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The Theory of Construction of Finite Semigroups I

By Takayuki TAMURA

By a semigroup we mean a non-void set of elements x, y, z, $\cdots$ closed under an associative binary operation: $(xy)z=x(yz)$. If the number of elements of a semigroup S is finite, we call S a finite semigroup. The structure of a finite semigroup or a semigroup satisfying some conditions like finiteness was studied by Suschkewitsch, Clifford, Rees, and Schwarz [1], [2], [3], [4], [5]. However the classification of all finite semigroups and theory of construction of finite semigroups have not yet been discussed systematically. Though all types of semigroups of order 2, 3 and 4 were determined in [9], [10], the method was so unsystematic or inconvenient that it was not applicable to the general case. The author has investigated finite semigroups from the standpoint of a greatest semilattice decomposition, and this new method has been already applied to the case of order 5 [11].

Generally a semigroup is decomposed into a sum of special semigroups called s-indecomposable semigroups. Since s-decomposable semigroups are constructed out of a semilattice and s-indecomposable semigroups by means of the process of compositions, the study of finite semigroups is reduced to that of semilattices and s-indecomposable semigroups. Semilattices and s-indecomposable semigroups are subdivided into a few classes, and their construction must be considered in the different way from the case of s-decomposable semigroups.

In the series of these papers, we shall construct theoretically all finite s-decomposable semigroups and all finite s-indecomposable semigroups, except finite non-commutative simple groups, by induction with respect to the number of elements. Here, of course, the discussion on the construction of finite groups will be excluded, except only one case of c-indecomposable, non-simple groups.

The present paper is the first part of this series of the papers which consist of six parts:

I Greatest decomposition of a semigroup.

II Compositions of semigroups and finite s-decomposable semigroups.

III Finite c-indecomposable groups.

IV Finite unipotent semigroups.
V Finite c–indecomposable semigroups.
VI Finite s–indecomposable semigroups.

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I. Greatest Decomposition of a Semigroup

In the previous papers [12], [13], [14], we obtained the theory of greatest decomposition of a semigroup of special type. Afterwards the writer found that our discussion was connected with the theory of algebraic system due to Prof. K. Shoda [6], [7], and furthermore the writer has recently obtained that the theorems as to the greatest decomposition are proved by introducing a closure operation.

The purpose of this paper is to treat the problem of the greatest decomposition as extensively as possible, and to refer to a remarkable property of semilattice decomposition. In § 1 and § 2 we shall arrange lemmas and theorems on closure operations defined in a complete lattice, and in § 5 and § 6 we shall apply the results to the proof of the existence of the greatest μ–decomposition of a semigroup which is not necessarily finite. It is interesting that a semigroup S is decomposed into a union of s–indecomposable subsemigroups in the case of the greatest semilattice decomposirion of S (§ 9). The analogous property, however, is not always seen in the greatest idempotent decomposition and the greatest commutative decomposition (§ 10). Finally we shall refer to a classification of finite s–indecomposable semigroups (§ 11).

§ 1. Closure Operations (1)

In these paragraphs § 1—3, A is a complete lattice having an ordering $\leqq$ in which a symbol $\sup\limits_{\alpha} x_\alpha$ or $\bigvee\limits_{\alpha} x_\alpha$ denotes[1] the least upper bound of a family $\{x_\alpha\}$, and $\inf\limits_{\alpha} x_\alpha$ or $\bigwedge\limits_{\alpha} x_\alpha$ does[1] the greatest lower bound of $\{x_\alpha\}$. Consider a closure operation[2] φ in A, i.e. an operation which maps any $x \in A$ to $\varphi(x) \in A$ satisfying conditions

$$(1.1) \qquad x \leqq \varphi(x)\,, \qquad\qquad (1.2) \qquad x \leqq y \textit{ implies } \varphi(x) \leqq \varphi(y)\,.$$

Let C be the set of all φ–closed elements x by which we mean $x = \varphi(x)$. C is not empty because the greatest element of A belongs to C. The

1) Especially when $\alpha = 1, 2, \cdots, k$, we may write $x_1 \cup \cdots \cup x_k$, $x_1 \cap \cdots \cap x_k$.
2) Regarding closure operations, see [8].

following lemma is very familiar [8].

Lemma 1. *C is a complete lattice in which a greatest lower bound of any two elements coincides with one defined in A.*

The following corollary will be useful later.

Corollary 1. *All φ-closed elements x in which $a \leqq x$ form a complete lattice.*

Let Γ be the set of all closure operations in $A: \Gamma = \{\varphi_\gamma ; \gamma \in \mathfrak{C}\}$. Γ is a complete lattice in which an ordering $\varphi \leqq \psi$ is defined as $\varphi(x) \leqq \psi(x)$ for all x, so that a least upper bound $\sup_\alpha \varphi_\alpha$ (or denoted by $\bigvee_\alpha \varphi_\alpha$) and a greatest lower bound $\inf_\alpha \varphi_\alpha$ (or $\bigwedge_\alpha \varphi_\alpha$) of a family $\{\varphi_\alpha\}$, $\alpha \in \mathfrak{A} \subset \mathfrak{C}$, are given as

$$(1.3) \qquad (\sup_\alpha \varphi_\alpha)(x) = \sup_\alpha \varphi_\alpha(x), \qquad (\inf_\alpha \varphi_\alpha)(x) = \inf_\alpha \varphi_\alpha(x).$$

If a multiplication $\varphi\psi$ of φ and ψ is defined as $(\varphi\psi)(x) = \varphi(\psi(x))$,[3] then Γ forms a semigroup with respect to this multiplication. The following inequalities are easily shown by the above definitions.

(1.4) $\varphi \leqq \psi$ *implies* $\eta\varphi \leqq \eta\psi$ *and* $\varphi\eta \leqq \psi\eta$ *for every* $\eta \in \Gamma$. *So* $\varphi \leqq \psi$, $\eta \leqq \xi$ *imply* $\varphi\eta \leqq \psi\xi$.

(1.5) $\varphi \leqq \varphi\psi$, $\psi \leqq \varphi\psi$.

In particular, if $\varphi\psi = \psi$, we say that ψ is *φ-invariant.*

Lemma 2. *The following* (1.6), (1.7), (1.8) *and* (1.9) *are all equivalent.*

(1.6) $\psi \leqq \varphi = \varphi^2$, (1.7) $\varphi\psi = \psi\varphi = \varphi = \varphi^2$,

(1.8) $\psi\varphi = \varphi = \varphi^2$, (1.9) $\varphi\psi = \varphi = \varphi^2$.

Proof. We can prove them in the following way,

$$(1.6) \rightarrow (1.7) \begin{array}{l} \nearrow (1.8) \searrow \\ \searrow (1.9) \nearrow \end{array} (1.6)$$

(1.6)→(1.7): by (1.4) and (1.5), $\varphi \leqq \varphi\psi \leqq \varphi^2 = \varphi$ and $\varphi \leqq \psi\varphi \leqq \varphi^2 = \varphi$. (1.7)→(1.8) and (1.7)→(1.9) are trivial, (1.8)→(1.6) and (1.9)→(1.6) are due to (1.5).

Lemma 3. *Let* $\varphi = \sup_\alpha \varphi_\alpha$. *An element x is φ-closed if and only if*

3) Especially we denote $\underbrace{\varphi\varphi\cdots\varphi}_{n}$ by φ^n.

it is φ_α-closed for all α.

