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TOTALLY REAL SUBMANIFOLDS AND SYMMETRIC BOUNDED DOMAINS

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Introduction. Let $P_n(c)$ denote the complex projective n-space endowed with the Kählerian metric of constant holomorphic sectional curvature $c>0$. We consider an n-dimensional complete totally real submanifold M of $P_n(c)$ with parallel second fundamental form σ. The first named author [6] reduced the classification of such submanifolds to that of certain cubic forms of n-variables, and he classified completely those without Euclidean factor among such submanifolds. (Note that such a submanifold is always locally symmetric.)

In this note we shall give another way of the classification of these submanifolds. Let $D\subset \boldsymbol{C}^{n+1}$ be a symmetric bounded domain of tube type realized by the Harish-Chandra imbedding. We imbed the Shilov boundary $\hat{M}$ of D into the hypersphere $S^{2n+1}(c/4)$ of the radius $2/\sqrt{c}$ with respect to a suitable hermitian inner product of $\boldsymbol{C}^{n+1}$. Let $M\subset P_n(c)$ be the image of $\hat{M}$ under the Hopf fibering π: $S^{2n+1}(c/4)\rightarrow P_n(c)$. Then M is an n-dimensional complete totally real submanifold with parallel second fundamental form (Theorem 2.1), and conversely such a submanifold is obtained in this way (Theorem 3.1). The crucial point in the argument is that $M\subset P_n(c)$ has the parallel second fundamental form if and only if $\hat{M}=\pi^{-1}(M)\subset S^{2n+1}(c/4)$ has the parallel second fundamental form (Lemma 1.1). Thus we may use the classification (Ferus [3], Takeuchi [10]) of submanifolds in spheres with parallel second fundamental form.

As an application, we give a characterization of an n-dimensional compact totally real minimal submanifold M of $P_n(c)$ with $||\sigma||^2=n(n+1)c/4(2n-1)$. (Recall that $||\sigma||^2<n(n+1)c/4(2n-1)$ implies $\sigma=0$. cf. Chen-Ogiue [1].) Such a submanifold M is unique and nothing but the flat isotropic surface $M_0^2\subset P_2(c)$ with parallel second fundamental form constructed in Naitoh [5] (Theorem 4.5).

1. Hopf fiberings

Let $\boldsymbol{R}^{n+1}$ be the real Cartesian $(n+1)$-space with the standard inner pro-

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duct $\langle\ ,\ \rangle$. For a constant $k>0$, we denote by $S^n(k)$ the hypersphere of $\boldsymbol{R}^{n+1}$ with the radius $1/\sqrt{k}$ endowed with the Riemannian metric $\hat{g}$ induced from $\langle\ ,\ \rangle$.

Now we fix a positive integer m and a constant $c>0$, and denote by $P_m(c)$ the complex projective m-space $P_m(\boldsymbol{C})$ endowed with the Kählerian metric g of constant holomorphic sectional curvature c. We regard the complex Cartesian $(m+1)$-space $\boldsymbol{C}^{m+1}$ as a Euclidean $(2m+2)$-space by the inner product: $\langle z,w\rangle = \mathcal{R}e\, {}^t z\bar{w}$ for $z,w\in\boldsymbol{C}^{m+1}$. Then the Hopf fibering $\pi\colon S^{2m+1}(c/4)\to P_m(c)$ defined by $\pi(z)=[z]$, $[z]$ being the point of $P_m(\boldsymbol{C})$ with the homogeneous coordinate z, is a Riemannian submersion in the sense of O'Neill [7]. The complex structure tensors on $\boldsymbol{C}^{m+1}$ and $P_m(\boldsymbol{C})$ are denoted by J. We write $S=S^{2m+1}(c/4)$. Define a unit normal vector field ν for the imbedding $S\hookrightarrow\boldsymbol{C}^{m+1}$ by $\nu_q=(\sqrt{c}/2)q$ for $q\in S$, and put $V_q=\boldsymbol{R}(J\nu_q)$ and

$$H_q = \{z\in\boldsymbol{C}^{m+1};\ \langle z,q\rangle = \langle z,J\nu_q\rangle = 0\}$$

for $q\in S$. Then the subbundles $V(S)=\bigcup_{q\in S}V_q$ and $H(S)=\bigcup_{q\in S}H_q$ of the tangent bundle TS of S are the vertical and the horizontal distributions for the Riemannian submersion π, respectively, and thus we have an orthogonal Whitney sum: $TS=V(S)\oplus H(S)$. The complex structure J on $\boldsymbol{C}^{m+1}$ leaves each H_q invariant and $J_q|H_q$ corresponds to $J_{\pi(q)}$ on $P_m(\boldsymbol{C})$ under the linear isometry $\pi_*\colon H_q\to T_{\pi(q)}P_m(c)$. For a vector field X on S, its $V(S)$-component and $H(S)$-component will be denoted by $\mathcal{V}X$ and $\mathcal{H}X$, respectively. If $\mathcal{V}X=X$ (resp. $\mathcal{H}X=X$), X is said to be *vertical* (resp. *horizontal*). If X is horizontal and projectable to a vector field X_* on $P_m(\boldsymbol{C})$, it is called the *horizontal lift* of X_* and denoted by $X=h.l.\,X_*$. The Riemannian connections of S and $P_m(c)$ are denoted by ∇^S and $\bar{\nabla}$, respectively. Let A and T be the fundamental tensors for the Riemannian submersion π defined in O'Neill [7]. Then we have $T=0$, since each fibre of the Hopf fibering π is totally geodesic in S. For such a Riemannian submersion we have the following identities:

$$(1.1)\quad \nabla^S_V X = \mathcal{H}\nabla^S_V X\,,$$

$$(1.2)\quad \nabla^S_V V = A_X V + \mathcal{V}\nabla^S_X V\,,$$

$$(1.3)\quad \nabla^S_X Y = \mathcal{H}\nabla^S_X Y + A_X Y$$

for horizontal vector fields X,Y and a vertical vector field V on S. If further $X=h.l.\,X_*$ and $Y=h.l.\,Y_*$, then we have

$$(1.4)\quad \mathcal{H}\nabla^S_V X = A_X V\,,$$

$$(1.5)\quad \mathcal{H}\nabla^S_X Y = h.l.\,\bar{\nabla}_{X_*} Y_*\,.$$

The fundamental tensor A for our Hopf fibering π is given by

(1.6) $A_X(J\nu)=(\sqrt{c}/2)JX$,

(1.7) $A_XY=(\sqrt{c}/2)\langle X,JY\rangle J\nu$

for horizontal vector fields X,Y on S. For these identities (1.1)~(1.7), we refer the reader to O'Neill [7].

