n \leq N$, which implies

$$p_n(y) \geq (1/2) \cdot v(y) \geq (1/2)\alpha \qquad \text{for } n = 1, 2, \cdots, N,$$

Thus, we have

$$s(y) \geq \beta \qquad \text{on } W_x \cap \bar{V}_1^{(N)}, \quad \text{q.e.d.}.$$

3. Consequences

In this section, we shall give some theorems of Riesz type as an application of our main theorem.

Theorem 2.

Let φ be a non-constant Fatou map[7] *from X into X' and A' be a polar set*

7) Cf. [5], p. 52.

in an arbitrary compactification X'^* of X'. In case $X' \in \mathcal{H} - \mathcal{P}$, we assume further that there exists an open set E' each point of which is polar. In a compactification X^* of X, let A be a boundary set satisfying

$$\varphi^*(x) = \bigcap_{U \in \mathcal{U}_x} \overline{\varphi(U \cap X)} \subset A' \qquad \textit{for every } x \in A,$$

where $\mathcal{U}_x$ is a base of neighbourhoods of x in X^* and the closure is taken in X'^*

Then, the set A is polar.

Proof. If $X' \in \mathcal{P}$ the proof is easily carried out. We shall be concerned with the case $X' \in \mathcal{H} - \mathcal{P}$. Let $z' \in E' - A'$, and $\{D_n'\}$ be a base of neighbourhoods of z'.

Putting

$$A_n = \{x \in A;\ \varphi^*(x) \cap \overline{D_n'} = \phi\}\ ,$$

we have

$$A = \bigcup_{n=1}^{\infty} A_n\,.$$

Therefore, in order to prove the theorem it is enough to show that A_n is polar. Thus, we may assume $A' \cap D_1' = \phi$ (the hypothesis 1) of Theorem 1). Let u_0' be positive and harmonic in D_1' and p_0' be a finite continuous positive superharmonic function in D_1' not harmonic in a regular neighbourhood D_2' of z' ($\overline{D_2'} \subset D_1'$). Put

$$p_1' = (\hat{\mathrm{R}}_{p_0'}^{K'})_{D_1'}$$

where K' is a compact neighbourhood of z' contained in D_2', and set

$$f' = \begin{cases} 0 & \text{on } X' - D_2'\,, \\ (p_1' - \mathrm{H}_{p_1'}^{D_2'})/u_0' & \text{on } D_2'\,. \end{cases}$$

The f' is finite continuous and non-constant on X. Further we see that f' is a Wiener function on X'[8]. Since φ is a Fatou map $f' \circ \varphi$ is a Wiener function on X. From a theorem of Constantinescu-Cornea[9], except for a countable set of constants γ

$$\hat{\mathrm{R}}_1^{\varphi^{-1}(F_\gamma')}$$

is a potential, where

$$F_\gamma' = \{x' \in X';\ f'(x') = \gamma\}\ .$$

We note that F_γ' is non-polar and compact, and for each point y' of F_γ' the

8) Cf. [5], p. 16

9) Cf. [5], Theorem 2.6, p. 14.

intersection of every neighbourhood of y' with F_γ' is non-polar. We may take z' so that $z' \in F_\gamma'$

Let V be an intersection of a neighbourhood of A with X satisfying

$$\bar{V} \cap \varphi^{-1}(F_\gamma') = \phi.$$

The reduced function

$$v = \hat{\mathrm{R}}_1^V$$

is continuous in a neighbourhood of $\varphi^{-1}(F_\gamma')$ and $v=1$ on V. We assert that $\mathrm{R}_v^{\varphi^{-1}(F_\gamma')}$ is a potential. In fact, since, by hypothesis, $1 \in \mathcal{W}(X)$ (see §1, 4)), we have

$$1 = \mathrm{h}_1^X + q\,,$$

where q is a Wiener potential[10], therefore

$$|q| \leq p$$

for some potential p. Form

$$v = \hat{\mathrm{R}}_1^V \leqq \mathrm{h}_1^X + p$$

we infer

$$v \leq 1 + 2p$$

and

$$\hat{\mathrm{R}}_v^{\varphi^{-1}(F_\gamma')} \leq \hat{\mathrm{R}}_1^{\varphi^{-1}(F_\gamma')} + 2p\,. \tag{3.1}$$

