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## 5-COLORED KNOT DIAGRAM WITH FOUR COLORS

Dedicated to Professor Takao Matumoto on his sixtieth birthday

SHIN SATOH

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### Abstract

We study Fox 5-colorings for diagrams of 1- and 2-dimensional knots. We prove that any 5-colorable 1-knot has a non-trivially 5-colored diagram such that exactly four colors of five are assigned to the arcs of the diagram. Moreover, we prove that there is a 5-colorable 2-knot such that, for any non-trivially 5-colored diagram, all five colors are assigned to the sheets of the diagram.

### 1. Introduction

Let us observe the 5-colored diagrams of the knots  $4_1$ ,  $5_1$ , and  $7_4$  as shown in Fig. 1, where the pallet  $\mathbb{Z}_5 = \{0, 1, \dots, 4\}$  is used to provide a 5-coloring for each diagram. What is the common property of these 5-colorings?

Each 5-coloring in the figure uses exactly four colors  $1, \dots, 4$  except 0. Hence, it is natural to ask the question: *Which 5-colorable knot has a 5-colored diagram with exactly four colors?* The first aim of this note is to give the answer to this question as follows:

**Theorem 1.1.** *Any 5-colorable knot has a non-trivially 5-colored diagram with exactly four colors.*

Harary and Kauffman [5] study the minimal number of colors assigned to the arcs for all non-trivially  $p$ -colored diagrams of a  $p$ -colorable knot  $K$ , which is denoted by  $C_p(K)$ . Refer to [7] also. Theorem 1.1 implies that  $C_5(K) = 4$  for any 5-colorable knot  $K$ . We remark that, if  $p$  is a prime with  $p > 3$ , then any  $p$ -colorable knot  $K$  satisfies  $C_p(K) \geq 4$  (Lemma 2.1).

On the other hand, a  $p$ -coloring is also defined for a diagram of a 2-dimensional knot (a 2-sphere in  $\mathbb{R}^4$ ), which satisfies the property that any non-trivial  $p$ -coloring needs at least four colors for  $p > 3$ . Hence, we can ask a similar question to the 1-dimensional knot case concerning the minimal number of colors for all non-trivial 5-colorings. The second aim of this note is to prove the following:

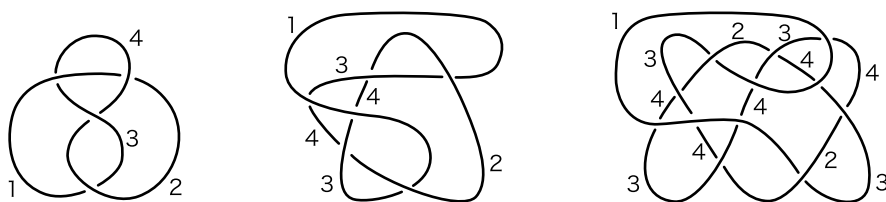


Fig. 1. 5-colored diagrams of  $4_1$ ,  $5_1$ , and  $7_4$ .

**Theorem 1.2.** *There is a 2-knot whose any non-trivially 5-colored diagram needs all of the five colors.*

For example, Theorem 1.2 holds for the 2-twist-spun figure-eight knot and  $(2, 5)$ -torus knot, which are both non-ribbon 2-knots. On the other hand, we have the following for the family of ribbon 2-knots:

**Proposition 1.3.** *Any 5-colorable ribbon 2-knot has a non-trivially 5-colored diagram with exactly four colors.*

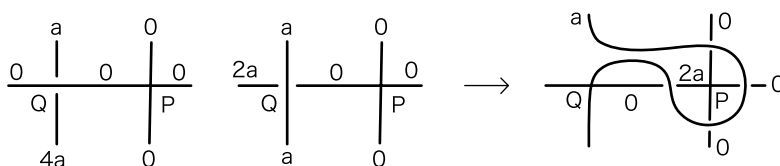
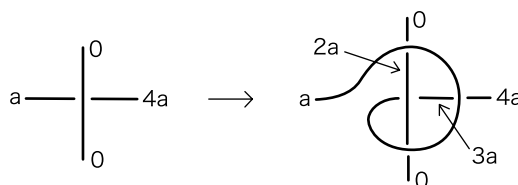
## 2. 5-colored 1-knot diagrams

