On locally finite iterative higher derivations

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ON LOCALLY FINITE ITERATIVE HIGHER DERIVATIONS

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Let $\Lambda$ be a commutative ring with unity. A higher derivation $\Delta=\{1, \Delta_1, \Delta_2, \cdots\}$ of $\Lambda$ is called locally finite if for any $a\in\Lambda$ there exists an index $j$ such that $\Delta_n(a)=0$ for all $n>j$. In a previous paper some properties of locally finite iterative higher derivations (abbreviated as lfih-derivations) and some applications of them were presented ([2]). In this paper the author gives another application of lfih-derivations, i.e., a characterization of two-dimensional polynomial ring. His proof supplies an alternative proof of Theorem 1 in [3], where the method is geometric while the present one is algebraic and elementary. As a Corollary a characterization of a line in an affine plane is given in terms of lfih-derivation where a line in an affine plane is meant a curve $C$ which can be taken as a coordinate axis of $\Lambda^2$. We call a curve $C:f(x, y)=0$ a quasi-line if the coordinate ring $k[x, y]/(f)$ is isomorphic to one-parameter polynomial ring. It is known that if the ground field is the complex number field $C$, then a quasi-line is always a line (cf. [1]). Combined with the present investigation it turns out that if the plane curve $C:f(x, y)=0$ is a quasi-line over $C$, then the derivation $D_f=(\partial f/\partial y)\frac{\partial}{\partial x}-(\partial f/\partial x)\frac{\partial}{\partial y}$ is locally nilpotent, i.e., the higher derivation $\left(1, D_f, \frac{1}{2!}D_f^2, \cdots\right)$ is a lfih-derivation and vice versa. The direct proof of this fact is expected very much.

Let $\Lambda$ be a commutative ring with unity. A higher derivation $\Delta=(1, \Delta_1, \Delta_2, \cdots)$ is a set of linear endomorphisms of $\Lambda$ into itself satisfying the conditions:

$\Delta_0(ab) = \sum_{i=0}^{\infty} \Delta_i(a)\Delta_{n-i}(b)$

where $\Delta_0$ denotes the identity mapping of $\Lambda$. Let $\Phi_\Delta$ be the homomorphism of the ring $\Lambda$ into $\Lambda[[T]]$ defined by

$\Phi_\Delta(a) = \sum_{i=0}^{\infty} D_i(a)T^i$.

We say that $\Delta$ is locally finite if $I_\Lambda \Phi_\Delta$ is contained in the polynomial ring $\Lambda[T]$, i.e., for any $a\in\Lambda$, there exists an integer $j$ such that $\Delta_n(a)=0$ for all $n>j$. $\Delta$ is called an iterative higher derivation if the additional conditions
are satisfied by $\Delta$. Let $a$ be an element of the ring $A$. We say that $a$ is a $\Delta$-constant if $\Delta_i(a) = 0$ for all $i \geq 1$. This is equivalent to saying that $\Phi_a(a) = a$. Sometimes we use the notation $\Delta^{-1}(0)$ to denote the ring of $\Delta$-constants, and $\Delta(a) = 0$ to denote $a$ being a $\Delta$-constant.

Lemma 1. Let $\Delta$ be a locally finite higher derivation of an integral domain $A$. Then the constant ring $B = \Delta^{-1}(0)$ is inertly embedded in $A$.

Proof. Let $b$ be an element of $B$ and let $b = cd$ be a decomposition of $b$ in $A$. Then we have $\phi(b) = \phi(c)\phi(d)$ where $\phi = \Phi^\Delta$. By assumption $\phi(b)$ is in $A$ and $\phi(c), \phi(d)$ are elements of a polynomial ring $A[T]$. Hence $\phi(c), \phi(d)$ are also in $A$. It means that $\phi(c) = c$ and $\phi(d) = d$, i.e., $c$ and $d$ are in $B$.

Theorem 1. Let $k$ be an algebraically closed field of arbitrary characteristic and $A$ be an integral domain containing $k$. Assume that $A$ satisfies the following conditions:

i) There exists a non-trivial $l$th-derivation $\Delta$ over $k$.

ii) The constant ring $A_0$ of $\Delta$ is a principal ideal domain finitely generated over $k$.

iii) Any prime element of $A_0$ remains prime in $A$.

