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ON LOCALLY FINITE ITERATIVE HIGHER DERIVATIONS

YOSHIKAZU NAKAI

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Let \( A \) be a commutative ring with unity. A higher derivation \( \Delta = \{1, \Delta_1, \Delta_2, \ldots\} \) of \( A \) is called locally finite if for any \( a \in A \) there exists an index \( j \) such that \( \Delta_n(a) = 0 \) for all \( n > j \). In a previous paper some properties of locally finite iterative higher derivations (abbreviated as \( \text{lfih-derivations} \)) and some applications of them were presented ([2]). In this paper the author gives another application of \( \text{lfih-derivations} \), i.e., a characterization of two-dimensional polynomial ring. His proof supplies an alternative proof of Theorem 1 in [3], where the method is geometric while the present one is algebraic and elementary. As a Corollary a characterization of a line in an affine plane is given in terms of \( \text{lfih-derivations} \), where a line in an affine plane is meant a curve \( C \) which can be taken as a coordinate axis of \( A^2 \). We call a curve \( C : f(x, y) = 0 \) a quasi-line if the coordinate ring \( k[x, y]/(f) \) is isomorphic to one-parameter polynomial ring. It is known that if the ground field is the complex number field \( \mathbb{C} \), then a quasi-line is always a line (cf. [1]). Combined with the present investigation it turns out that if the plane curve \( C : f(x, y) = 0 \) is a quasi-line over \( C \), then the derivation \( D_f = (\partial f/\partial y) \frac{\partial}{\partial x} - (\partial f/\partial x) \frac{\partial}{\partial y} \) is locally nilpotent, i.e., the higher derivation \( \left(1, \frac{1}{2!} D^2_f, \ldots\right) \) is a lfih-derivation and vice versa. The direct proof of this fact is expected very much.

Let \( A \) be a commutative ring with unity. A higher derivation \( \Delta = \{1, \Delta_1, \Delta_2, \ldots\} \) is a set of linear endomorphisms of \( A \) into itself satisfying the conditions:

\[
\Delta_0(ab) = \sum_{i=0}^{\infty} \Delta_i(a)\Delta_{m-i}(b)
\]

where \( \Delta_0 \) denotes the identity mapping of \( A \). Let \( \Phi_\Delta \) be the homomorphism of the ring \( A \) into \( A[[T]] \) defined by

\[
\Phi_\Delta(a) = \sum_{i=0}^{\infty} D_i(a)T^i.
\]

We say that \( \Delta \) is locally finite if \( I_{\infty} \Phi_\Delta \) is contained in the polynomial ring \( A[T] \), i.e., for any \( a \in A \), there exists an integer \( j \) such that \( \Delta_n(a) = 0 \) for all \( n > j \). \( \Delta \) is called an iterative higher derivation if the additional conditions
\[ \Delta_i \Delta_j = \binom{i+j}{i} \Delta_{i+j} \]

are satisfied by \( \Delta \). Let \( a \) be an element of the ring \( A \). We say that \( a \) is a \( \Delta \)-constant if \( \Delta_i(a) = 0 \) for all \( i \geq 1 \). This is equivalent to saying that \( \Phi_\Delta(a) = a \). Sometimes we use the notation \( \Delta^{-1}(0) \) to denote the ring of \( \Delta \)-constants, and \( \Delta(a) = 0 \) to denote \( a \) being a \( \Delta \)-constant.

**Lemma 1.** Let \( \Delta \) be a locally finite higher derivation of an integral domain \( A \). Then the constant ring \( B = \Delta^{-1}(0) \) is inertly embedded in \( A \).

**Proof.** Let \( b \) be an element of \( B \) and let \( b = cd \) be a decomposition of \( b \) in \( A \). Then we have \( \phi(b) = \phi(c) \phi(d) \) where \( \phi = \Phi_\Delta \). By assumption \( \phi(b) \) is in \( A \) and \( \phi(c), \phi(d) \) are elements of a polynomial ring \( A[T] \). Hence \( \phi(c), \phi(d) \) are also in \( A \). It means that \( \phi(c) = c \) and \( \phi(d) = d \), i.e., \( c \) and \( d \) are in \( B \).