Throughout this section, a *knot* means a circle embedded in  $\mathbb{R}^3$ . Any knot diagram  $D$  is regarded as a disjoint union of arcs obtained from the projected planar curves by cutting the lower paths at crossings. For an odd prime  $p$ , we consider an assignment of an element of  $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$  to each arc of  $D$ . It is called a  $p$ -coloring if  $a + c = 2b$  in  $\mathbb{Z}_p$  holds near each crossing, where the lower arcs are colored by  $a$  and  $c$  and the upper is colored by  $b$ . The color of the crossing is denoted by  $\{a \mid b \mid c\}$ . We say that a  $p$ -coloring is *trivial* if all arcs of  $D$  have the same color, and otherwise *non-trivial*.

**Lemma 2.1.** *If  $p > 3$ , then any non-trivial  $p$ -coloring for  $D$  needs at least four colors of  $0, 1, \dots, p-1$ .*

**Proof.** By definition,  $D$  has a crossing with the color  $\{a \mid b \mid c\}$  which does not satisfy  $a = b = c$ . Since  $a + c = 2b$ , we see that  $a, b, c$  are mutually different. Hence, any non-trivial  $p$ -coloring needs at least three colors.

Assume that exactly three colors are assigned to the arcs of  $D$ . Then it is easy to see that  $D$  has a pair of crossings whose colors are  $\{a \mid b \mid c\}$  and  $\{a \mid c \mid b\}$  for some mutually different  $a, b, c \in \mathbb{Z}_p$ . By the equations  $a + c = 2b$  and  $a + b = 2c$ , we have  $3(b - c) = 0$ . This is impossible for  $p > 3$  and  $b \neq c$ .  $\square$

Fig. 2. Eliminating a crossing with the color  $\{0 | 0 | 0\}$ .Fig. 3. Eliminating a crossing with the color  $\{a | 0 | 4a\}$ .

**Lemma 2.2.** Any 5-colorable knot has a non-trivially 5-colored diagram  $D$  with no crossing whose color is  $\{0 | 0 | 0\}$ .

*Proof.* Assume that  $D$  has a crossing of  $\{0 | 0 | 0\}$ . Then it is easy to see that  $D$  has an adjacent pair of crossings  $P$  and  $Q$  such that  $P$  is of  $\{0 | 0 | 0\}$  and  $Q$  is of  $\{a | 0 | 4a\}$  or  $\{0 | a | 2a\}$  for some  $a \neq 0$ . See the left or middle of Fig. 2. We deform the arc with the color  $a$  near  $Q$  which detours around  $P$  passing over the arcs. Then the color of  $P$  changes into  $\{2a | 2a | 2a\}$ , and new crossings are of  $\{0 | a | 2a\}$ . See the right of the figure. We repeat the deformation above if the obtained diagram still has a crossing of  $\{0 | 0 | 0\}$ .  $\square$

**Lemma 2.3.** Any 5-colorable knot has a non-trivially 5-colored diagram  $D$  with no crossing whose color is  $\{* | 0 | *\}$ .

*Proof.* We may assume that  $D$  has no crossing whose color is  $\{0 | 0 | 0\}$  by Lemma 2.2. Assume that  $D$  has a crossing of  $\{a | 0 | 4a\}$  for some  $a \neq 0$ . Then we deform the arc with the color  $a$  which detour around the crossing. See Fig. 3. Then the color of the original crossing changes into  $\{a | 2a | 3a\}$ , and new crossings are of  $\{0 | a | 2a\}$  and  $\{3a | 2a | 4a\}$ . We repeat the deformation above if the obtained diagram still has a crossing of  $\{a | 0 | 4a\}$  for some  $a \neq 0$ .  $\square$

*Proof of Theorem 1.1.* Let  $D$  be a non-trivially 5-colored diagram of the knot. By Lemma 2.3, we may assume that the upper arc of any crossing of  $D$  have a non-zero color. Hence, each arc with the color 0 connects a pair of crossings directly whose colors are  $\{0 | a | 2a\}$  and  $\{0 | b | 2b\}$  for some  $a, b \neq 0$ . According to  $b =$