Now let $f:(M,g)\to P_m(c)$ be an isometric immersion of a Riemannian manifold (M,g) into $P_m(c)$. The complex structure and the connection on the pull back $f^{-1}T(P_m(C))$ induced from J and $\bar{\nabla}$ are also denoted by J and $\bar{\nabla}$. Let $\hat{M}$ be the total space of the pull back $f^{-1}S$ of the principal $U(1)$-bundle $\pi: S\to P_m(C)$. The $U(1)$-bundle map $\hat{f}:\hat{M}\to S$ which covers f is also an immersion, and so we may define a Riemannian metric $\hat{g}$ on $\hat{M}$ in such a way that $\hat{f}:(\hat{M},\hat{g})\to S$ is an isometric immersion. Then the projection $\pi:(\hat{M},\hat{g})\to(M,g)$ is also a Riemannian submersion with $T=0$. Note that we have an orthogonal Whitney sum: $\hat{f}^{-1}(TS)=\hat{f}^{-1}V(S)\oplus\hat{f}^{-1}H(S)$. The connection on $\hat{f}^{-1}(TS)$ induced from ∇^S on TS and the complex structure on $\hat{f}^{-1}H(S)$ induced from J on $H(S)$ are also denoted by ∇^S and J, respectively. We define $V(\hat{M})=\hat{f}^{-1}V(S)$, which is the vertical distribution for the Riemannian submersion $\pi:(\hat{M},\hat{g})\to(M,g)$. Then the section $J\nu$ of $V(S)$ induces a section of $V(\hat{M})$, which will be also denoted by $J\nu$. Furthermore, regarding $T\hat{M}$ as a subbundle of $\hat{f}^{-1}(TS)$, we define $H(\hat{M})=T\hat{M}\cap\hat{f}^{-1}H(S)$, which is the horizontal distribution for π. Thus we have an orthogonal Whitney sum: $T\hat{M}=V(\hat{M})\oplus H(\hat{M})$. The second fundamental forms of $f:(M,g)\to P_m(c)$ and $\hat{f}:(\hat{M},\hat{g})\to S$ will be denoted by σ and $\hat{\sigma}$, respectively.

The isometric immersion $f:(M,g)\to P_m(c)$ is said to be *totally real* if $\langle J(T_pM),T_pM\rangle=\{0\}$ for each $p\in M$. This is the case if and only if

(1.8) $\langle JH_q(\hat{M}),H_q(\hat{M})\rangle=\{0\}$

for each $q\in\hat{M}$, where $H_q(\hat{M})$ denotes the fibre of $H(\hat{M})$ over q.

Lemma 1.1. *Let $f:(M,g)\to P_m(c)$ be a totally real isometric immersion and $\hat{f}:(\hat{M},\hat{g})\to S$ the isometric immersion induced from f in the above way. Then*

1) *f is minimal if and only if $\hat{f}$ is minimal;*

2) *(M,g) is complete if and only if $(\hat{M},\hat{g})$ is complete;*

3) *$f(M)$ is not contained in any complex hyperplane of $P_m(C)$ if and only if $\hat{f}(\hat{M})$ is not contained in any real hyperplane of C^{m+1};*

4) *Both $V(\hat{M})$ and $H(\hat{M})$ are parallel subbundles of $T\hat{M}$, i.e., they are invariant under the parallel translation of $(\hat{M},\hat{g})$ along any curve of $\hat{M}$;*

5) *Assume that the linear span $N_p^1(M)$ of $\sigma(T_pM,T_pM)$ is contained in $J(T_pM)$ for each $p\in M$. Then, σ is parallel if and only if $\hat{\sigma}$ is parallel.*

Proof. We shall prove first the following: Let ∇ and $\hat{\nabla}$ denote the Riemannian connections of (M,g) and $(\hat{M},\hat{g})$, respectively. Let X,Y be vector fields on

$\hat{M}$ which are horizontal lifts of vector fields X_*, Y_* on M, respectively. Then

(1.9) $\hat{\nabla}_X Y = h.l.\, \nabla_{X_*} Y_*$;

(1.10) $\hat{\sigma}(X,Y) = h.l.\sigma(X_*,Y_*)$;

(1.11) $\hat{\nabla}_X(J\nu) = \mathcal{V}\nabla^S_X(J\nu)$;

(1.12) $\hat{\sigma}(X,J\nu) = (\sqrt{c}/2)JX$;

(1.13) $\hat{\nabla}_{J\nu} X = 0$;

(1.14) $\hat{\nabla}_{J\nu}(J\nu) = 0$;

(1.15) $\hat{\sigma}(J\nu, J\nu) = 0$.

We have

$$\begin{aligned} \nabla^S_X Y &= \mathcal{H}\nabla^S_X Y + A_X Y && \text{by (1.3)} \\ &= \mathcal{H}\nabla^S_X Y + (\sqrt{c}/2)\langle X, JY\rangle J\nu && \text{by (1.7)} \\ &= \mathcal{H}\nabla^S_X Y = h.l.\,\overline{\nabla}_{X_*} Y_* && \text{by (1.8), (1.5)}. \end{aligned}$$

This implies (1.9), (1.10). We have

$$\begin{aligned} \nabla^S_X(J\nu) &= A_X(J\nu) + \mathcal{V}\nabla^S_X(J\nu) && \text{by (1.2)} \\ &= (\sqrt{c}/2)JX + \mathcal{V}\nabla^S_X(J\nu) && \text{by (1.6)}. \end{aligned}$$

This together with (1.8) implies (1.11), (1.12). We have

$$\nabla^S_{J\nu} X = \mathcal{H}\nabla^S_{J\nu} X = A_X(J\nu) \qquad \text{by (1.1), (1.4)}.$$

Thus, by (1.6) we obtain

(1.16) $\nabla^S_{J\nu} X = (\sqrt{c}/2)JX$.

This together with (1.8) implies (1.13). The equalities (1.14), (1.15) follow from $\nabla^S_{J\nu}(J\nu) = 0$.

1) Let η and $\hat{\eta}$ denote the mean curvature vectors of f and $\hat{f}$, respectively. Let dim $M = n$ and so dim $\hat{M} = n+1$. For an arbitrary $q \in \hat{M}$, choose an orthonormal basis $\{x_1, \cdots, x_n\}$ of $H_q(\hat{M})$ and put $x_{i*} = \pi_* x_i$, $1 \leqq i \leqq n$. Extend each x_{i*} to a vector field X_{i*} on M and let $X_i = h.l.\, X_{i*}$. Then, by (1.10), (1.15) we have

$$\begin{aligned} (n+1)\hat{\eta}_q &= \textstyle\sum_{i=1}^n \hat{\sigma}(X_i, X_i)_q + \hat{\sigma}(J\nu, J\nu)_q \\ &= \textstyle\sum_{i=1}^n (h.l.\,\sigma(X_{i*}, X_{i*}))_q = n(h.l.\,\eta)_q. \end{aligned}$$

This implies the assertion 1).