Proof. If $\varphi_\alpha(x)=x$ for all α, then $(\sup_\alpha \varphi_\alpha)(x) = \sup_\alpha (\varphi_\alpha(x)) = x$. Conversely if $\varphi(x)=x$, then $\varphi_\alpha(x) \leqq \varphi(x)=x$, so $\varphi_\alpha(x)=x$. Similarly we have

Lemma 4. *Let $\varphi=\varphi_1\varphi_2\cdots\varphi_k$. An element x is φ-closed if and only if it is φ_i-closed for $i=1, \cdots, k$.*

Now we define $\bar{\varphi}$ for $\varphi \in \Gamma$ in the following manner.

$$\bar{\varphi}(x) = \inf\{y\,;\ x \leqq y = \varphi(y)\}\ ,$$

which means the greatest lowest bound of the set $\{y\,;\ x \leqq y = \varphi(y)\}$. Immediately we see that $\bar{\varphi}(x)$ fulfils not only (1.1) and (1.2) but also

$$\bar{\varphi}^2(x) = \bar{\varphi}(x)^{4)} \quad \text{i.e.} \quad \bar{\varphi}^2 = \bar{\varphi}\,; \tag{1.10}$$

$\bar{\varphi}$ is an idempotent closure operation in A, and we shall soon see that $\bar{\varphi}(x)$ itself belongs to the set $\{y\,;\ x \leqq y = \varphi(y)\}$ because of (1.12). Since $\varphi(x)$ is a lower bound of the set $\{y\,;\ x \leqq y = \varphi(y)\}$ i.e. $\varphi(x) \leqq \varphi(y) = y$ (cf. (1.2)), we have

$$\varphi \leqq \bar{\varphi}\,. \tag{1.11}$$

By Lemma 2 and (1.10), we get

$$\psi \leqq \varphi \quad \textit{implies } \psi\bar{\varphi} = \bar{\varphi}\psi = \bar{\varphi}\,. \tag{1.12}$$

From this, $\bar{\varphi}(x) \in \{y\,;\ x \leqq y = \psi(y)\}$ and so $\bar{\psi}(x) \leqq \bar{\varphi}(x)$. Hence

$$\psi \leqq \varphi \quad \textit{implies } \ \bar{\psi} \leqq \bar{\varphi}\,. \tag{1.13}$$

Similarly, since $x \leqq \bar{\varphi}(x) = \bar{\varphi}^2(x)$, $\bar{\bar{\varphi}}(x) \leqq \bar{\varphi}(x)$; so

$$\bar{\bar{\varphi}} = \bar{\varphi}\,. \tag{1.14}$$

Thus the mapping $\varphi \rightarrow \bar{\varphi}$ is also an idempotent closure operation in Γ.

Lemma 5. *$\varphi^2=\varphi$ if and only if $\varphi=\bar{\varphi}$.*

Proof. If $\varphi^2=\varphi$, then $x \leqq \varphi(x) = \varphi(\varphi(x))$; and by the definition of $\bar{\varphi}$, we get $\bar{\varphi}(x) \leqq \varphi(x)$ for all x, so that $\varphi=\bar{\varphi}$ because of (1.11). The converse is clear as (1.10) shows.

Lemma 6. $$\bar{\varphi} = \inf\{\xi\,;\ \varphi \leqq \xi = \bar{\xi}\} \tag{1.15}$$

4) See [8].

$$= \inf\{\xi ;\ \varphi \leqq \xi = \xi^2\} \tag{1.16}$$
$$= \inf\{\xi ;\ \varphi\xi = \xi\varphi = \xi = \xi^2\} \tag{1.17}$$
$$= \inf\{\xi ;\ \varphi\xi = \xi = \xi^2\} \tag{1.18}$$
$$= \inf\{\xi ;\ \xi\varphi = \xi = \xi^2\} \tag{1.19}$$
$$= \inf\{\xi ;\ \varphi\xi = \xi\} \tag{1.20}$$
$$= \inf\{\xi ;\ \varphi\xi = \xi\varphi = \xi\} \tag{1.21}$$

Proof. $\bar{\varphi} = (1.15)$ is obvious since $\varphi \to \bar{\varphi}$ is considered as a closure operation in Γ. We get $(1.15) = (1.16)$ by Lemma 5, and $(1.16) = (1.17) = (1.18) = (1.19)$ by Lemma 2. We prove $\bar{\varphi} = (1.20)$ and $\bar{\varphi} = (1.21)$. For all ξ such that $\varphi\xi = \xi$, we have $\bar{\varphi} \leqq \xi$ because $x \leqq \xi(x) = \varphi(\xi(x))$; therefore $\bar{\varphi} \leqq (1.20)$. Meanwhile $\varphi\bar{\varphi} = \bar{\varphi}$ by (1.11) and (1.12), so that $\bar{\varphi} \geqq (1.20)$. Thus we have proved $\bar{\varphi} = (1.20)$. We can easily see $(1.20) \leqq (1.21) \leqq (1.17)$, while we have had $\bar{\varphi} = (1.20) = (1.17)$. Hence $\bar{\varphi} = (1.21)$.

§ 2. Closure Operations (2)

Let $\Gamma = \{\varphi_\gamma ;\ \gamma \in \mathfrak{C}\}$. For a family $\{\varphi_\alpha\}$, $\alpha \in \mathfrak{A} \subset \mathfrak{C}$, of elements of Γ, we define ζ as

$$\zeta(x) = \inf\{u ;\ x \leqq u = \varphi_\alpha(u) \quad \text{for all } \alpha \in \mathfrak{A}\}\ .$$

$\zeta(x)$ is easily proved to be an idempotent closure operation in A, consequently $\zeta = \bar{\zeta}$. As $\zeta(x) \geqq \inf\{u ;\ x \leqq u = \varphi_\alpha(u)\} = \bar{\varphi}_\alpha(x)$, we get

$$\text{(2.1)} \qquad \varphi_\alpha \leqq \bar{\varphi}_\alpha \leqq \zeta \leqq \bar{\zeta} \quad \textit{for all} \quad \alpha \ni \mathfrak{A}\ .$$

This ζ is denoted by $\bigvee_\alpha \varphi_\alpha$, especially when $\alpha = 1, 2$, it is denoted by $\zeta = \varphi_1 \vee \varphi_2$. Then it is easily shown that

$$\text{(2.2)} \qquad \textit{if} \quad \varphi_\alpha \leqq \psi_\alpha \quad \textit{for all} \quad \alpha,\ \bigvee_\alpha \varphi_\alpha \leqq \bigvee_\alpha \psi_\alpha$$

and particularly if $\varphi_\alpha = \eta$ for all α, then $\bigvee_\alpha \varphi_\alpha = \eta$. From (2.1)

$$\bigvee_\alpha \varphi_\alpha \leqq \bigvee_\alpha \bar{\varphi}_\alpha \leqq \zeta = \bigvee_\alpha \bar{\varphi}_\alpha\ .$$

Accordingly we have

Lemma 7. $$\bigvee_\alpha \varphi_\alpha = \bigvee_\alpha \bar{\varphi}_\alpha\ .$$