The assertion follows from the fact that the last two terms of (3.1) are potentials. Thus, the hypothesis of Theorem 1 is satisfied by

$$D_1',\ A',\ z',\ F_\gamma',\ v \ \text{and}\ \alpha = 1.$$

The filter $\mathcal{F}_x$ converging to x is the filter of the intersections of neighbourhoods of x with X. Hence, A is polar, q.e.d. .

Theorem 3.

Let φ be a non-constant harmonic map from X into X' and A' be a polar set in an arbitrary compactification X'^* of X'. In a resolutive compactification X^* of X, let A be a boundary set satisfying

$$\varphi^*(x) = \bigcap_{U \in \mathcal{U}_x} \overline{\varphi(U \cap X)} \subset A' \qquad \textit{for every } x \in A,$$

where $\mathcal{U}_x$ is a base of neighbourhoods of x in X^* and the closure is taken in X'^*.

Then, A is polar from inside, that is, every compact subset of A is polar.

10) Cf. [5], p. 16.

Proof. We shall consider only the case of $X' \in \mathcal{H} - \mathcal{P}$. Indeed, when $X' \in \mathcal{P}$, the map φ is a Fatou map and Theorem 3 is reduced to Theorem 2. Let B be an arbitrary compact subset of A, and let $z_0' \in E' - A'$, where E' is a non-polar set each point of which is polar. We can find a neighbourhood V of B in X^* and a relatively compact domain D' containing z_0' such that

$$\bar{V} \cap \overline{\varphi^{-1}(D')} = \phi . \tag{3.2}$$

In fact, since $z_0' \notin \varphi^*(x)$ for each $x \in B$, there exist an open neighbourhood U' of $\varphi^*(x)$ and an open neighbourhood $U_x'(z_0')$ of z_0' such that

$$\overline{U'} \cap \overline{U_x'}(z_0') = \phi .$$

For some open neighbourhood W_x of x in X^*, we have

$$\overline{\varphi(W_x \cap X)} \subset U' .$$

We may take a neighbourhood V_x of x in X^* so that

$$\bar{V}_x \subset W_x .$$

By the compactness of B, a finite set of such V_x covers B, i.e.,

$$B \subset \bigcup_{i=1}^{N} V_{x_i} = V .$$

It is easily seen that

$$D' = \bigcap_{i=1}^{N} U'_{x_i}(z_0')$$

and the set V defined above satisfy (3.2).
From (3.2), we know

$$\min(\hat{\mathrm{R}}_1^V, \hat{\mathrm{R}}_1^{\varphi^{-1}(D')})$$

is a potential[11].

Let F' be a compact neighbourhood of z_0' contained in D'. The funciton $v = \hat{\mathrm{R}}_1^V$ is continuous in a neighbourhood of $\varphi^{-1}(F')$. We shall show that $\hat{\mathrm{R}}_v^{\varphi^{-1}(F')}$ is a potential. For, as (3.1),

$$\hat{\mathrm{R}}_v^{\varphi^{-1}(F')} \leq \hat{\mathrm{R}}_1^{\varphi^{-1}(F')} + p_0 ,$$

where p_0 is a potential. On the other hand,

$$\hat{\mathrm{R}}_v^{\varphi^{-1}(F')} \leq v = \hat{\mathrm{R}}_1^V .$$

Therefore,

11) Cf. [5], Theorem 4.6, p. 37.

$$\hat{R}_v^{\varphi^{-1}(F')} \leq \min(\hat{R}_1^{\varphi^{-1}(F')} + p_0, \hat{R}_1^{\Gamma})$$
$$\leq \min(\hat{R}_1^{\varphi^{-1}(F')}, \hat{R}_1^{\Gamma}) + p_0 .$$

The hypothesis of Theorem 1 is satisfied by

$$D', \bigcup_{x \in B} \varphi^*(x), z_0', F', v \text{ and } \alpha = 1 .$$

Thus, B is polar, q.e.d. .

