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THE STRUCTURE OF UNIRULED MANIFOLDS WITH SPLIT TANGENT BUNDLE

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Abstract

In this paper we show that a uniruled manifold with a split tangent bundle admits almost holomorphic fibrations that are related to the splitting. We analyse these fibrations in detail in several special cases. This yields new results about the integrability of the direct factors and the universal covering of the manifold.

Contents

1. Introduction

A compact Kähler manifold X has a split tangent bundle if $T_X = V_1 \oplus V_2$, where V_1 and V_2 are subbundles of T_X. Initiated by Beauville's conjecture 1.6 on the universal covering of these manifolds [2], these manifolds have been studied by several authors during the last years ([12], [9], [4], [17]). One of the main themes of these papers is that uniruled manifolds with split tangent bundle play a distinguished role. For example if X is projective and not uniruled, then both V_1 and V_2 are integrable [17, Theorem 1.3], while for uniruled manifolds it is easy to construct examples where this is not the case.

The goal of this paper is to develop a structure theory for uniruled Kähler manifolds of arbitrary dimension. The main tool in this study will be the *rationally connected quotient map* (cf. Theorem 2.10 for the definition and properties). We will observe in Proposition 3.16 that if Z is a general fibre of the rationally connected quotient map of X, then

$$T_Z = (T_Z \cap V_1|_Z) \oplus (T_Z \cap V_2|_Z).$$

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In particular Z is a rationally connected manifold with a (maybe trivial) splitting of the tangent bundle. This "ungeneric position property" (cf. [16] for the terminology) puts us in a much better situation since we have the following description for rationally connected manifolds with split tangent bundle.

Theorem 1.1 ([17, Theorem 1.4]). *Let X be a rationally connected projective manifold such that $T_X = V_1 \oplus V_2$. If V_1 or V_2 is integrable, then there exists an isomorphism $X \simeq X_1 \times X_2$ such that $V_j = p_{X_j}^* T_{X_j}$ for $j = 1, 2$. In particular both V_1 and V_2 are integrable.*

So far there are no examples of rationally connected manifolds with split tangent bundle where the direct factors are not integrable. In fact I am fairly optimistic that such examples do not exist.

Conjecture 1.2. *Let X be a projective manifold with split tangent bundle $T_X = V_1 \oplus V_2$. If X is rationally connected, V_1 or V_2 is integrable.*

Using Theorem 1.1 we can show the existence of a meromorphic fibration on X that is related to the decomposition of the tangent bundle. More precisely we have the

Theorem 1.3. *Let X be a uniruled compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Let Z be a general fibre of the rationally connected quotient map, and suppose that $T_Z \cap V_1|_Z$ or $T_Z \cap V_2|_Z$ is integrable. Then for $i = 1, 2$ there exists an almost holomorphic fibration (cf.* Definition 2.9) *$\phi_i : X \dashrightarrow Y_i$ such that the general fibre F_i is rationally connected and*

$$T_{F_i} = (T_Z \cap V_i|_Z)|_{F_i} \subset V_i|_{F_i}.$$

If we specify to the case where one of the direct factors has rank 2 we obtain a more precise statement[1].

Theorem 1.4. *Let X be a uniruled compact Kähler manifold such that $T_X = V_1 \oplus V_2$ and* $\operatorname{rk} V_1 = 2$. *Let Z be a general fibre of the rationally connected quotient map, and suppose that $T_Z \cap V_1|_Z$ or $T_Z \cap V_2|_Z$ is integrable. Then there are three possibilities*:

1) $T_Z \cap V_1|_Z = V_1|_Z$. *Then the manifold X admits the structure of an analytic fibre bundle $X \to Y$ such that the general fibre is rationally connected and $T_{X/Y} = V_1$.*
2) $T_Z \cap V_1|_Z$ *is a line bundle. Then there exists an equidimensional map $\phi : X \to Y$ such that the general ϕ-fibre F is a rational curve and $T_F \subset V_1|_F$.*
3) $T_Z \subset V_2|_Z$.

[1] The situation where one of the direct factors has rank 1 is fully understood, cf. [4].

This result is an analogue to the classification of compact Kähler surfaces: such a surface S is rationally connected or admits a fibration $S \to C$ with general fibre a rational curve or is not covered by rational curves.

In the projective case, we then give two applications of this structure theory: the first application is to try to "contract the obstruction to being integrable", that is to construct a fibration $X \to Y$ such that Y and the general fibre F have a split tangent bundle with *integrable* direct factors. We attain this goal for a splitting in vector bundles of small rank by showing a special case of Conjecture 1.2 (cf. Lemma 4.21) and combining it with the structure Theorem 1.4.

Theorem 1.5. *Let X be a uniruled projective manifold such that $T_X = \bigoplus_{j=1}^{k} V_j$, where for all $j \in \{1,\ldots,k\}$ we have $\operatorname{rk} V_j \leq 2$. Let Z be a general fibre of the rationally connected quotient map. If $T_Z \cap V_j|_Z \neq 0$, the direct factor V_j is integrable.*

Furthermore the rationally connected quotient map can be realised as a flat fibration $\phi\colon X \to Y$ on a projective manifold Y such that

$$T_Y = \bigoplus_{j=1}^{k} T\phi(V_j).$$

In particular $T\phi(V_j)$ is an integrable subbundle of T_Y for every $j \in \{1,\ldots,k\}$ (cf. Notation 2.13 *for the precise definition of $T\phi(V_j)$).*

As a second application we go back to the origin of our study of manifolds with split tangent bundle which is the

Conjecture 1.6 (A. Beauville). *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$, where V_1 and V_2 are vector bundles. Let $\mu\colon \tilde{X} \to X$ be the universal covering of X. Then $\tilde{X} \simeq X_1 \times X_2$, where $p_{X_j}^* T_{X_j} \simeq \mu^* V_j$. If moreover V_j is integrable, then there exists an automorphism of $\tilde{X}$ such that we have an identity of subbundles of the tangent bundle $\mu^* V_j = p_{X_j}^* T_{X_j}$.*

This will be done in Section 5 where we obtain the

Theorem 1.7. *Let X be a uniruled projective manifold such that $T_X = V_1 \oplus V_2$ and* $\operatorname{rk} V_1 = 2$. *Let Z be a general fibre of the rationally connected quotient map, then one of the following holds.*

1) $T_Z \cap V_1|_Z \neq 0$. *If V_1 and V_2 are integrable, Conjecture* 1.6 *holds.*
2) $T_Z \cap V_1|_Z = 0$. *Then* $\det V_1^*$ *is pseudoeffective and V_2 is integrable.*

This result generalises and considerably simplifies the proof of [17, Theorem 1.5].