Then $A$ is a polynomial ring in one variable over $A_0$.

Proof. Let $A_i$ be the set of elements $\xi$ in $A$ such that $\Delta_n(\xi) = 0$ for $n > i$. $A_0$ is the ring of $\Delta$-constants and $A_i$’s are $A_0$-modules. It is proved in [2] that there exists an integer $s (\geq 0)$ such that

$$A_0 = A_1 = \cdots A_{s-1} \subseteq A_s = \cdots = A_{2p^s-1} \subseteq A_{2p^s} = \cdots$$

where $\subseteq$ denotes proper containment. The integer $mp^s$ is called the $m$-th jump index $(m=1, 2, \cdots)$. For simplicity we set $q = p^s$ and $M_s = A_{s+1}$. It is also proved in [2] that for any element $\xi$ in $M_1$, we have

$$\phi(\xi) = \xi + a_1T + \alpha_1T^p + \cdots + \alpha_1T^q$$

where $\alpha$’s are in $A_0$ and $\phi = \Phi^\Delta$. Let $I_1$ be the set of elements in $A_0$ which appear as coefficients of $T^s$ in $\phi(\xi)$ for some $\xi \in M_1$. It is easily seen that $I_1$ is an ideal of $A_0$. Similarly let $I_s$ be the set of elements which appear as coefficients of $T^{qs}$ in $\phi(\xi)$ for some $\xi \in M_s$. Then $I_s$ is also an ideal of $A_0$. Let $a_n$ be a generator of the $I_s$ and let $x$ be an element of $M_1$ such that

$$\phi(x) = x + \cdots + a_1T^q.$$
We shall prove simultaneously the following

\[ (1)_n \quad (a_n) = (a^*_n), \]
\[ (2)_n \quad M_n = A_0 + A_0 x + \cdots + A_0 x^n, \quad (n = 1, 2, \ldots) \]

by induction on \( n \). First we shall remark that \((1)_n\) implies \((2)_n\). In fact let \( \xi \) be in \( M_n \). Then \( \Delta_{a_n}(\xi) \) is in \( I_n = (a_n) \). From \((1)_n\) it follows that there exists a constant \( c \) in \( A_0 \) such that \( \Delta_{a_n}(\xi) = ca_n^*n \). Then \( \phi(\xi - cx^n) \) is of degree \( < nq \), hence \( \xi - cx^n \in M_{n-1} \). Now assume \((1)_n, (2)_n\) and we shall prove \((1)_{n+1}\). Since \( a_{n+1}^* \in I_{n+1} = (a_{n+1}) \), there is a constant \( c \) in \( A_0 \) such that \( a_{n+1}^* = ca_{n+1} \). Let \( \xi \) be an element of \( M_{n+1} \) such that

\[ \phi(\xi) = a_{n+1}^* T^{(n+1)q}. \]

Then \( \phi(c\xi - x^{n+1}) \) is of degree \( < (n+1)q \), hence \( c\xi - x^{n+1} \in M_n \). By \((2)_n\) there are \( b_i \)'s in \( A_0 \) such that

\[ c\xi = x^{n+1} + \sum_{i=0}^{q-1} b_i x^i. \]

We shall show that \( c \) is a unit of \( A_0 \). Assume that \( c \) is a non-unit in \( A_0 \). Let \( f \) be a prime element which divides \( c \). Taking the residue class modulo \( fA \) we get an algebraic relation

\[ x^{n+1} + b_i x^i = 0. \]

By assumption (iii) \( f \) is also a prime element of \( A \). Hence \( A/fA \) is an integral domain. Since \( k \) is algebraically closed and \( A_0 \) is finitely generated over \( k \), we have \( A_0/fA_0 = k \). Hence there exists \( \gamma \) in \( k \) such that \( x = \gamma \). It means that \( x - \gamma = fy \) with some \( y \in A \). Then we have \( \phi(x - \gamma) = f\phi(y) \), i.e., \( \Delta_\gamma(x) = f\Delta_\gamma(y) \). Since \( \Delta_\gamma(y) \in I = (a_i) = (\Delta_\gamma(x)) \) we get a contradiction. Thus we have proven \((1)_{n+1}\). Since \( A = \bigcup_{n=1}^{\infty} M_n \), we obtain the desired result \( A = A_0[x] \).