4. The Martin compactification and the Dirichlet problem

4.1. The definition of the Martin space in an axiomatic system was indicated briefly in [2]. For the sake of completeness, we shall define the Martin space. We list up the hypothesis for the harmonic space X we shall consider here:

$$\begin{cases} \text{Axioms 1, 2 and 3 of Brelot.} \\ X \text{ is non-compact and has a countable basis of open sets,} \\ X \in \mathcal{P}, \\ 1 \in \mathcal{W}(X), \\ \text{Proportionality axiom, i.e., potentials with one point support} \\ \qquad\qquad \{y\} \text{ are all proportional, for every } y \in X . \\ \text{Existence of a completely determining domain } \delta_0. \end{cases}$$

Let S^+ be the set of all non-negative superharmonic functions and E be the set of all potentials with one point support. By Hervé [9], we know that under Hervé's topology the positive cone S^+ is metrizable and has a compact base A, and $E \cap A$ is homeomorphic to X.

The *Martin space* X^* is defined to be the closure of $E \cap A$ in A. Extreme points of A are contained in X^*. Every $s \in S^+$ can be expressed by a regular Borel measure ν on X^* (Choquet-Hervé):

$$s(y) = \int p(y) d\nu(p) .$$

Clearly X^* is a compactification of X, i.e., X^* is a compact space containing X as a dense open set. $\Delta = X^* - X$ is called the *Martin boundary of* X and the set of extreme points of the set of all positive harmonic functions u with $u(y_0) = 1$ is denoted by Δ_1. Δ_1 is a G_δ-set. A harmonic function u can be expressed uniquely by a regular Borel measure μ on Δ_1:

$$u(y) = \int_{\Delta_1} w(y) d\mu(w) .$$

μ is called the canonical measure of u. The point of Δ_1 is a positive minimal[12].

12) Cf. [11].

harmonic function w on X with $w(y_0)=1$. For the sake of convenience, the point of Δ are indicated by the letter x, whereas they are harmonic functions in A. We use also the letter w_x to emphasize the fact that a point $x\in\Delta$ is a harmonic function on X. Thus, the kernels $k(x, y)=w_x(y)$ are defined for $(x, y)\in\Delta\times X$ and they correspond to the original Martin kernels[12]. The construction of Martin spaces in an axiomatic setting by means of Martin kernels is given in [10], but we shall not use kernels in this paper.

4.2. Here we discuss the Dirichlet problem with respect to the Martin compactification.

Theorem 4. *The Martin compactification is resolutive.*

Proof. In order to prove the theorem, it is sufficient to show that every continuous function f on X^* is harmonizable[13]. We may suppose $0\leq f\leq 1$. Put

$$\tag{4.1} \begin{cases} A_i=\{x\in\Delta_1;\ (i-1/2)/n<f(x)\leq(i+1/2)/n\} \\ G_i=\{y\in X; f(y)<(3i-2)/3n\}\cup\{y\in X; f(y)>(3i+2)/3n\}. \end{cases} \qquad (i=0, 1, \cdots, n)$$

By the hypothesis $1\in\mathcal{W}(X)$, $u=\mathrm{h}_1^X$ is positive and harmonic. The canonical measure of u is denoted by μ_u:

$$u(y)=\int_{\Delta_1} w_x(y)d\mu_u(x).$$

Set

$$u_i(y)=\int_{A_i} w_x(y)d\mu_u(x) \qquad (i=0, 1, \cdots, n).$$

It is readily seen that $\hat{\mathrm{R}}_{w_x}^{G_i}$ is a potential for every $x\in A_i$, therefore the regularization of

$$\mathrm{R}_{u_i}^{G_i}(y)=\int_{A_i}\mathrm{R}_{w_x}^{G_i}(y)d\mu_u(x)$$

is a potential.

