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ON LOCALLY FINITE ITERATIVE HIGHER DERIVATIONS

YOSHIKAZU NAKAI

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Let $A$ be a commutative ring with unity. A higher derivation $\Delta=\{1, \Delta_1, \Delta_2, \ldots\}$ of $A$ is called locally finite if for any $a \in A$ there exists an index $j$ such that $\Delta_n(a)=0$ for all $n > j$. In a previous paper some properties of locally finite iterative higher derivations (abbreviated as lfih-derivations) and some applications of them were presented ([2]). In this paper the author gives another application of lfih-derivations, i.e., a characterization of two-dimensional polynomial ring. His proof supplies an alternative proof of Theorem 1 in [3], where the method is geometric while the present one is algebraic and elementary. As a Corollary a characterization of a line in an affine plane is given in terms of lfih-derivations where a line in an affine plane is meant a curve $C$ which can be taken as a coordinate axis of $A^2$. We call a curve $C:f(x, y)=0$ a quasi-line if the coordinate ring $k[x, y]/(f)$ is isomorphic to one-parameter polynomial ring. It is known that if the ground field is the complex number field $C$, then a quasi-line is always a line (cf. [1]). Combined with the present investigation it turns out that if the plane curve $C:f(x, y)=0$ is a quasi-line over $C$, then the derivation $D_f=(\partial f/\partial y)\frac{\partial}{\partial x}-(\partial f/\partial x)\frac{\partial}{\partial y}$ is locally nilpotent, i.e., the higher derivation $\left(1, D_f, \frac{1}{2!}D_f^2, \ldots\right)$ is a lfih-derivation and vice versa. The direct proof of this fact is expected very much.

Let $A$ be a commutative ring with unity. A higher derivation $\Delta=\{1, \Delta_1, \Delta_2, \ldots\}$ is a set of linear endomorphisms of $A$ into itself satisfying the conditions:

$$\Delta_n(ab) = \sum_{i=0}^{n} \Delta_i(a)\Delta_{n-i}(b)$$

where $\Delta_0$ denotes the identity mapping of $A$. Let $\Phi_\Delta$ be the homomorphism of the ring $A$ into $A[[T]]$ defined by

$$\Phi_\Delta(a) = \sum_{i=0}^{n} D_i(a)T^i.$$ 

We say that $\Delta$ is locally finite if $I_m\Phi_\Delta$ is contained in the polynomial ring $A[T]$, i.e., for any $a \in A$, there exists an integer $j$ such that $\Delta_n(a)=0$ for all $n > j$. $\Delta$ is called an iterative higher derivation if the additional conditions
\[ \Delta_i \Delta_j = \binom{i+j}{i} \Delta_{i+j} \]

are satisfied by \( \Delta \). Let \( a \) be an element of the ring \( A \). We say that \( a \) is a \( \Delta \)-constant if \( \Delta_i(a) = 0 \) for all \( i \geq 1 \). This is equivalent to saying that \( \Phi(\Delta(a)) = a \). Sometimes we use the notation \( \Delta^{-1}(0) \) to denote the ring of \( \Delta \)-constants, and \( \Delta(a) = 0 \) to denote \( a \) being a \( \Delta \)-constant.

**Lemma 1.** Let \( \Delta \) be a locally finite higher derivation of an integral domain \( A \). Then the constant ring \( B = \Delta^{-1}(0) \) is inertly embedded in \( A \).

Proof. Let \( b \) be an element of \( B \) and let \( b = cd \) be a decomposition of \( b \) in \( A \). Then we have \( \phi(b) = \phi(c)\phi(d) \) where \( \phi = \Phi^\Delta \). By assumption \( \phi(b) \) is in \( A \) and \( \phi(c), \phi(d) \) are elements of a polynomial ring \( A[T] \). Hence \( \phi(c), \phi(d) \) are also in \( A \). It means that \( \phi(c) = c \) and \( \phi(d) = d \), i.e., \( c \) and \( d \) are in \( B \).

**Theorem 1.** Let \( k \) be an algebraically closed field of arbitrary characteristic and let \( A \) be an integral domain containing \( k \). Assume that \( A \) satisfies the following conditions:

1. There exists a non-trivial \( \mathfrak{I} \)-th derivation \( \Delta \) over \( k \).
2. The constant ring \( A_0 \) of \( \Delta \) is a principal ideal domain finitely generated over \( k \).
3. Any prime element of \( A_0 \) remains prime in \( A \).