In the same way as Lemma 6,

Lemma 8. $$\zeta = \bigvee_\alpha \varphi_\alpha = \inf\{\xi ;\ \varphi_\alpha \leqq \xi = \bar{\xi} \quad \text{for all } \alpha\} \tag{2.3}$$
$$= \inf\{\xi ;\ \varphi_\alpha \leqq \xi = \xi^2 \quad \text{for all } \alpha\} \tag{2.4}$$

$$= \inf \{\xi ;\ \varphi_\alpha \xi = \xi \varphi_\alpha = \xi = \xi^2 \text{ for all } \alpha\} \quad (2.5)$$
$$= \inf \{\xi ;\ \varphi_\alpha \xi = \xi = \xi^2 \text{ for all } \alpha\} \quad (2.6)$$
$$= \inf \{\xi ;\ \xi \varphi_\alpha = \xi = \xi^2 \text{ for all } \alpha\} \quad (2.7)$$
$$= \inf \{\xi ;\ \varphi_\alpha \xi = \xi \text{ for all } \alpha\} \quad (2.8)$$
$$= \inf \{\xi ;\ \varphi_\alpha \xi = \xi \varphi_\alpha = \xi \text{ for all } \alpha\} \quad (2.9)$$

and the equalities are valid even if $\bar{\varphi}_\alpha$ takes place of φ_α.

Moreover we prove

Lemma 9. $\zeta = \overline{\sup_\alpha \varphi_\alpha} = \overline{\sup_\alpha \bar{\varphi}_\alpha}$.

Proof. Since $\varphi_\alpha \leqq \sup_\alpha \varphi_\alpha \leqq \zeta$ from (2.1), $\varphi_\alpha \leqq \bar{\varphi}_\alpha \leqq \overline{\sup_\alpha \varphi_\alpha} \leqq \bar{\zeta} = \zeta$, where we see $\overline{\sup_\alpha \varphi_\alpha} \in \{\xi ;\ \varphi_\alpha \leqq \xi = \bar{\xi}$ for all $\alpha\}$, so that $\overline{\sup_\alpha \varphi_\alpha} \geqq \zeta$ (See (2.3)). Therefore we get $\zeta = \overline{\sup_\alpha \varphi_\alpha} = \overline{\sup_\alpha \bar{\varphi}_\alpha}$ where the last term is obtained from Lemma 7.

Now $\sup_{\substack{\alpha_1, \cdots, \alpha_k \\ k \text{ fixed}}} \varphi_{\alpha_1}^{n_1} \cdots \varphi_{\alpha_k}^{n_k}$ denotes a least upper bound of the set of all $\varphi_{\alpha_1}^{n_1} \cdots \varphi_{\alpha_k}^{n_k}$ where $\alpha_1, \cdots, \alpha_k$ vary throughout $\mathfrak{A}$ but the positive numbers $k, n_1, \cdots, n_k$ are fixed. Also $\sup_{\substack{k, \alpha_1, \cdots, \alpha_k \\ \{n_i\} \text{ fixed}}} \varphi_{\alpha_1}^{n_1} \cdots \varphi_{\alpha_k}^{n_k}$ denotes a least upper bound of the set of all elements $\varphi_{\alpha_1}^{n_1} \cdots \varphi_{\alpha_k}^{n_k}$ where $\alpha_1, \cdots, \alpha_k$, and k vary but the sequence $n_1, \cdots, n_k, \cdots$ of positive numbers are fixed. We easily see

Lemma 10. $\zeta = \overline{\sup_{\substack{\alpha_1, \cdots, \alpha_k \\ k \text{ fixed}}} \varphi_{\alpha_1}^{n_1} \cdots \varphi_{\alpha_k}^{n_k}} = \overline{\sup_{\substack{k, \alpha_1, \cdots, \alpha_k \\ \{n_i\} \text{ fixed}}} \varphi_{\alpha_1}^{n_1} \cdots \varphi_{\alpha_k}^{n_k}}$.

We add that these equalities hold even if the range where $\alpha_1, \cdots, \alpha_k$ vary satisfies one of the following conditions:

(1) $\alpha_i \neq \alpha_j, \quad i \neq j$.

(2) $(\alpha_1, \cdots, \alpha_k)$ is a combination of k different elements which are taken from $\mathfrak{A}$. Of course we can replace $\bar{\varphi}_{\alpha_i}$ for φ_{α_i} in the equalities of Lemma 10. Particularly, for a finite number of φ_α,

Lemma 11. (2.10) $\overline{\varphi \vee \psi} = \overline{\bar{\varphi} \vee \bar{\psi}} = \overline{\varphi \cup \psi} = \overline{\varphi\psi} = \overline{\psi\varphi} = \overline{\bar{\varphi}\bar{\psi}} = \overline{\bar{\psi}\bar{\varphi}}$.

(2.11) $\overline{\bigvee_{1 \leqq \alpha \leqq k} \varphi_\alpha} = \overline{\sup_{1 \leqq \alpha \leqq k} \varphi_\alpha} = \overline{\varphi_1 \varphi_2 \cdots \varphi_k}$.

(2.12) $\bar{\varphi} = \overline{\sup_k \varphi^k} = \overline{\sup_k \varphi^{nk}}$.

Finally we shall discuss a relation between $\overline{\varphi\psi}$ and $\bar{\varphi}\bar{\psi}$. Since $\bar{\varphi} \leqq \overline{\varphi\psi}$ and $\bar{\psi} \leqq \overline{\varphi\psi}$, generally it holds that $\bar{\varphi}\bar{\psi} \leqq \overline{\varphi\psi}$ by (1.4) and (1.10).

However the equality $\bar{\varphi}\bar{\psi}=\overline{\varphi\psi}$ does not necessarily hold.

Lemma 12. *The following conditions are equivalent:*

(2.13) $\bar{\varphi}\bar{\psi}$ *is* ψ*-invariant*: $\psi(\bar{\varphi}\bar{\psi})=\bar{\varphi}\bar{\psi}$,

(2.14) $\varphi \vee \psi \leqq \bar{\varphi}\bar{\psi}$,

(2.15) $\bar{\varphi}\bar{\psi}=\overline{\varphi\psi}$,

(2.16) $\bar{\varphi}\bar{\psi}=\bar{\psi}\bar{\varphi}$.

Proof. (2.13)→(2.14): As $\varphi(\bar{\varphi}\bar{\psi})=(\varphi\bar{\varphi})\bar{\psi}=\bar{\varphi}\bar{\psi}$ by (1.12), $\bar{\varphi}\bar{\psi}$ is ψ-invariant as well as φ-invariant. According to (2.8), $\varphi \vee \psi \leqq \bar{\varphi}\bar{\psi}$. (2.14)→(2..15): By (2.10) $\varphi \vee \psi=\overline{\varphi\psi} \leqq \bar{\varphi}\bar{\psi}$. Hence $\bar{\varphi}\bar{\psi}=\overline{\varphi\psi}$. (2.15)→(2.16): By (2.10) and (2.15), $\bar{\varphi}\bar{\psi}=\overline{\varphi\psi}=\overline{\psi\varphi}=\bar{\psi}\bar{\varphi}$. Lastly (2.16)→(2.13): $\psi(\bar{\varphi}\bar{\psi})=\psi(\bar{\psi}\bar{\varphi})=(\psi\bar{\psi})\bar{\varphi}=\bar{\psi}\bar{\varphi}=\bar{\varphi}\bar{\psi}$. Thus the proof has been accomplished.