From

$$(3i-2)/3n\cdot[u_i-\mathrm{R}_{u_i}^{G_i}]\leq fu_i\leq(3i+2)/3n\cdot u_i+\mathrm{R}_{u_i}^{G_i} \qquad \text{on } X$$

we derive

$$\sum_{i=0}^{n}(3i-2)/3n\cdot[u_i-\mathrm{R}_{u_i}^{G_i}]\leq\sum_{i=0}^{n}fu_i=fu\leq\sum_{i=0}^{n}(3i+2)/3n\cdot u_i+\sum_{i=0}^{n}\mathrm{R}_{u_i}^{G_i}.$$

We have $1=u+q$, where q is a Wiener potential, thus $|q|\leq p$ for some potential p.

13) Cf. [5], Theorem 4.4, p.35.

From

$$\bar{h}^X_{fp} \leq \bar{h}^X_p = 0$$

it is derived

$$(4.2)\qquad \begin{cases} \bar{h}^X_{fq} \leq \bar{h}^X_{f|q|} \leq \bar{h}^X_{fp} = 0\,, \\ \underline{h}^X_{fq} \geq \underline{h}^X_{f(-p)} = -\bar{h}^X_{fp} = 0\,, \end{cases}$$

that is, $h^X_{fq}=0$. Inequalities

$$\underline{h}^X_{fu} = \underline{h}^X_{fu} + \underline{h}^X_{fq} \leq \underline{h}^X_{f(u+q)} = \underline{h}^X_f \leq \bar{h}^X_f = \bar{h}^X_{f(q+u)} \leq \bar{h}^X_{fu} + \bar{h}^X_{fq} = \bar{h}^X_{fu}$$

imply

$$\bar{h}^X_f - \underline{h}^X_f \leq \bar{h}^X_{fu} - \underline{h}^X_{fu} \leq \sum_{i=0}^{n} (4/3n)u_i = (4/3n)\cdot u\,.$$

Since n is arbitrary, we know that f is harmonizable, q.e.d..

Corollary. *For any continuous function f on X^*, f and $f h^X_1$ are harmonizable and*

$$h^X_f = h^X_{fh^X_1}\,.$$

Theorem 5. *For any continuous function f on X^*, we have*

$$h^X_{fu} = (\mathcal{D}_{f,u})u\,,$$

where $u = h^X_1$ and $\mathcal{D}_{f,u}$ is the u-Dirichlet solution for the boundary function f with respect to the Martin compactification[14].

This is a straightforward consequence from the construction of upper and lower solutions of the relative Dirichlet problem. We omit the proof.

By Theorem 4, there exists a harmonic measure ω_y such that

$$H_f(y) = h^X_f(y) = \int f d\omega_y\,,$$

for every continuous function f on X^*, where H_f is the Dirichlet solution for the boundary function f with respect to X^*.

Combining this with the results of Gowrisankaran[15], we know that for any continuous function f on Δ

$$\int f(x) d\omega_y(x) = \int f(x) w_x(y)\, d\mu_u(x)\,,$$

that is,

14) Cf. [7], p. 349.

15) Cf. [8], Theorem 6, p. 464 and Theorem 7, p. 465.

$$d\omega_y(x) = w_x(y)d\mu_u(x) .$$

This implies that ω_y and μ_u are mutually absolutely continuous for each $y\in X$. For, since $w_x(y_0)=1$ for every $x\in\Delta_1$, we can find a constant $C>0$ so that $1/C<w_x(y)=w_x(y)/w_x(y_0)<C$ for all $x\in\Delta_1$. The measure theoretic properties of boundary sets (for example, the nullity of the exceptional sets) are therefore unaltered for μ_u and ω_y. In view of this fact, we shall denote a harmonic measure by ω without suffix.

Theorem 6. (i) $u=h_1^X$ *has a fine limit* 1 $d\omega$*-almost everywhere on* Δ.