2. Notation and basic results

We work over the complex field $\mathbb{C}$. For standard definitions in complex algebraic geometry we refer to [15] or [19], for positivity notions of vector bundles we follow the definitions from [25]. Manifolds and varieties are always supposed to be irreducible.

A fibration is a proper surjective morphism $\phi\colon X \to Y$ with connected fibres from a complex manifold onto a normal complex variety Y. The ϕ-smooth locus is the largest Zariski open subset $Y^* \subset Y$ such that for every $y \in Y^*$, the fibre $\phi^{-1}(y)$ is a smooth variety of dimension $\dim X - \dim Y$. The ϕ-singular locus is its complement. A fibre is always a fibre in the scheme-theoretic sense, a set-theoretic fibre is the reduction of the fibre.

Let us recall some basic definitions from the theory of rational curves.

DEFINITION 2.8. Let X be a compact Kähler variety. A rational curve is a non-constant morphism $f : \mathbb{P}^1 \to X$.

The manifold is uniruled if through a general point of X there exists a rational curve. It is rationally connected if for two general points there exists a rational curve through these two points.

REMARK. By a theorem of Campana [6], a rationally connected compact Kähler manifold is projective, in particular Theorem 1.1 applies in the Kähler situation.

DEFINITION 2.9. A meromorphic map $\phi\colon X \dashrightarrow Y$ from a compact Kähler manifold to a normal Kähler variety is almost holomorphic if there exist non-empty open subsets $X^* \subset X$ and $Y^* \subset Y$ such that $\phi|_{X^*} : X^* \to Y^*$ is a fibration. In particular for $y \in Y$ a general point, the fibre $\phi^{-1}(y)$ exists in the usual sense and is compact.

The importance of almost holomorphic maps is due to the fact that every compact Kähler manifold admits such a fibration that separates the rationally connected part and the non-uniruled part: the *rationally connected quotient map* [21] or *MRC-fibration* [23] or *rational quotient map* [10]:

Theorem 2.10 ([22, Theorem 5.4], [8, Theorem 1.1], [13]).[2] *Let X be a compact Kähler manifold. Then there exists an almost holomorphic fibration $\phi : X \dashrightarrow Y$ onto a normal compact Kähler variety Y such that the general fibre is rationally connected and the variety Y is not uniruled. This map is unique up to meromorphic equivalence of fibrations* (*cf.* [7, 1.1.2] *for the definition*), *and is called rationally connected quotient map.*

[2] The statement in [22] is in the algebraic setting, but the same proof goes through in the compact Kähler category: the main technical tool [8, Theorem 1.1] holds in this larger generality.

The rationally connected quotient map has the following universal property: let $\psi\colon X \dashrightarrow Z$ be an almost holomorphic fibration such that the general fibre is rationally connected. Then ϕ factors through ψ, i.e. there exists an almost holomorphic fibration $\tau : Z \dashrightarrow Y$ such that $\phi = \tau \circ \psi$.

2.1. Foliation theory. We recall some basic statements about holomorphic foliations, for more details we refer to [5, 16]. Let X be a compact Kähler manifold. A subbundle $V \subset T_X$ is integrable if it is closed under the Lie bracket. We recall that the Lie bracket

$$[\,.\,,\ .\,]\colon V \times V \to T_X$$

is a bilinear antisymmetric mapping that is not $\mathcal{O}_X$-linear but induces an $\mathcal{O}_X$-linear map $\bigwedge^2 V \to T_X/V$ that is zero if and only if V is integrable. In particular

$$H^0\left(X, \mathcal{H}om\left(\bigwedge^2 V, T_X/V\right)\right) = 0$$

implies that V is integrable. In general we will show this vanishing property using a dominating family of subvarieties $(Z_s)_{s\in S}$ of X (i.e. through a general point of X passes at least one member of the family) such that a general member of the family satisfies

$$H^0\left(Z_s, \mathcal{H}om\left(\bigwedge^2 V, T_X/V\right)\bigg|_{Z_s}\right) = H^0\left(Z_s, \left(\left(\bigwedge^2 V\right)^* \otimes (T_X/V)\right)\bigg|_{Z_s}\right) = 0.$$

Since an antiample vector bundle does not have any global sections, we will use this frequently in the following form.

Lemma 2.11. *Let X be a compact Kähler manifold, and let $V \subset T_X$ be a subbundle. Let $(Z_s)_{s\in S}$ be a dominating family of X such that for a general member Z_s of the family, the vector bundle $\bigwedge^2 V|_{Z_s}$ is ample and $(T_X/V)|_{Z_s}$ is trivial. Then V is integrable.*

By the Frobenius theorem an integrable subbundle V of T_X induces a foliation on X, i.e. for every $x \in X$ there exists an analytic neighbourhood U and a submersion $U \to W$ such that $T_{U/W} = V|_U$. These submersions are called the distinguished maps of the foliation and the fibres are the so-called plaques. The foliation induces an equivalence relation on X, two points being equivalent if and only if they can be connected by chains of smooth (open) curves C_i such that $T_{C_i} \subset V|_{C_i}$. An equivalence class is called a leaf of the foliation. A subset of X is V-saturated if it is a union of leaves.

The next proposition, which is a corollary of the global stability theorem for foliations on Kähler manifold (cf. [27] for a short proof) gives a first idea why rationally connected manifolds are so useful in this context.

Proposition 2.12. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Suppose that $V_1 \subset T_X$ is integrable and that one leaf is compact and rationally connected. Then X has the structure of an analytic fibre bundle $X \to Y$ over a compact Kähler manifold Y such that $T_{X/Y} = V_1$ and the fibres are rationally connected.*

Proof. By [17, Corollary 2.11] there exists a submersion $X \to Y$ onto a compact Kähler manifold Y such that $T_{X/Y} = V_1$ and the fibres are rationally connected. The arguments of [17, Lemma 3.19] (which do not use the projectiveness hypothesis made there) then establish that the submersion is locally trivial. □

2.2. Pushing forward subsheaves of the tangent bundle. For the applications in the projective case it will be crucial to track the behaviour of the splitting under certain morphisms. Let us first give a precise definition of the push forward of a subsheaf of the tangent bundle, this definition just formalizes the idea of looking at the natural tangent map.