2) This follows from the compactness of the fibre $U(1)$ of π.

3) Assume that $\hat{f}(\hat{M})$ is contained in a real hyperplane of $\boldsymbol{C}^{m+1}$. Then

there exist $a\in \boldsymbol{C}^{m+1}-\{0\}$ and $k\in \boldsymbol{R}$ and that $\langle \hat{f}(\hat{M}),a\rangle=\{k\}$. Take a point $q\in \hat{M}$ and let ${}^t(\hat{f}(q))\bar{a}=re^{\sqrt{-1}\phi}$ so that $k=r\cos\phi$. For each $\varepsilon=e^{\sqrt{-1}\theta}\in U(1)$, $\theta\in \boldsymbol{R}$, we have

$$k=\langle \hat{f}(q\varepsilon),a\rangle=\langle \hat{f}(q)\varepsilon,a\rangle=\mathcal{Re}\{{}^t(\hat{f}(q))\bar{a}\varepsilon\}$$
$$=\mathcal{Re}(re^{\sqrt{-1}(\phi+\theta)})=r\cos(\phi+\theta)\,.$$

We have therefore $r=0$, and hence $\langle \hat{f}(\hat{M}),a\rangle=\{0\}$. Now for each $\varepsilon\in U(1)$ we have $\langle \hat{f}(\hat{M}),a\varepsilon\rangle=\langle \hat{f}(\hat{M})\bar{\varepsilon},a\rangle=\langle \hat{f}(\hat{M}\bar{\varepsilon}),a\rangle=\{0\}$. Thus $f(M)$ is contained in the complex hyperplane $\{[z]\in P_m(\boldsymbol{C});\, {}^tza=0\}$ of $P_m(\boldsymbol{C})$. If conversely $f(M)$ is contained in a complex hyperplane $\{[z]\in P_m(\boldsymbol{C});\, {}^tza=0\}$, $a\in \boldsymbol{C}^{m+1}-\{0\}$, then $\hat{f}(\hat{M})$ is contained in the real hyperplane $\{z\in \boldsymbol{C}^{m+1};\, \langle z,a\rangle=0\}$ of $\boldsymbol{C}^{m+1}$.

4) Equalities (1.11), (1.14) and (1.9), (1.13) imply that $V(\hat{M})$ and $H(\hat{M})$ are invariant, respectively, under the covariant differentiation by any vector field on M. Thus the assertion 4) follows.

5) Let $\nabla^\perp$ and $\hat{\nabla}^\perp$ be the normal connections on the normal bundles NM and $N\hat{M}$, respectively, and let ∇^* and $\hat{\nabla}^*$ be the coveriant derivations on $T^*M\otimes T^*M\otimes NM$ and $T^*\hat{M}\otimes T^*\hat{M}\otimes N\hat{M}$, respectively, where T^*M and $T^*\hat{M}$ denote the cotangent bundles. Let X,Y,Z be the horizontal lifts of vector fields X_*, Y_*,Z_* on M, respectively. Then

$$\text{(a)}\quad (\hat{\nabla}^*\hat{\sigma})(J\nu,J\nu,J\nu)=\hat{\nabla}^\perp_{J\nu}\hat{\sigma}(J\nu,J\nu)-2\hat{\sigma}(\hat{\nabla}_{J\nu}(J\nu),J\nu)$$
$$=0 \qquad \text{by (1.15), (1.14)}\,.$$

$$\text{(b)}\quad (\hat{\nabla}^*\hat{\sigma})(X,J\nu,J\nu)=\hat{\nabla}^\perp_X\hat{\sigma}(J\nu,J\nu)-2\hat{\sigma}(\hat{\nabla}_X(J\nu),J\nu)$$
$$=-2\hat{\sigma}(\mathcal{V}\nabla^S_X(J\nu),J\nu) \qquad \text{by (1.15), (1.11)}$$
$$=0 \qquad \text{by (1.15)}\,.$$

$$(\hat{\nabla}^*\hat{\sigma})(J\nu,X,Y)=\hat{\nabla}^\perp_{J\nu}\hat{\sigma}(X,Y)-\hat{\sigma}(\hat{\nabla}_{J\nu}X,Y)-\hat{\sigma}(X,\hat{\nabla}_{J\nu}Y)$$
$$=\hat{\nabla}^\perp_{J\nu}\hat{\sigma}(X,Y) \qquad \text{by (1.13)}\,.$$

Here $\hat{\sigma}(X,Y)=\mathcal{h.l.}\,\sigma(X_*,Y_*)$ by (1.10). Therefore we have

$$\nabla^S_{J\nu}\hat{\sigma}(X,Y)=(\sqrt{c}/2)J\hat{\sigma}(X,Y)=(\sqrt{c}/2)\mathcal{h.l.}\,J\sigma(X_*,Y_*)\,,$$

since (1.16) holds also for the horizontal lift X of a vector field X_* on $P_m(\boldsymbol{C})$. Now the assumption $N^1_p(M)\subset J(T_p(M))$ implies that $J\sigma(X_*,Y_*)$ is tangent to M, and hence $\nabla^S_{J\nu}\hat{\sigma}(X,Y)$ is tangent to $H(M)$. Thus $\hat{\nabla}^\perp_{J\nu}\hat{\sigma}(X,Y)=0$, and hence

$$\text{(c)}\quad (\hat{\nabla}^*\hat{\sigma})(J\nu,X,Y)=0\,.$$

Moreover, by (1.9) ,(1.10) we have

$$(\hat{\nabla}^*\hat{\sigma})(X,Y,Z)=\hat{\nabla}^\perp_X\hat{\sigma}(Y,Z)-\hat{\sigma}(\hat{\nabla}_XY,Z)-\hat{\sigma}(Y,\hat{\nabla}_XZ)$$
$$=\hat{\nabla}^\perp_X\hat{\sigma}(Y,Z)-\mathcal{h.l.}\,\sigma(\nabla_{X_*}Y_*,Z_*)-\mathcal{h.l.}\,\sigma(Y_*,\nabla_{X_*}Z_*)\,.$$

Here $\hat{\sigma}(Y,Z)=h.l.\sigma(Y_*,Z_*)$ by (1.10), and thus $\nabla^S_X\hat{\sigma}(Y,Z)=h.l.\bar{\nabla}_{X_*}\sigma(Y_*,Z_*)$ by (1.5). Therefore $\hat{\nabla}^{\perp}_X\hat{\sigma}(Y,Z)=h.l.\nabla^{\perp}_{X_*}\sigma(Y_*,Z_*)$. Thus we obtain

$$\text{(d)}\quad (\hat{\nabla}^*\hat{\sigma})(X,Y,Z)=h.l.(\nabla^*\sigma)(X_*,Y_*,Z_*)\,.$$

Now (a), (b), (c), (d) imply the assertion 5), since $\hat{\nabla}^*\hat{\sigma}$ is symmetric trilinear in virtue of the Codazzi equation. q.e.d.