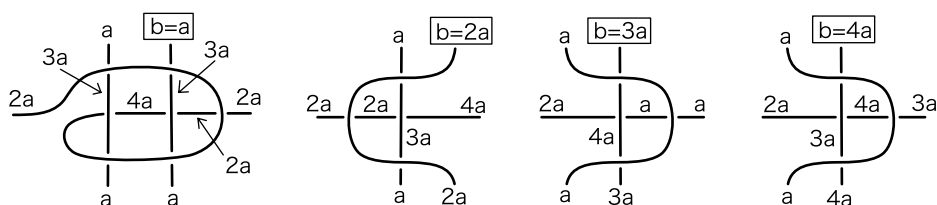
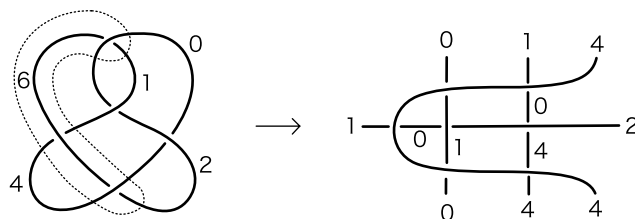


Fig. 4. Eliminating an arc colored by 0.

Fig. 5. A 7-colored diagram of  $5_2$  with four colors.

$a, 2a, 3a, 4a$ , we deform the arc as shown in Fig. 4 so that the arc with 0 is eliminated. We repeat the deformation above if the obtained diagram still has an arc whose color is 0.  $\square$

REMARK 2.4. (i) The argument as above can be easily applied to the families of 5-colored virtual knot diagrams [6] and virtual arc diagrams [9].

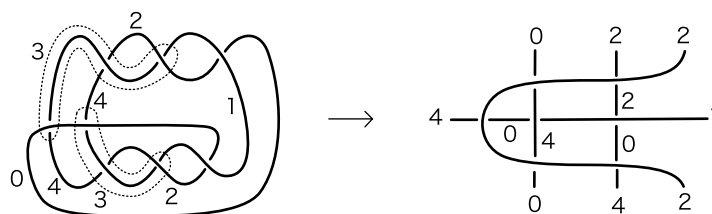
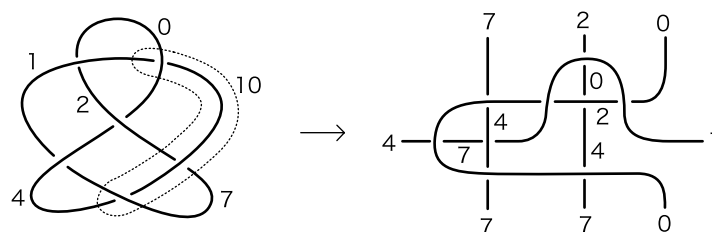
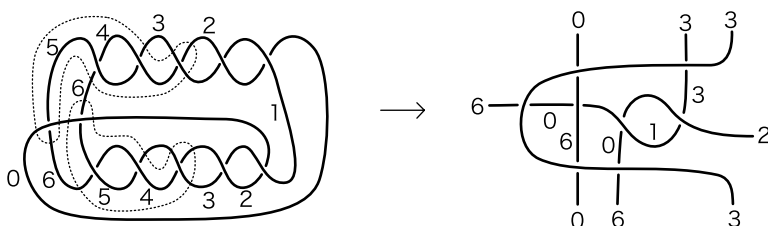
(ii) In the proof of Theorem 1.1, we eliminate the color 0 from a 5-colored diagram. By adding  $a \neq 0$  to the color of each arc, we can easily eliminate the color  $a$  instead of 0.

For the cases of 7- and 11-colorings, we have just several examples as follows:

EXAMPLE 2.5. By Lemma 2.1, any non-trivial 7-coloring requires at least four colors assigned to the arcs of a diagram.

Consider the 7-colored diagram of the knot  $5_2$  with five colors 0, 1, 2, 4, 6 as shown in the left of Fig. 5. We deform a neighborhood of the arc with 6 as in the right, so that we can eliminate the color 6 without introducing new colors except 0, 1, 2, 4. Hence, we have  $C_7(5_2) = 4$ .

Similarly, consider the 7-colored diagram of the  $(2, 7)$ -torus knot  $T_{2,7}$  with five colors 0, 1, 2, 3, 4 as shown in the left of Fig. 6. We deform neighborhoods of the arcs with the color 3 as in the right, so that we obtain a diagram colored by 0, 1, 2, 4. Hence, we have  $C_7(T_{2,7}) = 4$ . (Kauffman and Lopes [7] conjectured  $C_p(T_{2,p}) = (p + 3)/2$ .)