NOTATION 2.13. Let $\phi\colon X \to Y$ be a fibration between quasiprojective manifolds. The canonical map $\phi^*\Omega_Y \to \Omega_X$ induces a a generically surjective sheaf homomorphism $T\phi\colon T_X \to \phi^* T_Y$. In particular for $\mathcal{S} \subset T_X$ a quasicoherent subsheaf, we have an inclusion $T\phi(\mathcal{S}) \subset \phi^* T_Y$. Since ϕ is proper, by Grauert's theorem the push-forward $\phi_*(T\phi(\mathcal{S})) \subset T_Y$ is a quasicoherent subsheaf.

For the convenience of the reader, we will denote by

$$T\phi(\mathcal{S}) \subset T_Y$$

the saturation of $\phi_*(T\phi(\mathcal{S}))$ in T_Y [26, III, 1.6]. With this notation $T\phi(\mathcal{S})$ is a reflexive subsheaf of T_Y.

Let X be a projective manifold such that $T_X = V_1 \oplus V_2$. Suppose that X is the blow-up $\mu\colon X \to X'$ of a projective manifold X' along a smooth submanifold Z. Since in the complement of the exceptional locus we have an isomorphism $\mu^*\Omega_{X'} \simeq \Omega_X$, we can consider the reflexive sheaves $T\mu(V_i)$ as subsheaves of $T_{X'}$ and it is clear that they induce a splitting in the complement of Z. Since Z has codimension at least 2, the splitting extends to X', that is

$$T_{X'} = T\mu(V_1) \oplus T\mu(V_2).$$

Furthermore we have an easy lemma relating the universal coverings of X and X'.

Lemma 2.14. *Let X be a projective manifold such that $T_X = V_1 \oplus V_2$. Suppose that X is the blow-up $\mu\colon X \to X'$ of a projective manifold X' along a smooth submanifold. Then we have a splitting $T_{X'} = T\mu(V_1) \oplus T\mu(V_2)$. If $T\mu(V_1)$ and $T\mu(V_2)$ are integrable and* Conjecture 1.6 *holds for X', then the conjecture holds for X.*

Proof. The proof is exactly the same as in [17, Proposition 4.24] and we refrain from repeating the lengthy argument. □

Lemma 2.15. *Let X be a projective manifold such that $T_X = V_1 \oplus V_2$. Let $\phi\colon X \to Y$ be a fibration onto a projective manifold Y that makes X into a $\mathbb{P}^1$- or conic bundle. Then for $j = 1, 2$, the reflexive sheaf $T\phi(V_j) \subset T_Y$ is a subbundle of T_Y and*

$$T_Y = T\phi(V_1) \oplus T\phi(V_2).$$

Proof. If ϕ is a $\mathbb{P}^1$-bundle the morphism is smooth, so [17, Lemma 4.22] applies. If ϕ is a conic bundle it is well-known that the set $D \subset Y$ such that for $y \in D$, the fibre $\phi^{-1}(y)$ is not reduced, has codimension at least 2 [28, Proposition 1.8.5]. Therefore [17, Lemma 4.22] applies again. □

3. The rationally connected quotient map

In this section we show Proposition 3.16 which is the crucial observation of this paper. The moral idea behind the statement is that the rationally connected quotient map reflects the existence of an 'positive' subsheaf $\mathcal{S}$ of the tangent bundle T_X. Proposition 3.16 can then be seen as a translation of the basic fact that a direct sum of sheaves is 'positive' (e.g. ample) if and only if both direct factors are 'positive' (e.g. ample). Once we have established this technical statement, we can use Theorem 1.1 to show Theorem 1.3 and with some extra effort Theorem 1.4.

Proposition 3.16. *Let X be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$. Let $X \dashrightarrow Y$ be an almost holomorphic fibration such that the general fibre is rationally connected. Then the general fibre Z satisfies*

$$T_Z = (T_Z \cap V_1|_Z) \oplus (T_Z \cap V_2|_Z).$$

Proof. The general fibre has a trivial normal bundle $N_{Z/X}$ and is rationally connected, so

$$\mathrm{Hom}(T_Z, N_{Z/X}) \simeq H^0\big(Z, \Omega_Z \otimes \mathcal{O}_Z^{\oplus \dim X - \dim Z}\big) = 0.$$

Fix now an arbitrary $x \in Z$, and let t be an element of the vector space $T_{Z,x}$. The decomposition $T_{X,x} = V_{1,x} \oplus V_{2,x}$ induces a decomposition $t = v_1 + v_2$ with $v_j \in V_{j,x}$. Furthermore for $j = 1, 2$ we have a decomposition $v_j = t_j + n_j$ with $t_j \in T_{Z,x}$ and $n_j \in N_{Z/X,x}$. Since $\mathrm{Hom}(T_Z, N_{Z/X}) = 0$, the composition of the maps $t \mapsto v_j \mapsto n_j$ is zero. Therefore $n_j = 0$ for $j = 1, 2$, so we have $t = t_1 + t_2$. Moreover by construction $t_j \in (T_{Z,x} \cap V_{j,x})$, this shows the claim. □

REMARK. The reader will have noticed that the proof does not really use the rational connectedness of Z, but merely the cohomological condition $h^0(Z, \mathcal{H}om(T_Z, N_{Z/X})) =$

$h^0(Z, \Omega_Z) = 0$. In fact the proposition is part of a more "ungeneric position" theory describing fibre spaces with split tangent bundle that is developed in [16]. A similar cohomological condition was used in [2, 4.4.] to show a more special result.

Proof of Theorem 1.3. Let Z be a general fibre of the rationally connected quotient map of X. By Proposition 3.16 we have

$$T_Z = (T_Z \cap V_1|_Z) \oplus (T_Z \cap V_2|_Z).$$

By hypothesis one of the intersections $T_Z \cap V_1|_Z$ or $T_Z \cap V_2|_Z$ is integrable. Therefore by Theorem 1.1 the general fibre Z is isomorphic to a product $Z_1 \times Z_2$ such that $T_Z \cap V_j|_Z = p_{Z_j}^* T_{Z_j}$ for $j = 1, 2$.