2. Shilov boundaries of symmetric bounded domains of tube type

We fix a positive integer n and a constant $c>0$. Let us consider an object $\mathfrak{d}=(D_1,\cdots,D_s;c_1,\cdots,c_s)$, $s\geqq 1$, where

(i) D_i, $1\leqq i\leqq s$, is an irreducible symmetric bounded domain of tube type, and $\Sigma_i \dim_C D_i=n+1$;

(ii) c_i, $1\leqq i\leqq s$, is a positive constant, and $\Sigma_i 1/c_i=1/c$.

We shall associate to such an object $\mathfrak{d}$ a totally real isometric imbedding f: $(M,g)\rightarrow P_m(c)$ of an n-dimensional complete connected Riemannian manifold (M,g) with parallel second fundamental form.

Let $D=D_1\times\cdots\times D_s$ be the direct product of the D_i's, $1\leqq i\leqq s$. It is also a symmetric bounded domain of tube type. Note that $\dim_C D=n+1$ in virtue of (i). The identity component G of the group of holomorphisms of D is semi-simple and with the trivial center. Therefore it is identified with the group of inner automorphisms of $\mathfrak{g}=\text{Lie } G$, the Lie algebra of G, and hence it is also identified with a closed subgroup of the group G_C of inner automorphisms of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. Fix a point $o\in D$ and put

$$K=\{\phi\in G;\,\phi o=o\}\,,\qquad \mathfrak{k}=\text{Lie } K\,.$$

Then the subspace

$$\mathfrak{p}=\{X\in\mathfrak{g};\,B(X,\mathfrak{k})=\{0\}\}$$

of $\mathfrak{g}$, where B denotes the Killing form of $\mathfrak{g}_C$, is invariant under the adjoint action of K, and it is identified with the tangent space T_oD of D at o. Let H be the unique element of the center of $\mathfrak{k}$ such that $ad\,H|\mathfrak{p}$ coincides with the complex structure J of D on $\mathfrak{p}=T_oD$. Then the complexification $\mathfrak{p}_C$ of $\mathfrak{p}$ is decomposed to the direct sum: $\mathfrak{p}_C=\mathfrak{p}_C^++\mathfrak{p}_C^-$ of K-invariant subspaces $\mathfrak{p}_C^{\pm}$ defined by

$$\mathfrak{p}_C^{\pm}=\{X\in\mathfrak{p}_C;\,[H,X]=\pm\sqrt{-1}\,X\}\,.$$

Note that the linear map $\iota\colon\mathfrak{p}\rightarrow\mathfrak{p}_C^+$ defined by $\iota(X)=(1/2)(X-\sqrt{-1}\,[H,X])$ is a K-equivariant C-linear isomorphism of $(\mathfrak{p},J)$ onto $\mathfrak{p}_C^+$. Denoting by τ the complex conjugation of $\mathfrak{g}_C$ with respect to the compact real form $\mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$, we define a K-invariant hermitian inner product $(\ ,\ )_\tau$ on $\mathfrak{p}_C^+$ by

$(X,Y)_\tau=-B(X,\tau Y)$ for $X,Y\in\mathfrak{p}_C^+$. We define then a K-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}_C^+$, regarded as a real vector space, by $\langle X,Y\rangle=2\mathcal{Re}(X,Y)_\tau$ for $X,Y\in\mathfrak{p}_C^+$. Then we have

$$\text{(2.1)}\quad \langle \iota X,\iota Y\rangle = B(X,Y) \qquad \text{for} \quad X,Y\in\mathfrak{p}\,.$$

Let $c\in G_u$, G_u being the connected subgroup of G_C generated by $\mathfrak{g}_u$, denote the standard Cayley transform for D(cf. Takeuchi [9]), and define an involutive automorphism θ of G_C by $\theta(x)=c^2xc^{-2}$ for $x\in G_C$. The differential $Ad\,c^2$ of θ will be also denoted by θ. Then we have $\theta\tau=\tau\theta$, $\theta\mathfrak{k}=\mathfrak{k}$ and $\theta H=-H$. We may define an anti-linear endomorphism $X\to\bar{X}$ of $\mathfrak{p}_C^+$ by $\bar{X}=\tau\theta X$, so that

$$\mathfrak{p}^+ = \{X\in\mathfrak{p}_C^+;\bar{X}=X\}$$

is a real form of $\mathfrak{p}_C^+$. Let now $F\colon D\hookrightarrow\mathfrak{p}_C^+$ be the Harish-Chandra imbedding for D, and $S\subset\partial D\subset\mathfrak{p}_C^+$ the Shilov boundary of D. The groups G,K or $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{p}_C^+$ etc. are the direct products or the direct sums of respective objects for D_i, $1\leqq i\leqq s$, which will be denoted by the same notation but with the suffix i. Then F is the product imbedding $F_1\times\cdots\times F_s$ of Harish-Chandra imbeddings $F_i\colon D_i\hookrightarrow\mathfrak{p}_{iC}^+$ for D_i, and S is the direct product $S_1\times\cdots\times S_s$ of Shilov boundaries $S_i\subset\partial D_i\subset\mathfrak{p}_{iC}^+$ of D_i. The group K acts transitively on S and S is a compact connected manifold with dim $S=\dim_C D=n+1$. Let $X_i^0\in S_i$ be the standard base point of S_i (cf. Takeuchi [9]). Then

(2.2) eigenvalues of $ad(\iota^{-1}(\sqrt{-1}\,X_i^0))$ on $\mathfrak{g}_i$ are $0,2,-2$.

Put $X^0=X_1^0+\cdots+X_s^0\in S$ and

$$K_0 = \{k\in K;kX^0 = X^0\}\ .$$

Then (K,K_0) is a symmetric pair with respect to θ and S is identified with the quotient manifold K/K_0. If we set

$$\mathfrak{s} = \{X\in\mathfrak{k};\theta X = -X\}\ ,$$

and $\psi(X)=[X,\sqrt{-1}\,X^0]$ for $X\in\mathfrak{s}$, then ψ defines a linear isomorphism of $\mathfrak{s}$ onto $\mathfrak{p}^+$. In particular, we have

$$\text{(2.3)}\quad [\mathfrak{s},\sqrt{-1}\,X^0] = \mathfrak{p}^+\,.$$

For these properties of symmetric bounded domains of tube type, we refer the reader to Korányi-Wolf [4], Takeuchi [9].