**Remark.** If \( A \) is a UFD, then the condition (iii) is automatically satisfied.

**Theorem 2.** Let \( k \) be as in Theorem 1, and let \( A \) be a finitely generated normal integral domain over \( k \) such that

(i) \( \dim A = 2 \)
(ii) \( A^* = k^* \) where \( * \) denotes the set of units.
(iii) Either \( A \) is UFD or \( Q(A) \) is unirational over \( k \).

Let \( \Delta \) be a non-trivial \( \ell fth \)-derivation of \( A \) over \( k \). Then the constant ring \( A_0 \) of \( \Delta \) is a polynomial ring over \( k \). More precisely let \( f \) be an irreducible element in \( A_0 \). Then \( A_0 = k[f] \).

**Proof.** \( A_0 \) is not reduced to \( k \) because there exists an element \( u \) in \( A_0 \) and
a variable $t$ over $A_0$ such that $A[u^{-1}]=A_0[u^{-1}][t]$. (cf. Appendix, [2]). Let $f$ be an element of $A_0\setminus k$ which is irreducible in $A$. The existence of such an element $f$ is assured by the Lemma 1. We shall show that $A_0=k[f]$. Since $A_0[u^{-1}]=A[u^{-1}][tA[u^{-1}]]$, $A_0[u^{-1}]$ is a finitely generated integral domain over $k$. In case $A$ is a UFD, $A_0[u^{-1}]$ is also a UFD owing to the Lemma 1. Moreover the transcendence degree of the quotient field $K$ of $A_0$ is 1. Hence $K$ is a purely transcendental extension of $k$. If $A$ is not a UFD we assumed that $Q(A)$ is unirational. Then by the generalized Lüroth's theorem $K$ is also a one-dimensional purely transcendental extension of $k$. Let $B$ be the integral closure of $k[f]$ in $K$. Then $B$ is also finitely generated over $k$ and $B^*={k^*}$ because $B$ is contained in $A$. Hence there exists an element $t$ in $B$ such that $B=k[t]$. Since $f$ is contained in $B$ we can write $f=\lambda(t)$. But $f$ is irreducible in $A$, hence degree of $\lambda$ in $t$ must be 1. It proves that $k[t]=k[f]=B$. Now assume $A_0\neq B$. Since $A_0$ and $B$ have the same quotient field, $A_0$ contains an element of the form $\gamma(f)/s(f)$ where $(\gamma(f), s(f))=1$ and $\deg s(f)\geq 1$. Then $A_0$ must contain a non-constant unit. This is against the assumption (ii).

Combining these theorems we have the following

**Theorem 3.** Let $k$ be an algebraically closed field of arbitrary characteristic and let $A$ be a finitely generated integral domain over $k$. Assume that $A$ satisfies the following conditions:

(i) $\dim A=2$
(ii) $A^*={k^*}$
(iii) $A$ is UFD.

Assume that $A$ has a non-trivial $k$th-derivation $\Delta$ over $k$. Then $A$ is a two-dimensional polynomial ring over $k$. More precisely if the constant ring $A_0$ of $\Delta$ is written as $k[f]$, then $A=k[f, g]$ for some other element $g$ in $A$.

The assumption (iii) is essential as is shown in the following

**Example 1.** Let $A={C}[x, y, y(x-1)/x]$. Then as is easily seen $A^*={C^*}$ and $A$ has a locally nilpotent derivation $D$ such that

$$Dx=2y-1, \quad Dy=y(x-1)/x.$$

By a simple calculation we see $D^{-1}(0)=k[y(x-1)/x]$. The element $y(x-1)/x$ is not a prime element in $A$. Hence $A$ is neither UFD nor a polynomial ring.

(*) This example is due to K. Yoshida.
As an application of Theorem 3 we give a necessary and sufficient condition for a plane curve \( C: f(x, y)=0 \) to be a line. We recollect here some definitions. A plane curve \( C: f(x, y)=0 \) defined over a field \( k \) is called a quasi-line over \( k \) if the coordinate ring \( A=k[x, y]/(f) \) is isomorphic to a polynomial ring in one variable. \( C \) is called a line if there exists another curve \( \Gamma: g(x, y)=0 \) such that we have \( k[x, y]=k[f, g] \). (**) 

**Theorem 4.** Let \( k \) be an algebraically closed field and let \( C: f(x, y)=0 \) be an irreducible curve over \( k \). Then the following conditions are equivalent to each other.