If $T_Z \cap V_1 = 0$ the identity map $X \to X$ satisfies the statement, so we suppose without loss of generality that $T_Z \cap V_1$ is not zero. Since this holds for a general fibre, the submanifolds $Z_1 \times z$ for $z \in Z_2$ form a dominant family of submanifolds of X. Let $Y_1^* \subset \mathcal{C}(X)$ be the open subset parametrizing the family in the cycle space $\mathcal{C}(X)$ [14, Chapter VIII], let $\Gamma_1 \subset Y_1^* \times X$ be the graph of the family, and let $q_1 \colon \Gamma_1 \to Y_1^*$ and $p_1 \colon \Gamma_1 \to X$ be the natural projections. By construction p_1 is dominant and an isomorphism on its image $p_1(\Gamma_1)$. Since q_1 is a fibration, we have a fibration $\phi_1^* := q_1 \circ p_1^{-1} \colon p_1(\Gamma_1) \to Y_1^*$. Let Y_1 be the normalisation of the closure of Y_1^* in $\mathcal{C}(X)$, then we obtain the stated almost holomorphic fibration $\phi_1 \colon X \dashrightarrow Y_1$. The general fibre F_1 of this map is just a member of the family $Z_1 \times z$, so clearly $T_{F_1} \subset V_1|_{F_1}$ and F_1 is rationally connected. The statement for $T_Z \cap V_2$ follows analogously. □

REMARK. It is clear from examples that in general the constructed fibration is not a holomorphic map, so we might think about resolving the indeterminacies by blowing-up $X' \to X$. It is not obvious and would be interesting to see if this can be done in a way such that X' has a split tangent bundle.

Proposition 3.17. *Let X be a uniruled compact Kähler manifold with split tangent bundle $T_X = V_1 \oplus V_2$. Let Z be a general fibre of the rationally connected quotient map, and suppose that $T_Z \cap V_1|_Z$ or $T_Z \cap V_2|_Z$ is integrable. If $T_Z \cap V_1|_Z = V_1|_Z$, the manifold X has the structure of an analytic fibre bundle $X \to Y$ such that $T_{X/Y} = V_1$.*

Proof. By Theorem 1.3 the condition $T_Z \cap V_1|_Z = V_1|_Z$ implies that there exists an almost holomorphic map $\phi_1 \colon X \dashrightarrow Y_1$ such that the general fibre F_1 is rationally connected and satisfies

$$T_{F_i} = (T_Z \cap V_1|_Z)|_{F_i} = V_1|_{F_1}.$$

It follows that V_1 is integrable and has a rationally connected leaf. We conclude with Proposition 2.12. □

In view of Proposition 3.17 it is clear that Theorem 1.4 follows as soon as we have understood the geometry when $T_Z \cap V_1|_Z$ is a line bundle. Since we will consider this situation also in the next section, we state this case as a

Proposition 3.18. *Let X be a uniruled compact Kähler manifold with split tangent bundle $T_X = V_1 \oplus V_2$ where* $\operatorname{rk} V_1 = 2$. *Suppose that the general fibre Z of the rationally connected quotient map satisfies* $\operatorname{rk}(T_Z \cap V_1|_Z) = 1$. *Then there exists an equidimensional map $\phi\colon X \to Y$ on a compact Kähler variety such that the general fibre F is a rational curve that satisfies $T_F \subset V_1|_F$.*

Proof. The line bundle $T_Z \cap V_1|_Z$ is integrable, so by Theorem 1.3 there exists an almost holomorphic map $\phi_1\colon X \dashrightarrow Y_1$ such that the general fibre F_1 is rationally connected and satisfies

$$T_{F_1} = (T_Z \cap V_1|_Z)|_{F_1} \subset V_1|_{F_1}.$$

Since $\operatorname{rk}(T_Z \cap V_1|_Z) = 1$, the general fibre is a smooth rational curve such that

$$T_X|_{F_1} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\dim X - 1}.$$

Since $h^0(F_1, N_{F_1/X}) = \dim X - 1$ and $h^1(F_1, N_{F_1/X}) = 0$, the corresponding open subvariety of the cycle space $\mathcal{C}(X)$ is smooth of dimension $\dim X - 1$. We denote by Y its closure in $\mathcal{C}(X)$ and endow it with the reduced structure. Denote by $\Gamma \subset Y \times X$ the reduction of the graph over Y. Denote furthermore by $p_X\colon \Gamma \to X$ and $p_Y\colon \Gamma \to Y$ the restrictions of the projections to the graph.

STEP 1. *We show that p_X is finite.* We argue by contradiction, then by the analytic version of Zariski's main theorem there are fibres of positive dimension. Let $x \in X$ be a point such that $p_X^{-1}(x)$ has a component of positive dimension. Let $\Delta \subset p_Y(p_X^{-1}(x))$ be an irreducible component of dimension $k > 0$. Then $\Gamma_\Delta := p_Y^{-1}(\Delta)$ has dimension $k+1$. Consider now the foliation induced by $p_X^* V_1$ on $\Gamma \subset Y \times X$. Since a general p_Y-fibre is contained in a $p_X^* V_1$-leaf and this is a closed condition, every fibre $p_Y^{-1}(y)$ is contained in a $p_X^* V_1$-leaf. So for $y \in \Delta$, the set $p_X(p_Y^{-1}(y))$ is contained in $\mathfrak{V}_1^x$, the V_1-leaf through x. It follows that $S := p_X(p_Y^{-1}(\Delta))$ is contained set-theoretically in $\mathfrak{V}_1^x$. Since p_X is injective on the fibres of p_Y, and $p_Y^{-1}(\Delta)$ has dimension $k+1 \geq 2$, the subvariety S has dimension at least 2. Since $\operatorname{rk} V_1 = 2$, it has dimension 2 and $S = \mathfrak{V}_1^x$ (at least set-theoretically). So $\mathfrak{V}_1^x$ is a compact leaf and is covered by a family of rational cycles that intersect in the point x. Hence $\mathfrak{V}_1^x$ is rationally connected, so by Proposition 2.12 there exists a submersion $\psi\colon X \to Z$ such that $T_{X/Z} = V_1$ and the fibres are rationally connected. By the universal property of the rationally connected quotient the general ψ-fibre is contracted by the rationally connected quotient map. This implies $\operatorname{rk}(T_Z \cap V_1|_Z) = \operatorname{rk} V_1$, a contradiction.