Theorem 1. *If* $\bar{\varphi}\bar{\psi}$ *is* ψ*-invariant, then* $\bar{\varphi} \vee \bar{\psi}=\overline{\varphi \cup \psi}=\overline{\varphi\psi}=\bar{\varphi}\bar{\psi}$.

§ 3. Completely Additive Closure Operation.

Let $A=\{x_\beta;\ \beta \in \mathfrak{M}\}$ be a complete lattice. If a closure operation φ in A satisfies $\varphi(\sup_\beta x_\beta) \leqq \sup_\beta \varphi(x_\beta)$, $\beta \in \mathfrak{N} \subset \mathfrak{M}$, being equivalent to $\varphi(\sup_\beta x_\beta)=\sup_\beta \varphi(x_\beta)$, then φ is called *completely additive*. Let Δ be the set of all completely additive closure operations in A.

Lemma 13. (3.1) $\varphi \in \Delta$, $\psi \in \Delta$ *imply* $\varphi\psi \in \Delta$.

(3.2) $\varphi_\alpha \in \Delta$, $\alpha \in \mathfrak{A} \subset \mathfrak{C}$ *imply* $\sup_\alpha \varphi_\alpha \in \Delta$.

(3.3) *Completely distributive*: $\psi(\sup_\alpha \varphi_\alpha)=\sup_\alpha (\psi\varphi_\alpha)$,

$$(\sup_\alpha \varphi_\alpha)\psi=\sup_\alpha (\varphi_\alpha\psi),$$

particularly, $\psi(\varphi_1 \cup \varphi_2)=\psi\varphi_1 \cup \psi\varphi_2$,

$$(\varphi_1 \cup \varphi_2)\psi=\varphi_1\psi \cup \varphi_2\psi.$$

Proof. As far as (3.1) and (3.2) are concerned, it is sufficient to prove the complete additivity

(3.1): $(\varphi\psi)(\sup_\beta x_\beta)=\varphi(\psi(\sup_\beta x_\beta))=\varphi(\sup_\beta \psi(x_\beta))=\sup_\beta(\varphi\psi(x_\beta))$.

(3.2): $(\sup_\alpha \varphi_\alpha)(\sup_\beta x_\beta)=\sup_\alpha(\varphi_\alpha(\sup_\beta x_\beta))=\sup_\alpha(\sup_\beta \varphi_\alpha(x_\beta))$

$=\sup_\beta(\sup_\alpha(\varphi_\alpha(x_\beta)))=\sup_\beta((\sup_\alpha \varphi_\alpha)(x_\beta))$.

(3.3): For any $x \in A$, $(\psi(\sup_\alpha \varphi_\alpha))(x)=\psi(\sup_\alpha \varphi_\alpha(x))=\sup_\alpha((\psi\varphi_\alpha)(x))$

$= (\sup_\alpha(\psi\varphi_\alpha))(x)$. Similarly $((\sup_\alpha \varphi_\alpha)\psi)(x) = (\sup_\alpha (\varphi_\alpha\psi))(x)$.

Theorem 2. *If* $\varphi_\alpha \in \Delta$, $\alpha \in \mathfrak{A}$, *then* $\sup_\alpha \varphi_\alpha = \overline{\sup_\alpha \varphi_\alpha}$.

Proof. By the completely distributive law and Lemma 10,

$$(\sup_\alpha \varphi_\alpha)(\sup_\alpha \varphi_\alpha) = \sup_\alpha((\sup_\alpha \varphi_\alpha)\varphi_\alpha) = \sup_\alpha(\sup_\gamma \varphi_\gamma\varphi_\alpha) = \sup_{\gamma,\alpha} \varphi_\gamma\varphi_\alpha = \sup_\alpha \varphi_\alpha.$$

According to Lemma 5, we have $\sup_\alpha \varphi_\alpha = \overline{\sup_\alpha \varphi_\alpha}$.

Corollary 2. *If* $\varphi \in \Delta$, *then* $\bar{\varphi} = \sup_n \varphi^n$. (See (2.12))

§ 4. Examples

Example 1. Let E be an abstract set and Ω be the system[5] of all subsets X of E. Ω is a complete lattice under the inclusion relation of X. Let f be a mapping which associates an element x of E with a subset $f(x)$ of E. An operation which associates $X \in \Omega$ with $\varphi(X) \in \Omega$ is defined as follows:

$$\varphi(X) = X \cup \sum_{x \in X} f(x)\ ^{6)}.$$

Then φ fulfils not only (1.1) and (1.2) but also complete additivity:

$$\varphi(\sum_\alpha X_\alpha) = \sum_\alpha X_\alpha \cup \sum_{x \in \Sigma X_\alpha} f(x) = \sum_\alpha X_\alpha \cup \sum_\alpha \sum_{x \in X_\alpha} f(x) = \sum_\alpha (X_\alpha \cup \sum_{x \in X_\alpha} f(x)) = \sum_\alpha \varphi(X_\alpha).$$

f is called a fundamental mapping on E, and φ is called a closure operation constructed from f.

Example 2. Let E and Ω remain as they are in the former example. Let E^n be a product set i.e. the set $\{(x_1, \cdots, x_n)\,;\ x_i \in E, i=1, \cdots, n\}$. If we are given a mapping f which associates $x \in E^n$ with a subset $f(x)$ of E, and if φ is defined as

$$\varphi(X) = X \cup \sum_{x \in X^n} f(x) \quad \text{where } X^n = \{(x_1, \cdots, x_n)\,;\ x_i \in X, i = 1, \cdots, n\},$$

then φ is a closure operation in Ω. This f is called a fundamental mapping on E^n, and φ is called a closure operation constructed from f.

Remark. Generally there is no dependency between the two conditions: $\bar{\varphi}\bar{\psi} = \bar{\psi}\bar{\varphi}$ and $\varphi\psi = \psi\varphi$. The following two examples show this fact.

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5) Empty set is contained in Ω.

6) Σ and $\cup$ mean set union.

Example 3. Let I be the set of all positive integers. The system Ω of all subsets of I is a complete lattice under the inclusion relation. Two closure operations φ and ψ in Ω are defined as follows.

$$\varphi(X) = X \cup \sum_{x \in X} f(x) \qquad \text{where } f(x) = \{x+2\},^{7)}$$

$$\psi(X) = \begin{cases} X & \text{if } X \text{ is finite,} \\ X \cup \sum_{x \in X} g(x) & \text{if } X \text{ is infinite,} \end{cases}$$

where $g(x) = \{x+1\}$. It is easily shown that $\varphi \in \Delta$ and $\psi \in \Gamma$.

If X is finite, $\qquad \varphi(\psi(X)) = \varphi(X) = \psi(\varphi(X))$,

if X is infinite, $\qquad \varphi(\psi(X)) = \sum_{x \in X} \{x,\ x+1,\ x+2,\ x+3\} = \psi\varphi(X)$ [8].