**Theorem 1.** Let \( k \) be an algebraically closed field of arbitrary characteristic and let \( A \) be an integral domain containing \( k \). Assume that \( A \) satisfies the following conditions:

i) There exists a non-trivial \( \Delta \)-derivation \( \Delta \) over \( k \).

ii) The constant ring \( A_0 \) of \( \Delta \) is a principal ideal domain finitely generated over \( k \).

iii) Any prime element of \( A_0 \) remains prime in \( A \).

Then \( A \) is a polynomial ring in one variable over \( A_0 \).

**Proof.** Let \( A_i \) be the set of elements \( \xi \) in \( A \) such that \( \Delta_n(\xi) = 0 \) for \( n > i \). \( A_0 \) is the ring of \( \Delta \)-constants and \( A_i \)'s are \( A_0 \)-modules. It is proved in [2] that there exists an integer \( s \) (\( \geq 0 \)) such that

\[
A_0 = A_1 = \cdots A_{s'-1} \subset A_{s'} = \cdots = A_{2s'-1} \subset A_{2s'} = \cdots
\]

where \( \subset \) denotes proper containment. The integer \( mp^t \) is called the \( m \)-th jump index (\( m = 1, 2, \ldots \)). For simplicity we set \( q = p^t \) and \( M_n = A_{nq} \). It is also proved in [2] that for any element \( \xi \) in \( M_1 \), we have

\[
\phi(\xi) = \xi + \alpha_0 T + \alpha_1 T^p + \cdots + \alpha_t T^q
\]

where \( \alpha \)'s are in \( A_0 \) and \( \phi = \phi_\Delta \). Let \( I_1 \) be the set of elements in \( A_0 \) which appear as coefficients of \( T^s \) in \( \phi(\xi) \) for some \( \xi \in M_1 \). It is easily seen that \( I_1 \) is an ideal of \( A_0 \). Similarly let \( I_n \) be the set of elements which appear as coefficients of \( T^{ns} \) in \( \phi(\xi) \) for some \( \xi \in M_n \). Then \( I_n \) is also an ideal of \( A_0 \). Let \( a_n \) be a generator of the \( I_n \) and let \( x \) be an element of \( M_1 \) such that

\[
\phi(x) = x + \cdots + a_1 T^q \]
We shall prove simultaneously the following

1. \((\alpha_n) = (\alpha^2)\),
2. \(M_n = A_0 + A_0x + \cdots + A_0x^n\), \((n=1, 2, \ldots)\)

by induction on \(n\). First we shall remark that (1) implies (2). In fact let \(\xi \in M_n\). Then \(\Delta_n(\xi)\) is in \(I_n = (a_n)\). From (1) it follows that there exists a constant \(c\) in \(A_0\) such that \(\Delta_n(\xi) = ca^2\). Then \(\phi(\xi - cx^m)\) is of degree \(< nq\), hence \(\xi - cx^m \in M_{n-1}\). Now assume (1), (2), and we shall prove (1). Since \(a^{n+1} \in I_{n+1} = (a_{n+1})\), there is a constant \(c\) in \(A_0\) such that \(a^{n+1} = ca_{n+1}\). Let \(\xi\) be an element of \(M_{n+1}\) such that

\[\phi(\xi) = \alpha_{n+1}T^{(n+1)q}\]

Then \(\phi(c\xi - x^{n+1})\) is of degree \(<(n+1)q\), hence \(c\xi - x^{n+1} \in M_n\). By (2) there are \(b_i\)'s in \(A_0\) such that

\[c\xi = x^{n+1} + \sum_{i=0}^n b_i x^i\]

We shall show that \(c\) is a unit of \(A_0\). Assume that \(c\) is a non-unit in \(A_0\). Let \(f\) be a prime element which divides \(c\). Taking the residue class modulo \(fA\) we get an algebraic relation

\[x^{n+1} + \sum_{i=1}^n b_i x^i = 0\]