Now let $\dim_C D_i=n_i+1$ and put $a_i=1/\sqrt{2c_i(n_i+1)}$, $1\leqq i\leqq s$. We define an $(n+1)$-dimensional compact connected submanifold $\hat{M}$ of $\mathfrak{p}_C^+$ by

$$\hat{M} = a_1S_1\times\cdots\times a_sS_s\,,$$

and endow it with the Riemannian metric $\hat{g}$ induced from $\langle\ ,\ \rangle$. We write

$\hat{M}_i=a_iS_i\subset\mathfrak{p}^+_{iC}$, $1\leqq i\leqq s$. If we put $E_i=\sqrt{-1}a_iX_i^0\in\mathfrak{p}^+_{iC}$ and $E=E_1+\cdots+E_s\in\mathfrak{p}^+_C$, then E_i belongs to $\hat{M}_i$, since each D_i is a circular domain in $\mathfrak{p}^+_{iC}$, and hence E belongs to $\hat{M}$. Thus we have $\hat{M}_i=K_iE_i$ and $\hat{M}=KE$. Note that we have also

(2.4) $\quad K_0=\{k\in K;\, kE=E\}$,

and hence $\hat{M}$ is identified with K/K_0. Moreover, (2.1), (2.2) imply

$$\langle\sqrt{-1}X_i^0,\ \sqrt{-1}X_i^0\rangle=4\dim\mathfrak{p}_i=8\dim_C D_i=8(n_i+1)$$

and hence $\langle E_i,E_i\rangle=4/c_i$, thus $\langle E,E\rangle=\Sigma_i\langle E_i,E_i\rangle=\Sigma_i 4/c_i=4/c$ in virtue of (ii). Therefore, identifying $\mathfrak{p}^+_{iC}$ with C^{n_i+1} by an orthonormal basis of $\mathfrak{p}^+_{iC}$ with respect to $2(\ ,\ )_\tau$, and thus identifying $\mathfrak{p}^+_C$ with C^{n+1}, we have

$$\hat{M}_i\subset S^{2n_i+1}(c_i/4),\qquad 1\leqq i\leqq s\,,$$

and

$$\hat{M}=\hat{M}_1\times\cdots\times\hat{M}_s\subset S^{2n+1}(c/4)\,.$$

Furthermore, the property (2.2) implies that each inclusion $\hat{M}_i\hookrightarrow S^{2n_i+1}(c_i/4)$ is a standard minimal isometric imbedding of an irreducible symmetric R-space $\hat{M}_i$ in the sense of Takeuchi [10]. Thus, by Takeuchi [10] the inclusion $\hat{f}$: $(\hat{M},\hat{g})\to S^{2n+1}(c/4)$ is an isometric imbedding with parallel second fundamental form such that $\hat{f}(\hat{M})$ is not contained in any real hyperplane of C^{n+1}. Here the identity component $I^0(\hat{M})$ of the group of isometries of $(\hat{M},\hat{g})$ may be identified with K. Moreover, $\hat{f}$ is minimal if and only if

(2.5) $\quad c_i(n_i+1)=c(n+1)\qquad$ for each i, $1\leqq i\leqq s$.

Now let π: $S^{2n+1}(c/4)\to P_n(c)$ be the Hopf fibering and put $M=\pi(\hat{M})$. It is a compact connected submanifold of $P_n(C)$ since it is a K-orbit in $P_n(C)$. We endow M with the Riemannian metric g induced from that of $P_n(c)$, and denote by f: $(M,g)\to P_n(c)$ the inclusion. Since the connected subgroup Z of K generated by $\boldsymbol{R}H$ acts on $\mathfrak{p}^+_C$ by $U(1)=\{\varepsilon I;\ \varepsilon\in C,\ |\varepsilon|=1\}$, we have $\pi^{-1}(M)=\hat{M}$. Therefore we have dim $M=n$. Thus we are in the position of **1** with $m=n$.

Theorem 2.1. *Let f: $(M,g)\to P_n(c)$ be the isometric imbedding associated to $\mathfrak{d}=(D_1,\cdots,D_s;\,c_1,\cdots,c_s)$ in the above way. Then*

1) *f is totally real and has the parallel second fundamental form. In particular, (M,g) is locally symmetric;*

2) *f is minimal if and only if $c_i\dim_C D_i=c(n+1)$ for each i, $1\leqq i\leqq s$;*

3) *The dimension of the Euclidean factor of the locally symmetric space (M,g) is equal to $s-1$;*

4) *(M,g) has no Euclidean factor if and only if $s=1$ and $\dim_C D_1\geqq2$. In*

this case, (M,g) *is irreducible and* f *is minimal*;

5) (M,g) *is flat if and only if* $s=n+1$ *and* $\dim_C D_i=1$, *i.e.,* D_i *is the unit disk, for each* i, $1\leqq i\leqq n+1$.

Proof. We prove first that f is totally real. Since K acts on $P_n(c)$ as isometric holomorphisms of $P_n(c)$, f is K-equivariant and M is a K-orbit, we need only to prove the property (1.8) for $q=E$. By (2.4) the tangent space $T_E\hat{M}$ is identified with $\mathfrak{s}$. Moreover, by (2.3) we have $[\mathfrak{s},E]=\mathfrak{p}^+$, and hence $T_E\hat{M}$ is identified with $\mathfrak{p}^+$. In particular we have $\sqrt{-1}\,E=[H,E]\in\mathfrak{p}^+$, since $H\in\mathfrak{s}$. Thus, if we put

$$\mathfrak{h}=\{X\in\mathfrak{p}^+;\langle X,\sqrt{-1}\,E\rangle=\{0\}\}\ ,$$

it is identified with $H_E(\hat{M})$. Now $\langle\mathfrak{p}^+,\sqrt{-1}\,\mathfrak{p}^+\rangle=\{0\}$ implies $\langle\mathfrak{h},\sqrt{-1}\,\mathfrak{h}\rangle=\{0\}$. We have therefore the required property: $\langle H_E(\hat{M}),JH_E(\hat{M})\rangle=\{0\}$.

The assertion that σ is parallel is an immediate consequence of Lemma 1.1,5), since $NM=J(TM)$ in our case. The assertion 2) follows from Lemma 1.1,1) and (2.5). The assertions 3),4),5), except for the minimality for f in 4), follow from the following observations:

(a) the dimension of Euclidean factor of M=the one of $\hat{M}-1$;

(b) the dimension of Euclidean factor of $\hat{M}_i=1$;

(c) the number of irreducible factors of $\hat{M}_i=\begin{cases}1 & \text{if } \dim_C D_i\geqq 2\,,\\ 0 & \text{if } \dim_C D_i=1\,.\end{cases}$

The minimality of f in 4) follows from 2). q.e.d.

