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## ON LOCALLY FINITE ITERATIVE HIGHER DERIVATIONS

## YOSHIKAZU NAKAI

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Let A be a commutative ring with unity. A higher derivation  $\underline{\Delta} = \{1, \Delta_1, \dots, \Delta_n\}$  $\Delta_2, \dots$  of A is called locally finite if for any  $a \in A$  there exists an index j such that  $\Delta_n(a) = 0$  for all n > j. In a previous paper some properties of locally finite iterative higher derivations (abbreviated as lfih-derivations) and some applications of them were presented ([2]). In this paper the author gives another application of lfih-derivations, i.e., a characterization of two-dimensional polynomial ring. His proof supplies an alternative proof of Theorem 1 in [3], where the method is geometric while the present one is algebraic and elementary. As a Corollary a characterization of a line in an affine plane is given in terms of lfih-derivation where a line in an affine plane is meant a curve C which can be taken as a coordinate axis of  $A^2$ . We call a curve C: f(x, y) = 0 a quasi-line if the coordinate ring k[x, y]/(f) is isomorphic to one-parameter polynomial ring. It is known that if the ground field is the complex number field C, then a quasiline is always a line (cf. [1]). Combined with the present investigation it turns out that if the plane curve C: f(x, y) = 0 is a quasi-line over C, then the derivation  $D_f = (\partial f / \partial y) \frac{\partial}{\partial x} - (\partial f / \partial x) \frac{\partial}{\partial y}$  is locally nilpotent, i.e., the higher derivation  $\left(1, D_f, \frac{1}{2!}D_f^2, \cdots\right)$  is a line-derivation and vice versa. The direct proof of this fact is expected very much.

Let A be a commutative ring with 1. A higher derivation  $\underline{\Delta} = (1, \Delta_1, \Delta_2, \cdots)$  is a set of linear endomorphisms of A into itself satisfying the conditions:

$$\Delta_n(ab) = \sum_{i=0}^n \Delta_i(a) \Delta_{n-i}(b)$$

where  $\Delta_0$  denotes the identity mapping of A. Let  $\Phi_{\Delta}$  be the homomorphism of the ring A into A[[T]] defined by

$$\Phi_{\underline{\Delta}}(a) = \sum_{i=0}^{\infty} D_i(a) T^i$$
.

We say that  $\underline{\Delta}$  is locally finite if  $I_m \Phi_{\underline{\Delta}}$  is contained in the polynomial ring A[T], i.e., for any  $a \in A$ , there exists an integer j such that  $\Delta_n(a)=0$  for all n>j.  $\underline{\Delta}$  is called an iterative higher derivation if the additional conditions

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$$\Delta_i \Delta_j = \binom{i+j}{i} \Delta_{i+j}$$

are satisfied by  $\underline{\Delta}$ . Let *a* be an element of the ring *A*. We say that *a* is a  $\underline{\Delta}$ -constant if  $\Delta_i(a)=0$  for all  $i \geq 1$ . This is equivalent to saying that  $\Phi_{\underline{\Delta}}(a)=a$ . Sometimes we use the notation  $\underline{\Delta}^{-1}(0)$  to denote the ring of  $\underline{\Delta}$ -constants, and  $\underline{\Delta}(a)=0$  to denote *a* being a  $\underline{\Delta}$ -constant.

**Lemma 1.** Let  $\underline{\Delta}$  be a locally finite higher derivation of an integral domain A. Then the constant ring  $B = \underline{\Delta}^{-1}(0)$  is inertly embedded in A.

Proof. Let b be an element of B and let b=cd be a decomposition of b in A. Then we have  $\phi(b)=\phi(c)\phi(d)$  where  $\phi=\Phi_{\Delta}$ . By assumption  $\phi(b)$  is in A and  $\phi(c)$ ,  $\phi(d)$  are elements of a polynomial ring A[T]. Hence  $\phi(c)$ ,  $\phi(d)$  are also in A. It means that  $\phi(c)=c$  and  $\phi(d)=d$ , i.e., c and d are in B.