By assumption (iii) \(f\) is also a prime element of \(A\). Hence \(A/fA\) is an integral domain. Since \(k\) is algebraically closed and \(A_0\) is finitely generated over \(k\), we have \(A_0/fA_0 = k\). Hence there exists \(\gamma\) in \(k\) such that \(x = \gamma\). It means that \(x - \gamma = fy\) with some \(y \in A\). Then we have \(\phi(x - \gamma) = f\phi(y)\), i.e., \(\Delta_\phi(x) = f\Delta_\phi(y)\). Since \(\Delta_\phi(y) \in I_1 = (a_1) = (\Delta_\phi(x))\) we get a contradiction. Thus we have proven (1). Since \(A = \bigcup_{n=1} A_n\), we obtained the desired result \(A = A_0[x]\).

Remark. If \(A\) is a \(UFD\), then the condition (iii) is automatically satisfied.

**Theorem 2.** Let \(k\) be as in Theorem 1, and let \(A\) be a finitely generated normal integral domain over \(k\) such that

(i) \(\text{dim} A = 2\)

(ii) \(A^* = k^*\) where \(^*\) denotes the set of units.

(iii) Either \(A\) is \(UFD\) or \(Q(A)\) is unirational over \(k\).

Let \(\Delta\) be a non-trivial \(\ell\)-th derivation of \(A\) over \(k\). Then the constant ring \(A_0\) of \(\Delta\) is a polynomial ring over \(k\). More precisely let \(f\) be an irreducible element in \(A_0\). Then \(A_0 = k[f]\).

Proof. \(A_0\) is not reduced to \(k\) because there exists an element \(u\) in \(A_0\) and
a variable \( t \) over \( A_0 \) such that \( A[u^{-1}] = A_0[u^{-1}][t] \). (cf. Appendix, [2]). Let \( f \) be an element of \( A_0 \setminus k \) which is irreducible in \( A \). The existence of such an element \( f \) is assured by the Lemma 1. We shall show that \( A_0 = k[f] \). Since \( A_0[u^{-1}] = A[u^{-1}][tA[u^{-1}], A_0[u^{-1}] \] is a finitely generated integral domain over \( k \).

In case \( A \) is a \( UFD \), \( A_0[u^{-1}] \) is also a \( UFD \) owing to the Lemma 1. Moreover the transcendence degree of the quotient field \( K \) of \( A_0 \) is 1. Hence \( K \) is a purely transcendental extension of \( k \). If \( A \) is not a \( UFD \) we assumed that \( Q(A) \) is unirational. Then by the generalized Lüroth's theorem \( K \) is also a one-dimensional purely transcendental extension of \( k \). Let \( B \) be the integral closure of \( k[f] \) in \( K \). Then \( B \) is also finitely generated over \( k \) and \( B^* = k^* \) because \( B \) is contained in \( A \). Hence there exists an element \( t \) in \( B \) such that \( B = k[t] \). Since \( f \) is contained in \( B \) we can write \( f = \lambda(t) \). But \( f \) is irreducible in \( A \), hence degree of \( \lambda \) in \( t \) must be 1. It proves that \( k[t] = k[f] = B \). Now assume \( A_0 \neq B \). Since \( A_0 \) and \( B \) have the same quotient field, \( A_0 \) contains an element of the form \( \gamma(f)/s(f) \) where \( (\gamma(f), s(f)) = 1 \) and \( \deg s(f) \geq 1 \). Then \( A_0 \) must contain a non-constant unit. This is against the assumption (ii).

Combining these theorems we have the following

**Theorem 3.** Let \( k \) be an algebraically closed field of arbitrary characteristic and let \( A \) be a finitely generated integral domain over \( k \). Assume that \( A \) satisfies the following conditions:

(i) \( \dim A = 2 \)

(ii) \( A^* = k^* \)

(iii) \( A \) is \( UFD \).

Assume that \( A \) has a non-trivial \( \text{lih} \)-derivation \( \Delta \) over \( k \). Then \( A \) is a two-dimensional polynomial ring over \( k \). More precisely if the constant ring \( A_0 \) of \( \Delta \) is written as \( k[f] \), then \( A = k[f, g] \) for some other element \( g \) in \( A \).