STEP 2. *Construction of ϕ.* Since p_X is birational by construction and finite, it is bijective by the analytic version of Zariski's main theorem. Since X is smooth and Γ reduced this shows that p_X is an isomorphism. Since p_Y is equidimensional, $\phi := p_Y \circ p_X^{-1} \colon X \to Y$ is equidimensional. $\square$

Proof of Theorem 1.4. By Proposition 3.16, the general fibre Z of the rationally connected quotient map satisfies

$$T_Z = (V_1|_Z \cap T_Z) \oplus (V_2|_Z \cap T_Z).$$

Since $\operatorname{rk} V_1 = 2$, there are three cases.

If $V_1|_Z \cap T_Z = V_1|_Z$, we conclude with Proposition 3.17.

If $0 \subsetneq V_1|_Z \cap T_Z \subsetneq T_Z$, the intersection has rank 1. Proposition 3.18 shows that we are in the second case of the statement.

If $V_1|_Z \cap T_Z = 0$, clearly $T_Z = T_Z \cap V_2|_Z \subset V_2|_Z$. $\square$

4. The projective case

The main setback of Theorem 1.4 is that in the second case it is not clear if the base of the constructed fibration is smooth, yet the smoothness is crucial to show that the splitting of the tangent bundle pushes down to the base. In order to improve our analysis of this fibration we have to use the theory of Mori contractions, this forces us leave the Kähler world. In Lemma 4.21 we will then show the integrability of at least one direct factor for a splitting in vector bundles of rank 2. For uniruled varieties, the statement does not generalise to a splitting in vector bundles of higher rank. Nevertheless the lemma provides some first evidence for Conjecture 1.2 which it establishes for manifolds of dimension four.

A Mori contraction of a projective manifold X is a morphism with connected fibres $\phi \colon X \to Y$ to a normal variety Y such that the anticanonical bundle $-K_X$ is ϕ-ample. We say that the contraction is elementary if the relative Picard number $\rho(X/Y)$ is equal to one. The contraction is said to be of fibre type if $\dim Y < \dim X$; otherwise it is birational.

Lemma 4.19. *Let X be a projective manifold, and let $\phi \colon X \to Y$ be an equidimensional fibration of relative dimension 1 on a normal variety Y such that the general fibre F is a rational curve. Then there exists a factorisation $\phi = \tilde{\phi} \circ \mu$, where $\mu \colon X \to \tilde{X}$ is a birational morphism onto a projective manifold $\tilde{X}$ and $\tilde{\phi} \colon \tilde{X} \to Y$ makes $\tilde{X}$ into a $\mathbb{P}^1$- or conic bundle. Furthermore μ is a composition of blow-ups of projective manifolds along submanifolds of codimension 2, and Y is smooth.*

Proof. We argue by induction on the relative Picard number $\rho(X/Y)$. If $\rho(X/Y) = 1$, the anticanonical divisor $-K_X$ is ϕ-ample and the contraction is elementary, so by Ando's theorem ϕ induces a $\mathbb{P}^1$-bundle or a conic bundle structure. In both cases Y is smooth.

Suppose now that $\rho(X/Y) > 1$. Since the general fibre is a rational curve, the canonical divisor is not ϕ-nef. It follows from the relative contraction theorem [20, Theorem 4-1-1] that there exists an elementary contraction $\mu\colon X \to \tilde{X}$ that is a Y-morphism, i.e. there exists a morphism $\tilde{\phi}\colon \tilde{X} \to Y$ such that $\phi = \tilde{\phi} \circ \mu$. Since ϕ is equidimensional of relative dimension 1, it follows that all the μ-fibres have dimension at most 1. Thus μ is of fibre type of relative dimension 1 or of birational type.

We claim that μ is not of fibre type. We argue by contradiction and suppose that $\dim X = \dim \tilde{X} + 1$. Then $\dim \tilde{X} = \dim Y$, so $\tilde{\phi}$ is a birational morphism. Since $\rho(\tilde{X}/Y) = \rho(X/Y) - 1 > 0$, the map $\tilde{\phi}$ is not an isomorphism, so by Zariski's main theorem there exists a fibre $\tilde{\phi}^{-1}(y)$ of positive dimension. Since μ is of fibre type, we see that $\phi^{-1}(y) = \mu^{-1}(\tilde{\phi}^{-1}(y))$ has dimension at least 2, a contradiction.

Hence μ is a birational contraction such that all the fibres have dimension at most 1. Recall now the Ionescu-Wiśniewski inequality [18, Theorem 0.4], [29, Theorem 1.1]

$$\dim E + \dim G \geq \dim X,$$

where E is the exceptional locus of the birational contraction μ and G is any μ-fibre. It follows that the contraction is divisorial, i.e. $\dim E = \dim X - 1$ and *all* the fibres have dimension at most 1. By Ando's theorem [1, Theorem 2.1] we know that $\tilde{X}$ is smooth and μ is the blow-up of $\tilde{X}$ along a smooth submanifold of codimension 2. Now $\rho(\tilde{X}/Y) = \rho(X/Y) - 1$ and $\tilde{\phi}$ is equidimensional of relative dimension 1 over Y, so the statement follows by the induction hypothesis. $\square$

REMARK. In order to generalise the proof to the compact Kähler case it would be necessary to establish a relative contraction theorem for projective morphisms between compact Kähler varieties. Unfortunately the Mori theory for compact Kähler manifolds is not yet at this stage, in particular there seem to be no statements for the relative situation.

Corollary 4.20. *Let X be a uniruled projective manifold such that $T_X = V_1 \oplus V_2$ and $\operatorname{rk} V_1 = 2$. Let Z be a general fibre of the rationally connected quotient map, and suppose that $T_Z \cap V_1|_Z$ or $T_Z \cap V_2|_Z$ is integrable. Suppose that $T_Z \cap V_1|_Z \neq 0$. Then X admits a flat fibration $\phi\colon X \to Y$ onto a smooth projective manifold Y such that*

$$T_Y = T\phi(V_1) \oplus T\phi(V_2).$$

Proof. If $T_Z \cap V_1|_Z = V_1|_Z$ we conclude with the first case of Theorem 1.4.

