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5-COLORED KNOT DIAGRAM WITH FOUR COLORS

Dedicated to Professor Takao Matumoto on his sixtieth birthday

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Abstract

We study Fox 5-colorings for diagrams of 1- and 2-dimensional knots. We prove that any 5-colorable 1-knot has a non-trivially 5-colored diagram such that exactly four colors of five are assigned to the arcs of the diagram. Moreover, we prove that there is a 5-colorable 2-knot such that, for any non-trivially 5-colored diagram, all five colors are assigned to the sheets of the diagram.

1. Introduction

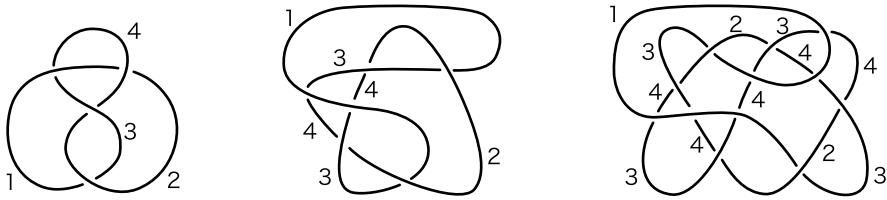
Let us observe the 5-colored diagrams of the knots 4_1 , 5_1 , and 7_4 as shown in Fig. 1, where the pallet $\mathbb{Z}_5 = \{0, 1, \dots, 4\}$ is used to provide a 5-coloring for each diagram. What is the common property of these 5-colorings?

Each 5-coloring in the figure uses exactly four colors $1, \dots, 4$ except 0. Hence, it is natural to ask the question: *Which 5-colorable knot has a 5-colored diagram with exactly four colors?* The first aim of this note is to give the answer to this question as follows:

Theorem 1.1. *Any 5-colorable knot has a non-trivially 5-colored diagram with exactly four colors.*

Harary and Kauffman [5] study the minimal number of colors assigned to the arcs for all non-trivially p -colored diagrams of a p -colorable knot K , which is denoted by $C_p(K)$. Refer to [7] also. Theorem 1.1 implies that $C_5(K) = 4$ for any 5-colorable knot K . We remark that, if p is a prime with $p > 3$, then any p -colorable knot K satisfies $C_p(K) \geq 4$ (Lemma 2.1).

On the other hand, a p -coloring is also defined for a diagram of a 2-dimensional knot (a 2-sphere in \mathbb{R}^4), which satisfies the property that any non-trivial p -coloring needs at least four colors for $p > 3$. Hence, we can ask a similar question to the 1-dimensional knot case concerning the minimal number of colors for all non-trivial 5-colorings. The second aim of this note is to prove the following:

Fig. 1. 5-colored diagrams of 4_1 , 5_1 , and 7_4 .

Theorem 1.2. *There is a 2-knot whose any non-trivially 5-colored diagram needs all of the five colors.*

For example, Theorem 1.2 holds for the 2-twist-spun figure-eight knot and $(2, 5)$ -torus knot, which are both non-ribbon 2-knots. On the other hand, we have the following for the family of ribbon 2-knots:

Proposition 1.3. *Any 5-colorable ribbon 2-knot has a non-trivially 5-colored diagram with exactly four colors.*

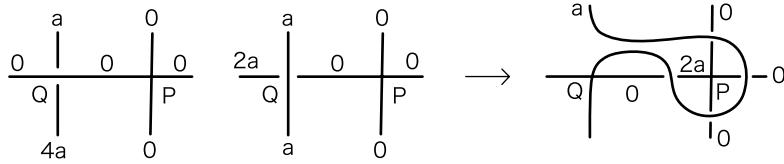
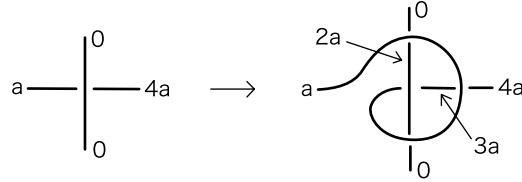
2. 5-colored 1-knot diagrams