The assumption (iii) is essential as is shown in the following

**Example 1.** Let \( A = C[x, y, \frac{y(y-1)}{x}] \). Then as is easily seen \( A^* = C^* \) and \( A \) has a locally nilpotent derivation \( D \) such that

\[
Dx = 2y - 1, \quad Dy = \frac{y(y-1)}{x}.
\]

By a simple calculation we see \( D^{-1}(0) = k[\frac{y(y-1)}{x}] \). The element \( \frac{y(y-1)}{x} \) is not a prime element in \( A \). Hence \( A \) is neither \( UFD \) nor a polynomial ring.

(*) This example is due to K. Yoshida.
As an application of Theorem 3 we give a necessary and sufficient condition for a plane curve \( C: f(x, y)=0 \) to be a line. We recollect here some definitions. A plane curve \( C: f(x, y)=0 \) defined over a field \( k \) is called a quasi-line over \( k \) if the coordinate ring \( A=k[x, y]/(f) \) is isomorphic to a polynomial ring in one variable. \( C \) is called a line if there exists another curve \( \Gamma: g(x, y)=0 \) such that we have \( k[x, y]=k[f, g] \). 

**Theorem 4.** Let \( k \) be an algebraically closed field and let \( C: f(x, y)=0 \) be an irreducible curve over \( k \). Then the following conditions are equivalent to each other.

(i) \( C \) is a line

(ii) There is a \( l \)-th derivation \( \Delta \) such that \( \Delta(f)=0 \).

(iii) \( C_u: f(x, y)−u=0 \) is a quasi-line over \( k(u) \) where \( u \) is an indeterminate.

Proof. The implication (i)→(ii), (i)→(iii) is obvious. (ii)→(i) follows from Theorem 2 and 3. It remains to show that (iii) implies (i). Assume (iii). Since \( k(u)[x, y]/(f−u) \) is isomorphic to \( k(f)[x, y] \), there exists an element \( t \) in \( k[x, y] \) such that \( k(f)[x, y]=k(f)[t] \). Let \( \Delta' \) be the \( l \)-th derivation of \( k(f)[t] \) over \( k(f) \) such that

\[
\Delta'(t^m) = \binom{m}{n}t^{m-n}
\]

Then there exists an element \( a \) in \( k[f] \) such that \( a\Delta'=\Delta \) sends \( k[x, y] \) into itself, where \( a\Delta' \) is higher derivation

\[
a\Delta' = (1, a\Delta', a^2\Delta', \ldots, a^s\Delta', \ldots).
\]

Clearly \( \Delta(f)=0 \) and \( f \) is a prime element in \( k[x, y] \). Hence \( \Delta^{-1}(0)=k[f] \) and by Theorem 3, \( f \) is a line.

In case where the characteristic of \( k \) is zero we can say more. First we prove a Lemma.

**Lemma 2.** Let \( C: f(x, y)=0 \) be a line in a plane. Then \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \)=1.

Proof. Since \( C \) is a line, there exists a curve \( \Gamma: g(x, y)=0 \) such that \( k[x, y]=k[f, g] \). Then there exists \( F(X, Y) \) and \( G(X, Y) \) in \( k[X, Y] \) such that

\[
F(f, g) = x
\]
\[
G(f, g) = y.
\]

Then we have

(**) In [4] our "line" and "quasi-line" are called "embedded line" and "line" respectively.
Theorem 5. Let $k$ be an algebraically closed field of characteristic zero and let $C: f(x, y)=0$ be an irreducible curve over $k$. Then $C$ is a line if and only if the derivation
\[ D_f = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \]
is locally nilpotent.

Proof. Assume that $C: f(x, y)=0$ is a line. Let $\Gamma: g(x, y)=0$ be a curve such that $k[f, g]=k[x, y]$. Then there exists a locally nilpotent derivation $\Delta$ of $k[x, y]$ such that $\Delta f=0$ and $\Delta g=1$. Since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ form a basis of derivations of $k[x, y]$ we can write
\[ \Delta = a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \] with $a, b \in k[x, y]$.