1. \( C \) is a line
2. There is a \( \ell \)-th derivation \( \Delta \) such that \( \Delta(f)=0 \).
3. \( C_u: f(x, y)—u=0 \) is a quasi-line over \( k(u) \) where \( u \) is an indeterminate.

Proof. The implication (i)\( \rightarrow \) (ii), (i)\( \rightarrow \) (iii) is obvious (ii)\( \rightarrow \) (i) follows from Theorem 2 and 3. It remains to show that (iii) implies (i). Assume (iii). Since \( k(u)[x, y]/(f—u) \) is isomorphic to \( k(f)[x, y] \), there exists an element \( t \) in \( k[x, y] \) such that \( k(f)[x, y]=k(f)[t] \). Let \( \Delta' \) be the \( \ell \)-th derivation of \( k(f)[t] \) over \( k(f) \) such that

\[
\Delta'(t^m) = \binom{m}{n}t^{m-n}
\]

Then there exists an element \( a \) in \( k[f] \) such that \( a\Delta'=\Delta \) sends \( k[x, y] \) into itself, where \( a\Delta' \) is higher derivation

\[
a\Delta' = (1, a\Delta', a^2\Delta', \ldots, a^\ell\Delta', \ldots).
\]

Clearly \( \Delta(f)=0 \) and \( f \) is a prime element in \( k[x, y] \). Hence \( \Delta^{-1}(0)=k[f] \) and by Theorem 3, \( f \) is a line.

In case where the characteristic of \( k \) is zero we can say more. First we prove a Lemma.

**Lemma 2.** Let \( C: f(x, y)=0 \) be a line in a plane. Then \( \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)=1 \).

Proof. Since \( C \) is a line, there exists a curve \( \Gamma: g(x, y)=0 \) such that \( k[x, y]=k[f, g] \). Then there exists \( F(X, Y) \) and \( G(X, Y) \) in \( k[X, Y] \) such that

\[
F(f, g) = x \\
G(f, g) = y .
\]

Then we have

(**) In [4] our "line" and "quasi-line" are called "embedded line and line" respectively.
\[
\frac{\partial F}{\partial f} \frac{\partial f}{\partial x} + \frac{\partial F}{\partial g} \frac{\partial g}{\partial x} = 1 \tag{1}
\]
\[
\frac{\partial F}{\partial y} \frac{\partial f}{\partial y} + \frac{\partial F}{\partial g} \frac{\partial g}{\partial y} = 0 \tag{2}
\]
\[
\frac{\partial G}{\partial f} \frac{\partial f}{\partial x} + \frac{\partial G}{\partial g} \frac{\partial g}{\partial x} = 0 \tag{3}
\]
\[
\frac{\partial G}{\partial y} \frac{\partial f}{\partial y} + \frac{\partial G}{\partial g} \frac{\partial g}{\partial y} = 1 \tag{4}
\]

Now assume \( \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \subset m \) for some maximal ideal \( m \). Then from (2) either \( \frac{\partial F}{\partial g} \) or \( \frac{\partial g}{\partial y} \) is contained in \( m \). The first case cannot occur because of (1) and the second case contradicts (4). Thus \( \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \) is a unit ideal.

**Theorem 5.** Let \( k \) be an algebraically closed field of characteristic zero and let \( C: f(x, y) = 0 \) be an irreducible curve over \( k \). Then \( C \) is a line if and only if the derivation
\[
D_f = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}
\]
is locally nilpotent.