3. Classification of totally real submanifolds with parallel second fundamental form

Let $\mathfrak{d}=(D_1,\cdots,D_s;c_1,\cdots,c_s)$ and $\mathfrak{d}'=(D_1',\cdots,D_t';c_1',\cdots,c_t')$ satisfy conditions (i), (ii) in **2**. They are said to be *equivalent*, denoted by $\mathfrak{d}\sim\mathfrak{d}'$, if $s=t$ and there exists a permutation p of s-letters $\{1,2,\cdots,s\}$ such that $D'_{p(i)}$ is isomorphic to D_i and $c'_{p(i)}=c_i$ for each i, $1\leqq i\leqq s$. The set of all equivalence classes of $\mathfrak{d}=(D_1,\cdots,D_s;c_1,\cdots,c_s)$ with (i), (ii) will be denoted by $\mathcal{D}_{n,c}$. Let $\mathrm{Aut}(P_n(c))$ denote the group of isometric holomorphisms of $P_n(c)$. It is isomorphic to the projective unitary group $PU(n+1)$ of degree $n+1$ in the natural way. We denote by $\mathcal{S}_{n,c}$ the set of all $\mathrm{Aut}(P_n(c))$-congruence classes of n-dimensional complete connected totally real submanifolds M of $P_n(c)$ with parallel second fundamental form. Then from the naturality of Harish-Chandra imbedding our correspondence $\mathfrak{d}\to M$ in **2** induces a map $\mathcal{D}_{n,c}\to\mathcal{S}_{n,c}$.

Theorem 3.1. 1) *The map* $\mathcal{D}_{n,c}\to\mathcal{S}_{n,c}$ *is a bijection.*

2) *Let* $f:(M,g)\to P_n(c)$ *be a totally real isometric immersion of an n-dimensional complete connected Riemannian manifold* (M,g) *with parallel second fundamental*

form. Then there exist an n-dimensional complete connected totally real submanifold $\iota: M' \hookrightarrow P_n(c)$ *with parallel second fundamental form and an isometric covering* $f': M \to M'$ *such that* $f = \iota \circ f'$.

Proof. 1) *Surjectivity*: Let $M \subset P_n(c)$ be an n-dimensional complete connected totally real submanifold with parallel second fundamental form. We use the notation in **1** with $m=n$. Then, by Lemma 1.1 $\hat{M} = \pi^{-1}(M) \subset S^{2n+1}(c/4)$ is complete, connected, with parallel second fundamental form and not contained in any real hyperplane of $\boldsymbol{C}^{n+1}$. Moreover we have $\pi(\hat{M}) = M$. Thus, by Theorem 4.1 of Takeuchi [10]

$$\hat{M} = \hat{M}_1 \times \cdots \times \hat{M}_s \subset S^{m_1}(c_1/4) \times \cdots \times S^{m_s}(c_s/4) \subset S^{2n+1}(c/4),$$

where each $\hat{M}_i \subset S^{m_i}(c_i/4)$, $c_i > 0$, is an irreducible symmetric R-space, $\Sigma_i m_i + s = 2n+2$ and $\Sigma_i 1/c_i = 1/c$. Here the group $K = I^0(\hat{M})$ is identified with the identity component of the group $\{\phi \in O(\boldsymbol{C}^{n+1}); \phi\hat{M} = \hat{M}\}$. Since $\hat{M}$ is invariant under the subgroup $Z = \{\varepsilon I; \varepsilon \in \boldsymbol{C}, |\varepsilon| = 1\}$ of $O(\boldsymbol{C}^{n+1})$, Z is a closed subgroup of K. Let $p: \tilde{M} \to \hat{M}$ be the universal Riemannian covering of $\hat{M}$. Then, by Lemma 1.1,4) $\tilde{M}$ is the Riemannian product $\tilde{V} \times \tilde{H}$ of maximal integral submanifolds $\tilde{V}$ and $\tilde{H}$ in $\tilde{M}$ of distributions $p^{-1}V(\hat{M})$ and $p^{-1}H(\hat{M})$, respectively. Since $\tilde{V}$ is a flat line, it is contained in the Euclidean part of $\tilde{M}$. Thus, if we identify Lie $I^0(\hat{M})$ with a Lie subalgebra of Lie $I^0(\tilde{M})$, Lie Z = Lie $I^0(\tilde{V})$ is contained in the center of Lie $I^0(\hat{M})$ = Lie K. Therefore Z is contained in the center of K, which implies that K is a subgroup of the unitary group $U(n+1)$. It follows that each irreducible symmetric pair $(\mathfrak{g}_i, \mathfrak{k}_i)$ associated to $\hat{M}_i$ is of hermitian type. Moreover, each $\mathfrak{g}_i$ has a semi-simple element E_i such that $ad\, E_i$ has just three distinct real eigenvalues. This is the case if and only if each irreducible symmetric bounded domain D_i associated to $(\mathfrak{g}_i, \mathfrak{k}_i)$ is of tube type. Here we have $2\dim_{\boldsymbol{C}} D_i = m_i + 1$, and hence $\Sigma_i \dim_{\boldsymbol{C}} D_i = n+1$. Therefor, $M \subset P_n(c)$ is obtained from $\mathfrak{d} = (D_1, \cdots, D_s; c_1, \cdots, c_s)$ by the construction in **2**. This proves the surjectivity of our map.

Injectivity: Let $M \subset P_n(c)$ and $M' \subset P_n(c)$ be associated to $\mathfrak{d} = (D_1, \cdots, D_s; c_1, \cdots, c_s)$ and $\mathfrak{d}' = (D'_1, \cdots, D'_t; c'_1, \cdots, c'_t)$, respectively. Various objects in the construction of M' will be denoted by the same notation as for M but with primes. Suppose that there exists $\phi \in \mathrm{Aut}(P_n(c)) = PU(n+1)$ with $\phi M = M'$. Then we have a $\boldsymbol{C}$-linear isometry $\hat{\phi}: \mathfrak{p}^+_{\boldsymbol{C}} \to \mathfrak{p}'^+_{\boldsymbol{C}}$ with respect to $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ such that $\hat{\phi}\hat{M} = \hat{M}'$ and $\hat{\phi}$ induces ϕ. Then the homomorphism $\hat{\phi}_K: K = I^0(\hat{M}) \to K' = I^0(\hat{M}')$ defined by $\hat{\phi}_K(k) = \hat{\phi} \circ k \circ \hat{\phi}^{-1}$ is an isomorphism. The differential $(\hat{\phi}_K)_*: \mathfrak{k} \to \mathfrak{k}'$ of $\hat{\phi}_K$ will be denoted by $\hat{\phi}_{\mathfrak{k}}$. Moreover, the $\boldsymbol{C}$-linear isomorphism $\hat{\phi}_{\mathfrak{p}}: (\mathfrak{p}, J) \to (\mathfrak{p}', J')$ with $\hat{\phi} \circ \iota = \iota' \circ \hat{\phi}_{\mathfrak{p}}$ is a linear isometry with respect to B and B', and it satisfies

$$(3.1) \qquad \hat{\phi}_{\mathfrak{p}}(kX) = \hat{\phi}_K(k)\,(\hat{\phi}_{\mathfrak{p}} X) \qquad \text{for} \quad k \in K,\ X \in \mathfrak{p}\,.$$

