Ito, N.
Osaka J. Math.
7 (1970), 231-251

ON FINITE GROUPS WITH GIVEN CONJUGATE TYPES II*

Dedicated to Professor Keizo Asano on his sixtieth birthday

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(Received December 11, 1969)

Let $\mathfrak{G}$ be a finite group. Let $\{n_1, \cdots, n_r\}$ be the set of integers each of which is the index of the centralizer of some element of $\mathfrak{G}$ in $\mathfrak{G}$. We may assume that $n_1 > n_2 > \cdots > n_r = 1$. Then the vector $(n_1, \cdots, n_r)$ is called the conjugate type vector of $\mathfrak{G}$. A group with the conjugate type vector $(n_1, \cdots, n_r)$ is said to be a group of type $(n_1, \cdots, n_r)$.

In an earlier paper [5] we have proved that any group of type $(n_1, 1)$ is nilpotent. In the present paper we want to prove the following theorem.

Theorem. *Any group of type* $(n_1, n_2, 1)$ *is solvable.****

At few critical points the proof requires heavy group-theoretical apparatus.

NOTATION AND DEFINATION. Let $\mathfrak{G}$ be a finite group. $Z(\mathfrak{G})$ is the center of $\mathfrak{G}$. $Z_2(\mathfrak{G})$ is the second center of $\mathfrak{G}$. $D(\mathfrak{G})$ is the commutator subgroup of $\mathfrak{G}$. $\Phi(\mathfrak{G})$ is the Frattini subgroup of $\mathfrak{G}$. Let p be a prime. $O_p(\mathfrak{G})$ is the largest normal p-subgroup of $\mathfrak{G}$. $F(\mathfrak{G})$ is the Fitting subgroup of $\mathfrak{G}(F(\mathfrak{G}) = \prod_p O_p(\mathfrak{G}))$. Let $\mathfrak{X}$ be a finite set. $|\mathfrak{X}|$ is the number of elements in $\mathfrak{X}$. $|\mathfrak{X}|_p$ is the highest power of p dividing $|\mathfrak{X}|$. $\pi(\mathfrak{G})$ is the set of prime divisors of $|\mathfrak{G}|$. If $\mathfrak{X} \subseteqq \mathfrak{G}$ and is non-empty, then $C(\mathfrak{X})$ is the centralizer of $\mathfrak{X}$ in $\mathfrak{G}$. If $\mathfrak{X} = \{X\}$, $C(\mathfrak{X}) = C(X)$. $N(\mathfrak{X})$ is the normalizer of $\mathfrak{X}$ in $\mathfrak{G}$. Let $\mathfrak{X}$ be a subgroup of $\mathfrak{G}$ and $\mathfrak{Y}$ a subgroup of $\mathfrak{X}$. If $G^{-1}\mathfrak{Y}G \subseteqq \mathfrak{X}$ $(G \in \mathfrak{G})$ implies that $G^{-1}\mathfrak{Y}G = \mathfrak{Y}$, we say that $\mathfrak{Y}$ is weakly closed in $\mathfrak{X}$ with respect to $\mathfrak{G}$. $\mathfrak{G}$ is called a Frobenius group, if $\mathfrak{G}$ is a product of a normal subgroup $\mathfrak{N}$ and a subgroup $\mathfrak{H}$ such that no elements $(\neq E)$ of $\mathfrak{N}$ and $\mathfrak{H}$ commute one another. Let Σ be a group of automorphisms of $\mathfrak{G}$. If every element $\sigma \neq 1$ of Σ leaves no element $(\neq E)$ of $\mathfrak{G}$ fixed, Σ is called regular. If all the Sylow subgroups of $\mathfrak{G}$ are cyclic, then $\mathfrak{G}$ is called a Z-group. $PGL(2, q)$ and $PSL(2, q)$ denote the projective general and special linear groups of degree 2 over the field of q-elements.

* This is a continuation of [5].

** This research is partially supported by N.S.F. Grant GP 9584.

*** A part of the theorem, namely in the form of Proposition 2.2 was known at the time of [5].

A proper subgroup $\mathfrak{F}$ of $\mathfrak{G}$ is called fundamental, if there exists an element X of $\mathfrak{G}$ such that $\mathfrak{F}=C(X)$. A fundamental subgroup $\mathfrak{F}$ is called free, if $\mathfrak{F}$ is not contained in and does not contain any other fundamental subgroup of $\mathfrak{G}$. $\mathfrak{G}$ is called of type F, if all the fundamental subgroups of $\mathfrak{G}$ are free.

Let $\mathfrak{G}$ be a group of type $(n_1, n_2, 1)$. If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are fundamental subgroups of $\mathfrak{G}$ such that $\mathfrak{F}_1 \supsetneqq \mathfrak{F}_2$, then $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are called fundamental subgroups of $\mathfrak{G}$ of type 1 and of type 2 respectively.

1. Preliminaries

Let $\mathfrak{G}$ be a group of type $(n_1, n_2, 1)$ which is a counter-example of the least order against the theorem. Then $\mathfrak{G}$ is non-solvable.

Proposition 1.1. (Burnside). $|\pi(\mathfrak{G})| \geq 3$.

Proof. ([3], p. 492]).

Proposition 1.2. $Z(\mathfrak{G}) \subseteqq \Phi(\mathfrak{G})$.

Proof. Otherwise, there exists a proper subgroup $\mathfrak{H}$ of $\mathfrak{G}$ such that $\mathfrak{G}=Z(\mathfrak{G})\mathfrak{H}$. Let X be an element of $\mathfrak{H}$. Since $C(X) \supseteqq Z(\mathfrak{G})$, we have that $\mathfrak{G}: C(X)=\mathfrak{H}: \mathfrak{H} \cap C(X)$. Hence $\mathfrak{H}$ is a group of type $(n_1, n_2, 1)$. By the choice of $\mathfrak{G}$ $\mathfrak{H}$ is solvable. Then $\mathfrak{G}$ is solvable against the assumption.

Proposition 1.3. *For every prime divisor p of $|\mathfrak{G}|$ there exists a p-element X such that $C(X) \neq \mathfrak{G}$.*

Proof. Otherwise, a Sylow p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is contained in $Z(\mathfrak{G})$. By a theorem of Zassenhaus ([3], p. 126) there exists a Sylow p-complement of $\mathfrak{G}$. Hence $\mathfrak{P} \not\subseteqq \Phi(\mathfrak{G})$. This contradicts Proposition 1.2.

Proposition 1.4. (Cf. [5], Proposition 1.1). *Let $\mathfrak{F}$ be a free fundamental subgroup of $\mathfrak{G}$. Then $\mathfrak{F}$ is either* (i) *abelian, or* (ii) *a non-abelian p-subgroup for some prime p, or* (iii) *a direct product of a non-abelian p-subgroup and the Sylow p-complement $\mathfrak{E}_p \neq \mathfrak{E}$ of $Z(\mathfrak{G})$.*

2. Case where $\mathfrak{G}$ is of type F

In this section we assume that $\mathfrak{G}$ is of type F

Proposition 2.1. *$\mathfrak{G}$ contains no fundamental subgroup of prime power order.*

Proof. Let $\mathfrak{F}$ be a fundamental p-subgroup of $\mathfrak{G}$. Let $q(\neq p)$ be a prime divisor of $|\mathfrak{G}|$ (Cf. Proposition 1.1) and let $X(\neq E)$ be an element of the center of a Sylow q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$. Then $C(X)$ contains $\mathfrak{Q}$. Since $Z(\mathfrak{G}) \subseteqq \mathfrak{F}$,

$Z(\mathfrak{G})$ is a p-group. Now let $\mathfrak{F}_1$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}_1| \neq |\mathfrak{F}|$. Then $\mathfrak{F}_1$ contains a Sylow q-subgroup of $\mathfrak{G}$ for every $q(\neq p)$. By Propositions 1.1 and 1.4 $\mathfrak{F}_1$ is abelian. Let $\mathfrak{P}$ be a Sylow p-subgroup of $\mathfrak{G}$. Then $\mathfrak{G}=\mathfrak{F}_1\mathfrak{P}$, and hence $\mathfrak{G}$ is solvable (For instance, [4]). This is a contradiction.

Proposition 2.2. *$\mathfrak{G}$ contains a fundamental subgroup which is of the form* (iii) *in Proposition* 1.4.

Proof. Assume the contrary. Then by Propositions 1.4 and 2.1 all the fundamental subgroups of $\mathfrak{G}$ are abelian. The intersection of any two distinct fundamental subgroups of $\mathfrak{G}$ is equal to $Z(\mathfrak{G})$. Hence $\mathfrak{G}/Z(\mathfrak{G})$ admits an abelian normal partition whose components are factor groups of fundamental subgroups of $\mathfrak{G}$ by $Z(\mathfrak{G})$. Then by a theorem of Suzuki ([6], Theorems 2 and 3) $\mathfrak{G}/Z(\mathfrak{G})$ has the following structures: If $C(XZ(\mathfrak{G}))$ is nilpotent for every involution $XZ(\mathfrak{G})$ of $G/Z(\mathfrak{G})$, then $\mathfrak{G}/Z(\mathfrak{G})$ is isomorphic to $PSL(2, q)$. If $\mathfrak{G}/Z(\mathfrak{G})$ contains an involution $XZ(\mathfrak{G})$ with non-nilpotent $C(XZ(\mathfrak{G}))$, then $\mathfrak{G}/Z(\mathfrak{G})$ is isomorphic to $PGL(2, q)$.

First assume that q is even. Then $PSL(2, q)$ $(=PGL(2, q))$ contains an involution whose centralizer is a 2-group. Hence $\mathfrak{G}$ contains a 2-element X such that $C(X)/Z(\mathfrak{G})$ is a 2-group. Let $\mathfrak{F}$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}| \neq |C(X)|$. By Proposition 1.3 $|\mathfrak{F}/Z(\mathfrak{G})|$ must be divisible by every odd prime divisor of $|\mathfrak{G}|$. But $PSL(2, q)$ contains no elements of order ab, where $a(\neq 1)$ and $b(\neq 1)$ are divisors of $q+1$ and $q-1$ respectively. This is a contradiction. So $q=p^m$ is odd. $PSL(2, q)$ and $PGL(2, q)$ contain p-elements whose centralizers are p-groups. Hence $\mathfrak{G}$ contains a p-element X such that $C(X)/Z(\mathfrak{G})$ is a p-group. Let $\mathfrak{F}$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}| \neq |C(X)|$. By Proposition 1.3 $|\mathfrak{F}/Z(\mathfrak{G})|$ must be divisible by every prime divisor of $|\mathfrak{G}|$ other than p. Let a and b be odd prime divisors of $q+1$ and $q-1$ respectively. Then $PSL(2, q)$ and $PGL(2, q)$ contains no element of order ab. This is a contradiction. Therefore $q+1$ or $q-1$ is a power of 2. But then $PSL(2, q)$ and $PGL(2, q)$ contain 2-elements whose centralizers are 2-groups. Hence $\mathfrak{G}$ contains a 2-element Y such that $C(Y)/Z(\mathfrak{G})$ is a 2-group. Since $|C(Y)=|\mathfrak{F}|$, this is a contradiction.