**Theorem 1.** Let k be an algebraically closed field of arbitrary characteristic and let A be an integral domain containing k. Assume that A satisfies the following conditions:

i) There exists a non-trivial lfih-derivation  $\triangle$  over k.

ii) The constant ring  $A_0$  of  $\Delta$  is a principal ideal domain finitely generated over k.

iii) Any prime element of  $A_0$  remains prime in A.

Then A is a polynomial ring in one variable over  $A_0$ .

Proof. Let  $A_i$  be the set of elements  $\xi$  in A such that  $\Delta_n(\xi)=0$  for n>i.  $A_0$  is the ring of  $\Delta$ -constants and  $A_i$ 's are  $A_0$ -modules. It is proved in [2] that there exists an integer s ( $\geq 0$ ) such that

$$A_{0} = A_{1} = \cdots A_{p^{s}-1} \subset A_{p^{s}} = \cdots = A_{2p^{s}-1} \subset A_{2p^{s}} = \cdots$$

where  $\subset$  denotes proper containment. The integer  $mp^s$  is called the *m*-th jump index  $(m=1, 2, \cdots)$ . For simplicity we set  $q=p^s$  and  $M_n=A_{nq}$ . It is also proved in [2] that for any element  $\xi$  in  $M_1$ , we have

$$\phi(\xi) = \xi + \alpha_0 T + \alpha_1 T^p + \dots + \alpha_s T^q$$

where  $\alpha$ 's are in  $A_0$  and  $\phi = \phi_{\Delta}$ . Let  $I_1$  be the set of elements in  $A_0$  which appear as coefficients of  $T^q$  in  $\phi(\xi)$  for some  $\xi \in M_1$ . It is easily seen that  $I_1$ is an ideal of  $A_0$ . Similarly let  $I_n$  be the set of elements which appear as coefficients of  $T^{nq}$  in  $\phi(\xi)$  for some  $\xi \in M_n$ . Then  $I_n$  is also an ideal of  $A_0$ . Let  $a_n$  be a generator of the  $I_n$  and let x be an element of  $M_1$  such that

$$\phi(x) = x + \cdots + a_1 T^q \, .$$

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We shall prove simultaneously the following

$$\begin{array}{ll} (1)_n & (a_n) = (a_1^n) , \\ (2)_n & M_n = A_0 + A_0 x + \dots + A_0 x^n , \quad (n=1,\,2,\,\dots) \end{array}$$

by induction on *n*. First we shall remark that  $(1)_n$  implies  $(2)_n$ . In fact let  $\xi$  be in  $M_n$ . Then  $\Delta_{nq}(\xi)$  is in  $I_n = (a_n)$ . From  $(1)_n$  it follows that there exists a constant *c* in  $A_0$  such that  $\Delta_{nq}(\xi) = ca_1^n$ . Then  $\phi(\xi - cx^n)$  is of degree  $\langle nq$ , hence  $\xi - cx^n \in M_{n-1}$ . Now assume  $(1)_n$ ,  $(2)_n$  and we shall prove  $(1)_{n+1}$ . Since  $a_1^{n+1} \in I_{n+1} = (a_{n+1})$ , there is a constant *c* in  $A_0$  such that  $a_1^{n+1} = ca_{n+1}$ . Let  $\xi$  be an element of  $M_{n+1}$  such that

$$\phi(\xi) = + \cdots + a_{n+1} T^{(n+1)q}$$

Then  $\phi(c\xi - x^{n+1})$  is of degree  $\langle (n+1)q$ , hence  $c\xi - x^{n+1} \in M_n$ . By (2)<sub>n</sub> there are  $b_i$ 's in  $A_0$  such that

$$c\xi = x^{n+1} + \sum_{i=0}^n b_i x^n \, .$$

We shall show that c is a unit of  $A_0$ . Assume that c is a non-unit in  $A_0$ . Let f be a prime element which divides c. Taking the residue class modulo fA we get an algebraic relation