Proof. Assume that \( C: f(x, y) = 0 \) is a line. Let \( \Gamma: g(x, y) = 0 \) be a curve such that \( k[f, g] = k[x, y] \). Then there exists a locally nilpotent derivation \( \Delta \) of \( k[x, y] \) such that \( \Delta f = 0 \) and \( \Delta g = 1 \). Since \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) form a basis of derivations of \( k[x, y] \) we can write
\[
\Delta = a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \quad \text{with } a, b \in k[x, y].
\]
Since \( \Delta f = 0 \) we have
\[
a \frac{\partial f}{\partial x} - b \frac{\partial f}{\partial y} = 0 \tag{1}
\]
Let \( a \frac{\partial f}{\partial y} = b \frac{\partial f}{\partial y} = \lambda, \) i.e., \( a = \lambda \frac{\partial f}{\partial y}, \, b = \lambda \frac{\partial f}{\partial x} \). Then we have \( \Delta = \lambda D_f \).

We show that \( \lambda \in k[x, y] \). From Lemma 2 it follows that \( \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = 1 \) for some \( \alpha, \beta \) in \( k[x, y] \). Hence \( \lambda = b\alpha + a\beta \in k[x, y] \). On the other hand the existence of \( g \in k[x, y] \) such that \( \Delta g = 1 \) implies \((a, b) = 1 \). Since \( \lambda \) is a common
divisor of $a$ and $b$ we see that $\lambda \in k^*$. This means that $D_f$ is locally nilpotent. The "if" part of the Theorem is immediate from Theorem 3.

According to S. Abhyankar and T. Moh a quasi-line is a line in case of characteristic zero ([1]). In the case where the characteristic of $k$ is a positive prime integer $p$ there is a counter example.

**Example 2***

A curve $C: f(x, y)=0$ such that

$$f(x, y) \equiv y^p - x - x^{pq}$$

is a quasi-line but not a line where $p$ is the characteristic of $k$ and $q$ is an integer $\geq 2$ not divisible by $p$.

**Proof.** If we set

$$u = y - (y^p - x^q)^{\frac{1}{q}}$$

then $x \equiv u^p$ and $y \equiv u + u^{pq}$ modulo $f(x, y)$. Hence $f(x, y)=0$ is a quasi-line. To see that $c$ is not a line it suffices to show that there is no locally finite higher derivation killing $f$. Assume the contrary and let $\Delta$ be a lfih-derivation killing $f$ and $\phi = \Phi_{\Delta}$. Let

$$\phi(x) = x + \sum a_i T^i$$

$$\phi(y) = y + \sum b_i T^i.$$ 

From $\phi(f) = f$ we get

$$(y^p + \sum b_i T^{pq}) - (x + \sum a_i T^i) - (x^p + \sum a_i T^{pq})^q = y^p - x - x^{pq} \quad \cdots(1)$$

First we easily see that $a_i = 0$ if $i \not\equiv 0 \pmod{p}$. We set $a_{pq} = \alpha_i$. Then we have

$$(y^p + \sum b_i T^{pq}) - (x + \sum \alpha_i T^{pq}) - (x^p + \sum \alpha_i T^{pq})^q = y^p - x - x^{pq}$$

First we remark that

$$\alpha_i \in A^p$$

for any $i$ where $A = k[x, y]$. Now assume that $n \geq 1$. We compute the coefficient of $T^{p^2 n(q-1)}$. Since $T^{p^2 n(q-1)}$ does not appear in the middle term we have the relation:

$$b_{pq}^{p^2 n(q-1)} = \sum \alpha_{t_1} \cdots \alpha_{t_q} + q x^p \alpha_n^{p(q-1)}$$

From (2) $\alpha_{t_1} \cdots \alpha_{t_q}$, $\alpha_n^{p(q-1)}$ are in $A^p$. Hence $x^p$ must also be in $A^p$. This is

**** This example is a generalization of the one given in [4].
impossible. This proves \( n=0 \), i.e., \( x \) must be a \( \Delta \)-constant. Hence \( y \) is also a \( \Delta \)-constant. Thus there is no non-trivial \( \text{ihih} \)-derivation \( \Delta \) such that \( \Delta(f) = 0 \).

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\textbf{References}


\textit{Added in Proof.} In Theorem 3 we assumed that \( k \) is algebraically closed. This assumption is essential as is shown in the following Example. Let \( B = \mathbb{R}[X, Y]/X^2 + Y^2 + 1 \). Then \( B \) is a UFD and satisfies \( B^* = \mathbb{R}^* \). The ring \( A = B[Z] \) satisfies all the requirement in Theorem 3, but \( A \) is not a polynomial ring of two variables over the field \( \mathbb{R} \).