Let $\mathfrak{F}_0=\mathfrak{P}_0\times\mathfrak{C}_p$ be a fundamental subgroup of $\mathfrak{G}$ which is of the form (iii) in Proposition 1.4; namely, $\mathfrak{P}_0$ is the non-abelian Sylow p-subgroup of $\mathfrak{F}_0$ and $\mathfrak{C}_p\neq\mathfrak{C}$ is the Sylow p-complement of $Z(\mathfrak{G})$. So in this section p is fixed henceforth.

Proposition 2.3. *Let $\mathfrak{F}_1$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Then $\mathfrak{F}_1$ is abelian.*

Proof. By Propositions 2.1 and 1.3, otherwise, $\mathfrak{F}_1$ is a direct product of a non-abelian q-subgroup and the Sylow q-complement $\mathfrak{C}_q \neq \mathfrak{C}$ of $Z(\mathfrak{G})$. By Propositions 1.1 and 1.3 there exist a prime divisor r of $|\mathfrak{G}|$ distinct from p and q and an r-element X of $\mathfrak{G}$ such that $C(X) \neq \mathfrak{G}$. Then we obtain that $|C(X)|_r > |\mathfrak{F}_0|_r$ and $|C(X)|_r > |\mathfrak{F}_1|_r$. This is a contradiction.

Proposition 2.4. *Let $\mathfrak{F}_1$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Let q be a prime divisor of $|\mathfrak{G}|$ distinct from p. Let $\mathfrak{Q}_1$ be the Sylow q-subgroup of $\mathfrak{F}_1$ and $\mathfrak{Q}$ a Sylow q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}_1$. If $\mathfrak{Q}$ is abelian, then $\mathfrak{Q}=\mathfrak{Q}_1$. If $\mathfrak{Q}$ is non-abelian, then $\mathfrak{Q}:\mathfrak{Q}_1=q$. Hence q^2 does not divide $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ and $\mathfrak{G}:N(\mathfrak{F}_1)$ is a power of p.*

Proof. If $\mathfrak{Q}$ is abelian, then by Proposition 1.3 $\mathfrak{Q}=\mathfrak{Q}_1$. So let us assume that $\mathfrak{Q}$ is non-abelian. Then by Proposition 2.3 $\mathfrak{Q}_1 \subsetneqq \mathfrak{Q}$, and hence $Z(\mathfrak{Q}) = \mathfrak{Q} \cap Z(\mathfrak{G})$. Let X be an element of $Z_2(\mathfrak{Q})$ such that $X \notin Z(\mathfrak{Q})$ and $X^q \in Z(\mathfrak{Q})$. Then $C(X) \supseteqq D(\mathfrak{Q})$. Take any element Y of $\mathfrak{Q}$. Then $Y^{-1}XY = XZ$ with $Z \in Z(\mathfrak{Q})$. Therefore $C(X)=C(XZ)=C(Y^{-1}XY)=Y^{-1}C(X)Y$ and Y^q belongs to $C(X)$. Let $\mathfrak{Q}_2$ be the Sylow q-subgroup of $C(X)$. Then $N(C(X))$ contains $\mathfrak{Q}$ and $\mathfrak{Q}/\mathfrak{Q}_2$ is an elementary abelian q-group.

By Propositions 1.1 and 1.3 there exist a prime divisor r of $|C(X)|$ and the Sylow r-subgroup $\mathfrak{R}_2$ of $C(X)$ such that $\mathfrak{R}_2 \supsetneqq Z(\mathfrak{G}) \cap \mathfrak{R}_2$. We show that $\mathfrak{Q}/\mathfrak{Q}_2$ can be considered as a regular automorphism group of $\mathfrak{R}_2/Z(\mathfrak{G}) \cap \mathfrak{R}_2$. In fact, let us assume that there exist $W \in \mathfrak{Q}$ and $V \in \mathfrak{R}_2$ such that $W \notin \mathfrak{Q}_2$, $V \notin Z(\mathfrak{G}) \cap \mathfrak{R}_2$ and $W^{-1}VW = VU$ with $U \in Z(\mathfrak{G}) \cap \mathfrak{R}_2$. Then $[V, W]^q = [V, W^q] = E = U^q$, which implies that $U=E$. Therefore W belongs to $C(V)=C(X)$, which is a contradiction. Hence $\mathfrak{Q}/\mathfrak{Q}_2$ is cyclic ([3], p. 499), and $\mathfrak{Q}:\mathfrak{Q}_2=q$. Therefore $\mathfrak{Q}:\mathfrak{Q}_1=q$ and $\mathfrak{Q}_1$ is normal in $\mathfrak{Q}$. Finally we show that $\mathfrak{Q} \subseteqq N(\mathfrak{F}_1)$. Assume the contrary. Then there exist an element A of $\mathfrak{Q}$ and a Sylow r-subgroup $\mathfrak{R}_1$ of $\mathfrak{F}_1$ such that $\mathfrak{R}_1 \neq A^{-1}\mathfrak{R}_1 A$. Then an abelian group $\mathfrak{F}_1 = C(\mathfrak{Q}_1)$ contains $\mathfrak{R}_1$ and $A^{-1}\mathfrak{R}_1 A$ as its Sylow r-subgroups. Hence $\mathfrak{R}_1 = A^{-1}\mathfrak{R}_1 A$, which is a contradiction.

Proposition 2.5. *Let $\mathfrak{F}_1$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Then $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is not a p-group and has a square-free order.*

Proof. In order to prove that $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ is square-free, it suffices to show that p^2 does not divide $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ (Proposition 2.4). Let $\mathfrak{P}$, $\bar{\mathfrak{P}}_1$, $\mathfrak{P}_1$ and $\mathfrak{S}_p$ be Sylow p-subgroups of $\mathfrak{G}$, $N(\mathfrak{F}_1)$, $\mathfrak{F}_1$ and $Z(\mathfrak{G})$ such that $\mathfrak{P} \supseteqq \bar{\mathfrak{P}}_1 \supseteqq \mathfrak{P}_1 \supseteqq \mathfrak{S}_p$.

If $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is a p-group, then by Proposition 2.4 we have that $\mathfrak{G}=\mathfrak{P}\mathfrak{F}_1$. Since $\mathfrak{F}_1$ is abelian (Proposition 2.3), $\mathfrak{G}$ is solvable (For instance, [4]). This is a contradiction.

Now assume that $\bar{\mathfrak{P}}_1 : \mathfrak{P}_1 \geq p^2$. If $\mathfrak{P}_1 \supsetneqq \mathfrak{S}_p$, then $C(\mathfrak{P}_1) = \mathfrak{F}_1$. Thus $\bar{\mathfrak{P}}_1$ is non-abelian. Then $Z(\bar{\mathfrak{P}}_1) = Z(\mathfrak{G}) \cap \mathfrak{P}$, and $\mathfrak{P}_1$ contains an element X such that $X \in Z_2(\bar{\mathfrak{P}}_1)$, $X \notin Z(\bar{\mathfrak{P}}_1)$ and $X^p \in Z(\bar{\mathfrak{P}}_1)$. As in the proof of Proposition 2.4 we obtain that $\bar{\mathfrak{P}}_1/\mathfrak{P}_1$ is an elementary abelian p-group. By Propositions 1.1 and 1.3 there exist a prime divisor q of $|\mathfrak{F}_1|$ distinct from p and the Sylow q-subgroup $\mathfrak{Q}_1$ of $\mathfrak{F}_1$ such that $\mathfrak{Q}_1 \neq Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. As in the proof of Proposition 2.4 we can show that $\bar{\mathfrak{P}}_1/\mathfrak{P}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. Hence $\bar{\mathfrak{P}}_1/\mathfrak{P}_1$ is cyclic and $\bar{\mathfrak{P}}_1 : \mathfrak{P}_1 = p$, which is against the assumption. So we can assume that $\mathfrak{P}_1 = \mathfrak{S}_p$. As above $\bar{\mathfrak{P}}_1/\mathfrak{P}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. Hence if p is odd, then $\bar{\mathfrak{P}}_1/\mathfrak{P}_1$ is cyclic. So $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is a Z-group. We already know that there exists a prime divisor r of $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ distinct from p. Let $\mathfrak{X}/\mathfrak{F}_1$ be a Hall $\{p, r\}$-subgroup of $N(\mathfrak{F}_1)/\mathfrak{F}_1$. By Propositions 1.1 and 1.3 there exist a prime divisor q of $|\mathfrak{F}_1|$ distinct from p and r and a Sylow q-subgroup $\mathfrak{Q}_1$ of $\mathfrak{F}_1$ such that $\mathfrak{Q}_1 \neq Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. Then $\mathfrak{X}/\mathfrak{F}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. So $\mathfrak{X}/\mathfrak{F}_1$ contains an element $Y\mathfrak{F}_1$ of order pr ([3], p. 499). Then $|C(Y)|_p > |\mathfrak{F}_1|_p$ and $|C(Y)|_r > |\mathfrak{F}_0|_r$. This is a contradiction. If $p=2$ and $\bar{\mathfrak{P}}_1/\mathfrak{P}_1$ is cyclic, we get a contradiction as above. Thus we may assume that $\bar{\mathfrak{P}}_1/\mathfrak{P}_1$ is a (generalized) quaternion group. Let $I\mathfrak{F}_1$ be an involution of $N(\mathfrak{F}_1)/\mathfrak{F}_1$. Then for every element K of $\mathfrak{Q}_1$ we get that $IKI \equiv K^{-1}$ (mod. $\mathfrak{Q}_1 \cap Z(\mathfrak{G})$). Let $W\mathfrak{F}_1$ be an element of order r of $N(\mathfrak{F}_1)/\mathfrak{F}_1$. Then we obtain that $IW^{-1}KWI \equiv W^{-1}K^{-1}W \equiv W^{-1}IKIW$ (mod. $\mathfrak{Q}_1 \cap Z(\mathfrak{G})$). Put $\mathfrak{Y}/\mathfrak{Q}_1 \cap Z(\mathfrak{G}) = C(\mathfrak{Q}_1/\mathfrak{Q}_1 \cap Z(\mathfrak{G}))$. Then $\mathfrak{Y} : \mathfrak{F}_1$ equals a power of q and $\mathfrak{Y}$ is normal in $N(\mathfrak{F}_1)$. Since $|WI\mathfrak{Y}| = 2r$, $|C(WI)/Z(\mathfrak{G})| \equiv 0 \pmod{2r}$. Then $|C(WI)|_2 > |\mathfrak{F}_1|_2$ and $|C(WI)|_r > |\mathfrak{F}_0|_r$. This is a contradiction.