$$\bar{x}^{n+1} + \bar{b}_i \bar{x}^n = 0.$$

By assumption (iii) f is also a prime element of A. Hence A/fA is an integral domain. Since k is algebraically closed and  $A_0$  is finitely generated over k, we have  $A_0/fA_0 = k$ . Hence there exists  $\gamma$  in k sub that  $\bar{x} = \gamma$ . It means that  $x = \gamma = fy$  with some  $y \in A$ . Then we have  $\phi(x - \gamma) = f\phi(y)$ , i.e.,  $\Delta_q(x) = f\Delta_q(y)$ . Since  $\Delta_q(y) \in I_1 = (a_1) = (\Delta_q(x))$  we get a contradiction. Thus we have proven  $(1)_{n+1}$ . Since  $A = \bigcup_{n=1}^{\infty} M_n$ , we obtain the desired result  $A = A_0[x]$ .

REMARK. If A is a UFD, then the condition (iii) is automatically satisfied.

**Theorem 2.** Let k be as in Theorem 1, and let A be a finitely generated normal integral domain over k such that

- (i)  $\dim A=2$
- (ii)  $A^* = k^*$  where \* denotes the set of units.
- (iii) Either A is UFD or Q(A) is unirational over k.

Let  $\underline{\Delta}$  be a non-trivial lfih-derivation of A over k. Then the constant ring  $A_0$  of  $\underline{\Delta}$  is a polynomial ring over k. More precisely let f be an irreducible element in  $A_0$ . Then  $A_0 = k[f]$ .

Proof.  $A_0$  is not reduced to k because there exists an element u in  $A_0$  and

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a variable t over  $A_0$  such that  $A[u^{-1}] = A_0[u^{-1}][t]$ . (cf. Appendix, [2]). Let f be an element of  $A_0 \setminus k$  which is irreducible in A. The existence of such an element f is assured by the Lemma 1. We shall show that  $A_0 = k[f]$ . Since  $A_0[u^{-1}] \simeq A[u^{-1}]/tA[u^{-1}], A_0[u^{-1}]$  is a finitely generated integral domain over k. In case A is a UFD,  $A_0[u^{-1}]$  is also a UFD owing to the Lemma 1. Moreover the transcednence degree of the quotient field K of  $A_0$  is 1. Hence K is a purely transcendental extension of k. If A is not a UFD we assumed that Q(A)is unirational. Then by the generalized Luroth's theorem K is also a onedimensional purely transcendental extension of k. Let B be the integral closure of k[f] in K. Then B is also finitely generated over k and  $B^* = k^*$  because B is contained in A. Hence there exists an element t in B such that B = k[t]. Since f is contained in B we can write  $f = \lambda(t)$ . But f is irreducible in A, hence degree of  $\lambda$  in t must be 1. It proves that k[t] = k[f] = B. Now assume  $A_0 \neq B$ . Since  $A_0$  and B have the same quotient field,  $A_0$  contains an element of the form  $\gamma(f)/s(f)$  where  $(\gamma(f), s(f))=1$  and deg  $s(f)\geq 1$ . Then  $A_0$  must contain a non-constant unit. This is against the assumption (ii).

Combining these theorems we have the following

**Theorem 3.** Let k be an algebraically closed field of arbitrary characteristic and let A be a finitely generated integral domain over k. Assume that A satisfies the following conditions:

- (i)  $\dim A=2$
- (ii)  $A^* = k^*$
- (iii) A is UFD.

Assume that A has a non-trivial lfih-derivation  $\Delta$  over k. Then A is a twodimensional polynomial ring over k. More precisely if the constant ring  $A_0$  of  $\Delta$  is written as k[f], then A = k[f, g] for some other element g in A.

The assumption (iii) is essential as is shown in the following

EXAMPLE 1.<sup>(\*)</sup> Let  $A = C \left[ x, y, \frac{y(y-1)}{x} \right]$ . Then as is easily seen  $A^* = C^*$ 

and A has a locally nilpotent derivation D such that

$$Dx = 2y-1, Dy = \frac{y(y-1)}{x}.$$

By a simple calculation we see  $D^{-1}(0) = k \left[ \frac{y(y-1)}{x} \right]$ . The element  $\frac{y(y-1)}{x}$  is not a prime element in A. Hence A is neither UFD nor a polynomial ring.

