Nakai, Y.
Osaka J. Math.
15 (1978), 655-662

ON LOCALLY FINITE ITERATIVE HIGHER DERIVATIONS

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(Received June 1, 1977)

Let A be a commutative ring with unity. A higher derivation $\underline{\Delta}=\{1, \Delta_1, \Delta_2, \cdots\}$ of A is called locally finite if for any $a\in A$ there exists an index j such that $\Delta_n(a)=0$ for all $n>j$. In a previous paper some properties of locally finite iterative higher derivations (abbreviated as lfih-derivations) and some applications of them were presented ([2]). In this paper the author gives another application of lfih-derivations, i.e., a characterization of two-dimensional polynomial ring. His proof supplies an alternative proof of Theorem 1 in [3], where the method is geometric while the present one is algebraic and elementary. As a Corollary a characterization of a line in an affine plane is given in terms of lfih-derivation where a line in an affine plane is meant a curve C which can be taken as a coordinate axis of A^2. We call a curve $C:f(x, y)=0$ a quasi-line if the coordinate ring $k[x, y]/(f)$ is isomorphic to one-parameter polynomial ring. It is known that if the ground field is the complex number field $\boldsymbol{C}$, then a quasi-line is always a line (cf. [1]). Combined with the present investigation it turns out that if the plane curve $C:f(x, y)=0$ is a quasi-line over $\boldsymbol{C}$, then the derivation $D_f=(\partial f/\partial y)\dfrac{\partial}{\partial x}-(\partial f/\partial x)\dfrac{\partial}{\partial y}$ is locally nilpotent, i.e., the higher derivation $\left(1, D_f, \dfrac{1}{2!}D_f^2, \cdots\right)$ is a lfih-derivation and vice versa. The direct proof of this fact is expected very much.

Let A be a commutative ring with 1. A higher derivation $\underline{\Delta}=(1, \Delta_1, \Delta_2, \cdots)$ is a set of linear endomorphisms of A into itself satisfying the conditions:

$$\Delta_n(ab)=\sum_{i=0}^{n}\Delta_i(a)\Delta_{n-i}(b)$$

where Δ_0 denotes the identity mapping of A. Let $\Phi_{\underline{\Delta}}$ be the homomorphism of the ring A into $A[[T]]$ defined by

$$\Phi_{\underline{\Delta}}(a)=\sum_{i=0}^{\infty}D_i(a)T^i\,.$$

We say that $\underline{\Delta}$ is locally finite if $I_m\Phi_{\underline{\Delta}}$ is contained in the polynomial ring $A[T]$, i.e., for any $a\in A$, there exists an integer j such that $\Delta_n(a)=0$ for all $n>j$. $\underline{\Delta}$ is called an iterative higher derivation if the additional conditions

$$\Delta_i\Delta_j = \binom{i+j}{i}\Delta_{i+j}$$

are satisfied by $\underline{\Delta}$. Let a be an element of the ring A. We say that a is a $\underline{\Delta}$-constant if $\Delta_i(a)=0$ for all $i\geqq 1$. This is equivalent to saying that $\Phi_{\underline{\Delta}}(a)=a$. Sometimes we use the notation $\underline{\Delta}^{-1}(0)$ to denote the ring of $\underline{\Delta}$-constants, and $\underline{\Delta}(a)=0$ to denote a being a $\underline{\Delta}$-constant.

Lemma 1. *Let $\underline{\Delta}$ be a locally finite higher derivation of an integral domain A. Then the constant ring $B=\underline{\Delta}^{-1}(0)$ is inertly embedded in A.*

Proof. Let b be an element of B and let $b=cd$ be a decomposition of b in A. Then we have $\phi(b)=\phi(c)\phi(d)$ where $\phi=\Phi_{\underline{\Delta}}$. By assumption $\phi(b)$ is in A and $\phi(c)$, $\phi(d)$ are elements of a polynomial ring $A[T]$. Hence $\phi(c)$, $\phi(d)$ are also in A. It means that $\phi(c)=c$ and $\phi(d)=d$, i.e., c and d are in B.