Since $\Delta f=0$ we have
\[ a \frac{\partial f}{\partial x} - b \frac{\partial f}{\partial y} = 0 \] \hspace{1cm} (1)

Let $a \frac{\partial f}{\partial y} = b \frac{\partial f}{\partial y} = \lambda$, i.e., $a=\lambda \frac{\partial f}{\partial y}$, $b=\lambda \frac{\partial f}{\partial x}$. Then we have $\Delta = \lambda D_f$.

We show that $\lambda \in k[x, y]$. From Lemma 2 it follows that $\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = 1$ for some $\alpha, \beta$ in $k[x, y]$. Hence $\lambda = b \alpha + a \beta \in k[x, y]$. On the other hand the existence of $g \in k[x, y]$ such that $\Delta g=1$ implies $(a, b)=1$. Since $\lambda$ is a common
divisor of $a$ and $b$ we see that $\lambda \in k^*$. This means that $D_f$ is locally nilpotent. The "if" part of the Theorem is immediate from Theorem 3.

According to S. Abhyankar and T. Moh a quasi-line is a line in case of characteristic zero ([1]). In the case where the characteristic of $k$ is a positive prime integer $p$ there is a counter example.

**Example 2(*** A curve $C: f(x, y) = 0$ such that

$$f(x, y) \equiv y^p - x - x^{pq}$$

is a quasi-line but not a line where $p$ is the characteristic of $k$ and $q$ is an integer $\geq 2$ not divisible by $p$.

**Proof.** If we set

$$u = y - (y^p - x^q)^t$$

then $x \equiv u^p$ and $y \equiv u + u^{pq}$ modulo $f(x, y)$. Hence $f(x, y) = 0$ is a quasi-line. To see that $c$ is not a line it suffices to show that there is no locally finite higher derivation killing $f$. Assume the contrary and let $\Delta$ be a lfih-derivation killing $f$ and $\phi = \Phi_\Delta$. Let

$$\phi(x) = x + \sum_i a_i T^i$$
$$\phi(y) = y + \sum_i b_i T^i.$$ 

From $\phi(f) = f$ we get

$$(y^p + \sum_i b_i^p T^p)^t - (x + \sum_i a_i T^i)^t - (x^p + \sum_i a_i^p T^p)^t = y^p - x - x^{pq} \quad \ldots (1)$$

First we easily see that $a_i = 0$ if $i \equiv 0 \pmod{p}$. We set $a_{p^i} = a_i$. Then we have

$$(y^p + \sum_i b_i^p T^p)^t - (x + \sum_i \alpha_i T^{p^i})^t - (x^p + \sum_i \alpha_i^p T^{p^i})^t = y^p - x - x^{pq}$$

First we remark that

$$\alpha_i \in A^p \quad (2)$$

for any $i$ where $A = k[x, y]$. Now assume that $n \geq 1$. We compute the coefficient of $T^{p^k(q-1)}$. Since $T^{p^k(q-1)}$ does not appear in the middle term we have the relation:

$$b_{p^k(q-1)} = \sum_{i_1 \cdots i_k} \alpha_{i_1} \cdots \alpha_{i_k} + qx^{e(q^k(q-1))}$$

From (2) $\alpha_{i_1} \cdots \alpha_{i_k}$, $\alpha_n^{(q-1)}$ are in $A^p$. Hence $x^p$ must also be in $A^p$. This is

(*** This example is a generalization of the one given in [4].)
impossible. This proves \( n=0 \), i.e., \( x \) must be a \( \Delta \)-constant. Hence \( y \) is also a \( \Delta \)-constant. Thus there is no non-trivial \( \mathfrak{l}_{\text{ih}} \)-derivation \( \Delta \) such that \( \Delta(f) = 0 \).

\textbf{Osaka University}

\underline{References}


\textit{Added in Proof.} In Theorem 3 we assumed that \( k \) is algebraically closed. This assumption is essential as is shown in the following Example. Let \( B=R[X, Y]/(X^2+Y^2+1) \). Then \( B \) is a UFD and satisfies \( B^*=R^* \). The ring \( A=B[Z] \) satisfies all the requirement in Theorem 3, but \( A \) is not a polynomial ring of two variables over the field \( R \).