Hence $\varphi\psi = \psi\varphi$. However $\bar{\varphi}\bar{\psi} \neq \bar{\psi}\bar{\varphi}$. In fact, for example,

$$\bar{\varphi}\bar{\psi}(\{2, 4, 6\}) = \bar{\varphi}(\{2, 4, 6\}) = \{2n\,;\ n \geqq 1\} ,$$
$$\bar{\psi}\bar{\varphi}(\{2, 4, 6\}) = \bar{\psi}(\{2n\,;\ n \geqq 1\}) = \{n\,;\ n \geqq 1\} .$$

Example 4. Let I, Ω, and $g(x)$ remain as they are in Example 3. Two operators ξ and η are defined as

$$\xi(X) = X \cup \sum_{x \in X} g(x), \quad \eta(X) = X \cup \sum_{x \in X} h(x) \text{ where } h(x) = \{2x\} .$$

Of course $\xi, \eta \in \Delta$. Then $\bar{\xi}\bar{\eta} = \bar{\eta}\bar{\xi}$ though $\xi\eta \neq \eta\xi$.

In fact

$$\xi\eta(\{1, 2\}) = \xi(\{1, 2, 4\}) = \{1, 2, 3, 4, 5\} ,$$
$$\eta\xi(\{1, 2\}) = \eta(\{1, 2, 3\}) = \{1, 2, 3, 4, 6\} ,$$

hence $\qquad \xi\eta \neq \eta\xi ,$

while, for any finite or infinite subset $\{x_i\}$, $i = 1, 2, \cdots$,

$$\bar{\xi}\bar{\eta}(\{x_i\}) = \bar{\xi}(\{x_i,\ 2^n x_i\,;\ n \geqq 1,\ i = 1, 2, \cdots\}) = \{x_i,\ x_i + n\,;\ n \geqq 1,\ i = 1, 2, \cdots\},$$
$$\bar{\eta}\bar{\xi}(\{x_i\}) = \bar{\eta}(\{x_i,\ x_i + n\,;\ n \geqq 1,\ i = 1, 2, \cdots\}) = \{x_i,\ x_i + n\,;\ n \geqq 1,\ i = 1, 2, \cdots\},$$

therefore $\qquad \bar{\xi}\bar{\eta} = \bar{\eta}\bar{\xi} .$

§ 5. Decomposition of a Semigroup

If a semigroup S is homomorphic to a semigroup T, then S is divided into disjoint classes of elements which are mapped to the same element of T. This is called a decomposition of S. All decomposition form a

7) $\{x+2\}$ shows a set of only one element $x+2$.

8) $\{x, x+1, x+2, x+3\}$ is a set of four elements $x, x+1, x+2, x+3$.

partially ordered set under the ordering $\delta_1 \leqq \delta_2$ meaning that a decomposition δ_2 is a refinement of a decomposition δ_1.

Let S be a semigroup and $F(S)$ be the free semigroup generated by elements of S. X denotes a set of relations in $F(S)$ i.e. a set of formulas showing which elements are equivalent. X is also regarded as a set of ordered pairs of elements of $F(S)$. Let $F(S, X)$ be a factor semigroup of $F(S)$ which is got by using the relations, which compose X, finitely many times. According to [6], [7],

Fundamental Theorem. *$F(S)$ is homomorphic to S, and $F(S, A)$ is isomorphic to S for a suitable A. If $B \subset C$, then $F(S, B)$ is homomorphic to $F(S, C)$, and the converse is also true.*

Let $F(S, A)$ be isomorphic to S. If a semigroup S is homomorphic to a semigroup T, then a decomposition of $F(S)$ is obtained, i.e. there is a set X of relations in $F(S)$ such that $F(S, X)$ is isomorphic to T and $A \subset X$. We can restrict X to be one which fulfils

(1) $(x, x) \in X$ for all $x \in F(S)$,

(2) $(x, y) \in X$ implies $(y, x) \in X$,

(3) $(x, y) \in X$, $(y, z) \in X$ imply $(x, z) \in X$,

(4) $(x, y) \in X$ implies $(ax, ay) \in X$, $(xa, ya) \in X$ for every $a \in F(S)$.

Such an X is called a congruent set of relations. Thus the decomposition of S is associated with congruent set X of relations in $F(S)$.

In these paragraphs § 5 and § 6 E denotes the set of all ordered pairs of elements of $F(S)$: $E = \{(x, y)\ ;\ x, y \in F(S)\}$, and Ω denotes the system of all subsets X of E. We define four fundamental mappings f_1, f_2, f_3, and f_4 as follows:

$$f_1((x, y)) = \{(x, x),\ (y, y)\}\ ,\quad f_2((x, y)) = \{(y, x)\}\ ,$$

$$f_3((x_1, y_1), (x_2, y_2)) = \begin{cases} \text{empty} & \text{if } y_1 \neq x_2\,, \\ \{(x_1, y_2)\} & \text{if } y_1 = x_2\,, \end{cases}$$

$$f_4((x, y)) = \{(ax, ay),\ (xa, ya)\ ;\ \text{for any } a \in F(S)\}$$

where f_1, f_2, and f_4 are fundamental mappings on E, and f_3 is a fundamental mapping on E^2 (See Example 2). Denote by c_1, c_2, c_3 and c_4 the closure operations in Ω constructed from f_1, f_2, f_3, and f_4 respectively.

Let $c_0 = c_1 \cup c_2 \cup c_3 \cup c_4$. Immediately we have

Lemma 14. *X is a congruent set of relations in $F(S)$ if and only if X is c_0-closed.*[9]

9) See § 1.

Let A be a congruent set of relations in $F(S)$ and let S be isomorphic to $F(S, A)$. The system of all decompositions of S and the system of all congruent sets X containing A are isomorphic as partially ordered sets, in which the ordering of the latter is considered as set inclusion relation. Since all congruent sets which contain A form a complete lattice by Corollary 1, we have

Theorem 3. *All decompositions of a semigroup form a complete lattice.*

This theorem has been also established in [6], [7].

§ 6. μ-Decomposition

Let us consider a power-product of variables $x_1, x_2, \cdots, x_n$ varying over S and constant elements $a, b, \cdots$ of S where we may have no constant element at all but we must contain a variable at least. Such a form is termed a monomial of $x_1, \cdots, x_n$ and is denoted by $f(x_1, \cdots, x_n)$ etc. or by f if we need not specify variables. For example, $f(x)=ax$, $f(x_1, x_2, x_3)=a_4x_1^3x_2^2bx_1x_3^3x_2$.

Now T_δ denotes a factor semigroup of S due to a decomposition δ of S. Consider a set $\mathfrak{D}$ of all decompositions δ of S such that T_δ satisfies the following conditions:

There is a system of monomials of variables $x_{j1}, \cdots, x_{jn}$ and constant elements $a_\lambda'(\lambda=1, \cdots, k)$ in T_δ,

$$p_{ji}(x_{j1}, \cdots, x_{jn}), \quad q_{ji}(x_{j1}, \cdots, x_{jn}), \quad i=1, \cdots, m_j, \quad m_j \text{ depending on } j,$$
$$r_j(x_{j1}, \cdots, x_{jn}), \quad s_j(x_{j1}, \cdots, x_{jn}), \quad j=1, \cdots, l$$

such that, for each j $(j=1, \cdots, l)$,

$$p_{ji}(x_{j1}, \cdots, x_{jn}) = q_{ji}(x_{j1}, \cdots, x_{jn}), \quad i=1, \cdots, m_j,$$

imply
$$r_j(x_{j1}, \cdots, x_{jn}) = s_j(x_{j1}, \cdots, x_{jn}),$$

where the constant elements $a_\lambda'(\lambda=1, \cdots, k)$ contained here shall be homomorphic images of the fixed elements $a_\lambda(\lambda=1, \cdots, k)$ of S respectively, and a_λ and the forms of the monomials p_{ji}, q_{ji}, r_j, s_j do not depend on δ. Then a decomposition δ is called $\mu(p_{ji}, q_{ji}, r_j, s_j)$-decomposition of S or μ-decomposition of S, and the set $\mathfrak{D}$ is called the set of μ-decompositions, and we say that a factor semigroup T_δ fulfils μ-condition. If T_δ satisfies two μ-conditions μ_1, μ_2 at the same time, then T_δ is said to satisfy an *adjoint condition* $\mu_1 \cup \mu_2$ of μ_1 and μ_2.