Theorem 1. *Let k be an algebraically closed field of arbitrary characteristic and let A be an integral domain containing k. Assume that A satisfies the following conditions:*

i) *There exists a non-trivial lfih-derivation $\underline{\Delta}$ over k.*

ii) *The constant ring A_0 of $\underline{\Delta}$ is a principal ideal domain finitely generated over k.*

iii) *Any prime element of A_0 remains prime in A.*

Then A is a polynomial ring in one variable over A_0.

Proof. Let A_i be the set of elements ξ in A such that $\Delta_n(\xi)=0$ for $n>i$. A_0 is the ring of $\underline{\Delta}$-constants and A_i's are A_0-modules. It is proved in [2] that there exists an integer s ($\geqq 0$) such that

$$A_0 = A_1 = \cdots A_{p^s-1} \subset A_{p^s} = \cdots = A_{2p^s-1} \subset A_{2p^s} = \cdots$$

where $\subset$ denotes proper containment. The integer mp^s is called the m-th jump index ($m=1, 2, \cdots$). For simplicity we set $q=p^s$ and $M_n=A_{nq}$. It is also proved in [2] that for any element ξ in M_1, we have

$$\phi(\xi) = \xi+\alpha_0T+\alpha_1T^p+\cdots+\alpha_sT^q$$

where α's are in A_0 and $\phi=\phi_{\underline{\Delta}}$. Let I_1 be the set of elements in A_0 which appear as coefficients of T^q in $\phi(\xi)$ for some $\xi\in M_1$. It is easily seen that I_1 is an ideal of A_0. Similarly let I_n be the set of elements which appear as coefficients of T^{nq} in $\phi(\xi)$ for some $\xi\in M_n$. Then I_n is also an ideal of A_0. Let a_n be a generator of the I_n and let x be an element of M_1 such that

$$\phi(x) = x+\cdots+a_1T^q .$$

We shall prove simultaneously the following

$$(1)_n \quad (a_n) = (a_1^n)\,,$$
$$(2)_n \quad M_n = A_0+A_0x+\cdots+A_0x^n\,, \qquad (n=1, 2, \cdots)$$

by induction on n. First we shall remark that $(1)_n$ implies $(2)_n$. In fact let ξ be in M_n. Then $\Delta_{nq}(\xi)$ is in $I_n=(a_n)$. From $(1)_n$ it follows that there exists a constant c in A_0 such that $\Delta_{nq}(\xi)=ca_1^n$. Then $\phi(\xi-cx^n)$ is of degree $<nq$, hence $\xi-cx^n\in M_{n-1}$. Now assume $(1)_n$, $(2)_n$ and we shall prove $(1)_{n+1}$. Since $a_1^{n+1}\in I_{n+1}=(a_{n+1})$, there is a constant c in A_0 such that $a_1^{n+1}=ca_{n+1}$. Let ξ be an element of M_{n+1} such that

$$\phi(\xi) = +\cdots+a_{n+1}T^{(n+1)q}\,.$$

Then $\phi(c\xi-x^{n+1})$ is of degree $<(n+1)q$, hence $c\xi-x^{n+1}\in M_n$. By $(2)_n$ there are b_i's in A_0 such that

$$c\xi = x^{n+1}+\sum_{i=0}^{n} b_i x^n\,.$$

We shall show that c is a unit of A_0. Assume that c is a non-unit in A_0. Let f be a prime element which divides c. Taking the residue class modulo fA we get an algebraic relation

$$\bar{x}^{n+1}+\bar{b}_i\bar{x}^n = 0\,.$$

By assumption (iii) f is also a prime element of A. Hence A/fA is an integral domain. Since k is algebraically closed and A_0 is finitely generated over k, we have $A_0/fA_0=k$. Hence there exists γ in k suh that $\bar{x}=\gamma$. It means that $x-\gamma=fy$ with some $y\in A$. Then we have $\phi(x-\gamma)=f\phi(y)$, i.e., $\Delta_q(x)=f\Delta_q(y)$. Since $\Delta_q(y)\in I_1=(a_1)=(\Delta_q(x))$ we get a contradiction. Thus we have proven $(1)_{n+1}$. Since $A=\bigcup_{n=1}^{\infty} M_n$, we obtain the desired result $A=A_0[x]$.