Throughout this section, a *knot* means a circle embedded in \mathbb{R}^3 . Any knot diagram D is regarded as a disjoint union of arcs obtained from the projected planar curves by cutting the lower paths at crossings. For an odd prime p , we consider an assignment of an element of $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ to each arc of D . It is called a p -coloring if $a+c=2b$ in \mathbb{Z}_p holds near each crossing, where the lower arcs are colored by a and c and the upper is colored by b . The color of the crossing is denoted by $\{a \mid b \mid c\}$. We say that a p -coloring is *trivial* if all arcs of D have the same color, and otherwise *non-trivial*.

Lemma 2.1. *If $p > 3$, then any non-trivial p -coloring for D needs at least four colors of $0, 1, \dots, p-1$.*

Proof. By definition, D has a crossing with the color $\{a \mid b \mid c\}$ which does not satisfy $a=b=c$. Since $a+c=2b$, we see that a, b, c are mutually different. Hence, any non-trivial p -coloring needs at least three colors.

Assume that exactly three colors are assigned to the arcs of D . Then it is easy to see that D has a pair of crossings whose colors are $\{a \mid b \mid c\}$ and $\{a \mid c \mid b\}$ for some mutually different $a, b, c \in \mathbb{Z}_p$. By the equations $a+c=2b$ and $a+b=2c$, we have $3(b-c)=0$. This is impossible for $p > 3$ and $b \neq c$. \square

Fig. 2. Eliminating a crossing with the color $\{0 | 0 | 0\}$.Fig. 3. Eliminating a crossing with the color $\{a | 0 | 4a\}$.

Lemma 2.2. *Any 5-colorable knot has a non-trivially 5-colored diagram D with no crossing whose color is $\{0 | 0 | 0\}$.*

Proof. Assume that D has a crossing of $\{0 | 0 | 0\}$. Then it is easy to see that D has an adjacent pair of crossings P and Q such that P is of $\{0 | 0 | 0\}$ and Q is of $\{a | 0 | 4a\}$ or $\{0 | a | 2a\}$ for some $a \neq 0$. See the left or middle of Fig. 2. We deform the arc with the color a near Q which detours around P passing over the arcs. Then the color of P changes into $\{2a | 2a | 2a\}$, and new crossings are of $\{0 | a | 2a\}$. See the right of the figure. We repeat the deformation above if the obtained diagram still has a crossing of $\{0 | 0 | 0\}$. \square

Lemma 2.3. *Any 5-colorable knot has a non-trivially 5-colored diagram D with no crossing whose color is $\{ * | 0 | * \}$.*

Proof. We may assume that D has no crossing whose color is $\{0 | 0 | 0\}$ by Lemma 2.2. Assume that D has a crossing of $\{a | 0 | 4a\}$ for some $a \neq 0$. Then we deform the arc with the color a which detour around the crossing. See Fig. 3. Then the color of the original crossing changes into $\{a | 2a | 3a\}$, and new crossings are of $\{0 | a | 2a\}$ and $\{3a | 2a | 4a\}$. We repeat the deformation above if the obtained diagram still has a crossing of $\{a | 0 | 4a\}$ for some $a \neq 0$. \square

Proof of Theorem 1.1. Let D be a non-trivially 5-colored diagram of the knot. By Lemma 2.3, we may assume that the upper arc of any crossing of D have a non-zero color. Hence, each arc with the color 0 connects a pair of crossings directly whose colors are $\{0 | a | 2a\}$ and $\{0 | b | 2b\}$ for some $a, b \neq 0$. According to $b =$