If $0 \subsetneq T_Z \cap V_1|_Z \subsetneq T_Z$ we are in the second case of Theorem 1.4, so there exists an equidimensional fibration $X \to Y$ such that the general fibre is a rational curve. Since X is projective, there exists by Lemma 4.19 a factorisation $\phi = \tilde{\phi} \circ \mu$, where $\mu\colon X \to \tilde{X}$ is birational morphism onto a projective manifold $\tilde{X}$ and $\tilde{\phi}\colon \tilde{X} \to Y$ makes $\tilde{X}$ into a

$\mathbb{P}^1$- or conic bundle. Since μ is a composition of smooth blow-ups a repeated application of Lemma 2.14 shows that

$$T_{\tilde{X}} = T\mu(V_1) \oplus T\mu(V_2).$$

We can now apply Lemma 2.15 to $\tilde{\phi}$ to see that for $i = 1, 2$

$$T\tilde{\phi}(T\mu(V_i)) = T\phi(V_i)$$

is a subbundle of T_Y such that $T_Y = T\phi(V_1) \oplus T\phi(V_2)$. □

Lemma 4.21. *Let X be a uniruled projective manifold such that $T_X = \bigoplus_{j=1}^k V_j$, where for all $j = 1, \dots, k$ we have $\operatorname{rk} V_j \leq 2$. Then one of the direct factors is integrable.*

In particular if $\dim X \leq 4$, one of the direct factors is integrable.

Proof. The statement is trivial if one direct factor has rank 1, so we suppose that all the direct factors have rank 2. Let $f : \mathbb{P}^1 \to X$ be a general minimal rational curve on X [10, Chapter 4], then

$$\bigoplus_{j=1}^k f^*V_j = f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus b}.$$

We may suppose up to renumbering that $f^*V_1 \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ where $c = 0$ or 1. It follows that for $i \geq 2$, we have $f^*V_i \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$ or $f^*V_i \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ or $f^*V_i \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, in particular

$$H^1(\mathbb{P}^1, f^*V_i^*) = 0, \quad \forall i \geq 2.$$

By [9, Lemma 0.4], we have $c_1(V_i) \in H^1(X, V_i^*)$, so $c_1(f^*V_i) \in H^1(\mathbb{P}^1, f^*V_i^*)$ is zero for $i \geq 2$. So $f^* \det V_i \simeq \mathcal{O}_{\mathbb{P}^1}$, since f^*V_i is nef this implies $f^*V_i \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ for $i \geq 2$. This shows that $\bigwedge^2 V_1|_{f(\mathbb{P}^1)}$ is ample and $(T_X/V_1)|_{f(\mathbb{P}^1)} = \bigoplus_{j \geq 2} V_j|_{f(\mathbb{P}^1)}$ is trivial. By Lemma 2.11 this implies the integrability of V_1. □

Proof of Theorem 1.5. Let Z be a general fibre of the rationally connected quotient map, then an application of Proposition 3.16 to all the possible decompositions $W_1 := V_i$ and $W_2 := \bigoplus_{j=1, j \neq i}^k V_j$ implies

$$T_Z = \bigoplus_{j=1}^k (T_Z \cap V_j|_Z).$$

Moreover we have $\operatorname{rk}(T_Z \cap V_j|_Z) \leq 2$ for all $j \in \{1, \dots, k\}$, so one of the direct factors is integrable by Lemma 4.21. We can now apply Theorem 1.1 inductively to see that all the direct factors $T_Z \cap V_j|_Z$ are integrable and Z splits in a product.

STEP 1. *Integrability of the direct factors*. Suppose that $T_Z \cap V_i|_Z$ is not zero, then there are two possibilities. Either $T_Z \cap V_i|_Z = V_i|_Z$, so the integrability of V_i is immediate from the integrability of $T_Z \cap V_i|_Z$; or $T_Z \cap V_i|_Z \subsetneq V_i|_Z$, then V_i has rank 2 and the splitting of Z in a product yields a dominant family of rational curves such that a general member C satisfies $T_C \subset V_i|_C$ and the normal bundle $N_{C/X}$ is trivial. Since

$$T_X|_C = V_i|_C \oplus \bigoplus_{j=1, j\neq i}^{k} V_j|_C = T_C \oplus N_{C/X}$$

and $\operatorname{rk} V_i = 2$, this implies that $\left(\bigwedge^2 V_i\right)\big|_C$ is ample and $(T_X/V_i)|_C \simeq \bigoplus_{j=1, j\neq i}^{k} V_j|_C$ is trivial. By Lemma 2.11 this implies the integrability of V_i.

STEP 2. *Structure of the rationally connected quotient map*. We proceed by induction on the dimension of X, the case $\dim X = 1$ is trivial. Up to renumbering we can suppose that the intersection $T_Z \cap V_1|_Z$ is not empty.

If V_1 has rank 1, we have $T_Z \cap V_1|_Z = V_1|_Z$, so V_1 is integrable and the general leaf is rationally connected. Thus by Proposition 2.12 there exists a submersion $\psi\colon X \to Y'$ such that $T_{X/Y'} = V_1$. Hence $T'_Y = \bigoplus_{j=2}^{k} T\psi(V_j)$. If Y' is not uniruled we are done, otherwise apply the induction hypothesis to Y'.

If V_2 has rank 2, we apply Corollary 4.20 to obtain a flat fibration $\psi\colon X \to Y'$ onto a projective manifold Y' such that the general fibre is rationally connected and

$$T_{Y'} = T\psi(V_1) \oplus T\psi\left(\bigoplus_{j=2}^{k} V_j\right) = \bigoplus_{j=1}^{k} T\psi(V_j).$$

If Y' is not uniruled we are done, otherwise apply the induction hypothesis to Y'.

STEP 3. *Integrability of the images*. Let $\phi\colon X \to Y$ be the map constructed in Step 2. Then

$$T_Y = \bigoplus_{j=1}^{k} T\phi(V_j)$$

and Y is not uniruled. Apply Lemma 5.22 below to all the possible decompositions $W_1 := T\phi(V_i)$ and $W_2 := \bigoplus_{j=1, j\neq i}^{k} T\phi(V_j)$ to see that for all $i \in \{1, \dots, k\}$, the subbundle $T\phi(V_i)$ is integrable. □

5. An application to the universal covering

This section is essentially devoted to the proof of Theorem 1.7. The basic strategy is to prove Conjecture 1.6 by a reduction to the case of non-uniruled varieties and induction on the rank of the direct factors. Before we come to the proof we have to show a refinement of [17, Theorem 1.3].