Proposition 2.6. *Let $\mathfrak{F}_1$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Let q be a prime divisor of $|\mathfrak{G}|$ distinct from p. Let $\mathfrak{Q}_1$ be the Sylow q-subgroup of $\mathfrak{F}_1$ and $\mathfrak{Q}$ a Sylow q-subgroup of $\mathfrak{G}$ containing $\mathfrak{Q}_1$. If $\mathfrak{Q}_1$ is not weakly closed in $\mathfrak{Q}$ with respect to $\mathfrak{G}$, then $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G}^2)$ is an elementary abelian q-group of order q^2.*

Proof. This is obvious by Proposition 2.4.

Proposition 2.7. *Let $\mathfrak{F}_1$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Let q be a prime divisor of $|\mathfrak{G}|$ distinct from p. Let $\mathfrak{Q}$, $\mathfrak{Q}_1$ and $\mathfrak{S}_q$ be Sylow q-subgroups of $\mathfrak{G}$, $\mathfrak{F}_1$ and $Z(\mathfrak{G})$ such that $\mathfrak{Q} \supseteq \mathfrak{Q}_1 \supseteq \mathfrak{S}_q$. Now if $\mathfrak{G}$ contains a normal subgroup $\mathfrak{H}$ of index q, then $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ is an elementary abelian group of order q^2.*

Proof. Since $|\mathfrak{F}_0|_q \neq |\mathfrak{S}_q|$, $\mathfrak{Q}_1 \supsetneqq \mathfrak{S}_q$ by Proposition 1.3. Let $\hat{\mathfrak{F}}_0$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\hat{\mathfrak{F}}_0| = |\mathfrak{F}_0|$. By Proposition 1.2 $\mathfrak{H}$ contains $Z(\mathfrak{G})$. Thus $\mathfrak{H}$ contains $\hat{\mathfrak{F}}_0$. So if for every pair of fundamental subgroups $\mathfrak{F}_1$ and $\hat{\mathfrak{F}}_1$ such that $|\mathfrak{F}_1| = |\hat{\mathfrak{F}}_1| \neq |\mathfrak{F}_0|$ we have that $\mathfrak{H} \cap \mathfrak{F}_1 = \mathfrak{H} \cap \hat{\mathfrak{F}}_1$, then $\mathfrak{H}$ is of type $(n_1', n_2', 1)$. By the minimality of $\mathfrak{G}$ $\mathfrak{H}$ is solvable, and hence $\mathfrak{G}$ is solvable against the assumption. So there exists a pair of fundamental subgroups $\mathfrak{F}_1$ and $\hat{\mathfrak{F}}_1$ such that $|\mathfrak{F}_1|=|\hat{\mathfrak{F}}_1|\neq|\mathfrak{F}_0|$ and $\mathfrak{H}\supseteq\mathfrak{F}_1$ and $\hat{\mathfrak{F}}_1:\mathfrak{H}\cap\hat{\mathfrak{F}}_1=q$. This implies, in particular, that $\mathfrak{Q}$ is non-abelian. Let $\hat{\mathfrak{Q}}_1$ be the Sylow q-subgroup of $\hat{\mathfrak{F}}_1$. We may assume that $\hat{\mathfrak{Q}}_1\subseteq\mathfrak{Q}$. Then $\mathfrak{Q}_1\neq\hat{\mathfrak{Q}}_1$ and $\mathfrak{Q}_1\cap\hat{\mathfrak{Q}}_1\subseteq Z(\mathfrak{Q})=\mathfrak{Q}\cap Z(\mathfrak{G})$. Therefore $\mathfrak{Q}/\mathfrak{Q}\cap Z(\mathfrak{G})$ is an elementary abelian group of order q^2.

Proposition 2.8. *Let $\mathfrak{F}_1$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}_1|\neq|\mathfrak{F}_0|$. Then $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ cannot be divisible by three distinct prime numbers q, r and s which are distinct from 2 and p.*

Proof. Assume that $q>r>s$. Let $\mathfrak{X}/\mathfrak{F}_1$ be a Hall $\{r, s\}$-subgroup of $N(\mathfrak{F}_1)/\mathfrak{F}_1$ (cf. Proposition 2.5). Let $\mathfrak{Q}_1$ be a Sylow q-subgroup of $\mathfrak{F}_1$. Then $\mathfrak{Q}_1\neq\mathfrak{Q}_1\cap Z(\mathfrak{G})$ (cf. Proposition 1.3). Now $\mathfrak{X}/\mathfrak{F}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/\mathfrak{Q}_1\cap Z(\mathfrak{G})$ (cf. Proof of Proposition 2.4). So $\mathfrak{X}/\mathfrak{F}_1$ is cyclic ([3], p. 499). The same argument holds for any Hall $\{r, t\}$-or $\{s, t\}$-subgroup of $N(\mathfrak{F}_1)/\mathfrak{F}_1$, where t is a prime divisor of $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ distinct from r and s. Thus $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is cyclic. Let $\mathfrak{S}$ and $\mathfrak{S}_1$ be Sylow s-subgroups of $N(\mathfrak{F}_1)$ and $\mathfrak{F}_1$ respectively. Then $\mathfrak{S}$ is a Sylow s-subgroup of $\mathfrak{G}$. If $\mathfrak{S}_1$ is weakly closed in $\mathfrak{S}$ with respect to $\mathfrak{G}$, then since s is odd and $\mathfrak{S}_1$ is abelian, $\mathfrak{G}$ contains a normal subgroup of index s ([2], p. 212). So by Propositions 2.6 and 2.7 we have that $\mathfrak{S}_1: Z(\mathfrak{G})\cap\mathfrak{S}_1=s$. Let $\mathfrak{Q}$ be a Sylow q-subgroup of $N(\mathfrak{F}_1)$. Then $\mathfrak{Q}/\mathfrak{Q}_1$ can be considered as a regular automorphism group of $\mathfrak{S}_1/Z(\mathfrak{G})\cap\mathfrak{S}_1$. Since $q>s$, this is a contradiction.

Proposition 2.9. *Let $\mathfrak{F}_1$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}_1|\neq|\mathfrak{F}_0|$. If $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ is even, then $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ cannot be divisible by two distinct prime numbers q, r which are distinct from 2 and p.*

Proof. This is obvious by the proof of Proposition 2.8.

REMARK. By Propositions 2.5, 2.8 and 2.9 we have that $N(\mathfrak{F}_1):\mathfrak{F}_1=q$ or pq or qr or pqr, where $q\neq r$ and $q\neq p\neq r$.

Proposition 2.10. *Let $\mathfrak{F}_1$ be a fundamental subgroup of $\mathfrak{G}$ such that $|\mathfrak{F}_1|\neq|\mathfrak{F}_0|$. Let $\mathfrak{P}$, $\mathfrak{P}_1$ and $\mathfrak{S}_p$ be Sylow p-subgroups of $\mathfrak{G}$, $\mathfrak{F}_1$ and $Z(\mathfrak{G})$ respectively, such that $\mathfrak{P}\supseteq\mathfrak{P}_1\supseteq\mathfrak{S}_p$. Then we have that $\mathfrak{P}_1=\mathfrak{S}_p$.*

Proof. Assume that $\mathfrak{P}_1\supsetneqq\mathfrak{S}_p$. Then since $\mathfrak{P}$ is not abelian (Proposition 2.2), $C(\mathfrak{P}_1)=\mathfrak{F}_1$ and $N(\mathfrak{F}_1)=N(\mathfrak{P}_1)\supseteq Z_2(\mathfrak{P})$. Let $\mathfrak{R}$ be the largest normal subgroup of $\mathfrak{G}$ contained in $N(\mathfrak{F}_1)$. Then since $\mathfrak{G}=\mathfrak{P}N(\mathfrak{F}_1)$ (Proposition 2.4), $\mathfrak{R}$ contains $Z_2(\mathfrak{P})$. Let X be an element of $Z_2(\mathfrak{P})$ not belonging to $Z(\mathfrak{G})$. Let $\mathfrak{Q}_1$ be a Sylow q-subgroup of $\mathfrak{F}_1$. If X belongs to $\mathfrak{F}_1$, then $\mathfrak{F}_1=C(X)$ contains $D(\mathfrak{P})$, and $N(\mathfrak{F}_1)=N(\mathfrak{P}_1)$ contains $\mathfrak{P}$. This is a contradiction. Thus X does not belong to $\mathfrak{F}_1$. So $XZ(\mathfrak{G})$ induces a regular automorphism on $\mathfrak{Q}_1/Z(\mathfrak{G})\cap\mathfrak{Q}_1$.

Hence $\mathfrak{R}$ contains $\mathfrak{Q}_1$. Therefore $\mathfrak{R}:\mathfrak{R}\cap\mathfrak{F}_1$ is a power of p, and $\mathfrak{Q}_1$ is the Sylow q-subgroup of the Fitting subgroup of $\mathfrak{R}$. Thus $\mathfrak{Q}_1$ is normal in $\mathfrak{G}$. Since $N(\mathfrak{Q}_1)=N(\mathfrak{F}_1)$, this is a contradiction.

Proposition 2.11. *p is odd.*

Proof. Assume that $p=2$. Let $IZ(\mathfrak{G})$ be an involution of $\mathfrak{G}/Z(\mathfrak{G})$ and put $C(IZ(\mathfrak{G}))=\dfrac{\mathfrak{X}}{Z(\mathfrak{G})}$. Then $\mathfrak{X}:C(I)$ is a power of 2. Since $|C(I)|=|\mathfrak{F}_0|$ (Proposition 2.10), $C(I)/Z(\mathfrak{G})$ is a 2-group. So $\dfrac{\mathfrak{X}}{Z(\mathfrak{G})}$ is a 2-group. Therefore by a theorem of Suzuki ([9], Theorem 2) $\mathfrak{G}/Z(\mathfrak{G})$ possesses one of the following properties: (a) $\mathfrak{G}/Z(\mathfrak{G})$ contains a normal Sylow 2-subgroup. (b) $\mathfrak{P}Z(\mathfrak{G})/Z(\mathfrak{G})$ is cyclic or (generalized) quaternion and if $X^{-1}\mathfrak{P}Z(\mathfrak{G})X\neq\mathfrak{P}Z(\mathfrak{G})$, then $X^{-1}\mathfrak{P}Z(\mathfrak{G})X\cap\mathfrak{P}Z(\mathfrak{G})=Z(\mathfrak{G})$, where $\mathfrak{P}$ is a Sylow 2-subgroup of $\mathfrak{G}$. (c) $\mathfrak{G}/Z(\mathfrak{G})$ contains two normal subgroups $\mathfrak{G}_1/Z(\mathfrak{G})$ and $\mathfrak{G}_2/Z(\mathfrak{G})$ ($\mathfrak{G}_1\supseteqq\mathfrak{G}_2$) such that (i) a Sylow 2-subgroup of $\mathfrak{G}_2/Z(\mathfrak{G})$ is normal, (ii) $\mathfrak{G}:\mathfrak{G}_1=$is odd, and (iii) $\mathfrak{G}_1/\mathfrak{G}_2$ is isomorphic to $PSL(2, q)$ (q is a Fermat or a Mersenne prime) or $PSL(2, 3^2)$ or $PSL(2, 2^m)$ ($m\geq 2$) or $S(q)$ or $PSU(3, q)$ ($q>2$) or $PSL(3, q)$ ($q>2$) or M_q; where $S(q)$, $PSU(3, q)$, $PSL(3, q)$ and M_q denote the Suzuki group, the 3-dimensional projective special unitary group, the 3-dimensional special linear group and the linear fractional group over the non-commutative nearfield of 9 elements respectively.