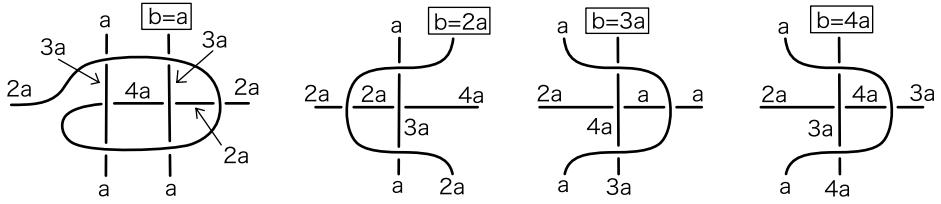
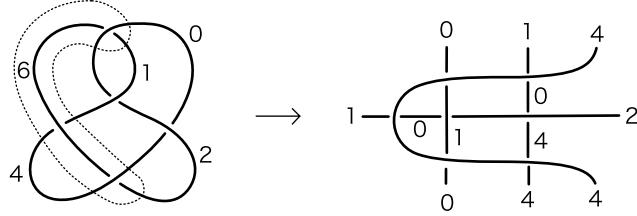


Fig. 4. Eliminating an arc colored by 0.

Fig. 5. A 7-colored diagram of 5_2 with four colors.

$a, 2a, 3a, 4a$, we deform the arc as shown in Fig. 4 so that the arc with 0 is eliminated. We repeat the deformation above if the obtained diagram still has an arc whose color is 0. \square

REMARK 2.4. (i) The argument as above can be easily applied to the families of 5-colored virtual knot diagrams [6] and virtual arc diagrams [9].

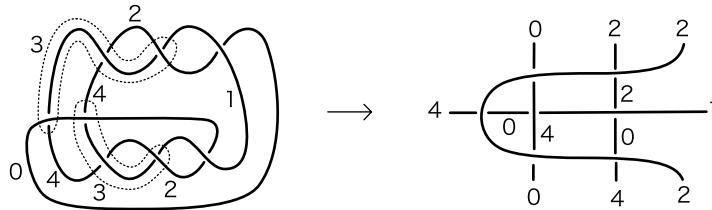
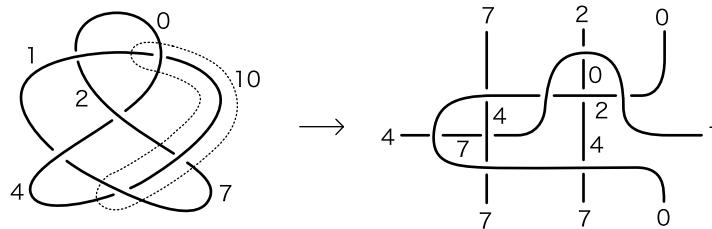
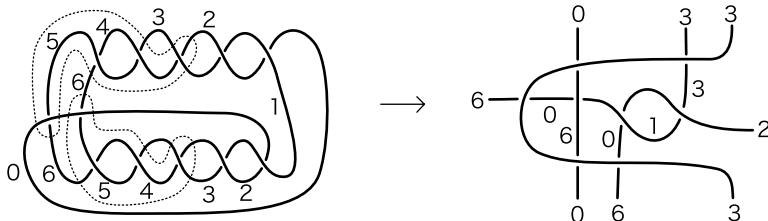
(ii) In the proof of Theorem 1.1, we eliminate the color 0 from a 5-colored diagram. By adding $a \neq 0$ to the color of each arc, we can easily eliminate the color a instead of 0.

For the cases of 7- and 11-colorings, we have just several examples as follows:

EXAMPLE 2.5. By Lemma 2.1, any non-trivial 7-coloring requires at least four colors assigned to the arcs of a diagram.

Consider the 7-colored diagram of the knot 5_2 with five colors 0, 1, 2, 4, 6 as shown in the left of Fig. 5. We deform a neighborhood of the arc with 6 as in the right, so that we can eliminate the color 6 without introducing new colors except 0, 1, 2, 4. Hence, we have $C_7(5_2) = 4$.