REMARK. If A is a *UFD*, then the condition (iii) is automatically satisfied.

Theorem 2. *Let k be as in Theorem* 1, *and let A be a finitely generated normal integral domain over k such that*

(i) *$\dim A=2$*
(ii) *$A^*=k^*$ where $*$ denotes the set of units.*
(iii) *Either A is UFD or $Q(A)$ is unirational over k.*

Let $\underline{\Delta}$ be a non-trivial lfih-derivation of A over k. Then the constant ring A_0 of $\underline{\Delta}$ is a polynomial ring over k. More precisely let f be an irreducible element in A_0. Then $A_0=k[f]$.

Proof. A_0 is not reduced to k because there exists an element u in A_0 and a variable t over A_0 such that $A[u^{-1}]=A_0[u^{-1}][t]$. (cf. Appendix, [2]). Let f be an element of $A_0\backslash k$ which is irreducible in A. The existence of such an element f is assured by the Lemma 1. We shall show that $A_0=k[f]$. Since $A_0[u^{-1}]\cong A[u^{-1}]/tA[u^{-1}]$, $A_0[u^{-1}]$ is a finitely generated integral domain over k. In case A is a UFD, $A_0[u^{-1}]$ is also a UFD owing to the Lemma 1. Moreover the transcednence degree of the quotient field K of A_0 is 1. Hence K is a purely transcendental extension of k. If A is not a UFD we assumed that $Q(A)$ is unirational. Then by the generalized Lüroth's theorem K is also a one-dimensional purely transcendental extension of k. Let B be the integral closure of $k[f]$ in K. Then B is also finitely generated over k and $B^*=k^*$ because B is contained in A. Hence there exists an element t in B such that $B=k[t]$. Since f is contained in B we can write $f=\lambda(t)$. But f is irreducible in A, hence degree of λ in t must be 1. It proves that $k[t]=k[f]=B$. Now assume $A_0\neq B$. Since A_0 and B have the same quotient field, A_0 contains an element of the form $\gamma(f)/s(f)$ where $(\gamma(f), s(f))=1$ and $\deg s(f)\geqq 1$. Then A_0 must contain a non-constant unit. This is against the assumption (ii).

Combining these theorems we have the following

Theorem 3. *Let k be an algebraically closed field of arbitrary characteristic and let A be a finitely generated integral domain over k. Assume that A satisfies the following conditions:*

(i) *$\dim A=2$*
(ii) *$A^*=k^*$*
(iii) *A is UFD.*

Assume that A has a non-trivial lfih-derivation $\underline{\Delta}$ over k. Then A is a two-dimensional polynomial ring over k. More precisely if the constant ring A_0 of $\underline{\Delta}$ is written as $k[f]$, then $A=k[f, g]$ for some other element g in A.

The assumption (iii) is essential as is shown in the following

EXAMPLE 1.(*) Let $A=\boldsymbol{C}\left[x, y, \dfrac{y(y-1)}{x}\right]$. Then as is easily seen $A^*=\boldsymbol{C}^*$ and A has a locally nilpotent derivation D such that

$$Dx = 2y-1,\ Dy = \frac{y(y-1)}{x}.$$

By a simple calculation we see $D^{-1}(0)=k\left[\dfrac{y(y-1)}{x}\right]$. The element $\dfrac{y(y-1)}{x}$ is not a prime element in A. Hence A is neither UFD nor a polynomial ring.

(*) This example is due to K. Yoshida.