If $\mathfrak{G}/Z(\mathfrak{G})$ has Property (a), then $\mathfrak{P}$ is normal in $\mathfrak{G}$. Since $\mathfrak{G}=\mathfrak{P}N(\mathfrak{F}_1)$, $\mathfrak{G}/\mathfrak{P}\simeq N(\mathfrak{F}_1)/\mathfrak{P}\cap N(\mathfrak{F}_1)$. So $\mathfrak{G}$ is solvable against the choice of $\mathfrak{G}$ (Proposition 2.5). Suppose that $\mathfrak{G}/Z(\mathfrak{G})$ has Property (b). Since $\mathfrak{P}_0$ is non-abelian (Proposition 2.2), $\mathfrak{P}Z(\mathfrak{G})/Z(\mathfrak{G})$ is (generalized) quaternion. So $\mathfrak{P}$ contains two elements A and B such that $A^{2^m}\equiv E$ $BA^{-1}B\equiv A^{-1}$, $B^2\equiv A^{2^{m-1}}$ (mod $Z(\mathfrak{P})$). Put $BA^{-1}B=A^{-1}Z$, $Z\in Z(\mathfrak{P})$. Then since $C(B^2)$ contains A, we get that $C(B^2)\supsetneqq C(B)$. Since $B^2\notin Z(\mathfrak{G})$ and $\mathfrak{G}$ is of type F, this is a contradiction. So $\mathfrak{G}/Z(\mathfrak{G})$ has Property (c).

Suppose that $\mathfrak{G}_2\neq Z(\mathfrak{G})$. Let $\mathfrak{P}_2$ be the Sylow 2-subgroup of $\mathfrak{G}_2$ and let $\mathfrak{Q}$ be a Sylow q-subgroup of $N(\mathfrak{F}_1)$ not contained in $\mathfrak{F}_1$ (Proposition 2.5). If $\mathfrak{P}_2\subsetneqq Z(\mathfrak{G})$, then $\mathfrak{Q}/\mathfrak{Q}\cap Z(\mathfrak{G})$ can be considered as a regular automorphism group of $\mathfrak{P}_2/\mathfrak{P}_2\cap Z(\mathfrak{G})$ (Proposition 2.10). So $\mathfrak{Q}/\mathfrak{Q}\cap Z(\mathfrak{G})$ is cyclic ([3], p. 499), and $\mathfrak{Q}$ is abelian. Then $\mathfrak{Q}$ is contained in $\mathfrak{F}_1$ (Proposition 1.3), which is a contradiction. Thus $\mathfrak{P}_2$ is contained in $Z(\mathfrak{G})$ and $\mathfrak{G}_2/Z(\mathfrak{G})$ has an odd order. But then $\mathfrak{P}/\mathfrak{P}\cap Z(\mathfrak{G})$ can be considered as a regular automorphism group of $\mathfrak{G}_2/Z(\mathfrak{G})$. So $\mathfrak{P}/\mathfrak{P}\cap Z(\mathfrak{G})$ is cyclic or (generalized) quaternion. This leads to a contradiction, as above. Thus we get that $\mathfrak{G}_2=Z(\mathfrak{G})$.

It can be easily checked that $PSU(3, q)$ and $PSL(3, q)$ ($q>2$) contain involutions whose centralizers are not 2-groups. Thus $\mathfrak{G}_1/Z(\mathfrak{G})$ is not isomorphic to $PSU(3, q)$ nor $PSL(3, q)$ ($q>2$). Now assume that $\mathfrak{G}\neq\mathfrak{G}_1$. Then it can be easily checked that $\mathfrak{G}/Z(\mathfrak{G})$ contains an element of even order, which is not a power of 2. Thus we get that $\mathfrak{G}=\mathfrak{G}_1$. By the proof of Proposition 2.2 we can assume that $\mathfrak{G}/Z(\mathfrak{G})$ is isomorphic to $S(q)$ or M_9. $S(q)$ contains no element of order ab, where a and b are prime divisors of q^2+1 and $q-1$ respectively (cf. [7]). M_9 contains no element of order 15. This contradicts Proposition 1.3.

Proposition 2.12. *A Sylow 2–subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ is not abelian.*

Proof. If $\mathfrak{Q}$ is abelian, then we may assume that $\mathfrak{Q}$ is contained in $\mathfrak{F}_1$. By a theorem of Feit-Thompson [1] $\mathfrak{Q}\neq\mathfrak{C}$. If $X^{-1}Z(\mathfrak{G})\mathfrak{Q}X/Z(\mathfrak{G})\cap Z(\mathfrak{G})\mathfrak{Q}/Z(\mathfrak{G})\neq Z(\mathfrak{G})$, then choose an element Y of $XZ^{-1}(\mathfrak{G})\mathfrak{Q}X\cap\mathfrak{Q}$ not belonging to $Z(\mathfrak{G})$. $C(Y)$ contains $X^{-1}\mathfrak{Q}X$ and $\mathfrak{Q}$. Since $C(Y)$ is abelian (Proposition 2.3), we get that $X^{-1}\mathfrak{Q}X=\mathfrak{Q}$. So by a theorem of Suzuki ([8], Theorem 2) $\mathfrak{G}/Z(\mathfrak{G})$ possesses one of the following properties: (a) $\mathfrak{G}/Z(\mathfrak{G})$ contains a normal Sylow 2–subgroup. (b) $\mathfrak{Q}Z(\mathfrak{G})/Z(\mathfrak{G})$ is cyclic or (generalized) quaternion. (c) $\mathfrak{G}/Z(\mathfrak{G})$ contains two normal subgroups $\mathfrak{G}_1/Z(\mathfrak{G})$ and $\mathfrak{G}_2/Z(\mathfrak{G})$ such that (i) $\mathfrak{G}/\mathfrak{G}_1$ and $\mathfrak{G}_2/Z(\mathfrak{G})$ have odd orders and (ii) $\mathfrak{G}_1/\mathfrak{G}_2$ is isomorphic to $PSL(2, q)$ $(q>3)$, $PSU(3, q)$ $(q>2)$ or $S(q)$.

If $\mathfrak{G}/Z(\mathfrak{G})$ has Property (a), then $\mathfrak{Q}$ is normal in $\mathfrak{G}$. Then $N(\mathfrak{Q})=N(\mathfrak{F}_1)=\mathfrak{G}$, which implies the solvability of $\mathfrak{G}$ (Proposition 2.5). This is a contradiction. If $\mathfrak{G}/Z(\mathfrak{G})$ has Property (b), then, since $\mathfrak{Q}$ is abelian, $\mathfrak{Q}Z(\mathfrak{G})/Z(\mathfrak{G})$ is cyclic. Take a prime divisor r of $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ and an r-element R of $N(\mathfrak{F}_1)$ not belonging to $\mathfrak{F}_1$. Then $R\mathfrak{F}_1$ induces a regular automorphism of $\mathfrak{Q}/\mathfrak{Q}\cap Z(\mathfrak{G})$, which is a contradiction. So $\mathfrak{G}/Z(\mathfrak{G})$ has Property (c).

Suppose that $\mathfrak{G}_2\neq Z(\mathfrak{G})$. Let $\mathfrak{P}_2$ be a Sylow p-subgroup of $\mathfrak{G}_2$. If $\mathfrak{P}_2\supsetneqq Z(\mathfrak{G})$, then we may assume that $N(\mathfrak{P}_2)$ contains $\mathfrak{Q}$. Thus $\mathfrak{Q}/\mathfrak{Q}\cap Z(\mathfrak{G})$ can be considered as a regular automorphism of $\mathfrak{P}_2/Z(\mathfrak{G})\cap\mathfrak{P}_2$. So $\mathfrak{Q}/\mathfrak{Q}\cap Z(\mathfrak{G})$ is cyclic, which leads to a contradiction as above. Thus $\mathfrak{P}_2$ is contained in $N(\mathfrak{F}_1)$ and $\mathfrak{G}_2$ is solvable. If $\mathfrak{G}_2$ is contained in $\mathfrak{F}_1$, then $\mathfrak{F}_1=C(\mathfrak{G}_2)$ is normal in $\mathfrak{G}$, which implies the solvability of $\mathfrak{G}$. This is a contradiction. So $\mathfrak{G}_2$ is not contained in $\mathfrak{F}_1$. Take an element X of $\mathfrak{G}_2$ not belonging to $\mathfrak{F}_1$. Then $X\mathfrak{F}_1$ induces a regular automorphism of $\mathfrak{Q}/\mathfrak{Q}\cap Z(\mathfrak{G})$. Hence $\mathfrak{G}_2$ contains $\mathfrak{Q}$, which is a contradiction. Thus we get that $\mathfrak{G}_2=Z(\mathfrak{G})$.

It can be easily checked that Sylow 2–subgroups of $PSU(3, q)$, $(q>2)$ and $S(q)$ are non-abelian. Thus $\mathfrak{G}_1/Z(\mathfrak{G})$ is isomorphic to $PSL(2, q)$. Now if q is odd, then a Sylow 2–subgroup of $PSL(2, q)$ is dihedral and contains its own centralizer (in $PSL(2, q)$). Since $\mathfrak{Q}$ is abelian, we get that $\mathfrak{Q}/\mathfrak{Q}\cap Z(\mathfrak{G})$ is elementary abelian of order 4. If q is even, then a Sylow 2–subgroup of $PSL(2, q)$ is an elementary abelian 2–group of order q and coincides with its own centralizer (in $PSL(2, q)$). Since $\mathfrak{F}_1=C(\mathfrak{Q})$, we get that $\mathfrak{F}_1\cap\mathfrak{G}_1=\mathfrak{Q}Z(\mathfrak{G})$.