Similarly, consider the 7-colored diagram of the $(2, 7)$ -torus knot $T_{2,7}$ with five colors 0, 1, 2, 3, 4 as shown in the left of Fig. 6. We deform neighborhoods of the arcs with the color 3 as in the right, so that we obtain a diagram colored by 0, 1, 2, 4. Hence, we have $C_7(T_{2,7}) = 4$. (Kauffman and Lopes [7] conjectured $C_p(T_{2,p}) = (p+3)/2$.)

Fig. 6. A 7-colored diagram of $T_{2,7}$ with four colors.Fig. 7. A 11-colored diagram of 6_2 with five colors.Fig. 8. A 11-colored diagram of $T_{2,11}$ with five colors.

QUESTION 2.6. Does it hold $C_7(K) = 4$ for any 7-colorable knot K ?

EXAMPLE 2.7. Similarly to Lemma 2.1, it is easy to see that if $p > 7$, then any non-trivial p -coloring for a knot diagram needs at least five colors of $0, 1, \dots, p-1$ assigned to the arcs of a diagram, that is, $C_p(K) \geq 5$.

Consider the 11-colored diagram of the knot 6_2 with six colors $0, 1, 2, 4, 7, 10$ as shown in the left of Fig. 7. We deform a neighborhood of the arc with 10 as in the right, so that we obtain a diagram colored by $0, 1, 2, 4, 7$. Hence, we have $C_{11}(6_2) = 5$.

Similarly, consider the 11-colored diagram of the $(2, 11)$ -torus knot $T_{2,11}$ with seven colors $0, 1, 2, 3, 4, 5, 6$ as shown in the left of Fig. 8. We deform neighborhoods of the arcs with 5 and 6 as shown in the right, so that we obtain a diagram colored by $0, 1, 2, 3, 6$. Hence, we have $C_{11}(T_{2,11}) = 5$.

QUESTION 2.8. Does it hold $C_{11}(K) = 5$ for any 11-colorable knot K ?

3. 5-colored 2-knot diagrams

Throughout this section, a *2-knot* means a 2-dimensional sphere embedded in \mathbb{R}^4 smoothly. A *diagram* of a 2-knot K is a projection image $\pi(K)$ under a projection $p: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ equipped with crossing information. Refer to [3] for more details.

Any 2-knot diagram is regarded as a disjoint union of compact, connected surfaces, each of which is called a *sheet*. For an odd prime p , an assignment of an element of \mathbb{Z}_p to each sheet of the diagram is called a *p -coloring* if $a + c = 2b$ in \mathbb{Z}_p holds near each double point, where the lower sheets are colored by a and c and the upper is colored by b .

Let D be a p -colored diagram of a 2-knot. Consider a triple point of D , where the top sheet is colored by a , the middle sheets are colored by b_1 and b_2 , and the bottom sheets on both sides of the middle sheet with b_i are colored by c_{i1} and c_{i2} ($i = 1, 2$). We may assume that the bottom sheets colored by c_{1j} and c_{2j} are adjacent along the top sheet ($j = 1, 2$). See Fig. 9. We say that a triple point is *degenerated* with respect to the p -coloring if $a = b_i$ or $b_i = c_{ij}$ holds for some $i, j \in \{1, 2\}$, and otherwise *non-degenerated*. Hence, a triple point is non-degenerated if and only if $a \neq b_i \neq c_{ij}$ holds for any $i, j \in \{1, 2\}$. The notion of non-degeneracy was used in [10].

Lemma 3.1. *For a non-degenerated triple point with the colors as above, we have the following.*

- (i) *It holds that $c_{11} \neq c_{12}$ and $c_{21} \neq c_{22}$.*
- (ii) *It holds that $c_{11} \neq c_{22}$ and $c_{12} \neq c_{21}$.*
- (iii) *It holds that $c_{11} \neq c_{21}$ or $c_{12} \neq c_{22}$.*

Proof. We first remark that, since $b_1 + b_2 = 2a$ and $a \neq b_1, b_2$, it holds that $b_1 \neq b_2$.

(i) Since $c_{i1} + c_{i2} = 2b_i$ and $b_i \neq c_{ij}$ for $i, j = 1, 2$, we see that b_i, c_{i1}, c_{i2} are mutually different.

