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ON PRIME RIGHT IDEALS OF INTERMEDIATE RINGS OF A FINITE NORMALIZING EXTENSION

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Introduction

Throughout this paper, S will represent a ring extension of a ring R with common identity 1. Let I be a right ideal of R , and $b_R(I) = \{r \in R \mid Rr \subset I\}$. I is called a prime right ideal, provided that if X, Y are right ideals of R with $XY \subset I$, then either $X \subset I$ or $Y \subset I$. It is clear that every maximal right ideal is a prime right ideal. If I is a prime right ideal, then $b_R(I)$ is a prime ideal. Next, let R' be a ring. An R - R' -bimodule M is called a *torsionfree* R - R' -bimodule if $r_M(X) = l_M(Y) = 0$ for every essential ideal X of R and every essential ideal Y of R' , where $r_M(X)$ (resp. $l_M(Y)$) is the right (resp. left) annihilator of X (resp. Y) in M , and M is called a *finite normalizing* R - R' -bimodule if there exist elements a_1, a_2, \dots, a_n of M such that $M = \sum_{i=1}^n Ra_i$ and $Ra_i = a_iR'$ for $i = 1, 2, \dots, n$. Such a system $\{a_i\}_i$ is called a normalizing generating system of M . Finally S is a *finite normalizing extension* of R if S is a finite normalizing R - R -bimodule.

In [1], [2], [3], [4] and [6], "cutting down" theorems for a prime ideals were studied. In the previous paper [7], we have obtained a "cutting down" theorem for a prime right ideal of a finite normalizing extension under the hypothesis that the finite normalizing extension considered is torsionfree. The present objective is to reprove the same without the hypothesis "torsionfree"; we shall prove the following theorem.

Theorem. *Let S be an arbitrary finite normalizing extension of R , T a ring with $R \subset T \subset S$. If J is a prime right ideal of T , then there exist prime right ideals K_1, K_2, \dots, K_s of R such that $\bigcap_{i=1}^s K_i = J \cap R$. In this case, $b_R(J \cap R) = \bigcap_{i=1}^s b_R(K_i)$.*

1. Preliminaries

Throughout this paper, S will represent a finite normalizing extension of R , and T a ring with $R \subset T \subset S$.

Let P be a prime ideal of T . In studying P and T/P , one can usually reduce problems to the case in which (1) S is a prime ring, and (2) $A \cap T \not\subset P$ for

each non-zero ideal A of S ; this case being described as a standard setting for P . Actually, in view of [3, Proposition 2.2], there exists a prime ideal Q of S with $Q \cap T \subset P$ such that, with identifications of subrings, $R/(Q \cap R) \subset T/(Q \cap T) \subset S/Q$ gives a standard setting for $P/(Q \cap T)$.

To our end, we quote the following results from [1], [2], and [3].

Proposition 1.1 ([1, Theorem 2.11] and [3, Theorem 2.13 and Proposition 2.14]). *Let P be a prime ideal of T for which the case is a standard setting. Then*

- (1) R is a semiprime ring,
- (2) there exists a set $\{P_1, P_2, \dots, P_m\}$ of at most n (=the number of normalizing generators of S over R) prime ideals of R such that $\bigcap_{i=1}^m P_i = 0$ and the prime rings R/P_i are all isomorphic, and
- (3) there exists a subset $\{P_{i_k}\}$ of $\{P_1, P_2, \dots, P_m\}$ such that $P \cap R = \bigcap_k P_{i_k}$.

Proposition 1.2 ([1, Propositions 3.3 and 5.3, and Lemma 5.2]). *Let S be a prime ring. Then*

- (1) S embeds in the right Martindale quotient ring $Q(S)$ of S ,
- (2) there exist orthogonal idempotents f_1, f_2, \dots, f_m in $V_{Q(S)}(R)$ such that $f_1 + f_2 + \dots + f_m = 1$ and $r_R(f_i) = P_i$ for all $i = 1, 2, \dots, m$, and,
- (3) $f_i Q(S) f_j$ is a torsionfree $f_i R f_j$ -bimodule and $f_i S f_j$ is a torsionfree finite normalizing $f_i R f_j$ -bimodule.