If r is an odd prime divisor of $(q+1)(q-1)$ distinct from p, then let R be an r-element of $\mathfrak{G}_1$ not belonging to $Z(\mathfrak{G})$. We may assume that $\mathfrak{F}_1=C(R)$ and that $\mathfrak{F}_1 \supseteqq \mathfrak{Q}$. Since $\mathfrak{F}_1$ is abelian, this is a contradiction. So we must have that $(q+1)(q-1)=2^{\alpha}p^{\beta}$ with $\alpha, \beta \geqq 0$. Since $q>3$, if we put $q=l^m$, then $l \neq 2$ and $l \neq p$. Let L be an l-element of $\mathfrak{G}_1$ not belonging to $Z(\mathfrak{G})$. We may assume that $\mathfrak{F}_1=C(L)$ and that $\mathfrak{F}_1 \supseteqq \mathfrak{Q}$. Since $\mathfrak{F}_1$ is abelian, this is a contradiction.

REMARK. By the remark after Proposition 2.9 and by Proposition 2.12 we have that $N(\mathfrak{F}_1): \mathfrak{F}_1=2$ or $2p$ or $2q$ or $2pq$, where q is an odd prime distinct from p.

Proposition 2.13. *We have that* $N(\mathfrak{F}_1): \mathfrak{F}_1=2$ *or* $2q$.

Proof. If $N(\mathfrak{F}_1): \mathfrak{F}_1=2p$, then $N(\mathfrak{F}_1)/\mathfrak{F}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/\mathfrak{Q}_1 \cap Z(\mathfrak{G})$, where $\mathfrak{Q}_1(\neq \mathfrak{E})$ is a Sylow q-subgroup of $\mathfrak{F}_1$ (By Proposition 1.1 there exists such a prime q). Thus $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is cyclic and there exists an element of order $2p$ of $\mathfrak{G}/Z(\mathfrak{G})$. This is a contradiction (Proposition 2.10). The case $N(\mathfrak{F}_1): \mathfrak{F}_1=2pq$ can be treated in the same way.

Proposition 2.14. *For any subgroup* $\mathfrak{X}$ *of* $\mathfrak{G}$ *put* $\bar{\mathfrak{X}}=\mathfrak{X}Z(\mathfrak{G})/Z(\mathfrak{G})$. $N(\bar{\mathfrak{P}})$ *is a Frobenius group with* $\bar{\mathfrak{P}}$ *as its kernel, where* $\mathfrak{P}$ *is a Sylow* p-*subgroup of* $\mathfrak{G}$.

Proof. $\mathfrak{G}$ is not p-nilpotent. In fact, if so, $N(\mathfrak{F}_1)$ is normal in $\mathfrak{G}$ (Proposition 2.13), which implies the solvability of $\mathfrak{G}$ against the choice of $\mathfrak{G}$. Hence $\bar{\mathfrak{G}}$ also is not p-nilpotent. Thus by a theorem of Frobenius ([3], p. 436) there exists a non-trivial subgroup $\bar{\mathfrak{H}}$ of $\bar{\mathfrak{P}}$ such that $N(\bar{\mathfrak{H}})/C(\bar{\mathfrak{H}})$ is not a p-group. We choose $\bar{\mathfrak{H}}$ so that $|\bar{\mathfrak{H}}|$ is as big as possible. We show that $\bar{\mathfrak{H}}=\bar{\mathfrak{P}}$. Assume that $\bar{\mathfrak{H}} \subsetneqq \bar{\mathfrak{P}}$. First we notice that $C(\bar{\mathfrak{H}})$ is a p-group. In fact, otherwise, there exist a p-element X not belonging to $Z(\mathfrak{G})$ and an element Y which does not belong to $Z(\mathfrak{G})$ and has order prime to p, such that $XY=YX$. This contradicts Proposition 2.10. Then we get that $C(\bar{\mathfrak{H}}) \subseteqq \bar{\mathfrak{H}}$. Otherwise, notice that $N(C(\bar{\mathfrak{H}})\bar{\mathfrak{H}}) \supseteqq N(\bar{\mathfrak{H}})$ and $C(C(\bar{\mathfrak{H}})\bar{\mathfrak{H}}) \subseteqq C(\bar{\mathfrak{H}})$, which contradicts the choice of $\bar{\mathfrak{H}}$. Let $\bar{\mathfrak{Q}}$ be a Sylow q-subgroup of $N(\bar{\mathfrak{H}})$, where $q \neq p$, and consider $\bar{\mathfrak{H}}\bar{\mathfrak{Q}}$. Then the above argument shows that $\bar{\mathfrak{Q}}$ can be considered as a regular automorphism group of $\bar{\mathfrak{H}}$. Hence $\bar{\mathfrak{Q}}$ is cyclic or (generalized) quaternion ([3], p. 499). Suppose that $\bar{\mathfrak{Q}}$ is (generalized) quaternion. Then $\bar{\mathfrak{Q}}$ contains two elements A and B such that $|AZ(\mathfrak{G})|=4$ and $B^2=A^2Z_1, BA^{-1}B=A^{-1}Z_2$ with Z_1, $Z_2 \in Z(\mathfrak{G})$. Then $\mathfrak{G} \supsetneqq C(B^2) \supsetneqq C(B)$. Since $\mathfrak{G}$ is of type F, this is impossible. So $\bar{\mathfrak{Q}}$ is cyclic. By a theorem of Feit-Thompson [1] $N(\bar{\mathfrak{H}})$ is solvable. So let $\bar{\mathfrak{H}}^*$ and $\bar{\mathfrak{X}}$ be a Sylow p-subgroup and a Sylow p-complement of $N(\bar{\mathfrak{H}})$ respectively. By assumption on $\bar{\mathfrak{H}}$ we have that $\bar{\mathfrak{H}}=0_p(N(\bar{\mathfrak{H}}))$ and $\bar{\mathfrak{H}}^* \supsetneqq \bar{\mathfrak{H}}$. There exists a non-trivial cyclic subgroup $\bar{\mathfrak{Y}}$ of $\bar{\mathfrak{X}}$ such that $\bar{\mathfrak{H}}\bar{\mathfrak{Y}}$ is normal in $N(\mathfrak{H})$. By a theorem of Sylow we obtain that $N(\mathfrak{H})=\mathfrak{H}\cdot N(\bar{\mathfrak{Y}})\cap N(\mathfrak{H})$. Therefore there exist an abelian subgroup $\mathfrak{Y}$ which is not contained in $Z(\mathfrak{G})$ and has order prime to p and a p-element Z not belonging to $Z(\mathfrak{G})$ such that Z normalizes $\mathfrak{Y}$. Let Y be an element of $\mathfrak{Y}$ not belonging to $Z(\mathfrak{G})$. Then $C(Y)$ and $C(ZYZ^{-1})=Z^{-1}C(Y)Z$ contains $\mathfrak{Y}$. Thus we get that $C(Y)=Z^{-1}C(Y)Z$ (Proposition 2.3). This contradicts Proposition 2.13. So we must have that $\mathfrak{P}=\mathfrak{H}$.

Let $\bar{\mathfrak{X}}=\mathfrak{X}/Z(\mathfrak{G})$ be a Sylow p-complement of $N(\mathfrak{P})$. Then, as above, $\bar{\mathfrak{X}}$ can be considered as a regular automorphism group of $\mathfrak{P}$. Thus $N(\mathfrak{P})$ is a Frobenius group with $\mathfrak{P}$ as its kernel.

Proposition 2.15. *Let $\bar{X}$ be an element of $\bar{\mathfrak{G}}=\mathfrak{G}/Z(\mathfrak{G})$ whose order is divisible by p. Then $\bar{X}$ is a p-element.*

Proof. Otherwise, put $\bar{X}=\bar{Y}\bar{Z}=\bar{Z}\bar{Y}$, where $\bar{Y}$ is a p-element and $\bar{Z}$ is an element whose order is prime to p. We may assume that $\bar{Y}$ belongs to $\mathfrak{P}$ (in Proposition 2.14). Then $\bar{Z}^{-1}\mathfrak{P}\bar{Z}\neq\mathfrak{P}$ (Proposition 2.14) and $\bar{Z}^{-1}\mathfrak{P}\bar{Z}\cap\mathfrak{P}\ni\bar{Y}\neq\bar{E}$. Let $\bar{\mathfrak{D}}=\mathfrak{P}\cap\bar{W}^{-1}\mathfrak{P}\bar{W}$ be a maximal intersection of $\mathfrak{P}$ with other Sylow p-subgroups. Then $\bar{\mathfrak{D}}\neq\bar{\mathfrak{E}}$ and a Sylow p-subgroup of $N(\bar{\mathfrak{D}})$ is not normal in $N(\bar{\mathfrak{D}})$ ([10] p. 138). This leads to a contradiction as in the proof of Proposition 2.14.

Proposition 2.16. *Sylow p-subgroups of $\bar{\mathfrak{G}}$ are independent, namely if $\bar{X}^{-1}\mathfrak{P}\bar{X}\neq\mathfrak{P}$, then $\bar{X}^{-1}\mathfrak{P}\bar{X}\cap\mathfrak{P}=\bar{\mathfrak{E}}$.*

Proof. This is obvious from the proof of Proposition 2.15.

Proposition 2.17. *Let X be an element of $\mathfrak{G}$ not belonging to $Z(\mathfrak{G})$ whose order is prime to p. Then $C(X)$ is conjugate with $\mathfrak{F}_1$ in $\mathfrak{G}$.*

Proof. If there exists a prime divisor r of $|\mathfrak{F}_1|$ distinct from 2 and q, then let $\mathfrak{R}$ be a Sylow r-subgroup of $\mathfrak{F}_1$. Then $C(\mathfrak{R})=\mathfrak{F}_1$ and $\mathfrak{R}$ is a Sylow r-subgroup of $\mathfrak{G}$ (Proposition 2.13). $C(X)$ is abelian and contains $Y^{-1}\mathfrak{R}Y$ for some $Y\in\mathfrak{G}$. Thus $C(X)=C(Y^{-1}\mathfrak{R}Y)=Y^{-1}C(\mathfrak{R})Y=Y^{-1}\mathfrak{F}_1Y$. The same argument holds if $\mathfrak{F}_1$ contains a Sylow subgroup of $\mathfrak{G}$. Therefore by Proposition 2.13 we may assume that $\mathfrak{F}_1$ is a $\{2, q\}$-group and that $N(\mathfrak{F}_1):\mathfrak{F}_1=2q$. Let $\mathfrak{S}, \mathfrak{S}_1, \mathfrak{S}_X$ be Sylow 2-subgroups of $N(\mathfrak{F}_1)$, $\mathfrak{F}_1$ and $C(X)$ respectively. We may assume that $\mathfrak{S}\supseteqq\mathfrak{S}_1$, $\mathfrak{S}\supseteqq\mathfrak{S}_X$ and $\mathfrak{S}_1\neq\mathfrak{S}_X$. Since $\mathfrak{S}_1\cap\mathfrak{S}_X\subseteqq Z(\mathfrak{G})$, we obtain that $\mathfrak{S}_1: Z(\mathfrak{G})\cap\mathfrak{S}_1=2$. Now let $\mathfrak{Q}$ and $\mathfrak{Q}_1$ be Sylow q-subgroups of $N(\mathfrak{H}_1)$ and $\mathfrak{F}_1$ respectively. Then $\mathfrak{Q}/\mathfrak{Q}_1$ can be considered as a regular automorphism group of $\mathfrak{S}_1/Z(\mathfrak{G})\cap\mathfrak{S}_1$. This is a contradiction.