We give simple examples of μ-conditions:

Example 5. The semigroup $\{x\}$ composed of only one element x fulfils every μ-condition, because $p_{ji}=q_{ji}=r_j=s_j=x$.

Example 6. c-condition (commutativity): $x=x,\ y=y \rightarrow xy=yx$ where "$x=x,\ y=y$" means "every $x,\ y$".

Example 7. i-condition (idempotency): $x=x \rightarrow x=x^2$.

Example 8. s-condition (semilattice): this is an adjoint condition of i and c,

$$x=x,\ y=y \rightarrow xy=yx, \quad z=z \rightarrow z=z^2.$$

s-decomposition, c-decomposition, and i-decomposition will play an important rôle in our theory.

Example 9. rca-condition (right cancelation): $xz=yz \rightarrow x=y$.
lca-condition (left cancelation): $zx=zy \rightarrow x=y$.
ca-condition (cancelation): adjoint condition of rca and lca.

$$xz=yz \rightarrow x=y, \quad uv=uw \rightarrow v=w.$$

Example 10. u-condition (unipotency or non-potency):

$$x^2=x, \quad y^2=y \rightarrow x=y.$$

Example 11. $ru(a)$-condition is the condition that the homomorphic image a' of a fixed element a of S is a right unit of T_δ.

$$x=x \rightarrow xa'=x.$$

$lu(b)$-condition that the homomorphic image b' of a fixed element b of S is a left unit of T_δ. $x=x \rightarrow b'x=x$.

$u(a)$-condition is an adjoint condition of $lu(a)$ and $ru(a)$:

$$x=x \rightarrow xa'=x, \quad y=y \rightarrow a'y=y.$$

Example 12. $rz(a)$-condition that the homomorphic image a' of a fixed element a of S is a right zero of T_δ. $x=x \rightarrow xa'=a'$.
$lz(b)$-condition that the homomorphic image b' of a fixed element b of S is a left zero of T_δ: $x=x \rightarrow b'x=b'$.
$z(a)$-condition is an adjoint condition of $rz(a)$ and $lz(a)$.

$$x=x \rightarrow xa'=a', \quad y=y \rightarrow a'y=a'.$$

Remark. As we shall state in § 7, the following examples are not μ-conditions.

u-condition that T_δ has a zero,
z-condition that T_δ has a unit,
the condition that T_δ is a group.

Hereafter we shall prove that $\mathfrak{D}$ has the greatest by utilizing a closure operation given in the preceding paragraphs.

Consider the decompositions of S which make S homomorphic to semigroups satisfying μ-condition: $p_{ji}=q_{ji}$ $(i=1, \cdots, m_j)$ imply $r_j=s_j$, $j=1, \cdots, l$. First, let us define the fundamental mappings f_j $(j=1, \cdots, l)$ on E^{m_j} whose element is denoted by $((x_1, y_1), (x_2, y_2), \cdots, (x_{m_j}, y_{m_j}))$. If there are $x_{j1}, \cdots, x_{jn} \in F(S)$ such that the homomorphic images of x_i and $p_{ji}(x_{j1}, \cdots, x_{jn})$ into S are equal and the homomorphic images of y_i and $q_{ji}(x_{j1}, \cdots, x_{jn})$ into S are equal $(i=1, \cdots, m_j)$, in other words, $x_i= p_{ji}(x_{j1}, \cdots, x_{jn})$, $y_i=q_{ji}(x_{j1}, \cdots, x_{jn})$ where consider them as elements of S, then

$$f_j((x_1, y_1), (x_2, y_2), \cdots, (x_{m_j}, y_{m_j})) = \{(r_j(x_{j1}, \cdots, x_{jn}), s_j(x_{j1}, \cdots, x_{jn})\}^{10)};$$

otherwise

$$f_j((x_1, y_1), (x_2, y_2), \cdots, (x_{m_j}, y_{m_j})) = \emptyset \quad \text{(empty)}.$$

Now ψ_j $(j=1, \cdots, l)$ denote closure operations in Ω constructed from f_j $(j=1, \cdots, l)$ respectively, and futhermore let

$$\varphi_\mu = c_0\psi_l \cup c_0\psi_{l-1} \cup \cdots \cup c_0\psi_2 \cup c_0\psi_1$$

where $c_0=c_1 \cup c_2 \cup c_3 \cup c_4$ (cf. § 5), and φ_μ is called the *congruent closure μ-operation.*

It is easily seen that $F(S, X)$ satisfies μ-condition if and only if X is c_0-closed and ψ_j-closed $(j=1, \cdots, l)$. By Lemmas 3 and 4, X is c_0-closed and ψ_j-closed $(j=1, \cdots, l)$ if and only if X is φ_μ-closed.

Lemma 15. *$F(S, X)$ satisfies μ-condition if and only if X is φ_μ-closed.*

Remember that A is a congruent set of relations in $F(S)$ and S is isomorphic to $F(S, A)$, so we see that $\bar{\varphi}_\mu(A)$ is the least φ_μ-closed subset containing A. It follows that $F(S, \bar{\varphi}_\mu(A))$ gives the greatest μ-decomposition of $F(S)$ and of S. Hence we have

Theorem 4. *When μ is fixed, there is the greatest μ-decomposition of a semigroup.*

$F(S, \bar{\varphi}_\mu(A))$ will be often called the greatest μ-homomorphic image of S.

§ 7. Counter Examples

In this paragraph we give examples of decompositions which have no greatest.

10) We may have more than one system of elements $x_{j1}, \cdots, x_{jn}$ so that we consider the set of pairs $(r_j(x_{j1}, \cdots, x_{jn}), s_j(x_{j1}, \cdots, x_{jn}))$ of r_j and s_j.

Example 13. z-decomposition, the decomposition of S to a semigroup having a zero.

Let S be an additive semigroup of all positive integers. For example, the following decomposition is a z-decomposition.

$$z_n: S=\sum_{\tau=0}^{n-1} S_\tau \quad \text{where} \quad S_0=\{x;\ x\geqq n\},\ S_i=\{i\},\ i=1, 2, \cdots, n-1.$$

Suppose that there is the greatest z-decomposition $\bar{z}$ of S. Since S itself has no zero, there are two positive integers $i<j$ which belong to the same class in $\bar{z}$. On the other hand, considering the decomposition z_j, we have $\bar{z}\gneqq z_j$. This is a contradiction. Hence there is no greatest z-decomposition of S.

Example 14. u-decomposition, the decomposition of S to a semigroup having a unit. S is the same semigroup as Example 13. The following u_n is a u-decomposition of S.

$$u_n: \quad S=\sum_{\tau=0}^{n-1} S_\tau$$

where $S_\tau=\{x;\ x\equiv\tau \ (\text{mod. } n)\}$, and T is a group of positive integers modulo n. It is similarly proved that there is no greatest u-decomposition of S.