(ii) *any positive superharmonic function* v *has finite fine limit* $d\omega$*-almost everywhere on* Δ.

Proof. (i) is an immediate consequence of

$$1-p/u\leq 1/u\leq 1+p/u \tag{4.3}$$

and the fact that p/u has a fine limit zero $d\mu_u$-almost everywhere on Δ[16]. (4.3) is derived from

$$1 = u+q, \text{ where } |q|\leq p \text{ for some potential } p .$$

(ii) follows from (i) by a theorem of Gowrisankaran[17] and

$$v = (v/u)\cdot u .$$

5. The theorem concerning fine filters

In this section, let X^* be the Martin compactification of X. We shall be concerned with the theorem of Riesz type with respect to fine filters.

Theorem 7. *Let* φ *be a non-constant harmonic map from* X *into* X', *and* A' *be a polar set in an arbitrary compactification* X'^* *of* X'. *In the Martin compactification* X^* *of* X, *to each point* x *of a boundary set* A *we assign a filter* $\mathcal{F}_x$ *consisting of all sets* $E\subset X$ *such that* $X-E$ *is thin at* x[18]. *If we have*

$$\varphi^*(x) = \bigcap_{E\in\mathcal{F}_x} \overline{\varphi(E)}\subset A' \qquad \text{for every } x\in A ,$$

then A *is of harmonic measure zero, and consequently a polar set.*

Proof. We may of course assume $A\subset\Delta_1$. Let μ_u be the canonical measure of $u=\mathrm{h}_1^X$:

$$u(y) = \int_{\Delta_1} w_x(y)d\mu_u(x) .$$

16) Cf. [8], Theorem 2, p. 458.
17) Cf. [8], Theorem 8, p. 467.
18) Cf. [8], p. 456.

Suppose $\omega^*(A)>0$. $z_0'\in E'-A'$, where E' is a non-polar set each point of which is polar, and let $\{U_n'\}$ be a system of neighbourhoods of z_0'.

Putting

$$A_n=\{x\in A;\ \varphi^*(x)\subset A'-\overline{U_n'}\}\ ,$$

we have

$$A=\bigcup_{n=1}^{\infty}A_n\,.$$

Thus $\omega^*(A_{n_0})>0$ for some n_0. Let F' be a compact neighbourhood of z_0' contained in U'_{n_0} and

$$B=\{x\in\Delta_1;\ \varphi^{-1}(F')\text{ is thin at }x\}\ .$$

It is known B is $d\mu_u$-measurable[19] (therefore B is $d\omega$-measurable) and $A_{n_0}\subset B$. We define a positive harmonic function v

$$v=\int_B w_x\,d\mu_u(x)$$

$\hat{R}_v^{\varphi^{-1}(F')}$ is a potential[20]. The fine limit of v is 1 $d\omega$-almost everywhere on B. This is an easy consequnce of

$$(5.1)\qquad\begin{cases} v(y)=\int_\Delta \chi_B(x)w_x(y)d\mu_u(x)\,, & \\ \operatorname*{fine\,lim}_{y\to x_0}\left[\int_\Delta \chi_B(x)w_x(y)d\mu_u(x)\right]\Big/u(y)=\chi_B(x_0) & d\mu_u\text{-a.e.,}\\ \operatorname*{fine\,lim}_{y\to x}u(y)=1 & d\mu_u\text{-a.e.,}\end{cases}$$

where χ_B is the characteristic function of B on Δ.

Thus the hypothesis of Theorem 1 is satisfied by

$$U'_{n_0},\ A'-\overline{U'_{n_0}},\ z_0',\ F',\ v\quad\text{and}\quad \alpha=1/2\,.$$

Applying Theorem 1 to A_{n_0}, we have a positive superharmonic function s on X possessing the fine limit $+\infty$ at almost all points of A_{n_0}, which contradicts with (ii) of Theorem 6. Thus the proof is completed.

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19) Cf. [8], Lemma 2, P. 457.
20) Cf. [8], Corollary to Theorem 1, p. 458.

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