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<sup>(\*)</sup> This example is due to K. Yoshida.

As an application of Theorem 3 we give a necessary and sufficient condition for a plane curve C: f(x, y)=0 to be a line. We recollect here some definitions. A plane curve C: f(x, y)=0 defined over a field k is called a quasi-line over k if the coordinate ring A=k[x, y]/(f) is isomorphic to a polynomial ring in one variable. C is called a line if there exists another curve  $\Gamma: g(x, y)=0$  such that we have  $k[x, y]=k[f, g].^{(**)}$ 

**Theorem 4.** Let k be an algebraically closed field and let C: f(x, y)=0 be an irreducible curve over k. Then the following conditions are equivalent to each other.

- (i) C is a line
- (ii) There is a lfih-derivation  $\Delta$  such that  $\Delta(f)=0$ .
- (iii)  $C_u: f(x, y) u = 0$  is a quasi-line over k(u) where u is an indeterminate.

Proof. The implication (i) $\rightarrow$ (ii), (i) $\rightarrow$ (iii) is obvious (ii) $\rightarrow$ (i) follows from Theorem 2 and 3. It remains to show that (iii) implies (i). Assume (iii). Since k(u)[x, y]/(f-u) is isomorphic to k(f)[x, y], there exists an element t in k[x, y]such that k(f)[x, y]=k(f)[t]. Let  $\Delta'$  be the lfih-derivation of k(f)[t] over k(f) such that

$$\Delta'_n(t^m) = \binom{m}{n} t^{m-n}$$

Then there exists an element a in k[f] such that  $a\Delta' = \Delta$  sends k[x, y] into itself, where  $a\Delta'$  is higher derivation

 $a\underline{\Delta}' = (1, a\Delta_1', a^2\Delta_2', \cdots, a^n\Delta_n', \cdots).$ 

Clearly  $\Delta(f)=0$  and f is a prime element in k[x, y]. Hence  $\Delta^{-1}(0)=k[f]$  and by Theorem 3, f is a line.

In case where the characteristic of k is zero we can say more. First we prove a Lemma.

**Lemma 2.** Let C: 
$$f(x, y)=0$$
 be a line in a plane. Then  $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)=1$ .

Proof. Since C is a line, there exists a curve  $\Gamma: g(x, y)=0$  such that k[x, y]=k[f, g]. Then there exists F(X, Y) and G(X, Y) in k[X, Y] such that

$$F(f, g) = x$$
$$G(f, g) = y.$$

Then we have

<sup>(\*\*)</sup> In [4] our "line" and "quasi-line" are called "embedded line and me respectively.

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Now assume  $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \subseteq m$  for some maximal ideal m. Then from (2) either  $\frac{\partial F}{\partial g}$  or  $\frac{\partial g}{\partial y}$  is contained in m. The first case cannot occur because of (1) and the second case contradicts (4). Thus  $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$  is a unit ideal.

**Theorem 5.** Let k be an algebraically closed field of characteristic zero and let C: f(x, y)=0 be an irreducible curve over k. Then C is a line if and only if the derivation

$$D_f = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$$

is locally nilpotent.

Proof. Assume that C: f(x, y)=0 is a line. Let  $\Gamma: g(x, y)=0$  be a curve such that k[f,g]=k[x, y]. Then there exists a locally nilpotent derivation  $\Delta$  of k[x, y] such that  $\Delta f=0$  and  $\Delta g=1$ . Since  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$  form a basis of derivations of k[x, y] we can write

$$\Delta = a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \text{ with } a, b \in k[x, y].$$

Since  $\Delta f = 0$  we have

$$a\frac{\partial f}{\partial x} - b\frac{\partial f}{\partial y} = 0 \tag{1}$$

Let 
$$a \Big/ \frac{\partial f}{\partial y} = b \Big/ \frac{\partial f}{\partial y} = \lambda$$
, i.e.,  $a = \lambda \frac{\partial f}{\partial y}$ ,  $b = \lambda \frac{\partial f}{\partial x}$ . Then we have  $\Delta = \lambda D_f$ .