§ 8. A Relation between Two Kinds of Greatest Decompositions

Consider two μ-decompositions μ_1, μ_2 of a semigroup S, which correspond to the congruent closure μ-operations φ_{μ_1} and φ_{μ_2} in Ω respectively. Furthermore μ_3 denotes an adjoint μ-condition of μ_1 and μ_2, and φ_{μ_3} denotes the congruent closure μ-operation corresponding to μ_3. Of course $F(S, \bar{\varphi}_{\mu_1}(A))$ and $F(S, \bar{\varphi}_{\mu_2}(A))$ give the greatest μ_1, μ_2-decompositions of S respectively.

Does the greatest μ_2-decomposition of $F(S, \bar{\varphi}_{\mu_1}(A))$ coincide with the greatest μ_1-decomposition of $F(S, \bar{\varphi}_{\mu_2}(A))$? Also, do they give the greatest μ_3-decomposition of S? Lemma 12 answers these problems: if and only if $\bar{\varphi}_{\mu_2}\bar{\varphi}_{\mu_1}$ is φ_{μ_1}-invariant,

$$\bar{\varphi}_{\mu_2}\bar{\varphi}_{\mu_1}=\bar{\varphi}_{\mu_1}\bar{\varphi}_{\mu_2}=\bar{\varphi}_{\mu_3}.$$

We have the following theorem as special case.

Theorem 5. *If the conditions μ_1 and μ_2 are preserved by any homomorphism, then $\bar{\varphi}_{\mu_1}\bar{\varphi}_{\mu_2}=\bar{\varphi}_{\mu_2}\bar{\varphi}_{\mu_1}=\bar{\varphi}_{\mu_3}$, in other words, the greatest μ_2-decomposition of the greatest μ_1-homomorphic image of S equals the greatest μ_3-decomposition of S.*

Since commutativity and idempotency are preserved by homomorphism, we get the following theorem:

Theorem 6. *Let T and U be the greatest c-homomorphic image of S and the greatest i-homomorphic image of S respectively. Then the greatest i-decomposition of T coincides with the greatest c-decomposition of U, and consequently with the greatest s-decomposition of S.*

If the greatest μ-homomorphic image of S is a semigroup composed of only one element, then S is called *μ-indecomposable.*

Corollary 3. *If S is s-indecomposable, then either S is c-indecomposable or the greatest c-homomorphic image of S is i-indecomposable.*

We show that ca-decomposition and $z(a)$-decomposition do not fulfil the condition of Theorem 5. See Examples 6 and 12 regarding ca-condition and $z(a)$-condition.

Example 15. Let I be the additive semigroup of all positive integers, and let A be a semigroup of order 2 defined as

$$A=\{0, 1\}, \quad 0^2=1^2=01=10=0.$$

S denotes a direct product of I and A: $S=\{(i, j)\,;\ i\geqq 1,\ j=0, 1\}$ where the multiplication is

$$(i, j)(k, j')=(i+k, 0).$$

The greatest ca-homomorphic image of S is isomorphic to I and the element $(2, 0)$ of S is mapped to 2 of I. The greatest $z(2)$-homomorphic image T of I is isomorphic to A, where the inverse image of the zero of T is the subset $\{2, 3, 4, \cdots\}$ of I. On the other hand the greatest $z((2, 0))$-homomorphic image B of S is $\{\infty, (1, 0), (i, 1), i\geqq 1\}$ where ∞ is the image of $\{(i, 0), i\geqq 2\}$ and whose multiplication is given as $xy=\infty$ for all $x, y\in B$. We see that B is ca-indecomposable. Hence $\bar{\varphi}_{z((2,0))}\bar{\varphi}_{ca}\neq\bar{\varphi}_{ca}\bar{\varphi}_{z((2,0))}$. We remark that ca-condition is not necessarily preserved by homomorphism.

§ 9. Greatest s-Decomposition

In this paragraph we shall point out an important property of s-decomposition, i.e., we shall prove that a semigroup S is decomposed into a union of s-indecomposable subsemigroups in the greatest s-decomposition of S. We, however, can not now discuss it generally from a standpoint of congruent closure μ-operations.

We suppose that a subsemigroup S' of a semigroup S is decomposed into a semilattice T' of order $\geqq 2$: $S'=\sum_{\sigma\in T'} S'_\sigma$. Make $\sigma(x)$ of T' correspond to $x \in S'$ such that $x \in S'_{\sigma(x)}$. The ordering $\sigma \leqq \tau$ in T' means that $\sigma\rho = \tau$ for some $\rho \in T'$.

Lemma 16. (1) *If S' is an ideal of S, then $\sigma(x) \leqq \sigma(xa)$, $\sigma(x) \leqq \sigma(ax)$ for any $x \in S'$ and any $a \in S$.*

(2) *$\sigma(x) = \sigma(y)$ implies $\sigma(xa) = \sigma(ya) = \sigma(ax) = \sigma(ay)$ for any $x, y \in S'$ and any $a \in S$.*

Proof. Proof of (1). Suppose that $\sigma(x_0) \nleqq \sigma(x_0 a)$ for some $x_0 \in S'$ and some $a \in S$. Let $T_1 = \{\tau ; \tau \leqq \sigma(x_0 a)\}$, $T_0 = T' - T_1$. Then T_0 and T_1 are non-void subsemilattices of T', and clearly

$$\sigma(x_0 a) \in T_1, \quad \sigma(x_0) \in T_0, \quad T_0 T_1 \subset T_0, \quad T_1 T_0 \subset T_0,$$

so that T_0 and T_1 form a factor semilattice T^* of order 2. Accordingly S' is decomposed into T^*:

$$S' = S_0' \cup S_1', \quad S_0' \cap S_1' = \emptyset \quad \text{where} \quad T^* = \{0, 1\}, \quad S_0' = \{x ; \sigma(x) \in T_0\},$$
$$S_1' = \{x ; \sigma(x) \in T_1\} \quad \text{and} \quad x_0 \in S_0', \quad x_0 a \in S_1'.$$

Now let us consider an equality: $(x_0 a)z = x_0(az)$ where we choose $z \in S_1'$ especially. Since S' is an ideal of S, $az \in S'$ and so the right hand side of this equality shows $x_0(az) \in S_0'S' \subset S_0'$, while we see $(x_0 a)z \in S_1'$ in the left hand side. This contradiction makes it impossible that $\sigma(x_0) \nleqq \sigma(x_0 a)$. Utilizing an equality $(za)x_0 = z(ax_0)$ where $z \in S_1'$ and $\sigma(x_0) \nleqq \sigma(ax_0)$, we can prove similarly $\sigma(x) \leqq \sigma(ax)$.

Proof of (2). First we shall prove that $\sigma(x) = \sigma(y)$ implies $\sigma(xa) = \sigma(ay)$. Suppose that $\sigma_0 = \sigma(xa) \neq \sigma(ay) = \sigma_1$ for some $x, y \in S'$ and some $a \in S$. Since $\sigma(x) \leqq \sigma(xa)$, $\sigma(y) \leqq \sigma(ay)$ by (1), we have

$$\sigma\{(xa)y\} = \sigma(xa)\sigma(y) = \sigma(xa)\sigma(x) = \sigma(xa) = \sigma_0,$$
$$\sigma\{x(ay)\} = \sigma(x)\sigma(ay) = \sigma(y)\sigma(ay) = \sigma(ay) = \sigma_1,$$

contradicting the equality $\sigma\{(xa)y\} = \sigma\{x(ay)\}$. Hence it has been proved that $\sigma(x) = \sigma(y)$ implies $\sigma(xa) = \sigma(ay)$. If we take $x = y$ in particular, we get $\sigma(xa) = \sigma(ax)$, $\sigma(ya) = \sigma(ay)$. Thus (2) has been proved.