Proposition 1.3 ([2, Corollary 2.25 and Theorem 4.6]). *Let S be a prime ring, and $Q(S)$ the right Martindale quotient ring of S . Let f_i be as in Proposition 1.2 and put*

$$\begin{aligned} S_{ij} &= S \cap f_i Q(S) f_j = S \cap f_i S f_j, \\ T_{ij} &= T \cap f_i Q(S) f_j = T \cap f_i T f_j, \\ S_i &= S_{ij} + f_i R, \\ T_i &= T_{ii} + f_i R, \text{ and} \\ T^* &= \sum_{i,j=1}^m T_{ij} \quad (i, j = 1, 2, \dots, m). \end{aligned}$$

Then

- (1) T_i and S_i are rings,
- (2) $f_i R \subset T_i \subset S_i \subset f_i S f_i \subset f_i Q(S) f_i$,
- (3) $T_{ii} \subset S_{ii} \subset f_i S f_i$,
- (4) $T_{ii} T^* T \subset f_i T^* T \subset T$, $RT^* \subset T^*$, $f_i RT^* T \subset f_i T^* T$ and $T_i T^* T \subset f_i T^* T$,
- (5) T^* is an essential R - R -subbimodule of T , and
- (6) there exists a non-zero ideal U of S such that $0 \neq U \cap T \subset T^*$.

2. Proof of Theorem

Let J be a prime right ideal of T . Then $b_T(J)$ is a prime ideal of T . As

was claimed at the opening of §1, in order to prove our cutting down theorem, we may assume that S is prime and the setting is standard for $b_T(J)$. Thus throughout this section, we keep the notations employed in §1. Furthermore, we set $h_i(J) = \{t_i \in T_i \mid t_i f_i T^* T \subset J\}$, which is a right ideal of T_i .

Lemma 2.1. $h_i(J) = T_i$ if and only if $f_i T^* T \subset J$.

Proof. If $h_i(J) = T_i$, then we have $f_i T^* T \subset T_i f_i T^* T \subset J$. Conversely, if $f_i T^* T \subset J$, then $T_i T^* T \subset f_i T^* T \subset J$ (by Proposition 1.3). Hence $T_i \subset h_i(J)$, and so $h_i(J) = T_i$.

Lemma 2.2. The set $\{i \mid h_i(J) \neq T_i\}$ is not empty.

Proof. If $h_i(J) = T_i$ for all $i = 1, 2, \dots, m$, then we have $T^* \subset f_1 T^* T + f_2 T^* T + \dots + f_m T^* T \subset J$ (Lemma 2.1). But, by Proposition 1.3, there exists a non-zero ideal U of S such that $U \cap T \subset T^* \subset J$, which contradicts the setting being standard for $b_T(J)$.

We now reorder, if necessary, so that $f_i T^* T \not\subset J$ for $i = 1, 2, \dots, s$, and $f_i T^* T \subset J$ for $i = s+1, \dots, m$.

Lemma 2.3. $b_T(J) \cap R \subset \cap_{i=1}^s P_i$.

Proof. By Proposition 1.3, there exists a non-zero ideal U of S such that $0 \neq U \cap T \subset T^*$. Therefore, since the setting is standard for $b_T(J)$, for $i = 1, 2, \dots, s$, we have $(U \cap T) \cdot f_i T^* T \not\subset b_T(J)$, and so $TT^* f_i f_i T^* T \not\subset b_T(J)$. Let us set $Q'_{(i)} = \{t_i \in T_i \mid TT^* f_i t_i f_i T^* T \subset b_T(J)\}$ for each $i = 1, 2, \dots, s$. Then, as is well known, by the correspondence of prime ideals in a Morita context