Then \( A \) is a polynomial ring in one variable over \( A_0 \).

Proof. Let \( A_i \) be the set of elements \( \xi \) in \( A \) such that \( \Delta_n(\xi) = 0 \) for \( n > i \). \( A_0 \) is the ring of \( \Delta \)-constants and \( A_i \)'s are \( A_0 \)-modules. It is proved in [2] that there exists an integer \( s \geq 0 \) such that

\[ A_0 = A_1 = \cdots A_{s-1} \subset A_s = \cdots = A_{2s-1} \subset A_{2s} = \cdots \]

where \( \subset \) denotes proper containment. The integer \( mp^s \) is called the \( m \)-th jump index \( (m = 1, 2, \cdots) \). For simplicity we set \( q = p^s \) and \( M_1 = A_{qs} \). It is also proved in [2] that for any element \( \xi \) in \( M_1 \), we have

\[ \phi(\xi) = \xi + \alpha_0 T + \alpha_1 T^p + \cdots + \alpha_s T^{qs} \]

where \( \alpha \)'s are in \( A_0 \) and \( \phi = \phi_\Delta \). Let \( I_1 \) be the set of elements in \( A_0 \) which appear as coefficients of \( T^s \) in \( \phi(\xi) \) for some \( \xi \in M_1 \). It is easily seen that \( I_1 \) is an ideal of \( A_0 \). Similarly let \( I_s \) be the set of elements which appear as coefficients of \( T^{qs} \) in \( \phi(\xi) \) for some \( \xi \in M_s \). Then \( I_s \) is also an ideal of \( A_0 \). Let \( a_s \) be a generator of the \( I_s \) and let \( x \) be an element of \( M_1 \) such that

\[ \phi(x) = x + \cdots + a_1 T^{qs} \]
We shall prove simultaneously the following

\[ (1)_n \quad (a_n) = (a^*_n), \]

\[ (2)_n \quad M_n = A_0 + A_0 x + \cdots + A_0 x^n, \quad (n=1, 2, \ldots) \]

by induction on \( n \). First we shall remark that \((1)_n\) implies \((2)_n\). In fact let \( \xi \) be in \( M_n \). Then \( \Delta_{n\xi}(\xi) \) is in \( I_n = (a_n) \). From \((1)_n\) it follows that there exists a constant \( c \) in \( A_0 \) such that \( \Delta_{n\xi}(\xi) = ca^*_n \). Then \( \phi(\xi - cx^n) \) is of degree \(< nq \), hence \( \xi - cx^n \in M_{n-1} \). Now assume \((1)_n\), \((2)_n\) and we shall prove \((1)_{n+1}\). Since \( a^{n+1}_n \in I_{n+1} = (a_{n+1}) \), there is a constant \( c \) in \( A_0 \) such that \( a^{n+1}_n = ca_{n+1} \). Let \( \xi \) be an element of \( M_{n+1} \) such that

\[ \phi(\xi) = + \cdots + a^{n+1}_{n+1} T^{(n+1)n}. \]

Then \( \phi(c\xi - x^{n+1}) \) is of degree \(< (n+1)q \), hence \( c\xi - x^{n+1} \in M_n \). By \((2)_n\) there are \( b_i \)'s in \( A_0 \) such that

\[ c\xi = x^{n+1} + \sum_{i=0}^{n+1} b_i x^i. \]

We shall show that \( c \) is a unit of \( A_0 \). Assume that \( c \) is a non-unit in \( A_0 \). Let \( f \) be a prime element which divides \( c \). Taking the residue class modulo \( fA \) we get an algebraic relation

\[ x^{n+1} + \sum_{i=1}^{n+1} b_i x^i = 0. \]

By assumption (iii) \( f \) is also a prime element of \( A \). Hence \( A/fA \) is an integral domain. Since \( k \) is algebraically closed and \( A_0 \) is finitely generated over \( k \), we have \( A_0/fA_0 = k \). Hence there exists \( \gamma \) in \( k \) such that \( x = \gamma \). It means that \( x - \gamma = f y \) with some \( y \in A \). Then we have \( \phi(x - \gamma) = f \phi(y) \), i.e., \( \Delta_{\phi}(x) = f \Delta_{\phi}(y) \). Since \( \Delta_{\phi}(y) \in I_1 = (a_1) = (\Delta_{\phi}(x)) \) we get a contradiction. Thus we have proven \((1)_{n+1}\). Since \( A = \bigcup_{n+1} M_n \), we obtain the desired result \( A = A_0[x] \).