Theorem 7. *If an s-decomposition ξ_0 of a semigroup S, $S = \sum_{\tau \in T} S_\tau$, is greatest, then each class S_τ is an s-indecomposable semigroup. Conversely if each S_τ is s-indecomposable, then such an s-decomposition of S is greatest.*

Proof. Suppose that there is s-decomposable S_{τ_0}, $\tau_0 \in T$, in spite of the greatest s-decomposition ξ_0 of S. Let $\bar{S}_1 = \sum_{\substack{\tau \leqq \tau_0 \\ \tau \in T}} S_\tau$ and $\bar{S}_0 = S - \bar{S}_1$.

Then it is easily shown that $\bar{S}_0$ is an ideal of S, and S_{τ_0} is an ideal of $\bar{S}_1$. On the other hand, we have supposed that S_{τ_0} is decomposed into a semilattice T' of order $\geqq 2$: $S_{\tau_0}=\sum_{\sigma\in T'} S_\sigma'$. Now let σ_0 be an element which is not greatest in T'. By Lemma 16, if $a\in\bar{S}_1$, then either (i) or (ii) holds:

(i) $\qquad S'_{\sigma_0}a\subset S'_{\sigma_0}$.

(ii) $\qquad$ There is $\sigma\in T'$ such that $S'_{\sigma_0}a\subset S_\sigma'$, $\sigma_0<\sigma$.

Set $R=\{a\,;\,S'_{\sigma_0}a\subset S'_{\sigma_0}\}$ and $Q=S-R$. Of course $\emptyset\neq R\subset\bar{S}_1$, $\bar{S}_0\subset Q$. Then $a\in R$ and $b\in R$ imply that $S'_{\sigma_0}(ab)=(S'_{\sigma_0}a)b\subset S'_{\sigma_0}b\subset S'_{\sigma_0}$, hence R is a subsemigroup of S. Next we shall prove that Q is an ideal of S, namely that $a\in Q$ and $x\in S$ imply $ax\in Q$ and $xa\in Q$. When one at least of a and x belongs to $\bar{S}_0\subset Q$, we get $xa\in\bar{S}_0\subset Q$, $ax\in\bar{S}_0\subset Q$ because $\bar{S}_0$ is an ideal of S. It is sufficient to consider the case where $a\in\bar{S}_1\cap Q$ and $x\in\bar{S}_1$. By Lemma 16, we get

$$S'_{\sigma_0}(ax)=(S'_{\sigma_0}a)x\subset S'_\sigma x\subset S'_{\sigma'} \quad \text{for some} \quad \sigma,\ \sigma'\in T',\quad \sigma_0<\sigma\leqq\sigma';$$

and, if $x\in R$, $\;S'_{\sigma_0}(xa)=(S'_{\sigma_0}x)a\subset S'_{\sigma_0}a\subset S_\sigma'$ for some $\sigma\in T'$, $\sigma_0<\sigma$,

otherwise, $S'_{\sigma_0}(xa)=(S'_{\sigma_0}x)a\subset S'_{\sigma_1}a\subset S'_{\sigma_2}$ for some $\sigma_1,\sigma_2\in T'$, $\sigma_0<\sigma_1\leqq\sigma_2$.

Therefore we have proved $ax\in Q$, $xa\in Q$. Thus Q is an ideal of S so that the partition $S=Q\cup R$ is an s-decomposition. However we see that S_{τ_0} intersects with both Q and R, and that Q contains elements of some of classes S_τ other than S_{τ_0}. Hence this new s-decomposition, $S=Q\cup R$, is incomparable with the s-decomposition ξ_0: $S=\sum_{\tau\in T'} S_\tau$, contradicting the assumption that ξ_0 is greatest.

Conversely suppose that an s-decomposition ξ_0 in which every class is s-indecomposable is not greatest. Then, denote by ξ_0' the greatest s-decomposition of S, $\xi_0'>\xi_0$. It follows that a certain class S_{τ_0} in ξ_0 is decomposed into another semilattice of order $\geqq 2$ in the decomposition ξ_0'. However it is impossible because of s-indecomposability. Thus the proof of the theorem has been completed.

The following corollary is obtained easily.

Corollary 4. *Let f be a homomorphism of a semigroup S to a semilattice T which causes the greatest s-decomposition of S. Further let T' be any subsemilattice of T, and S' be an inverse image of T' by f. Then the contraction of f to S', $S'\to T'$ comes, to be the greatest s-decomposition of S'.*

§ 10. Counter Examples (2)

We shall show that Theorem 7 does not necessarily hold in the case of i-decomposition and c-decomposition.

Example 16. Let S be a semigroup of four elements $a, b, c, d,$ with multiplication by the following table:

	a	b	c	d
a	a	b	c	a
b	a	b	c	a
c	a	b	c	b
d	a	b	c	a

The greatest i-decomposition of S is $S=S_1 \cup S_2$ where $S_1=\{a, b, d\}$, $S_2=\{c\}$. However S_1 is not i- indecomposable. In fact, $S_1=\{a, d\} \cup \{b\}$.

Example 17. Let S be a semigroup with multiplication

	a	b	c	d
a	a	b	a	a
b	a	b	a	a
c	a	b	a	c
d	a	b	a	d

The greatest c-decomposition of S is $S=S_1 \cup S_2$ where $S_1=\{a, b, c\}$, $S_2=\{d\}$, while S_1 is not c-indecomposable, for $S_1=\{a, b\} \cup \{c\}$.

§ 11. Classification of Finte s-Indecomposable Semigroups

A finite commutative semigroup S is decomposed into the union of mutually disjoint finite commutative idempotent subsemigroups S_i:

$$S=\sum_{i=1}^{k} S_i ,$$

which gives the greatest s-decomposition of S (See [12], [10]). The finite unipotent semigroups are classified into the two kinds: one is a unipotent semigroup which contains a group of order $\geqq 2$, the other is a unipotent semigroup whose idempotent is a zero. The former is called a unipotent semigroup with group (or without zero), the latter a unipotent semigroup with zero or z-semigroup.

Lemma 17. *If a finite semigroup S is s-indecomposable and c-decomposable, then the greatest c-homomorphic image is unipotent.*

Proof. Suppose that the greatest c-homomorphic image T of S is not unipotent. T is decomposed into the sum of commutative unipotent semigroups:

$$T=\sum_{i=1}^{t} T_i , \quad t \geqq 2 ,$$

so that T is s-decomposable. This contradicts the assumption.

Corollary 5. *A finite s-indecomposable commutative semigroup is unipotent.*

Theorem 8. *Finite s-indecomposable semigroups are classified into the four categories*:

(1) *c-indecomposable semigroup except gronps,*
(2) *unipotent semigroup with zero (z-semigroup),*
(3) *unipotent semigroup without zero,*
(4) *c-decomposable, non-commutative, non-unipotent semigroup,*

where a group belongs to (3).

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