Now we count the number of elements in $\bar{\mathfrak{G}}$. Put $|\mathfrak{P}|=p^a$, $|N(\bar{\mathfrak{F}}_1)|=x$, $|\bar{\mathfrak{F}}_1|=y$, and $|N(\mathfrak{P})|=p^a z$. By Propositions 2.15 and 2.16 there exist $\frac{x}{z}(p^a-1)$

elements ($\neq \bar{E}$) of $\mathfrak{G}$ whose orders are prime to p. Thus we obtain that

$$(*) \qquad p^a = x\frac{x}{z}(p^a-1)+p^a(y-1)+1\,.$$

From $(*)$ we obtain that

$$x < \frac{x}{z}+y\,.$$

Since y and z are divisors of x, it is only possible when either $z=1$ or $y=x$. By Proposition 2.13 we have that $y\neq x$. By Proposition 2.14 we have that $z\neq 1$.

Thus $\mathfrak{G}$ cannot be of type F.

3. Case where G is not of type F

In this section $\mathfrak{G}$ is not of type F (See § 2). Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be fundamental subgroups of $\mathfrak{G}$ such that $\mathfrak{F}_1 \supsetneqq \mathfrak{F}_2$.

Proposition 3.1. *$|\mathfrak{F}_1|$ is divisible by every prime divisor p of $|\mathfrak{G}|$.*

Proof. This is obvious by Proposition 1.3.

Proposition 3.2. *If $\mathfrak{F}$ is a free fundamental subgroup of $\mathfrak{G}$ with $|\mathfrak{F}|=|\mathfrak{F}_1|$, then $\mathfrak{F}$ is abelian.*

Proof. If $\mathfrak{F}$ is of type (iii) in Proposition 1.4, then $|\mathfrak{F}_1|_q=|\mathfrak{S}_q|$, where $\mathfrak{S}_q$ is the Sylow q-subgroup of $Z(\mathfrak{G})$ with $q\neq p$. This contradicts Proposition 1.3.

Proposition 3.3. *$|\mathfrak{F}_2|$ is divisible by every prime divisor p of $|\mathfrak{G}|$.*

Proof. Suppose that there exists a prime divisor p of $|\mathfrak{G}|$ which does not divide $|\mathfrak{F}_2|$. Since $Z(\mathfrak{G})\subseteqq\mathfrak{F}_2$, $|Z(\mathfrak{G})|\not\equiv 0 \pmod{p}$. Let $X\neq E$ be an element of $Z(\mathfrak{P})$, where $\mathfrak{P}$ is a Sylow p-subgroup of $\mathfrak{G}$. Then we have that $|C(X)| = |\mathfrak{F}_1|$. If $C(X)$ is of type 1, then X belongs to a fundamental subgroup of type 2 contained in $C(X)$. Then $|\mathfrak{F}_2|\equiv 0 \pmod{p}$ against the assumption. So $C(X)$ is free, and by Proposition 3.2 $C(X)$ is abelian. Since $|C(X)|=|\mathfrak{F}_1|$ and $C(X)\supseteqq\mathfrak{P}$, we may assume that $\mathfrak{P}\subseteqq\mathfrak{F}_1$. But then $C(X)\supseteqq Z(\mathfrak{F}_1)$ and $C(X)=\mathfrak{F}_1$. This is a contradiction.

Proposition 3.4. *We may choose $\mathfrak{F}_2$ so that there exist (at most) two primes p and q such that $\mathfrak{F}_2$ is a direct product of a $\{p, q\}$-Hall subgroup and an abelian $\{p, q\}$-Hall complement.*

Proof. We can find a p-element X with $C(X)=\mathfrak{F}_1$ for some prime p. Assume that for any other prime q and for any q-element Y of $\mathfrak{F}_1$ we have that

$C(Y)\supseteqq\mathfrak{F}_1$. Then $\mathfrak{F}_1$ is a direct product of a Sylow p-subgroup and an abelian Sylow p-complement. Hence the same is true for $\mathfrak{F}_2$. So we may assume that there exists a prime $q(\neq \mathfrak{p}_1)$ and a q-element Y of $\mathfrak{F}_1$ such that $C(Y)\not\supseteqq\mathfrak{F}_1$. Then we can choose $C(XY)$ as $\mathfrak{F}_2$ with the claimed property.

Proposition 3.5. $|\mathfrak{F}_2/Z(\mathfrak{G})|$ *is divisible by every prime divisor* p *of* $|\mathfrak{G}|$.

Proof. Let $\mathfrak{P}_2$ be a Sylow p-subgroup of $\mathfrak{F}_2$. Assume that $\mathfrak{P}_2$ is contained in $Z(\mathfrak{G})$. Let $\mathfrak{P}_1$ be a Sylow p-subgroup of $\mathfrak{F}_1$ containing $\mathfrak{P}_2$. Then by Proposition 1.3 we have that $\mathfrak{P}_1\supsetneqq\mathfrak{P}_2$. Let Y be an element of $\mathfrak{P}_1$ not belonging to $\mathfrak{P}_2$. Then $|C(Y)|=|\mathfrak{F}_1|$ and, since $\mathfrak{F}_2\supseteqq Z(\mathfrak{F}_1)$, $C(Y)$ must be free. By Proposition 3.2 $C(Y)$ is abelian. Since $C(Y)\supseteqq Z(\mathfrak{F}_1)$, $\mathfrak{F}_1=C(Y)$. This is a contradiction.

Proposition 3.6. *Every fundamental subgroup* $\mathfrak{F}_2$ *of type* 2 *is nilpotent.*

Proof. If there exists a p-element X with $C(X)=\mathfrak{F}_2$, then $\mathfrak{F}_2$ is a direct product of a Sylow p-subgroup and an abelian Sylow p-complement of $\mathfrak{F}_2$ (cf. the proof of Proposition 3.4). So we may assume that there exists no element X of a prime power order such that $C(X)=\mathfrak{F}_2$.

Let X be a p-element of $\mathfrak{F}_1$ with $C(X)=\mathfrak{F}_1$, where $\mathfrak{F}_1\supsetneqq\mathfrak{F}_2$. Let Y be an element of the least order of $\mathfrak{F}_2$ such that $C(Y)=\mathfrak{F}_2$. Put $\pi(|Y|)=\{q, r, \cdots\}$. Then $|\pi(|Y|)|\geq 2$. Put $Y=Y_qY_r\cdots$, where $Y_q\neq E$, $Y_r\neq E\cdots$ are q-, r-, $\cdots$ elements which are commutative with each other. Then by assumption $C(Y_q)\supsetneqq\mathfrak{F}_2$ for each q in $\pi(|Y|)$. If $C(Y_q)=\mathfrak{G}$, then $\mathfrak{F}_2=C(Y)=C(r\prod_{r\neq q} qY_q)$, which contradicts the choice of Y. So we get that $|C(Y_q)|=|\mathfrak{F}_1|$. Assume that $q\neq p$. Then $\mathfrak{F}_1\supseteqq C(XY_q)\supseteqq\mathfrak{F}_2$. If for every $q\neq p$ we have that $\mathfrak{F}_1=C(XY_q)=C(Y_q)$, and if $\mathfrak{F}_1=C(Y_pY_q)$ provided that p belongs to $\pi(|Y|)$, then $\mathfrak{F}_1=\mathfrak{F}_2$, which is a contradiction. So we may assume that either for some q $C(XY_q)=\mathfrak{F}_2$ or $\mathfrak{F}_2=C(Y_pY_q)$. Thus, in any case, $\mathfrak{F}_2$ is a direct product of a Hall $\{p, q\}$-subgroup and an abelian Hall $\{p, q\}$-complement (Proposition 3.4).

Let $r\neq p$, q and let Z be an r-element of $\mathfrak{F}_2$ with $C(Z)\neq\mathfrak{G}$ (Proposition 3.5). Then we may assume that $C(Z)=\mathfrak{F}_1$. In fact, otherwise, $\mathfrak{F}_1\supsetneqq C(XZ)\supseteqq\mathfrak{F}_2$ and hence, $C(XZ)=\mathfrak{F}_2$. Then $\mathfrak{F}_2$ is a direct product of a Hall $\{p, r\}$-subgroup and an abelian Hall $\{p, r\}$-complement of $\mathfrak{F}_2$ (Proposition 3.4). Since $q\neq r$, $\mathfrak{F}_2$ is then nilpotent. So $C(Z)=\mathfrak{F}_1$. Then the above argument shows that there exists a prime $s\neq r$ such that $\mathfrak{F}_2$ is a direct product of a Hall $\{r, s\}$-subgroup and an abelian Hall $\{r, s\}$-complement. Since $\{p, q\}\neq\{r, s\}$, this implies that $\mathfrak{F}_2$ is nilpotent.

Proposition 3.7. *No Sylow subgroup* $(\neq\mathfrak{E})$ *of* $\mathfrak{G}$ *is contained in* $\mathfrak{F}_2$.

Proof. Let $\mathfrak{P}$ be a Sylow p-subgroup $(\neq\mathfrak{E})$ of $\mathfrak{G}$. Assume that $\mathfrak{P}$ is contained in $\mathfrak{F}_2$. Then every element of $\mathfrak{G}$ belongs to some conjugate subgroup of $C(\mathfrak{P})$ (Proposition 3.6). This implies that $\mathfrak{G}=C(\mathfrak{P})$ and $\mathfrak{P}\subseteqq Z(\mathfrak{G})$ contradicting Proposition 1.3.

REMARK. For every prime divisor p of $|\mathfrak{G}|$ we have that p^2 divides $|\mathfrak{G}|$. This is obvious by Propositions 3.5 and 3.7.

DEFINITION 3.8. Let $\mathfrak{F}_1=C(X)$ with a p-element X. Then $\mathfrak{F}_1$ is called-p-singular if $Z(\mathfrak{F}_1)/Z(\mathfrak{G})$ is a p-group.