As an application of Theorem 3 we give a necessary and sufficient condition for a plane curve $C: f(x, y)=0$ to be a line. We recollect here some aefinitions. A plane curve $C: f(x, y)=0$ defined over a field k is called a quasi-line over k if the coordinate ring $A=k[x, y]/(f)$ is isomorphic to a polynomial ring in one variable. C is called a line if there exists another curve $\Gamma: g(x, y)=0$ such that we have $k[x, y]=k[f, g]$.(**)

Theorem 4. *Let k be an algebraically closed field and let $C: f(x, y)=0$ be an irreducible curve over k. Then the following conditions are equivalent to each other.*

(i) *C is a line*

(ii) *There is a lfih-derivation $\underline{\Delta}$ such that $\underline{\Delta}(f)=0$.*

(iii) *$C_u: f(x, y)-u=0$ is a quasi-line over $k(u)$ where u is an indeterminate.*

Proof. The implication (i)→(ii), (i)→(iii) is obvious (ii)→(i) follows from Theorem 2 and 3. It remains to show that (iii) implies (i). Assume (iii). Since $k(u)[x, y]/(f-u)$ is isomorphic to $k(f)[x, y]$, there exists an element t in $k[x, y]$ such that $k(f)[x, y]=k(f)[t]$. Let $\underline{\Delta}'$ be the lfih-derivation of $k(f)[t]$ over $k(f)$ such that

$$\Delta_n'(t^m) = \binom{m}{n} t^{m-n}$$

Then there exists an element a in $k[f]$ such that $a\underline{\Delta}'=\underline{\Delta}$ sends $k[x, y]$ into itself, where $a\Delta'$ is higher derivation

$$a\underline{\Delta}' = (1, a\Delta_1', a^2\Delta_2', \cdots, a^n\Delta_n', \cdots).$$

Clearly $\underline{\Delta}(f)=0$ and f is a prime element in $k[x, y]$. Hence $\underline{\Delta}^{-1}(0)=k[f]$ and by Theorem 3, f is a line.

In case where the characteristic of k is zero we can say more. First we prove a Lemma.

Lemma 2. *Let $C: f(x, y)=0$ be a line in a plane. Then* $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)=1$.

Proof. Since C is a line, there exists a curve $\Gamma: g(x, y)=0$ such that $k[x, y]=k[f, g]$. Then there exists $F(X, Y)$ and $G(X, Y)$ in $k[X, Y]$ such that

$$F(f, g) = x$$
$$G(f, g) = y.$$

Then we have

(**) In [4] our "line" and "quasi-line" are called "embedded line" and "line" respectively.

$$\frac{\partial F}{\partial f}\frac{\partial f}{\partial x}+\frac{\partial F}{\partial g}\frac{\partial g}{\partial x} = 1\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

$$\frac{\partial F}{\partial f}\frac{\partial f}{\partial y}+\frac{\partial F}{\partial g}\frac{\partial g}{\partial y} = 0\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

$$\frac{\partial G}{\partial f}\frac{\partial f}{\partial x}+\frac{\partial G}{\partial y}\frac{\partial g}{\partial x} = 0\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

$$\frac{\partial G}{\partial f}\frac{\partial f}{\partial y}+\frac{\partial G}{\partial g}\frac{\partial g}{\partial y} = 1\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

Now assume $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\subseteq m$ for some maximal ideal m. Then from (2) either $\frac{\partial F}{\partial g}$ or $\frac{\partial g}{\partial y}$ is contained in m. The first case cannot occur because of (1) and the second case contradicts (4). Thus $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is a unit ideal.

Theorem 5. *Let k be an algebraically closed field of characteristic zero and let $C\colon f(x, y)=0$ be an irreducible curve over k. Then C is a line if and only if the derivation*

$$D_f = \frac{\partial f}{\partial y}\frac{\partial}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial}{\partial y}$$

is locally nilpotent.

Proof. Assume that $C\colon f(x, y)=0$ is a line. Let $\Gamma\colon g(x, y)=0$ be a curve such that $k[f, g]=k[x, y]$. Then there exists a locally nilpotent derivation Δ of $k[x, y]$ such that $\Delta f=0$ and $\Delta g=1$. Since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ form a basis of derivations of $k[x, y]$ we can write

$$\Delta = a\frac{\partial}{\partial x}-b\frac{\partial}{\partial y} \text{ with } a, b\in k[x, y]\,.$$