We define an $\boldsymbol{R}$-linear isomorphism $\Phi: \mathfrak{g}=\mathfrak{k}+\mathfrak{p}\rightarrow\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$ by $\Phi=\hat{\phi}_{\mathfrak{k}}+\hat{\phi}_{\mathfrak{p}}$. Then (3.1) implies

$$(3.2)\quad \Phi\circ ad\, X=(ad\,\Phi X)\circ\Phi \qquad \text{for}\quad X\in\mathfrak{k}\,.$$

We shall show that Φ is actually a Lie isomorphism. Since (3.2) holds, we need only to show

$$(3.3)\quad \Phi[X,Y]=[\Phi X,\Phi Y] \qquad \text{for}\quad X,Y\in\mathfrak{p}\,.$$

For each $Z\in\mathfrak{k}$ we have

$$\begin{aligned} B'([\Phi X,\Phi Y],\Phi Z) &= B'(\Phi X,[\Phi Y,\Phi Z]) \\ &= B'(\Phi X,\Phi[Y,Z]) && \text{by (3.2)} \\ &= B'(\hat{\phi}_{\mathfrak{p}}X,\hat{\phi}_{\mathfrak{p}}[Y,Z])=B(X,[Y,Z]) \\ &= B([X,Y],Z)=B'(\Phi[X,Y],\Phi Z) && \text{by (3.2)}\,. \end{aligned}$$

This implies (3.3). Now the naturality of Harish-Chandra imbedding implies $\mathfrak{d}\sim\mathfrak{d}'$. This proves the injectivity of our map.

2) Construct an isometric immersion $\hat{f}:(\hat{M},\hat{g})\rightarrow S^{2n+1}(c/4)$ from f in the same way as in **1**. Then, by Lemma 1.1 $(\hat{M},\hat{g})$ is complete and $\hat{f}$ has the parallel second fundamental form. Thus, by Theorem 4.1 of Takeuchi [10] the image $\hat{M}'=\hat{f}(\hat{M})$ is a complete submanifold of $S^{2n+1}(c/4)$ and the map $\hat{f}':\hat{M}\rightarrow\hat{M}'$ induced by $\hat{f}$ is an isometric covering. Therefore $M'=\pi(\hat{M}')$ is an n-dimensional complete connected submanifold of $P_n(c)$ and the induced map $f':M\rightarrow M'$ is an isometric covering. It is clear that M' is a totally real submanifold of $P_n(c)$ with parallel second fundamental form. This completes the proof. q.e.d.

EXAMPLE. Let D be the irreducible symmetric bounded domain of type (IV) with $\dim_C D=n+1, n\geqq 2$. Then the submanifold $M\subset P_n(c)$ corresponding to $\mathfrak{d}=(D;c)$ is the naturally imbedded real projective n-space $P^n(c/4)$ with constant sectional curvature $c/4$, which is totally geodesic in $P_n(c)$.

We define a convex subset $\boldsymbol{F}_{n,c}$ of $\boldsymbol{R}^n$ by

$$\boldsymbol{F}_{n,c}=\{\alpha=(\alpha_i)\in\boldsymbol{R}^n;\ \alpha_i\geqq 0(1\leqq i\leqq n),\ \alpha_1+2\alpha_2+\cdots+n\alpha_n<1/c\}\,.$$

For each $\alpha\in\boldsymbol{F}_{n,c}$ we define constants $c_1,\cdots,c_{n+1}$ with $0<c_1\leqq c_2\leqq\cdots\leqq c_{n+1}$ by the relations

$$(3.4)\quad \alpha_i=1/c_i-1/c_{i+1}\ (1\leqq i\leqq n)\ \text{and}\ \Sigma_i 1/c_i=1/c\,,$$

and put

$$\hat{M}_\alpha^{n+1}=S^1(c_1/4)\times\cdots\times S^1(c_{n+1}/4)\subset S^{2n+1}(c/4)\,.$$

Then, by Theorem 2.1,5) $M_\alpha^n=\pi(\hat{M}_\alpha^{n+1})\subset P_n(c)$ is an n-dimensional complete connected flat totally real submanifold with parallel second fundamental form. Let $\mathcal{F}_{n,c}$ denote the set of all $\mathrm{Aut}(P_n(c))$-congruence classes of such submanifolds. Then the correspondence $\alpha \rightarrow M^n_\alpha$ induces a map $\boldsymbol{F}_{n,c} \rightarrow \mathcal{F}_{n,c}$.

Theorem 3.2. 1) *The map $\boldsymbol{F}_{n,c} \rightarrow \mathcal{F}_{n,c}$ is a bijection.*

2) *An n-dimensional complete connected flat totally real minimal submanifold of $P_n(c)$ with parallel second fundamental form is unique up to the congruence relative to the group $Aut(P_n(c))$, and it is given by $M^n_0 \subset P_n(c)$.*

Proof. 1) By Theorem 2.1,5) and Theorem 3.1, $\mathcal{F}_{n,c}$ corresponds one to one to the set of all $(n+1)$-tuples $(c_1, \cdots, c_{n+1})$ with $0 < c_1 \leqq c_2 \leqq \cdots \leqq c_{n+1}$ and $\Sigma_i 1/c_i = 1/c$. But the latter set corresponds one to one to the set $\boldsymbol{F}_{n,c}$ by the relations (3.4).

2) By Theorem 2.1,2), $M^n_\alpha \subset P_n(c)$ is minimal if and only if $c_i = c(n+1)$ for each i, $1 \leqq i \leqq n$. This is the case if and only if $\alpha = 0$. q.e.d.

REMARK. The norm $||\sigma_\alpha||$ of the second fundamental form σ_α of $M^n_\alpha \subset P_n(c)$ is given by

$$||\sigma_\alpha||^2 = \{\Sigma_i c_i - (3n+1)c\}/4 .$$

In particular, we have $||\sigma_0||^2 = n(n-1)c/4$.