Proposition 3.9. *Let $\mathfrak{X}$ be a finite group and $\mathfrak{S}$ a Sylow p-subgroup of $\mathfrak{X}$. Let $\mathfrak{Y}$ be a p-subgroup of $\mathfrak{X}$ such that $\mathfrak{Y}\supseteqq D(\mathfrak{S})$. Then there exists a Sylow p-subgroup $\mathfrak{T}$ of $\mathfrak{X}$ such that $\mathfrak{T}\supseteqq\mathfrak{Y}\supseteqq D(\mathfrak{T})$.*

Proof. Let $\mathfrak{T}$ be a s Sylow p-subgroup of $\mathfrak{X}$ such that $\mathfrak{Y}\supseteqq D(\mathfrak{T})$ and $\mathfrak{Y}:\mathfrak{Y}\cap\mathfrak{T}$ is the least. We show that $\mathfrak{Y}=\mathfrak{Y}\cap\mathfrak{T}$. Assume that $\mathfrak{Y}:\mathfrak{Y}\cap\mathfrak{T}\neq 1$.

Since $\mathfrak{Y}\cap\mathfrak{T}\supseteqq D(\mathfrak{T})$, $N(\mathfrak{Y}\cap\mathfrak{T})$ contains $\mathfrak{T}$. Put $\mathfrak{Z}=\mathfrak{Y}\cap N(\mathfrak{Y}\cap\mathfrak{T})$. Then $\mathfrak{Z}\supsetneqq\mathfrak{Y}\cap\mathfrak{T}$. If $N(\mathfrak{Y}\cap\mathfrak{T})=\mathfrak{X}$, then $\mathfrak{Y}\cap\mathfrak{T}\supseteqq G^{-1}D(\mathfrak{T})G$ for all $G\in\mathfrak{X}$. This contradicts the assumption $\mathfrak{Y}:\mathfrak{Y}\cap\mathfrak{T}\neq 1$. So we must have that $N(\mathfrak{Y}\cap\mathfrak{T})\neq\mathfrak{X}$. Then by an induction argument with respect to $|\mathfrak{X}|$ we may assume that there exists a Sylow p-subgroup $\mathfrak{U}$ of $N(\mathfrak{Y}\cap\mathfrak{T})$ such that $\mathfrak{U}\supseteqq\mathfrak{Z}\supseteqq D(\mathfrak{U})$. But $\mathfrak{U}$ is a Sylow p-subgroup of $\mathfrak{X}$ and $\mathfrak{Y}\cap\mathfrak{U}\supseteqq\mathfrak{Z}\supsetneqq\mathfrak{Y}\cap\mathfrak{T}$. This is a contradiction.

Proposition 3.10. *Let $\mathfrak{F}_1$ be a fundamental subgroup of type* 1. *Let q be a prime divisor of $\mathfrak{G}:\mathfrak{F}_1$. If there exists no q-singular fundamental subgroup of $\mathfrak{G}$, then q^2 does not divide $\mathfrak{G}:\mathfrak{F}_1$.*

Proof. Let $\mathfrak{Q}$ and $\mathfrak{Q}_1$ be Sylow q-subgroups of $\mathfrak{G}$ and $\mathfrak{F}_1$ such that $\mathfrak{Q}\supsetneqq\mathfrak{Q}_1$. Then $Z(\mathfrak{Q})\subseteqq Z(\mathfrak{G})$ and $\mathfrak{Q}$ is not abelian by Proposition 1.2. Let X be an element of $Z_2(\mathfrak{Q})$ not belonging to $Z(\mathfrak{Q})$ and $X^q\in Z(\mathfrak{Q})$. Let Y be an element of $\mathfrak{Q}$. Then $Y^{-1}XY=XZ$ with $Z\in Z(\mathfrak{G})$. Thus $C(Y^{-1}XY)=Y^{-1}C(X)Y=C(X)$ and $Y^{-q}X^{-1}Y^qX=Y^{-1}X^{-q}YX^q=E$. Therefore $\mathfrak{Q}$ is contained in $N(C(X))$ and $\mathfrak{Q}/\mathfrak{Q}_X$ is an elementary abelian q-group, where $\mathfrak{Q}_X=\mathfrak{Q}\cap C(X)$ is a Sylow q-subgroup of $C(X)$. If $|C(X)|=|\mathfrak{F}_2|$, then by Proposition 3.6 or Proposition 1.4 $C(X)$ is nilpotent. If $\mathfrak{Q}/\mathfrak{Q}_X$ can be considered as a regular automorphism group of $\mathfrak{R}_X/Z(\mathfrak{G})\cap\mathfrak{R}_X$, where $\mathfrak{R}_X$ is a Sylow r-subgroup of $C(X)$ and $r\neq q$, then $\mathfrak{Q}/\mathfrak{Q}_X$ is cyclic ([3], p. 499) and $\mathfrak{Q}:\mathfrak{Q}_X=q$ (*Cf.* Proposition 3.5). If $\mathfrak{Q}/\mathfrak{Q}_X$ is not regular as an automorphism group of $\mathfrak{R}_X/Z(\mathfrak{G})\cap\mathfrak{R}_X$, there exists an r-element Y in $\mathfrak{R}_X$ not belonging to $\mathfrak{Z}(\mathfrak{G})$ such that a Sylow q-subgroup $\mathfrak{Q}_Y$ of $C(Y)$ contains $\mathfrak{Q}_X$ properly. By Proposition 3.9 we may assume that $\mathfrak{Q}\supseteqq\mathfrak{Q}_Y\supseteqq D(\mathfrak{Q})$. Let Z be an element of $\mathfrak{Q}$. Then $Z^{-1}\mathfrak{Q}Z=\mathfrak{Q}_Y$. Now put $\mathfrak{R}^*=\langle Z^{-1}YZ, Z\in\mathfrak{Q}\rangle$. Then $\mathfrak{R}^*$ is a $\mathfrak{Q}$-invariant subgroup of $\mathfrak{R}_X$ and $\mathfrak{Q}_Y=\mathfrak{Q}\cap C(\mathfrak{R}^*)$. Since a Sylow q-complement of $C(X)$ is abelian (*cf.* Proposition 3.4), $C(Y)$ is a fundamental subgroup of type 1.

Therefore $\mathfrak{Q}/\mathfrak{Q}_Y$ can be considered as a regular automorphism group of $\mathfrak{R}^*/\mathfrak{R}^* \cap Z(\mathfrak{G})$. Hence $\mathfrak{Q}/\mathfrak{Q}_Y$ is cyclic ([3], p. 499) and $\mathfrak{Q}:\mathfrak{Q}_Y=q$.

If $|C(X)|=|\mathfrak{F}_1|$ and if $C(X)$ is free, then $C(X)$ is abelian by Proposition 3.2. $\mathfrak{Q}/\mathfrak{Q}_X$ can be considered as a regular automorphism group of $\mathfrak{R}_X/\mathfrak{R}_X \cap Z(\mathfrak{G})$, where $\mathfrak{R}_X$ is a Sylow r-subgroup of $C(X)$ and $r\neq q$. Thus $\mathfrak{Q}/\mathfrak{Q}_X$ is cyclic and $\mathfrak{Q}:\mathfrak{Q}_X=q$. So we may assume that $C(X)$ is of type 1. By the assumption there exists an r-element Y such that $C(X)=C(Y)$, where $q\neq r$. Then $\mathfrak{Q}/\mathfrak{Q}_X$ can be considered as a regular automorphism group of $\mathfrak{R}_X \cap Z(C(X))/\mathfrak{R}_X \cap Z(\mathfrak{G})$, where $\mathfrak{R}_X$ is a Sylow r-subgroup of $C(X)$. Hence $\mathfrak{Q}/\mathfrak{Q}_X$ is cyclic and $\mathfrak{Q}:\mathfrak{Q}_X=q$.

Proposition 3.11. *Let $\mathfrak{F}_1=C(X)$ be p-singular, where X is p-element. Let Y be a q-element of $\mathfrak{F}_1$ not belonging to $Z(\mathfrak{G})$ (Cf. Proposition 3.5). Let $\mathfrak{R}_1$ and $\mathfrak{R}_Y$ be Sylow r-subgroups of $\mathfrak{F}_1$ and $C(XY)$ such that $\mathfrak{R}_1 \supseteq \mathfrak{R}_Y$. If $r\neq p$, then $\mathfrak{R}_1:\mathfrak{R}_Y \leq r$.*

Proof. By assumption Y does not belong to $Z(\mathfrak{F}_1)$, and thus $C(XY)$ is of type 2. Assume that $\mathfrak{R}_1 \supsetneqq \mathfrak{R}_Y$. Let Z be an element of $Z(\mathfrak{R}_1)$. Then $|C(XZ)|_r > |C(XY)|_r$ and $C(X)\supseteq C(XZ)$. Hence by assumption $C(X)=C(XZ)$ and Z belongs to $Z(\mathfrak{G})$. So $Z(\mathfrak{R}_1)\subseteq Z(\mathfrak{G})$ and $\mathfrak{R}_1$ is not abelian. Let W be an element of $Z_2(\mathfrak{R}_1)$ not belonging to $Z(\mathfrak{G})$ and such that $W^r \in Z(\mathfrak{G})$. Then $C(XW)$ is of type 2. Let $\mathfrak{R}_W$ be a Sylow r-subgroup of $C(XW)$. Then as in the beginning of the proof of Proposition 3.10 we have that $\mathfrak{R}_1 \subseteq N(C(XW))$ and $\mathfrak{R}_1/\mathfrak{R}_W$ is an elementary abelian r-group. By Proposition 3.6 $C(XW)$ is nilpotent. Let $\mathfrak{S}_W$ be a Sylow s-subgroup of $C(XW)$ with $s\neq r$. Then $\mathfrak{R}_1/\mathfrak{R}_W$ can be considered as a regular automorphism group of $\mathfrak{S}_W/\mathfrak{S}_W \cap Z(\mathfrak{G})$ (Proposition 3.5). Thus $\mathfrak{R}_1/\mathfrak{R}_W$ is cyclic ([3], p. 499) and $\mathfrak{R}_1:\mathfrak{R}_W=\mathfrak{R}_1:\mathfrak{R}_Y=r$.

Proposition 3.12. *Let $\mathfrak{F}_1$ be p-singular and $q\neq p$. Then q^2 does not divide $\mathfrak{F}_1:\mathfrak{F}_2$.*

Proof. This is obvious by Proposition 3.11.