Since $\Delta f=0$ we have

$$a\frac{\partial f}{\partial x}-b\frac{\partial f}{\partial y} = 0 \qquad (1)$$

Let $a\Big/\frac{\partial f}{\partial y}=b\Big/\frac{\partial f}{\partial y}=\lambda$, i.e., $a=\lambda\frac{\partial f}{\partial y}$, $b=\lambda\frac{\partial f}{\partial x}$. Then we have $\Delta=\lambda D_f$. We show that $\lambda\in k[x, y]$. From Lemma 2 it follows that $\alpha\frac{\partial f}{\partial x}+\beta\frac{\partial f}{\partial y}=1$ for some α, β in $k[x, y]$. Hence $\lambda=b\alpha+a\beta\in k[x, y]$. On the other hand the existence of $g\in k[x, y]$ such that $\Delta g=1$ implies $(a, b)=1$. Since λ is a common divisor of a and b we see that $\lambda \in k^*$. This means that D_f is locally nilpotent. The "if" part of the Theorem is immediate from Theorem 3.

According to S. Abhyankar and T. Moh a quasi-line is a line in case of characteristic zero ([1]). In the case where the characteristic of k is a positive prime integer p there is a counter example.

EXAMPLE 2(***) A curve $C: f(x, y)=0$ such that

$$f(x, y) \equiv y^{p^2} - x - x^{pq}$$

is a quasi-line but not a line where p is the characteristic of k and q is an integer ≥ 2 not divisible by p.

Proof. If we set

$$u = y - (y^p - x^q)^q$$

then $x \equiv u^{p^2}$ and $y \equiv u + u^{pq}$ modulo $f(x, y)$. Hence $f(x, y)=0$ is a quasi-line. To see that c is not a line it suffices to show that there is no locally finite higher derivation killing f. Assume the contrary and let Δ be a lfih-derivation killing f and $\phi = \Phi_\Delta$. Let

$$\phi(x) = x + \sum_i a_i T^i$$

$$\phi(y) = y + \sum_i b_i T^i .$$

From $\phi(f) = f$ we get

$$(y^{p^2} + \sum_i b_i^{p^2} T^{p^2 i}) - (x + \sum_i a_i T^i) - (x^p + \sum_i a_i^p T^{pi})^q = y^{p^2} - x - x^{pq} \quad \cdots\cdots (1)$$

First we easily see that $a_i = 0$ if $i \not\equiv 0 \pmod{p^2}$. We set $a_{p^2 i} = \alpha_i$. Then we have

$$(y^{p^2} + \sum_i b_i^{p^2} T^{p^2 i}) - (x + \sum_{i=1}^n \alpha_i T^{p^2 i}) - (x^p + \sum_{i=1}^n \alpha_i^p T^{p^3 i})^q = y^{p^2} - x - x^{pq}$$

First we remark that

$$\alpha_i \in A^p \tag{2}$$

for any i where $A = k[x, y]$. Now assume that $n \geqq 1$. We compute the coefficient of $T^{p^3 n(q-1)}$. Since $T^{p^3 n(q-1)}$ does not appear in the middle term we have the relation:

$$b_{pn(q-1)}^{p^2} = \sum_{i_1 + \cdots + i_q = n(q-1)} \alpha_{i_1}^p \cdots \alpha_{i_q}^p + q x^p \alpha_n^{p(q-1)}$$

From (2) $\alpha_{i_1}^p \cdots \alpha_{i_q}^p$, $\alpha_n^{p(q-1)}$ are in A^{p^2}. Hence x^p must also be in A^{p^2}. This is

(***) This example is a generalization of the one given in [4].

impossible. This proves $n=0$, i.e., x must be a $\underline{\Delta}$-constant. Hence y is also a $\underline{\Delta}$-constant. Thus there is no non-trivial lfih-derivation $\underline{\Delta}$ such that $\underline{\Delta}(f)=0$.

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Added in Proof. In Theorem 3 we assumed that k is algebraically closed. This assumption is essential as is shown in the following Example. Let $B=\boldsymbol{R}[X, Y]/X^2+Y^2+1$. Then B is a UFD and satisfies $B^*=\boldsymbol{R}^*$. The ring $A=B[Z]$ satisfies all the requirement in Theorem 3, but A is not a polynomial ring of two variables over the field $\boldsymbol{R}$.