4. Characterization of a flat totally real surface in $P_2(c)$

Let $f: (M,g) \rightarrow (\bar{M},\bar{g})$ be an isometric immersion of an n-dimensional Riemannian manifold (M,g) into an $(n+q)$-dimensional Riemannian manifold $(\bar{M},\bar{g})$ with $q \geqq 1$. The inner product and the norm of tensors defined by Riemannian metrics are denoted by $\langle\ ,\ \rangle$ and $||\ \ ||$, respectively. We denote by σ the second fundamental form of f, and by S_ξ the shape operator of f. They are related by $\langle S_\xi X, Y\rangle = \langle \sigma(X,Y), \xi\rangle$ for vector fields X, Y on M and a normal vector field ξ. We define a section $\tilde{\sigma}$ of the bundle $\mathrm{End}(NM)$ of endomorphisms of the normal bundle NM by $\tilde{\sigma} = \sigma \circ {}^t\sigma$, regarding σ as a homomorphism from $TM \otimes TM$ to NM. Moreover, we define a homomorphism $S^\perp$ from $TM \otimes TM$ to $\mathrm{End}(NM)$ by $S^\perp(X,Y)\xi = \sigma(X, S_\xi Y) - \sigma(Y, S_\xi X)$ for vector fields X, Y on M and a normal vector field ξ. Then we have the following

Lemma 4.1 (Simons [8], Chern-do Carmo-Kobayashi [2]). *Let p be an arbitrary point of M. Then we have an inequality*

$$||\tilde{\sigma}_p||^2 + ||S^\perp_p||^2 \leqq (2 - 1/q)||\sigma_p||^4 .$$

If the equality holds, then either $\sigma_p = 0$ or $\sigma_p \neq 0$, $N^1_p(M) = N_p M$ and $q \leqq 2$.

Now assume that $(\bar{M},\bar{g})$ is a Kählerian manifold $\boldsymbol{M}_m(c)$ of constant holomorphic sectional curvature c with $\dim_C \boldsymbol{M}_m(c) = m$, and that f is totally real in the sense that $\langle J(T_pM), T_pM\rangle=\{0\}$ for each $p\in M$, where J denotes the complex structure tensor of $M_m(c)$. Then we have an orthogonal Whitney sum: $NM=J(TM)\oplus J(TM)^{\perp}$, where $J(TM)^{\perp}$ denotes the orthogonal complement of $J(TM)$ in NM. We define a homomorphism σ_J from $TM\otimes TM$ to NM by $\sigma_J(X,Y)=J(TM)$-component of $\sigma(X,Y)$ with respect to the above decomposition, for vector fields X,Y on M. Let $\Delta=Tr_g\nabla^{*2}$ denote the Laplacian on NM. Then, from Simons' formula (Simons [8]) which describes $\Delta\sigma$ for a general minimal isometric immersion, we have the following lemma.

Lemma 4.2. *Let $f:(M,g)\to M_m(c)$ be a totally real minimal isometric immersion. Then*

$$(4.1)\quad \langle\Delta\sigma,\sigma\rangle=(n\|\sigma\|^2+\|\sigma_J\|^2)c/4-\|\tilde{\sigma}\|^2-\|S^{\perp}\|^2\,.$$

Proposition 4.3. *Let $f:(M,g)\to M_m(c)$, $c\leqq 0$, be a totally real minimal isometric immersion with parallel second fundamental form. Then f is totally geodesic.*

Proof. Since $\nabla^*\sigma=0$, we have by Lemma 4.2

$$(n\|\sigma\|^2+\|\sigma_J\|^2)c/4=\|\tilde{\sigma}\|^2+\|S^{\perp}\|^2 \quad\text{with}\quad c\leqq 0\,.$$

This implies $\tilde{\sigma}=0$, and hence $\sigma=0$. q.e.d.

Lemma 4.4. *Let $f:(M,g)\to M_n(c)$ be a totally real minimal isometric immersion of an n-dimensional Riemannian manifold (M,g). Then*

1) *We have an inequality*

$$(4.2)\quad -\langle\Delta\sigma,\sigma\rangle\leqq\{(2-1/n)\|\sigma\|^2-(n+1)c/4\}\|\sigma\|^2;$$

2) *If furthermore M is compact, then we have*

$$(4.3)\quad \int_M\|\nabla^*\sigma\|^2v_g\leqq\int_M\{(2-1/n)\|\sigma\|^2-(n+1)c/4\}\|\sigma\|^2v_g\,,$$

where v_g denotes the Riemannian measure of (M,g).

Proof. 1) Since $J(TM)=NM$ in our case, we have $\sigma_J=\sigma$. Thus the equality (4.1) reduces to $\langle\Delta\sigma,\sigma\rangle=(n+1)c\|\sigma\|^2/4-\|\tilde{\sigma}\|^2-\|S^{\perp}\|^2$. Now (4.2) follows from Lemma 4.1.

2) Integrating the equality: $(1/2)\Delta(\|\sigma\|^2)=\langle\Delta\sigma,\sigma\rangle+\|\nabla^*\sigma\|^2$, we obtain

$$\int_M\|\nabla^*\sigma\|^2v_g=-\int_M\langle\Delta\sigma,\sigma\rangle v_g\,.$$

Thus (4.2) implies (4.3). q.e.d.

Theorem 4.5. *Let $f:(M,g)\to P_n(c)$, $c>0$, be a totally real minimal isometric immersion of a compact connected Riemannian manifold (M,g) with dim $M=$*

$n \geqq 2$. *Suppose that the second fundamental form* σ *of* f *satisfies an inequality*

$$||\sigma||^2 \leqq n(n+1)c/4(2n-1)$$

everywhere on M. *Then either* f *is totally geodesic and it is an isometric covering to the naturally imbedded real projective n-space in* $P_n(c)$, *or* $n=2$, $||\sigma||^2=c/2$ $(=n(n+1)c/4(2n-1))$ *everywhere on* M *and* f *is an isometric covering to the flat surface* $M_0^2 \subset P_2(c)$ *defined in* **3** (*up to the congruence relative to* $Aut(P_n(c))$).

Proof. We have

$$(2-1/n)||\sigma||^2-(n+1)c/4 = (2-1/n)\{||\sigma||^2-n(n+1)c/4(2n-1)\} \leqq 0$$

from the assumption. It follows from (4.3) that

$$\{||\sigma||^2-n(n+1)c/4(2n-1)\}||\sigma||^2 = 0$$

everywhere and that σ is parallel. Assume that f is not totally geodesic. Then $||\sigma||^2=n(n+1)c/4(2n-1)$ everywhere, and hence $n=2$ by Lemma 4.1. Now we see from Theorem 3.1 that a 2-dimensional complete connected totally real minimal submanifold M' of $P_2(c)$ with parallel second fundamental form is congruent to M_0^2 unless it is totally geodesic. On the other hand, the second fundamental form σ_0 of $M_0^2 \subset P_2(c)$ satisfies $||\sigma_0||^2=c/2$ (cf. Remark in **3**). Thus we get the theorem. q.e.d.

REMARK. Our $M_0^2 \subset P_2(c)$ is nothing but the flat isotropic surface in $P_2(c)$ with parallel second fundamental form constructed in Naitoh [5].

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