(ii) Assume that $c_{11} = c_{22}$ (the case $c_{12} \neq c_{21}$ is similarly proved). Then it holds that $c_{12} = 2a - c_{22} = 2a - c_{11} = c_{21}$, and hence, $b_1 = (c_{11} + c_{12})/2 = (c_{22} + c_{21})/2 = b_2$. This contradicts to $b_1 \neq b_2$.

(iii) Assume that $c_{11} = c_{21}$ and $c_{12} = c_{22}$. Then it holds that $b_1 = (c_{11} + c_{12})/2 = (c_{21} + c_{22})/2 = b_2$, which contradicts to $b_1 \neq b_2$. \square

Lemma 3.2. *Let p be a prime with $p > 3$. If a p -colored diagram of a 2-knot has a non-degenerated triple point, then the p -coloring needs at least five colors assigned to the sheets of the diagram.*

Proof. It is sufficient to prove that there are at least five different colors in the set $\{a, b_i, c_{ij} \mid i, j = 1, 2\}$ near a non-degenerated triple point. By Lemma 3.1, we have

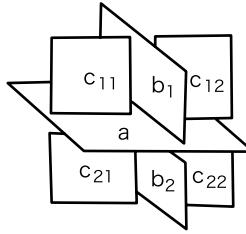


Fig. 9. Colors of sheets near a triple point.

two cases with respect to the colors of the bottom sheets (by changing the indices if necessary):

- (i) c_{11}, c_{12}, c_{21} , and c_{22} are four different colors.
- (ii) $c_{11} = c_{21}$, c_{12} , and c_{22} are three different colors.

For the case (i), since $a \neq c_{ij}$ holds for any $i, j = 1, 2$, we have five different colors $a, c_{11}, c_{12}, c_{21}$, and c_{22} .

Consider the case (ii). Since $c_{11} = c_{21} = a$, each of the triplets

$$\{a, b_1, b_2\}, \{a, c_{12}, c_{22}\}, \{a, b_1, c_{12}\}, \text{ and } \{a, b_2, c_{22}\}$$

consists of mutually different colors. Hence, to prove the lemma, it is sufficient to prove $b_1 \neq c_{22}$ and $b_2 \neq c_{12}$. Assume that $b_1 = c_{22}$ (the case $b_2 = c_{12}$ is similarly proved). Since $b_1 + b_2 = 2a$ and $c_{12} + c_{22} = c_{12} + b_1 = 2a$, we have $b_2 = c_{12}$. Hence, it holds that

$$c_{11} + c_{12} = a + b_2 = 2b_1$$

and

$$c_{21} + c_{22} = a + b_1 = 2b_2,$$

which induces $3(b_1 - b_2) = 0$. This is impossible for $p > 3$ and $b_1 \neq b_2$. \square

Let D be a diagram of a 2-knot K , and γ a (possibly trivial) p -coloring for D . By using the Mochizuki's 3-cocycle [8] of the dihedral quandle of order p , we can define a weight $W_p(t, \gamma) \in \mathbb{Z}_p$ for a triple point t of D in an appropriate manner. Take the sum $W_p(\gamma) = \sum_t W_p(t, \gamma)$ for all triple points of D . The cocycle invariant of the 2-knot K is defined by

$$\Phi_p(K) = \{W_p(\gamma) \mid \gamma: \text{any } p\text{-coloring for } D\}$$

as a multi-set [2]. The weight $W_p(t, \gamma)$ has the property that, if t is degenerated with respect to γ , then it holds that $W_p(t, \gamma) = 0$. In particular, if γ is a trivial p -coloring, then any t is degenerated, and hence, we have $W_p(\gamma) = 0$. In other words, if a p -coloring

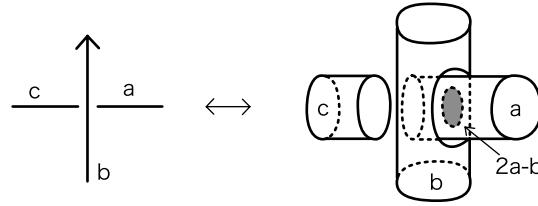


Fig. 10. Virtual arc presentation.

