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AN ELEMENTARY PROOF OF A THEOREM OF BREMNER

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In the paper [1] Bremner proved that the Diophantine equation

$$(1) \quad 3x^4 - 4y^4 - 2x^2 + 12y^2 - 9 = 0$$

has only two positive integer solutions $(x, y) = (1, 1)$ and $(3, 3)$, which was suggested by Enomoto, Ito and Noda in their research on tight 4-desings (see [2]). However, he used some of results of Cassels in biquadratic field $\mathbf{R}(\sqrt[4]{3})$ and the \mathfrak{p} -adic method of Skolem, so his proof is somewhat difficult. In 1983, Ko Chao and Sun Qi indicated that an elementary proof of Bremner's theorem would be significant (see [3]). Now such an elementary proof is given in this paper with nothing deeper than quadratic reciprocity used. We describe our method as follows.

Since (1) may be reduced to $(3x^2 - 1)^2 - 3(2y^2 - 3)^2 = 1$, we have $(3x^2 - 1) + (2y^2 - 3)\sqrt{3} = u_n + v_n\sqrt{3} = (2 + \sqrt{3})^n$, the latter equation denotes the general solution of the Pell's equation $U^2 - 3V^2 = 1$, n is an integer. Thus

$$(2) \quad 2y^2 = v_n + 3.$$

First we assume $n = 3m$. By

$$u_{3m} + v_{3m}\sqrt{3} = (u_m + v_m\sqrt{3})^3 = (u_m^3 + 9u_mv_m^2) + (3u_m^2v_m + 3v_m^3)\sqrt{3},$$

we get

$$v_{3m} = 3v_m(u_m^2 + v_m^2) = 3v_m(4v_m^2 + 1),$$

so that

$$2y^2 = 3(4v_m^3 + v_m) + 3,$$

which leads to

$$6y_1^2 = 4v_m^3 + v_m + 1 = (2v_m + 1)(2v_m^2 - v_m + 1),$$

where $y = 3y_1$, $y_1 > 0$. Since $(2v_m + 1, 2v_m^2 - v_m + 1) = 1$ and $2 \nmid (2v_m + 1)$, $3 \nmid (2v_m^2 - v_m + 1)$ we have

$$2y_2^2 = 2v_m^2 - v_m + 1, \quad y_2 | y_1, \quad y_2 > 0.$$

Thus

$$\begin{aligned} (4y_2)^2 &= (4v_m - 1)^2 + 7, \\ (4y_2 + 4v_m - 1)(4y_2 - 4v_m + 1) &= 7, \\ 4y_2 \pm 4v_m \mp 1 &= 7, \quad 4y_2 \mp 4v_m \pm 1 = 1, \end{aligned}$$

which gives $y_2=1$, $v_m=1$. Hence $m=1$, $n=3$. That is, if $3|n$ then (2) holds only when $n=3$, this case gives $(x, y)=(3, 3)$, a positive integer solution of (1).

Next we list the following relations which may be derived easily from the general solution of the Pell's equation:

$$\begin{aligned} (3) \quad u_{n+1} &= 4u_n - u_{n-1}, \quad u_0 = 1, \quad u_1 = 2, \\ (4) \quad v_{n+1} &= 4v_n - v_{n-1}, \quad v_0 = 0, \quad v_1 = 1, \\ (5) \quad v_{n+2k} &\equiv -v_n \pmod{u_k}. \end{aligned}$$

If $n \leq -2$ then $v_n + 3 < 0$, (2) cannot hold, so we only consider the cases $n \geq -1$. Since, by (2), $v_n \equiv 1 \pmod{2}$, then $n \equiv 1 \pmod{2}$ by (4). Take modulo 8 to (4) we find that if $n \equiv 1 \pmod{4}$ then $v_n \equiv 1 \pmod{8}$, leads to $2y^2 \equiv 4 \pmod{8}$, which is impossible, so that it is necessary for $n \equiv -1 \pmod{4}$.

Again, take modulo 37 to (4) we obtain a sequence with period 36 as follows (only the terms with foot indices of the form $4k-1$ are listed):

$n \pmod{36}$	-1	3	7	11	15	19	23	27	31
$v_n \pmod{37}$	-1	15	25	25	15	-1	13	7	13
$\left(\frac{v_n+3}{37}\right)$	-	-	+	+	-	-	+	+	+

Since (2) implies $\left(\frac{v_n+3}{37}\right) = \left(\frac{2y^2}{37}\right) = -1$, so according to the above table we can exclude $n \equiv 7, 11, 23, 27, 31 \pmod{36}$. Furthermore $n \equiv 3, 15 \pmod{36}$ belong to the case $3|n$, which has been solved in the previous paragraph, then may be excluded, so that there remain the cases $n \equiv -1, 19 \pmod{36}$.

Now by taking modulo 3 to (4) we can exclude $n \equiv 1 \pmod{6}$, so also $n \equiv 19 \pmod{36}$, since it implies $v_n \equiv 1 \pmod{3}$ and $2y^2 \equiv 1 \pmod{3}$, which is impossible. Thus the only case left is $n \equiv -1 \pmod{36}$.