Lemma 5.22. *Let X be a projective manifold with split tangent bundle $T_X = V_1 \oplus V_2$. Suppose that a general fibre Z of the rationally connected quotient map satisfies $T_Z \subset V_2|_Z$. Then V_2 is integrable and* $\det V_1^*$ *is pseudo-effective.*

Proof. STEP 1. Suppose that $\det V_1^*$ is pseudoeffective. Since V_1^* is a direct factor of Ω_X, the vector bundle $\det V_1 \otimes \bigwedge^{\operatorname{rk} V_1} \Omega_X$ has a trivial direct factor. If $\theta \in H^0\big(X, \det V_1 \otimes \bigwedge^{\operatorname{rk} V_1} \Omega_X\big)$ is the associated nowhere-vanishing $\det V_1$-valued form, and ζ a germ of any vector field, a local computation shows that $i_\zeta \theta = 0$ if and only if ζ is in V_2. An integrability criterion by Demailly [11, Theorem] shows that V_2 is integrable.

STEP 2. $\det V_1^*$ *is pseudoeffective*. We argue by contradiction, then by [3] there exists a birational morphism $\phi\colon X' \to X$ and a general intersection curve $C := D_1 \cap \cdots \cap D_{\dim X-1}$ of very ample divisors $D_1, \ldots, D_{\dim X-1}$ where $D_i \in |m_i H|$ for some ample divisor H such that $\phi^* \det V_1^* \cdot C < 0$. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = \phi^* V_1$$

be the Harder-Narasimham filtration with respect to the polarisation H, i.e. the graded pieces E_{i+1}/E_i are semistable with respect to H. Since $m_1, \ldots, m_{\dim X-1}$ can be arbitrarily high, we can suppose that the filtration commutes with restriction to C. Furthermore since C is general and E_1 a reflexive sheaf, the curve C is contained in the locus where E_1 is locally free. Since

$$\mu(E_1|_C) \geq \mu(\phi^* V_1|_C) = \frac{\phi^* \det V_1 \cdot C}{\operatorname{rk} \phi^* V_1} > 0$$

and $E_1|_C$ is semistable, it is ample by [24, p.62]. By [21, Corollary 1.5] this implies that E_1 is vertical with respect to the rationally connected quotient map of X', that is a general fibre Z' of the rationally connected quotient satisfies $E_1|_{Z'} \cap T_{Z'} = E_1|_{Z'}$. In particular the intersection $T_{Z'} \cap \phi^* V_1|_{Z'}$ is not zero. Since X' and X are birational, this implies that $T_Z \cap V_1|_Z$ is not zero, a contradiction. □

Proof of Theorem 1.7. If $T_Z \cap V_1|_Z = 0$, Proposition 3.16 shows that $T_Z \subset V_2|_Z$. Therefore we can conclude with Lemma 5.22.

If $T_Z \cap V_1|_Z = V_1|_Z$ there exists a submersion $X \to Y$ such that $T_{X/Y} = V_1$. Furthermore V_2 is an integrable connection on the submersion, so we conclude with the Ehresmann theorem [17, Theorem 3.17].

If $T_Z \cap V_1|_Z \subsetneq V_1|_Z$ is a line bundle there exists an equidimensional map $\phi\colon X \to Y$ of relative dimension one such that the general ϕ-fibre F is a rational curve and $T_F \subset V_1|_F$. Since X is projective there exists by Lemma 4.19 a factorisation $\phi = \tilde{\phi} \circ \mu$, where $\mu\colon X \to \tilde{X}$ is birational morphism onto a projective manifold $\tilde{X}$ and $\tilde{\phi}\colon \tilde{X} \to Y$ makes $\tilde{X}$ into a $\mathbb{P}^1$- or conic bundle. Furthermore μ is a composition of blow-ups of projective manifolds along submanifolds of codimension 2, so Lemma 2.14 implies

$$T_{\tilde{X}} = T\mu(V_1) \oplus T(\mu V_2).$$

By the same lemma it is sufficient to show the conjecture for $\tilde{X}$, so we can replace without loss of generality X by $\tilde{X}$ and suppose that the fibration ϕ makes X into a $\mathbb{P}^1$- or conic bundle over the projective manifold Y. Set $W_j := T\phi(V_j)$, then

$$T_Y = W_1 \oplus W_2$$

by Lemma 2.15 and W_1 has rank 1. Furthermore by ([17, Proposition 4.23.], see also [16, Corollary 4.3.9]) all the fibres of ϕ are reduced.

The manifold Y can't have the structure of a $\mathbb{P}^1$-bundle $Y \to M$ such that $T_{Y/M} = W_1$: this would yield a morphism $X \to M$ such that $T_{X/M} = V_1$ and the general fibre is rationally connected. This contradicts $T_Z \cap V_1|_Z \subsetneq V_1|_Z$. Therefore by [4, Theorem 1] the subbundle W_2 is integrable, and the universal covering $\mu\colon \tilde{Y} \to Y$ satisfies $\tilde{Y} \simeq Y_1 \times Y_2$ such that $\mu^* W_1 = p_{Y_1}^* T_{Y_1}$ and $\mu^* W_2 = p_{Y_2}^* T_{Y_2}$. Furthermore we have a commutative diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{\mu}"] \arrow[d, "\tilde{\phi}"] \arrow[dd, bend right, "q"'] & X \arrow[d, "\phi"] \\
\tilde{Y} \arrow[r, "\mu"] \arrow[d, "p_{Y_2}"] & Y \\
Y_2 &
\end{tikzcd}$$

where $\tilde{\mu}\colon \tilde{X} := X \times_Y \tilde{Y} \to X$ is étale. By construction the set-theoretical fibres of q are $\tilde{\mu}^* V_1$-leaves. Since ϕ has no multiple fibres, the fibration $\tilde{\phi}$ has no multiple fibres. Hence $q = p_{Y_2} \circ \tilde{\phi}$ does not have any multiple fibres, so the scheme-theoretical fibres are $\tilde{\mu}^* V_1$-leaves. This shows that q is a submersion with integrable connection $\tilde{\mu}^* V_2$. Since

$$T\tilde{\phi}(\tilde{\mu}^* V_2) = \mu^* W_2 = p_{Y_2}^* T_{Y_2},$$

there exists for every $\tilde{\mu}^* V_2$-leaf $\mathfrak{V}_2$ a $y_1 \in Y_1$ such that $\tilde{\phi}(\mathfrak{V}_2) = y_1 \times Y_2$. By Lemma 5.23 below the restriction of q to a $\tilde{\mu}^* V_2$ leaf is an étale covering, so we conclude with the Ehresmann theorem [17, Theorem 3.17]. $\square$

REMARK. Note that Theorem 1.7 generalises immediately to the compact Kähler case if we show that the map in the second case of Theorem 1.4 is flat.