Fig. 6. A 7-colored diagram of  $T_{2,7}$  with four colors.Fig. 7. A 11-colored diagram of  $6_2$  with five colors.Fig. 8. A 11-colored diagram of  $T_{2,11}$  with five colors.

QUESTION 2.6. Does it hold  $C_7(K) = 4$  for any 7-colorable knot  $K$ ?

EXAMPLE 2.7. Similarly to Lemma 2.1, it is easy to see that if  $p > 7$ , then any non-trivial  $p$ -coloring for a knot diagram needs at least five colors of  $0, 1, \dots, p-1$  assigned to the arcs of a diagram, that is,  $C_p(K) \geq 5$ .

Consider the 11-colored diagram of the knot  $6_2$  with six colors  $0, 1, 2, 4, 7, 10$  as shown in the left of Fig. 7. We deform a neighborhood of the arc with 10 as in the right, so that we obtain a diagram colored by  $0, 1, 2, 4, 7$ . Hence, we have  $C_{11}(6_2) = 5$ .

Similarly, consider the 11-colored diagram of the  $(2, 11)$ -torus knot  $T_{2,11}$  with seven colors  $0, 1, 2, 3, 4, 5, 6$  as shown in the left of Fig. 8. We deform neighborhoods of the arcs with 5 and 6 as shown in the right, so that we obtain a diagram colored by  $0, 1, 2, 3, 6$ . Hence, we have  $C_{11}(T_{2,11}) = 5$ .

QUESTION 2.8. Does it hold  $C_{11}(K) = 5$  for any 11-colorable knot  $K$ ?

### 3. 5-colored 2-knot diagrams

Throughout this section, a 2-knot means a 2-dimensional sphere embedded in  $\mathbb{R}^4$  smoothly. A *diagram* of a 2-knot  $K$  is a projection image  $\pi(K)$  under a projection  $p: \mathbb{R}^4 \rightarrow \mathbb{R}^3$  equipped with crossing information. Refer to [3] for more details.

Any 2-knot diagram is regarded as a disjoint union of compact, connected surfaces, each of which is called a *sheet*. For an odd prime  $p$ , an assignment of an element of  $\mathbb{Z}_p$  to each sheet of the diagram is called a  $p$ -coloring if  $a + c = 2b$  in  $\mathbb{Z}_p$  holds near each double point, where the lower sheets are colored by  $a$  and  $c$  and the upper is colored by  $b$ .

Let  $D$  be a  $p$ -colored diagram of a 2-knot. Consider a triple point of  $D$ , where the top sheet is colored by  $a$ , the middle sheets are colored by  $b_1$  and  $b_2$ , and the bottom sheets on both sides of the middle sheet with  $b_i$  are colored by  $c_{i1}$  and  $c_{i2}$  ( $i = 1, 2$ ). We may assume that the bottom sheets colored by  $c_{1j}$  and  $c_{2j}$  are adjacent along the top sheet ( $j = 1, 2$ ). See Fig. 9. We say that a triple point is *degenerated* with respect to the  $p$ -coloring if  $a = b_i$  or  $b_i = c_{ij}$  holds for some  $i, j \in \{1, 2\}$ , and otherwise *non-degenerated*. Hence, a triple point is non-degenerated if and only if  $a \neq b_i \neq c_{ij}$  holds for any  $i, j \in \{1, 2\}$ . The notion of non-degeneracy was used in [10].

**Lemma 3.1.** *For a non-degenerated triple point with the colors as above, we have the following.*

- (i) *It holds that  $c_{11} \neq c_{12}$  and  $c_{21} \neq c_{22}$ .*
- (ii) *It holds that  $c_{11} \neq c_{22}$  and  $c_{12} \neq c_{21}$ .*
- (iii) *It holds that  $c_{11} \neq c_{21}$  or  $c_{12} \neq c_{22}$ .*

*Proof.* We first remark that, since  $b_1 + b_2 = 2a$  and  $a \neq b_1, b_2$ , it holds that  $b_1 \neq b_2$ .

(i) Since  $c_{i1} + c_{i2} = 2b_i$  and  $b_i \neq c_{ij}$  for  $i, j = 1, 2$ , we see that  $b_i, c_{i1}, c_{i2}$  are mutually different.