γ satisfies $W_p(\gamma) \neq 0$, then there is a triple point t with $W_p(t, \gamma) \neq 0$ which implies that t is non-degenerated with respect to γ .

Proof of Theorem 1.2. Let K be the the 2-twist-spun figure-eight knot, and D a diagram of K . The cocycle invariant of K is calculated in [4] such that

$$\Phi_5(K) = \{0, \dots, 0 \text{ (5 times)}, 2, \dots, 2 \text{ (10 times)}, 3, \dots, 3 \text{ (10 times)}\}.$$

The number of 5-colorings for D is 25 which includes 5 trivial ones. Hence, for any non-trivial 5-coloring γ , it holds that $W_5(\gamma) = 2$ or 3. Since $W_5(\gamma) \neq 0$, D has a non-degenerated triple point with respect to γ . The proof is completed by Lemma 3.2. \square

REMARK 3.3. (i) Let K_{2n} be the $2n$ -twist-spun figure-eight knot. The cocycle invariant of K_{2n} is given by $\Phi_5(K_{2n}) = n \cdot \Phi_5(K_2) = \{0, \dots, 2n, \dots, 3n, \dots\}$ (cf. [1]). Hence, if n is not divisible by 5, then K_{2n} has the same property as in Theorem 1.2. (ii) Since the cocycle invariant of the 2-twist-spun $(2, 5)$ -torus knot K is

$$\Phi_5(K) = \{0, \dots, 0 \text{ (5 times)}, 1, \dots, 1 \text{ (10 times)}, 4, \dots, 4 \text{ (10 times)}\},$$

K has the same property as in Theorem 1.2.

A ribbon 2-knot is obtained by adding 1-handles to a trivial 2-link [11]. It is known that any ribbon 2-knot is presented by a virtual arc diagram [9]. Given an oriented virtual arc diagram A , we construct a diagram D of a ribbon 2-knot $\text{Tube}(A)$. In Fig. 10, we shows a part of D corresponding to a classical crossing of A . Moreover, it is easy to see that there is a one-to-one correspondence between the set of the 5-colorings for A and that for D .

Proof of Proposition 1.3. Let K be a 5-colorable ribbon 2-knot. We may assume that $K = \text{Tube}(A)$ for some virtual arc diagram A . Since K is 5-colorable, so is A . As mentioned in Remark 2.4 (i), we may assume that A has a non-trivial 5-coloring with exactly four colors 1, 2, 3, 4.

Consider the 5-colored diagram D of $K = \text{Tube}(A)$ corresponding to A . By the assumption for A , if D has a sheet colored by 0, then the sheet is the small one colored

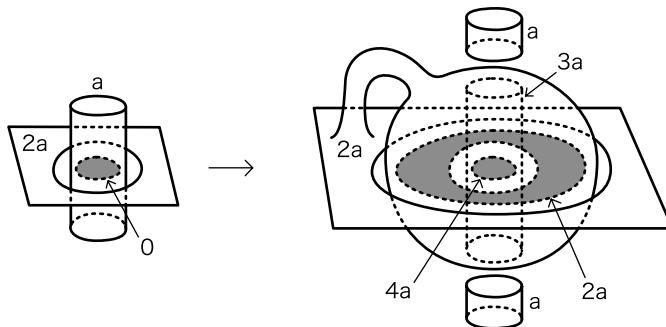


Fig. 11. Eliminating a sheet colored by 0.

by $2a - b$ ($= 0$) in Fig. 10. In a neighborhood of the sheet with 0, we deform the sheet with $2a$ ($= b$) as shown in Fig. 11 so that the color 0 is eliminated. The deformation is similar to the one in the most left of Fig. 4. This completes the proof. \square

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