**Remark.** If \( A \) is a \( UFD \), then the condition (iii) is automatically satisfied.

**Theorem 2.** Let \( k \) be as in Theorem 1, and let \( A \) be a finitely generated normal integral domain over \( k \) such that

(i) \( \dim A = 2 \)

(ii) \( A^* = k^* \) where \( ^* \) denotes the set of units.

(iii) Either \( A \) is \( UFD \) or \( Q(A) \) is unirational over \( k \).

Let \( \Delta \) be a non-trivial \( i \)-th-derivation of \( A \) over \( k \). Then the constant ring \( A_0 \) of \( \Delta \) is a polynomial ring over \( k \). More precisely let \( f \) be an irreducible element in \( A_0 \). Then \( A_0 = k[f] \).

**Proof.** \( A_0 \) is not reduced to \( k \) because there exists an element \( u \) in \( A_0 \) and
a variable \( t \) over \( A_0 \) such that \( A[u^{-1}] = A_0[u^{-1}][t] \). (cf. Appendix, [2]). Let \( f \) be an element of \( A_0 \setminus k \) which is irreducible in \( A \). The existence of such an element \( f \) is assured by the Lemma 1. We shall show that \( A_0 = k[f] \). Since \( A_0[u^{-1}] = A_0[u^{-1}][tA[u^{-1}], A_0[u^{-1}] \) is a finitely generated integral domain over \( k \). In case \( A \) is a \( UFD, A_0[u^{-1}] \) is also a \( UFD \) owing to the Lemma 1. Moreover the transcendence degree of the quotient field \( K \) of \( A_0 \) is 1. Hence \( K \) is a purely transcendental extension of \( k \). If \( A \) is not a \( UFD \) we assumed that \( Q(A) \) is unirational. Then by the generalized Lüroth's theorem \( K \) is also a one-dimensional purely transcendental extension of \( k \). Let \( B \) be the integral closure of \( k[f] \) in \( K \). Then \( B \) is also finitely generated over \( k \) and \( B^* = k^* \) because \( B \) is contained in \( A \). Hence there exists an element \( t \) in \( B \) such that \( B = k[t] \). Since \( f \) is contained in \( B \) we can write \( f = \lambda(t) \). But \( f \) is irreducible in \( A \), hence degree of \( \lambda \) in \( t \) must be 1. It proves that \( k[t] = k[f] = B \). Now assume \( A_0 \not= B \). Since \( A_0 \) and \( B \) have the same quotient field, \( A_0 \) contains an element of the form \( \gamma(f)/s(f) \) where \( (\gamma(f), s(f)) = 1 \) and \( \deg s(f) \geq 1 \). Then \( A_0 \) must contain a non-constant unit. This is against the assumption (ii).

Combining these theorems we have the following

**Theorem 3.** Let \( k \) be an algebraically closed field of arbitrary characteristic and let \( A \) be a finitely generated integral domain over \( k \). Assume that \( A \) satisfies the following conditions:

(i) \( \dim A = 2 \)

(ii) \( A^* = k^* \)

(iii) \( A \) is \( UFD \).

Assume that \( A \) has a non-trivial \( \partial \)-derivation \( \Delta \) over \( k \). Then \( A \) is a two-dimensional polynomial ring over \( k \). More precisely if the constant ring \( A_0 \) of \( \Delta \) is written as \( k[f] \), then \( A = k[f, g] \) for some other element \( g \) in \( A \).

The assumption (iii) is essential as is shown in the following

**Example 1.** Let \( A = \mathbb{C}[x, y, \frac{y(y-1)}{x}] \). Then as is easily seen \( A^* = \mathbb{C}^* \) and \( A \) has a locally nilpotent derivation \( D \) such that

\[
Dx = 2y - 1, \quad Dy = \frac{y(y-1)}{x}.
\]

By a simple calculation we see \( D^{-1}(0) = k\left[\frac{y(y-1)}{x}\right] \). The element \( \frac{y(y-1)}{x} \) is not a prime element in \( A \). Hence \( A \) is neither \( UFD \) nor a polynomial ring.