Suppose that $n \equiv -1 \pmod{36}$ and $n \neq -1$, we can write $n = -1 + (12k \pm 4) \cdot 3^r$, where $r \geq 2$. Let $m = 3^r$, by repeated application of (5) and the relations $v_{-n} = -v_n$, $v_{n \pm 1} = \pm u_n + 2v_n$, we get

$$\begin{aligned} v_n &\equiv v_{-1 \pm 4m} \pmod{u_{3m}}, \\ 2y^2 &\equiv v_{-1 \pm 4m} + 3 \equiv -v_{-1 \mp 2m} + 3 \equiv u_{2m} \pm 2v_{2m} + 3 \pmod{u_{3m}}. \end{aligned}$$

Since $u_{3m} = u_m(u_m^2 + 9v_m^2)$ and $2 \nmid m$ implies $u_m \equiv 2 \pmod{8}$, $v_m \equiv \pm 1 \pmod{8}$, $u_m^2 + 9v_m^2 \equiv 5 \pmod{8}$, so that

$$(6) \quad \left(\frac{u_{2m} \pm 2v_{2m} + 3}{u_m^2 + 9v_m^2} \right) = \left(\frac{2y^2}{u_m^2 + 9v_m^2} \right) = -1.$$

On the other hand, note that $u_{2m} = u_m^2 + 3v_m^2$, $v_{2m} = 2u_m v_m$, $u_m^2 - 3v_m^2 = 1$, we have

$$\begin{aligned} \left(\frac{u_{2m} + 2v_{2m} + 3}{u_m^2 + 9v_m^2} \right) &= \left(\frac{4u_m^2 + 4u_m v_m - 6v_m^2}{u_m^2 + 9v_m^2} \right) = \left(\frac{4u_m v_m - 42v_m^2}{u_m^2 + 9v_m^2} \right) = - \left(\frac{2u_m - 21v_m}{u_m^2 + 9v_m^2} \right) \\ &= - \left(\frac{9v_m^2 + u_m^2}{21v_m - 2u_m} \right) \quad (\text{note that } 21v_m - 2u_m > 0) \\ &= - \left(\frac{7}{21v_m - 2u_m} \right) \left(\frac{126v_m^2 + 14u_m^2}{21v_m - 2u_m} \right) \quad (\text{since } 7 \nmid u_m \text{ and } 21v_m - 2u_m \equiv \pm 1 \pmod{8}) \\ &= - \left(\frac{7}{21v_m - 2u_m} \right) \left(\frac{159u_m v_m}{21v_m - 2u_m} \right) \\ &= - \left(\frac{21v_m - 2u_m}{7 \cdot 159} \right) \left(\frac{\frac{1}{2}u_m}{21v_m - 2u_m} \right) \left(\frac{v_m}{21v_m - 2u_m} \right) \\ &= - \left(\frac{21v_m - 2u_m}{53} \right) \left(\frac{2u_m}{21} \right) \left(\frac{\frac{1}{2}u_m}{21v_m} \right) \left(\frac{v_m}{21v_m - 2u_m} \right) \\ &= - \left(\frac{2u_m - 21v_m}{53} \right) \left(\frac{u_m}{v_m} \right) \left(\frac{v_m}{21v_m - 2u_m} \right). \end{aligned}$$

If $v_m \equiv 1 \pmod{8}$, then $\left(\frac{v_m}{21v_m - 2u_m} \right) = \left(\frac{u_m}{v_m} \right)$; if $v_m \equiv -1 \pmod{8}$, then $\left(\frac{v_m}{21v_m - 2u_m} \right) = - \left(\frac{21v_m - 2u_m}{v_m} \right) = \left(\frac{u_m}{v_m} \right)$, the same as before. Hence, we obtain

$$(7) \quad \left(\frac{u_{2m} + 2v_{2m} + 3}{u_m^2 + 9v_m^2} \right) = - \left(\frac{2u_m - 21v_m}{53} \right).$$

Similarly we can show

$$(8) \quad \left(\frac{u_{2m} - 2v_{2m} + 3}{u_m^2 + 9v_m^2} \right) = - \left(\frac{2u_m + 21v_m}{53} \right).$$

Using the recurrent relations (3), (4) we take modulo 53 to $\{2u_n \pm 21v_n\}$, and obtain their residue sequences with the same period 9. Now $m \equiv 0 \pmod{9}$ implies both $2u_m \pm 21v_m \equiv 2 \pmod{53}$, so that (7) and (8) lead to

$$\left(\frac{u_{2m} \pm 2v_{2m} + 3}{u_m^2 + 9v_m^2} \right) = - \left(\frac{2}{53} \right) = 1,$$

which are contrary to (6).

Finally there remains $n=-1$, this case gives $(x, y)=(1, 1)$, another positive integer solution of (1), adding the previous one $(x, y)=(3, 3)$ given by the case $n=3$ we obtain all positive integer solutions of (1). This completes our proof.

Remark: Because in two equations $3x^2-1=u_n$ and $2y^2-3=v_n$ we only use the latter, so actually we have proved that the more general equation $x^2-3(2y^2-3)^2=1$ has only two positive integer solutions $(x, y)=(2, 1)$ and $(26, 3)$.

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References

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