$$C_i = \begin{pmatrix} T & TT^* f_i \\ f_i T^* T & T_i \end{pmatrix},$$

$Q'_{(i)}$ is a prime ideal of T_i corresponding to the prime ideal $b_T(J)$ of T . By [3, Proposition 2.11], we have $Q'_{(i)} \cap f_i R = 0$. Since $TT^* f_i \cdot (b_T(J) \cap R) \cdot f_i T^* T \subset T b_T(J) T \subset b_T(J)$, we obtain $f_i (b_T(J) \cap R) f_i \subset f_i R \cap Q'_{(i)} = 0$, and hence $b_T(J) \cap R \subset r_R(f_i) = P_i$. Hence $b_T(J) \cap R \subset \cap_{i=1}^s P_i$.

Lemma 2.4. If $i \leq s$, then $h_i(J)$ is a prime right ideal of T_i and $A \cap T_i \not\subset h_i(J)$ for each non-zero ideal A of S_i .

Proof. Let a, b be elements of T_i with $a T_i b \subset h_i(J)$ and $b \notin h_i(J)$. By Proposition 1.3, there exists a non-zero ideal U of S such that $0 \neq U \cap T \subset T^*$. Then, since $a \cdot f_i T^* T (U \cap T) f_i \cdot b f_i T^* T \subset a f_i T^* T T^* f_i b f_i T^* T \subset a T_i b f_i T^* T \subset J$, we have either $a f_i T^* T \subset J$ or $(U \cap T) b f_i T^* T \subset J$. But, noting that $U \cap T \not\subset J$ and $b f_i T^* T \not\subset J$, we get $(U \cap T) f_i b f_i T^* T \not\subset J$. Hence $a f_i T^* T \subset J$, and so $a \in h_i(J)$. We have thus seen that $h_i(J)$ is a prime right ideal of T_i . Next we claim that

$b_{T_i}(h_i(J)) \cap f_i R = 0$. Let $f_i r \in b_{T_i}(h_i(J)) \cap f_i R$ ($r \in R$). Then $f_i T^* T T^* f_i r f_i T^* T \subset T_i f_i r f_i T^* T \subset J$, and so, $f_i T^* T \not\subset J$ implies $T T^* f_i r f_i T^* T \subset J$, and therefore $T T^* f_i r f_i T^* T \subset b_T(J)$. Hence $f_i r \in Q'_{(i)} \cap f_i R = 0$, by the proof of Lemma 2.3 ([3, Prop. 2.11]). We have thus seen that $b_{T_i}(h_i(J)) \cap f_i R = 0$. Finally, if A is a non-zero ideal of S_i such that $A \cap T_i \subset h_i(J)$, then $0 \neq A \cap f_i R = (A \cap T_i) \cap f_i R \subset b_{T_i}(h_i(J)) \cap f_i R$ by [2, Proposition 2.20], But this contradicts $b_{T_i}(h_i(J)) \cap f_i R = 0$. This proves that $A \cap T_i \not\subset h_i(J)$ for each non-zero ideal A of S_i .

Lemma 2.5. *If I is a non-zero ideal of $f_i R$, then there exists a non-zero ideal A of S such that $A \cap f_i A f_i \cap T_i \subset I T_i$.*

Proof. If M is an R - R -subbimodule of T_i with $I T_i \cap M = 0$, then $I M \subset I T_i \cap M = 0$. Since $f_i R$ is a prime ring and T_i is $f_i R$ - $f_i R$ -torsionfree (Proposition 1.1 and 1.2), we have $M = 0$, and therefore $I T_i$ is an essential R - R -subbimodule of T_i . Now, choose a relative complement T_i^* of T_i in the R - R -bimodule $Q(S)$. Noting that $I T_i$ is R - R -essential in T_i , we see that $I T_i \oplus T_i^*$ is R - R -essential in $Q(S)$, so that $(I T_i \oplus T_i^*) \cap S$ is R - R -essential in S . Then, by [3, Corollary 2.25], there exists a non-zero ideal A of S such that $A \subset (I T_i \oplus T_i^*) \cap S (\subset I T_i \oplus T_i^*)$. Now, it is easy to see that $A \cap f_i A f_i \cap T_i \subset I T_i$.