(*) This example is due to K. Yoshida.
As an application of Theorem 3 we give a necessary and sufficient condition for a plane curve $C: f(x, y)=0$ to be a line. We recollect here some definitions. A plane curve $C: f(x, y)=0$ defined over a field $k$ is called a quasi-line over $k$ if the coordinate ring $A=k[x, y]/(f)$ is isomorphic to a polynomial ring in one variable. $C$ is called a line if there exists another curve $\Gamma: g(x, y)=0$ such that we have $k[x, y]=k[f, g]$.

**Theorem 4.** Let $k$ be an algebraically closed field and let $C: f(x, y)=0$ be an irreducible curve over $k$. Then the following conditions are equivalent to each other.

(i) $C$ is a line
(ii) There is a $1$-th derivation $\Delta$ such that $\Delta(f)=0$.
(iii) $C_u: f(x, y)\rightarrow u=0$ is a quasi-line over $k(u)$ where $u$ is an indeterminate.

Proof. The implication (i) $\rightarrow$ (ii), (i) $\rightarrow$ (iii) is obvious (ii) $\rightarrow$ (i) follows from Theorem 2 and 3. It remains to show that (iii) implies (i). Assume (iii). Since $k(u)[x, y]/(f-u)$ is isomorphic to $k(f)[x, y]$, there exists an element $t$ in $k[x, y]$ such that $k(f)[x, y]=k(f)[t]$. Let $\Delta'$ be the 1-th derivation of $k(f)[t]$ over $k(f)$ such that

$$\Delta'(t^m) = \left(m \atop n\right)t^{m-n}$$

Then there exists an element $a$ in $k[f]$ such that $a\Delta'=\Delta$ sends $k[x, y]$ into itself, where $a\Delta'$ is higher derivation

$$a\Delta' = (1, a\Delta', a^2\Delta', \ldots, a^n\Delta', \ldots).$$

Clearly $\Delta(f)=0$ and $f$ is a prime element in $k[x, y]$. Hence $\Delta^{-1}(0)=k[f]$ and by Theorem 3, $f$ is a line.

In case where the characteristic of $k$ is zero we can say more. First we prove a Lemma.

**Lemma 2.** Let $C: f(x, y)=0$ be a line in a plane. Then $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)=1$.

Proof. Since $C$ is a line, there exists a curve $\Gamma: g(x, y)=0$ such that $k[x, y]=k[f, g]$. Then there exists $F(X, Y)$ and $G(X, Y)$ in $k[X, Y]$ such that

$$F(f, g) = x$$
$$G(f, g) = y$$

Then we have

(**) In [4] our "line" and "quasi-line" are called "embedded line" and "line" respectively.
\[
\frac{\partial F}{\partial x} \frac{\partial f}{\partial y} + \frac{\partial F}{\partial y} \frac{\partial g}{\partial x} = 1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]
\[
\frac{\partial F}{\partial y} \frac{\partial f}{\partial x} + \frac{\partial F}{\partial x} \frac{\partial g}{\partial y} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)
\]
\[
\frac{\partial G}{\partial x} \frac{\partial f}{\partial y} + \frac{\partial G}{\partial y} \frac{\partial g}{\partial x} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3)
\]
\[
\frac{\partial G}{\partial y} \frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} \frac{\partial g}{\partial y} = 1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4)
\]

Now assume \( \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \subseteq m \) for some maximal ideal \( m \). Then from (2) either \( \frac{\partial f}{\partial y} \) or \( \frac{\partial g}{\partial x} \) is contained in \( m \). The first case cannot occur because of (1) and the second case contradicts (4). Thus \( \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \) is a unit ideal.

**Theorem 5.** Let \( k \) be an algebraically closed field of characteristic zero and let \( C: f(x, y)=0 \) be an irreducible curve over \( k \). Then \( C \) is a line if and only if the derivation

\[
D_f = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}
\]

is locally nilpotent.