(ii) Assume that  $c_{11} = c_{22}$  (the case  $c_{12} \neq c_{21}$  is similarly proved). Then it holds that  $c_{12} = 2a - c_{22} = 2a - c_{11} = c_{21}$ , and hence,  $b_1 = (c_{11} + c_{12})/2 = (c_{22} + c_{21})/2 = b_2$ . This contradicts to  $b_1 \neq b_2$ .

(iii) Assume that  $c_{11} = c_{21}$  and  $c_{12} = c_{22}$ . Then it holds that  $b_1 = (c_{11} + c_{12})/2 = (c_{21} + c_{22})/2 = b_2$ , which contradicts to  $b_1 \neq b_2$ .  $\square$

**Lemma 3.2.** *Let  $p$  be a prime with  $p > 3$ . If a  $p$ -colored diagram of a 2-knot has a non-degenerated triple point, then the  $p$ -coloring needs at least five colors assigned to the sheets of the diagram.*

*Proof.* It is sufficient to prove that there are at least five different colors in the set  $\{a, b_i, c_{ij} \mid i, j = 1, 2\}$  near a non-degenerated triple point. By Lemma 3.1, we have

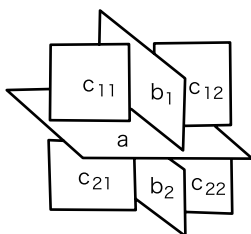


Fig. 9. Colors of sheets near a triple point.

two cases with respect to the colors of the bottom sheets (by changing the indices if necessary):

- (i)  $c_{11}$ ,  $c_{12}$ ,  $c_{21}$ , and  $c_{22}$  are four different colors.
- (ii)  $c_{11} = c_{21}$ ,  $c_{12}$ , and  $c_{22}$  are three different colors.

For the case (i), since  $a \neq c_{ij}$  holds for any  $i, j = 1, 2$ , we have five different colors  $a$ ,  $c_{11}$ ,  $c_{12}$ ,  $c_{21}$ , and  $c_{22}$ .

Consider the case (ii). Since  $c_{11} = c_{21} = a$ , each of the triplets

$$\{a, b_1, b_2\}, \{a, c_{12}, c_{22}\}, \{a, b_1, c_{12}\}, \text{ and } \{a, b_2, c_{22}\}$$

consists of mutually different colors. Hence, to prove the lemma, it is sufficient to prove  $b_1 \neq c_{22}$  and  $b_2 \neq c_{12}$ . Assume that  $b_1 = c_{22}$  (the case  $b_2 = c_{12}$  is similarly proved). Since  $b_1 + b_2 = 2a$  and  $c_{12} + c_{22} = c_{12} + b_1 = 2a$ , we have  $b_2 = c_{12}$ . Hence, it holds that

$$c_{11} + c_{12} = a + b_2 = 2b_1$$

and

$$c_{21} + c_{22} = a + b_1 = 2b_2,$$

which induces  $3(b_1 - b_2) = 0$ . This is impossible for  $p > 3$  and  $b_1 \neq b_2$ . □

Let  $D$  be a diagram of a 2-knot  $K$ , and  $\gamma$  a (possibly trivial)  $p$ -coloring for  $D$ . By using the Mochizuki's 3-cocycle [8] of the dihedral quandle of order  $p$ , we can define a weight  $W_p(t, \gamma) \in \mathbb{Z}_p$  for a triple point  $t$  of  $D$  in an appropriate manner. Take the sum  $W_p(\gamma) = \sum_t W_p(t, \gamma)$  for all triple points of  $D$ . The cocycle invariant of the 2-knot  $K$  is defined by

$$\Phi_p(K) = \{W_p(\gamma) \mid \gamma: \text{any } p\text{-coloring for } D\}$$

as a multi-set [2]. The weight  $W_p(t, \gamma)$  has the property that, if  $t$  is degenerated with respect to  $\gamma$ , then it holds that  $W_p(t, \gamma) = 0$ . In particular, if  $\gamma$  is a trivial  $p$ -coloring, then any  $t$  is degenerated, and hence, we have  $W_p(\gamma) = 0$ . In other words, if a  $p$ -coloring



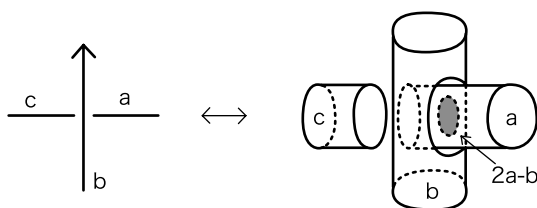


Fig. 10. Virtual arc presentation.