Corollary 2.6. *If $i \leq s$, then $h_i(J) \cap f_i R$ is a prime right ideal of $f_i R$ and $b_{f_i R}(h_i(J) \cap f_i R) = 0$.*

Proof. Let X, Y be right ideals of $f_i R$ such that $XY \subset h_i(J) \cap f_i R$ and $Y \not\subset h_i(J) \cap f_i R$. Then $f_i R Y$ is a non-zero ideal of $f_i R$, and so there exists a non-zero ideal A of S such that $A \cap f_i A f_i \cap T_i \subset f_i R Y T_i$ (Lemma 2.5). Since $A \cap f_i A f_i$ is a non-zero ideal of S_i ([2, Proposition 2.22]), $A \cap f_i A f_i \cap T_i \not\subset h_i(J)$ by Lemma 2.4. Therefore, since $X T_i (A \cap T_{ii}) \subset X (f_i R Y T_i \cap T_{ii}) \subset X f_i R Y T_i \subset h_i(J)$, we see that $X T_i \subset h_i(J)$, and therefore $X \subset h_i(J) \cap f_i R$. This proves that $h_i(J) \cap f_i R$ is a prime right ideal of $f_i R$. Next, suppose, to the contrary, that $b_{f_i R}(h_i(J) \cap f_i R) \neq 0$. Then, again by Lemma 2.5 and [2, Proposition 2.22], there exists a non-zero dideal B of S such that $B \cap f_i B f_i \cap T_i \subset b_{f_i}(h_i(J) \cap f_i R) T_i \subset h_i(J)$ and $B \cap f_i B f_i$ is a non-zero ideal of S_i . But this contradicts Lemma 2.4.

Now, we shall prove the final Lemma which implies our Theorem.

Lemma 2.7. *There exist prime right ideals K_1, K_2, \dots, K_s of R such that $\cap_{i=1}^s K_i = J \cap R$ and $b_R(K_i) = P_i$. In this case, $b_R(J \cap R) = \cap_{i=1}^s P_i$.*

Proof. We now set $K_i = \{r \in R \mid f_i r \in h_i(J) \cap f_i R\}$ ($i = 1, 2, \dots, s$). Then, by Corollary 2.6, we can easily see that K_i is a prime right ideal of R and $b_R(K_i) = r_R(f_i) = P_i$. If $r \in J \cap R$, then $f_i r f_i T^* T \subset r T^* T \subset J$, and so $f_i r \in h_i(J) \cap f_i R$. This implies that $J \cap R \subset \cap_{i=1}^s K_i$. Conversely, let r be an arbitrary element of $\cap_{i=1}^s K_i$. Then, for each $i \leq s$, $f_i r \in h_i(J) \cap f_i R$, and so $f_i r f_i T^* T \subset J$. On the other hand, noting that $f_i T^* T \subset J$ for $i \geq s+1$, it follows that $r T^* T \subset$

$\sum_{i=1}^s f_i r f_i T^* T + \sum_{i=s+1}^m f_i r f_i T^* T \subset J + \sum_{i=s+1}^m f_i T^* T \subset J$. In view of Proposition 1.3, there exists a non-zero ideal U of S such that $U \cap T \subset T^*$. Then $r \cdot (U \cap T) \subset r T^* T \subset J$ and $U \cap T \not\subset J$. (Note that the setting is standard for $b_T(J)$.) Hence $r \in J \cap R$. We have thus seen that $J \cap R = \bigcap_{i=1}^s K_i$. Furthermore $b_R(J \cap R) \subset \bigcap_{i=1}^s b_R(K_i) \subset \bigcap_{i=1}^s K_i = J \cap R$. This implies that $b_R(J \cap R) = \bigcap_{i=1}^s b_R(K_i) = \bigcap_{i=1}^s P_i$, and our proof is complete.