Lemma 5.23. *Let $\phi\colon X \to Y_1 \times Y_2$ be a proper surjective map from a complex manifold X onto a product of (not necessarily compact) complex manifolds such that* *the morphism $q := p_{Y_2} \circ \phi \colon X \to Y_2$ is a submersion that admits an integrable connection $V \subset T_X$. Suppose that for every V-leaf $\mathfrak{V}$, there exists a $y_1 \in Y_1$ such that $\phi(\mathfrak{V}) = y_1 \times Y_2$. Then the restriction of q to every V-leaf is an étale covering.*

The proof consists merely of rephrasing the classical proof of the Ehresmann theorem as in [5, V., §2, Proposition 1]. For the convenience of the reader we nevertheless include this technical exercise.

Proof. In this proof all fibres and intersections are set-theoretical.

Let $\mathfrak{V}$ be a V-leaf, and let $y_1 \in Y_1$ such that $\phi(\mathfrak{V}) = y_1 \times Y_2$. Since $p_{Y_2}|_{y_1 \times Y_2} \colon y_1 \times Y_2 \to Y_2$ is an isomorphism, it is sufficient to show that $\phi|_{\mathfrak{V}} \colon \mathfrak{V} \to y_1 \times Y_2$ is an étale map. Furthermore it is sufficient to show that for $y_1 \times y_2 \in y_1 \times Y_2$, there exists a disc $D \subset y_1 \times Y_2$ such that for $y \in D$, the fibre $\phi^{-1}(y)$ cuts each leaf of the restricted foliation $V|_{\phi^{-1}(D)}$ exactly in one point. Granting this for the moment, we show how this implies the result. The connected components of $\mathfrak{V} \cap \phi^{-1}(D)$ are leaves of $V|_{\phi^{-1}(D)}$ Let $\mathfrak{V}'$ be such a connected component. Since for $y \in D$, the intersection $\mathfrak{V}' \cap \phi^{-1}(y)$ is exactly one point, the restricted morphism $\phi|_{\mathfrak{V}'} \colon \mathfrak{V}' \to D$ is one-to-one and onto, so it is a biholomorphism. This shows that $\phi|_{\mathfrak{V} \cap \phi^{-1}(D)} \colon \mathfrak{V} \cap \phi^{-1}(D) \to D$ is a trivialisation of $\phi|_{\mathfrak{V}}$.

Let us now show the claim. Set $k := \operatorname{rk} V$ and $n := \dim X$, and set $Z := \phi^{-1}(y_1 \times Y_2)$. Since every V-leaf is sent on some $b \times Y_2$, the complex space Z is V-saturated. In particular if $\mathfrak{V} \subset Z$ is leaf, the restriction of a distinguished map $f_i \colon W_i \to \mathbb{D}^{n-k}$ to Z which we denote by $f_i|_{W_i \cap Z} \colon W_i \cap Z \to \mathbb{D}^{n-k}$, is a distinguished map for the foliation $V|_Z$ and a plaque of f_i is contained in $\mathfrak{V}$ if and only if it is a plaque of $f_i|_{W_i \cap Z}$.

STEP 1. *The local situation.* Let $x \in \phi^{-1}(y_1 \times y_2)$ be a point. Since q is a submersion with integrable connection V there exists coordinate neighbourhood $x \in W'_x \subset X$ with local coordinates $z_1, \ldots, z_k, z_{k+1}, \ldots, z_n$ and a coordinate neighbourhood $y_2 \in U_x \subset Y_2$ with coordinate $w_1, \ldots, w_k$ such that $q(W'_x) = U_x$ and $q|_{W'_x} \colon W_x \to U_x$ is given in these coordinates by

$$(z_1, \ldots, z_n) \to (z_1, \ldots, z_k).$$

Furthermore there exists a distinguished map $f_x \colon W'_x \to \mathbb{D}^{n-k}$ given in these coordinates by

$$(z_1, \ldots, z_n) \to (z_{k+1}, \ldots, z_n).$$

Since $x \in \phi^{-1}(y_1 \times y_2)$ and ϕ is equidimensional over a smooth base, so open, $\phi(W'_x)$ is a neighbourhood of $y_1 \times y_2$ in $Y_1 \times Y_2$. Since $p_{Y_2}|_{y_1 \times U_x} \colon y_1 \times U_x \to U_x$ is an isomorphism we can suppose that up to restricting U_x and W'_x a bit that

$$\phi(W'_x) \cap (y_1 \times Y_2) = y_1 \times U_x.$$

Set $W_x := W'_x \cap Z$, then $\phi|_Z(W_x) = y_1 \times U_x$. It then follows from this local description that $\phi|_{W_x} \colon W_x \to y_1 \times U_x$ has the property that for $y \in y_1 \times U_x$ the fibre $\phi^{-1}(y)$

intersects each plaque of the distinguished map $f_x|_{W_x}\colon W_x \to \mathbb{D}^k$ in exactly one point.

STEP 2. *Using the properness.* Since the fibre $\phi^{-1}(y_1 \times y_2)$ is compact, we can take a finite cover of the fibre by $W_i := W_{x_i}$ where $i = 1, \ldots, l$ and W_{x_i} is as in Step 1. For each $i \in \{1, \ldots, l\}$, the image $\phi(W_i)$ is a neighbourhood of $y_1 \times y_2 \in y_1 \times Y_2$. Let $D \subset \bigcap_{i=1}^{l} \phi(W_i)$ be a disc that contains $y_1 \times y_2$. If $\mathfrak{V}'$ is a leaf of $V|_{\phi^{-1}(D)}$, it is contained in some plaque P of W_i for some $i \in \{1, \ldots, l\}$. Since the plaques intersect each fibre at most in one point, $\phi|_{\mathfrak{V}'}\colon \mathfrak{V}' \to D$ is injective. The equality $P \cap \phi^{-1}(D) = \mathfrak{V}'$ then implies that

$$\phi(\mathfrak{V}') = \phi(P \cap \phi^{-1}(D)) = \phi(P) \cap D = D,$$

so $\phi|_{\mathfrak{V}'}\colon \mathfrak{V}' \to D$ is surjective. So $\mathfrak{V}'$ intersects each fibre exactly in one point. $\square$

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