We show that  $\lambda \in k[x, y]$ . From Lemma 2 it follows that  $\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = 1$  for some  $\alpha$ ,  $\beta$  in k[x, y]. Hence  $\lambda = b\alpha + a\beta \in k[x, y]$ . On the other hand the existence of  $g \in k[x, y]$  such that  $\Delta g = 1$  implies (a, b) = 1. Since  $\lambda$  is a common

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divisor of a and b we see that  $\lambda \in k^*$ . This means that  $D_f$  is locally nilpotent. The "if" part of the Theorem is immediate from Theorem 3.

According to S. Abhyankar and T. Moh a quasi-line is a line in case of characteristic zero ([1]). In the case where the characteristic of k is a positive prime integer p there is a counter example.

EXAMPLE  $2^{(***)}$  A curve C: f(x, y) = 0 such that

$$f(x, y) \equiv y^{p^2} - x - x^{pq}$$

is a quasi-line but not a line where p is the characteristic of k and q is an integer  $\geq 2$  not divisible by p.

Proof. If we set

$$u = y - (y^p - x^q)^q$$

then  $x \equiv u^{p^2}$  and  $y \equiv u + u^{pq}$  modulo f(x, y). Hence f(x, y) = 0 is a quasi-line. To see that c is not a line it suffices to show that there is no locally finite higher derivation killing f. Assume the contrary and let  $\Delta$  be a lfih-derivation killing f and  $\phi = \Phi_{\Delta}$ . Let

$$\phi(x) = x + \sum_{i} a_{i} T^{i}$$
$$\phi(y) = y + \sum_{i} b_{i} T^{i}.$$

From  $\phi(f) = f$  we get

$$(y^{p^{2}} + \sum_{i} b_{i}^{p^{2}} T^{p^{2}i}) - (x + \sum_{i} a_{i} T^{i}) - (x^{p} + \sum_{i} a_{i}^{p} T^{pi})^{q} = y^{p^{2}} - x - x^{pq} \cdots (1)$$

First we easily see that  $a_i=0$  if  $i \equiv 0 \pmod{p^2}$ . We set  $a_{p^2i}=\alpha_i$ . Then we have

$$(y^{p^2} + \sum_i b_i^{p^2} T^{p^2i}) - (x + \sum_{i=1}^n \alpha_i T^{p^2i}) - (x^p + \sum_{i=1}^n \alpha_i^p T^{p^3i})^q = y^{p^2} - x - x^{pq}$$

First we remark that

$$\alpha_i \in A^p \tag{2}$$

for any *i* where A = k[x, y]. Now assume that  $n \ge 1$ . We compute the coefficient of  $T^{p^3n(q-1)}$ . Since  $T^{p^3n(q-1)}$  does not appear in the middle term we have the relation:

$$b_{pn(q-1)}^{p^2} = \sum_{i_1 + \dots + i_q = n(q-1)} \alpha_{i_1}^p \cdots \alpha_{i_q}^p + qx^p \alpha_n^{p(q-1)}$$

From (2)  $\alpha_{i_1}^p \cdots \alpha_{i_q}^p$ ,  $\alpha_n^{p(q-1)}$  are in  $A^{p^2}$ . Hence  $x^p$  must also be in  $A^{p^2}$ . This is

<sup>(\*\*\*)</sup> This example is a generalization of the one given in [4].

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impossible. This proves n=0, i.e., x must be a  $\Delta$ -constant. Hence y is also a  $\Delta$ -constant. Thus there is no non-trivial lfih-derivation  $\Delta$  such that  $\Delta(f)=0$ .

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Added in Proof. In Theorem 3 we assumed that k is algebraically closed. This assumption is essential as is shown in the following Example. Let  $B = \mathbf{R}[X, Y]/X^2 + Y^2 + 1$ . Then B is a UFD and satisfies  $B^* = \mathbf{R}^*$ . The ring A = B[Z] satisfies all the requirement in Theorem 3, but A is not a polynomial ring of two variables over the field  $\mathbf{R}$ .