$\gamma$  satisfies  $W_p(\gamma) \neq 0$ , then there is a triple point  $t$  with  $W_p(t, \gamma) \neq 0$  which implies that  $t$  is non-degenerated with respect to  $\gamma$ .

Proof of Theorem 1.2. Let  $K$  be the 2-twist-spun figure-eight knot, and  $D$  a diagram of  $K$ . The cocycle invariant of  $K$  is calculated in [4] such that

$$\Phi_5(K) = \{0, \dots, 0 \text{ (5 times)}, 2, \dots, 2 \text{ (10 times)}, 3, \dots, 3 \text{ (10 times)}\}.$$

The number of 5-colorings for  $D$  is 25 which includes 5 trivial ones. Hence, for any non-trivial 5-coloring  $\gamma$ , it holds that  $W_5(\gamma) = 2$  or 3. Since  $W_5(\gamma) \neq 0$ ,  $D$  has a non-degenerated triple point with respect to  $\gamma$ . The proof is completed by Lemma 3.2.  $\square$

REMARK 3.3. (i) Let  $K_{2n}$  be the  $2n$ -twist-spun figure-eight knot. The cocycle invariant of  $K_{2n}$  is given by  $\Phi_5(K_{2n}) = n \cdot \Phi_5(K_2) = \{0, \dots, 2n, \dots, 3n, \dots\}$  (cf. [1]). Hence, if  $n$  is not divisible by 5, then  $K_{2n}$  has the same property as in Theorem 1.2. (ii) Since the cocycle invariant of the 2-twist-spun  $(2, 5)$ -torus knot  $K$  is

$$\Phi_5(K) = \{0, \dots, 0 \text{ (5 times)}, 1, \dots, 1 \text{ (10 times)}, 4, \dots, 4 \text{ (10 times)}\},$$

$K$  has the same property as in Theorem 1.2.

A ribbon 2-knot is obtained by adding 1-handles to a trivial 2-link [11]. It is known that any ribbon 2-knot is presented by a virtual arc diagram [9]. Given an oriented virtual arc diagram  $A$ , we construct a diagram  $D$  of a ribbon 2-knot  $\text{Tube}(A)$ . In Fig. 10, we show a part of  $D$  corresponding to a classical crossing of  $A$ . Moreover, it is easy to see that there is a one-to-one correspondence between the set of the 5-colorings for  $A$  and that for  $D$ .

Proof of Proposition 1.3. Let  $K$  be a 5-colorable ribbon 2-knot. We may assume that  $K = \text{Tube}(A)$  for some virtual arc diagram  $A$ . Since  $K$  is 5-colorable, so is  $A$ . As mentioned in Remark 2.4 (i), we may assume that  $A$  has a non-trivial 5-coloring with exactly four colors 1, 2, 3, 4.

Consider the 5-colored diagram  $D$  of  $K = \text{Tube}(A)$  corresponding to  $A$ . By the assumption for  $A$ , if  $D$  has a sheet colored by 0, then the sheet is the small one colored

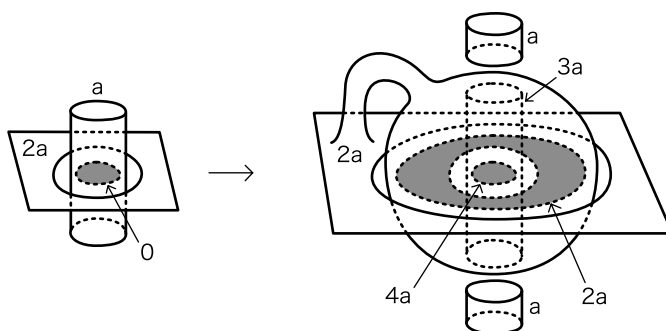


Fig. 11. Eliminating a sheet colored by 0.

by  $2a - b (= 0)$  in Fig. 10. In a neighborhood of the sheet with 0, we deform the sheet with  $2a (= b)$  as shown in Fig. 11 so that the color 0 is eliminated. The deformation is similar to the one in the most left of Fig. 4. This completes the proof.  $\square$

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