3. Examples

In §2, we have proved a “cutting down” theorem for a prime right ideal of a finite normalizing extension. On the other hand, from Heinicke and Robson [3, Theorem 2.12], we obtain another “cutting down” theorem for a prime right ideal. That is, if J is a prime right ideal of T , then there exist right ideals H_1, H_2, \dots, H_k of R such that $\bigcap_{i=1}^k H_i = J \cap R$ and each $H_i / (J \cap R)$ is a prime right R -module, where a right R -module N is called prime, provided that if $uN = 0$ for $0 \neq u \in N$ and an ideal I of R then $NI = 0$. In this section, we give some examples which show that these two expressions are essentially different.

In advance of giving examples, we claim the following: Let k be a field, $U = k[x_1, x_2, \dots]$ a polynomial ring over k in countable many indeterminate x_i , and $B = (x_1, x_2, \dots)$ the maximal two-sided ideal of U generated by x_1, x_2, \dots . Let us set $V = U/B^2$, $W = B/B^2$, $D = \begin{pmatrix} V & V \\ V & V \end{pmatrix}$ and $A = \begin{pmatrix} W & W \\ W & W \end{pmatrix}$. Then the unique maximal non-zero two-sided ideal A of D is neither left nor right D -finitely generated.

EXAMPLE 3.1. Let D be a ring containing a non-zero unique maximal two-sided ideal A of D which is neither left nor right D -finitely generated. Let M be a maximal right ideal of D with $b_D(M) = A$. Let

$$S = \begin{pmatrix} D & D & D \\ D & D & D \\ D & D & D \end{pmatrix}, R = \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix}, T = \begin{pmatrix} D & D & A \\ D & D & A \\ D & D & D \end{pmatrix} \text{ and } J = \begin{pmatrix} M & M & A \\ M & M & A \\ D & D & D \end{pmatrix}.$$

Then S is a finite normalizing extension of R , T is not a finite normalizing extension of R , and J is a prime right ideal of T . Let us set

$$K_1 = \begin{pmatrix} M & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \text{ and } K_2 = \begin{pmatrix} D & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & D \end{pmatrix}.$$

Then K_1 and K_2 are prime right ideal of R

with $J \cap R = K_1 \cap K_2$, and $K_i / (J \cap R)$ are prime right R -modules.

EXAMPLE 3.2. Let D, A and M be as in Example 3.1. Let us set

$$S = \begin{pmatrix} D & D & D & D \\ D & D & D & D \\ D & D & D & D \\ D & D & D & D \end{pmatrix}, R = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, T = \begin{pmatrix} D & D & D & A \\ D & D & D & A \\ D & D & D & A \\ D & D & D & D \end{pmatrix} \text{ and } J = \begin{pmatrix} M & M & M & A \\ M & M & M & A \\ M & M & M & A \\ D & D & D & D \end{pmatrix}.$$

Then S is a finite normalizing extension of R , T is not a finite normalizing extension of R , and J is a prime right ideal of T . Let us put

$$K_1 = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, \quad K_2 = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \quad \text{and} \quad K_3 = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & D \end{pmatrix}.$$

Then K_1 , K_2 and K_3 are the prime right ideals of R such that $J \cap R = K_1 \cap K_2 \cap K_3$. But the $K_i/(J \cap R)$ are not prime right R -modules. Next, we consider the following ideals:

$$H_1 = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, \quad H_2 = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \quad \text{and} \quad H_3 = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix}.$$

Then $H_1 \cap H_2 \cap H_3 = J \cap R$ and $H_i/(J \cap R)$ are prime right R -modules. But it is easy to see that none of H_i is a prime right ideal.

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