Proof. Assume that \( C: f(x, y)=0 \) is a line. Let \( \Gamma: g(x, y)=0 \) be a curve such that \( k[f, g] = k[x, y] \). Then there exists a locally nilpotent derivation \( \Delta \) of \( k[x, y] \) such that \( \Delta f = 0 \) and \( \Delta g = 1 \). Since \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) form a basis of derivations of \( k[x, y] \) we can write

\[
\Delta = a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \quad \text{with} \quad a, b \in k[x, y].
\]

Since \( \Delta f = 0 \) we have

\[
a \frac{\partial f}{\partial x} - b \frac{\partial f}{\partial y} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]

Let \( a \frac{\partial f}{\partial y} = b \frac{\partial f}{\partial y} = \lambda \), i.e., \( a = \lambda \frac{\partial f}{\partial y} \), \( b = \lambda \frac{\partial f}{\partial x} \). Then we have \( \Delta = \lambda D_f \).

We show that \( \lambda \in k[x, y] \). From Lemma 2 it follows that \( \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = 1 \) for some \( \alpha, \beta \in k[x, y] \). Hence \( \lambda = b \alpha + a \beta \in k[x, y] \). On the other hand the existence of \( g \in k[x, y] \) such that \( \Delta g = 1 \) implies \( (a, b) = 1 \). Since \( \lambda \) is a common
divisor of $a$ and $b$ we see that $\lambda \in k^*$. This means that $D_f$ is locally nilpotent. The "if" part of the Theorem is immediate from Theorem 3.

According to S. Abhyankar and T. Moh a quasi-line is a line in case of characteristic zero ([1]). In the case where the characteristic of $k$ is a positive prime integer $p$ there is a counter example.

**EXAMPLE 2(*** A curve $C: f(x, y)=0$ such that

$$f(x, y) \equiv y^p - x$$

is a quasi-line but not a line where $p$ is the characteristic of $k$ and $q$ is an integer $\geq 2$ not divisible by $p$.

Proof. If we set

$$u = y - (y^p - x^q)^q$$

then $x \equiv u^p$ and $y \equiv u + u^p$ modulo $f(x, y)$. Hence $f(x, y)=0$ is a quasi-line. To see that $c$ is not a line it suffices to show that there is no locally finite higher derivation killing $f$. Assume the contrary and let $\Delta$ be a lfih-derivation killing $f$ and $\phi = \Phi \Delta$. Let

$$\phi(x) = x + \sum_i a_i T^i$$

$$\phi(y) = y + \sum_i b_i T^i.$$ 

From $\phi(f)=f$ we get

$$(y^p + \sum_i b_i^p T^{pi}) - (x + \sum_i a_i T^i) - (y^p + \sum_i a_i^p T^{pi})^q = y^p - x - x^p$$

First we easily see that $a_i = 0$ if $i \equiv 0 \mod p^k$. We set $a_i^p = a_i$. Then we have

$$(y^p + \sum_i b_i^p T^{pi}) - (x + \sum_i (p_i + \sum_i a_i T^{pi})^q - (y^p + \sum_i \alpha_i T^{qi})^q = y^p - x - x^p$$

First we remark that

$$\alpha_i \in A^{p^k}$$

for any $i$ where $A = k[x, y]$. Now assume that $n \geq 1$. We compute the coefficient of $T^{p^k(\delta - 1)}$. Since $T^{p^k(\delta - 1)}$ does not appear in the middle term we have the relation:

$$b_{p^k(\delta - 1)} = \sum_{i_1, \ldots, i_q=1}^{p^k} \alpha_{i_1} \cdots \alpha_{i_q} + qx^p \alpha_n^{p^k(\delta - 1)}$$

From (2) $\alpha_{i_1} \cdots \alpha_{i_q}, \alpha_n^{p^k(\delta - 1)}$ are in $A^{p^k}$. Hence $x^p$ must also be in $A^{p^k}$. This is

(***) This example is a generalization of the one given in [4].
impossible. This proves \( n = 0 \), i.e., \( x \) must be a \( \Delta \)-constant. Hence \( y \) is also a \( \Delta \)-constant. Thus there is no non-trivial \( \text{ifih} \)-derivation \( \Delta \) such that \( \Delta(f) = 0 \).

\section*{References}


Added in Proof. In Theorem 3 we assumed that \( k \) is algebraically closed. This assumption is essential as is shown in the following Example. Let \( B = \mathbb{R}[X, Y]/X^2 + Y^2 + 1 \). Then \( B \) is a UFD and satisfies \( B^* = \mathbb{R}^* \). The ring \( A = B[Z] \) satisfies all the requirement in Theorem 3, but \( A \) is not a polynomial ring of two variables over the field \( \mathbb{R} \).