Proposition 3.13. *Assume that there exist no p-singular fundamental subgroups of $\mathfrak{G}$ for every p. If a Sylow q-subgroup $\mathfrak{Q}_2$ of a fundamental subgroup $\mathfrak{F}_2$ of type 2 is not abelian, then for every prime divisor r of $|\mathfrak{G}|$ distinct from q there exists a $\{q, r\}$-element X such that $\mathfrak{F}_2=C(X)$. In particular, a Sylow q-complement of $\mathfrak{F}_2$ is abelian.*

Proof. By Proposition 3.6 $\mathfrak{F}_2$ is nilpotent. Let $\mathfrak{F}_1=C(Y)$ is a fundamental subgroup of type 1 containing $\mathfrak{F}_2$. By assumption we may assume that Y is a p-element with $p\neq q$. If a Hall $\{p, q\}$-complement $\mathfrak{A}$ of $\mathfrak{F}_2$ contains an element Z not belonging to $Z(\mathfrak{F}_1)$, then $C(YZ)$ is of type 2 and contains $\mathfrak{Q}_2$. This implies that $\mathfrak{Q}_2$ is abelian against the assumption (*cf.* the proof of Proposition 3.4). So we must have that $\mathfrak{A}\subseteq Z(\mathfrak{F}_1)$. Then for every $r\neq p$, q there exists an r-element W such that $\mathfrak{F}_1=C(W)$ (Proposition 3.5). The above argument shows that a Hall $\{r, q\}$-complement of $\mathfrak{F}_2$ is contained in $Z(\mathfrak{F}_1)$. By Propositions 1.1 and 3.5 a Sylow q-complement of $\mathfrak{F}_2$ is contained in $Z(\mathfrak{F}_1)$.

Put $\mathfrak{F}_2=C(V)$ and $V=V_pV_q\cdots$, where V_p, $V_q\neq E$, $\cdots$ are p-, q-, $\cdots$ elements which are commutative with each other. Let U be an r-element such that $\mathfrak{F}_1=C(U)$. Then $\mathfrak{F}_1\supseteqq C(UV_q)\supseteqq\mathfrak{F}_2$. If $\mathfrak{F}_1=C(UV_q)$, then V_q belongs to $Z(\mathfrak{F}_1)$ and $\mathfrak{F}_1=\mathfrak{F}_2$ which is a contradiction. So $\mathfrak{F}_2=C(UV_q)$ as claimed.

Proposition 3.14. *Assume that there exist no p-singular fundamental subgroups of $\mathfrak{G}$ for every prime p. Then every fundamental subgroup $\mathfrak{F}_1$ of type* 1 *is nilpotent and n_1/n_2 is a prime power. Hence all the fundamental subgroups of $\mathfrak{G}$ are nilpotent.*

Proof. We show that for every element X of $\mathfrak{F}_1$ $\mathfrak{F}_1:\mathfrak{F}_1\cap C(X)=1$ or n_1/n_2. If $C(X)=\mathfrak{G}$, this is obvious. If $C(X)$ is free, then $C(X)$ is abelian (Propositions 3.5 and 1.4) and $C(X)$ contains $Z(\mathfrak{F}_1)$. This implies that $\mathfrak{F}_1$ contains $C(X)$, which is a contradiction. If $C(X)$ is of type 1, then we may assume that X is a p-element. By the assumption we can find a q-element Y such that $\mathfrak{F}_1=C(Y)$ and $p\neq q$. Then $C(XY)=C(X)\cap\mathfrak{F}_1$, which implies that $\mathfrak{F}_1:\mathfrak{F}_1\cap C(X)=1$ or n_1/n_2. So we may assume that $C(X)$ is of type 2. If $C(X)$ is abelian, then $C(X)$ contains $Z(\mathfrak{F}_1)$ and $C(X)$ is contained in $\mathfrak{F}_1$. Hence we may assume that a Sylow p-subgroup of $C(X)$ is not abelian for some p. Let $\mathfrak{F}_1=C(Y)$, where Y is a q-element. By Proposition 3.13 there exists a $\{p, r\}$-element $\bar{X}$ such that $C(X)=C(\bar{X})$ and $q\neq p$, r. Since Y belongs to $C(X)$ and $C(X)$ is nilpotent (Proposition 3.6), $Y\bar{X}=\bar{X}Y$. Thus by Proposition 3.13 we get that $C(\bar{X}Y)=C(\bar{X})$ is contained in $\mathfrak{F}_1$.

Hence by Theorem 1 of [5] $\mathfrak{F}_1$ is nilpotent and n_1/n_2 is a prime power.

Proposition 3.15. *There exists a p-singular fundamental subgroup of $\mathfrak{G}$ for some p.*

Proof. Assume the contrary. Then by Proposition 3.10 $\mathfrak{G}:\mathfrak{F}_1$ is square-free, and by Proposition 3.14 $\mathfrak{F}_1$ is nilpotent. We show that $\mathfrak{F}_1$ is normal in $\mathfrak{G}$, whence $\mathfrak{G}$ is solvable against the assumption. Now let $\mathfrak{P}_1$ and $\mathfrak{P}$ be Sylow p-subgroups of $\mathfrak{F}_1$ and $\mathfrak{G}$ such that $\mathfrak{P}_1\subseteqq\mathfrak{P}$. We show that $\mathfrak{P}\subseteqq N(\mathfrak{F}_1)$. We may assume that $\mathfrak{P}:\mathfrak{P}_1=p$. Put $\mathfrak{F}_1=\mathfrak{P}_1\times\bar{\mathfrak{P}}_1$, where $\bar{\mathfrak{P}}_1$ is a Sylow p-complement of $\mathfrak{F}_1$. Let X be an element of $\mathfrak{P}$ not belonging to $\mathfrak{P}_1$. Then $X^{-1}\mathfrak{F}_1X=\mathfrak{P}_1\times X^{-1}\bar{\mathfrak{P}}_1X$. Let Y be an element of $\mathfrak{P}_1$ not belonging to $Z(\mathfrak{G})$. Then $C(Y)$ is nilpotent (Proposition 3.14) and contains $\bar{\mathfrak{P}}_1$ and $X^{-1}\bar{\mathfrak{P}}_1X$ as Sylow p-complements. Hence $\bar{\mathfrak{P}}_1=X^{-1}\bar{\mathfrak{P}}_1X$, and X belongs to $N(\mathfrak{F}_1)$.

Proposition 3.16. *Assume that there exists a p-singular fundamental subgroup and that there exist no q-singular fundamental subgroups of $\mathfrak{G}$ for every* *prime q distinct from p. If a Sylow r-subgroup of a fundamental subgroup $\mathfrak{F}_2$ of type 2 is not abelian, then for every prime divisor s of $|\mathfrak{G}|$ distinct from r there exists a $\{r, s\}$-element X with $\mathfrak{F}_2=C(X)$. In particular, a Sylow r-complement of $\mathfrak{F}_2$ is abelian.*

Proof. By Proposition 3.6 $\mathfrak{F}_2$ is nilpotent. Let $\mathfrak{F}_1=C(Y)$ is a fundamental subgroup of type 1 containing $\mathfrak{F}_2$. The proof of Proposition 3.13 shows that the assertion is true if we can choose Y as an s-element with $s\neq r$. Then such a choice is possible, unless $r=p$ and $\mathfrak{F}_1$ is p-singular. So assume that $r=p$ and $\mathfrak{F}_1$ is p-singular. Let $\mathfrak{P}_2$ be a Sylow p-subgroup of $\mathfrak{F}_2$. Let $Z\neq E$ be a q-element of $\mathfrak{F}_2$ not belonging to $Z(\mathfrak{G})$ with $q\neq p$. Then $C(Z)$ contains $\mathfrak{P}_2$. If $C(Z)$ is free or of type 2, then $\mathfrak{P}_2$ is abelian against the assumption. Thus $C(Z)$ is of type 1 and we may assume that $C(Z)\supsetneqq\mathfrak{F}_2$. So $C(YZ)$ is of type 2 and contains a Hall $\{p. q\}$-complement $\mathfrak{A}$ of $\mathfrak{F}_2$. $\mathfrak{A}$ is abelian (Proposition 3.4). Let $W\neq E$ be an s-element of $\mathfrak{A}$ not belonging to $Z(\mathfrak{G})$ (By Proposition 3.5 such an element always exists). $C(W)$ cannot be free nor of type 2 as above. So $C(W)$ is of type 1 and contains $\mathfrak{F}_2$. So we can apply the proof of Proposition 3.13.

Proposition 3.17. *Assume that there exists a p-singular fundamental subgroup and that there exist no q-singular fundamental subgroups for every q distinct from p. If $\mathfrak{F}_1$ is not (p-) singular and of type 1, then $\mathfrak{F}_1$ is nilpotent and n_1/n_2 is a prime power.*

Proof. It is not difficult to check that the proof of Proposition 3.14 can be applied here.

Proposition 3.18. *Assume that there exists a p-singular fundamental subgroup and that there exist no q-singular fundamental subgroups for every q distinct from p. Then exists no non-singular fundamental subgroup of type 1.*

Proof. Assume the contrary and let $\mathfrak{F}_1$ be a non-singular fundamental subgroup of type 1. By Proposition 3.10 the prime to p part of $\mathfrak{G}$: $\mathfrak{F}_1$ is square-free. By Proposition 3.17 $\mathfrak{F}_1$ is nilpotent. By the proof of Proposition 3.15 $\mathfrak{G}: N(\mathfrak{F}_1)$ is a power of p.

First assume that $N(\mathfrak{F}_1)$ is solvable, and let $\mathfrak{H}$ be a Sylow p-complement of $N(\mathfrak{F}_1)$. $\mathfrak{H}$ is a Sylow p-complement of $\mathfrak{G}$. Put $\mathfrak{H}_1=\mathfrak{H}\cap\mathfrak{F}_1$. Then $\mathfrak{H}_1$ is a Sylow p-complement of $\mathfrak{F}_1$. Let $\mathfrak{P}_1$ be a Sylow p-subgroup of $\mathfrak{F}_1$. Then $\mathfrak{P}_1$ is normal in $\mathfrak{P}_1\mathfrak{H}$. Let $\mathfrak{P}$ be a Sylow p-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_1$. Since $\mathfrak{G}=\mathfrak{P}\mathfrak{H}$, $\mathfrak{P}$ contains a normal subgroup $\bar{\mathfrak{P}}_1$ of $\mathfrak{G}$ containing $\mathfrak{P}_1$. $\mathfrak{H}_1$ is a Sylow p-complement of $C(\mathfrak{P}_1)$. Hence if $\bar{\mathfrak{P}}_1\supsetneqq\mathfrak{P}_1$, then $\bar{\mathfrak{P}}_1\cap N(\mathfrak{H}_1)\supsetneqq\mathfrak{P}_1$. But for $X\in\bar{\mathfrak{P}}_1\cap N(\mathfrak{H}_1)$ and $Y\in\mathfrak{H}_1$, we have that $X^{-1}Y^{-1}XY\in\bar{\mathfrak{P}}_1\cap\mathfrak{H}_1=\mathfrak{E}$. Since $\mathfrak{F}_1$ is non-singular, we have that $C(\mathfrak{H}_1)\subseteqq\mathfrak{F}_1$, $\bar{\mathfrak{P}}_1\cap N(\mathfrak{H}_1)\subseteqq\mathfrak{F}_1$ and $\bar{\mathfrak{P}}_1\cap N(\mathfrak{H}_1)$

$\subseteqq \mathfrak{P}_1$, which is a contradiction. Hence we get that $\mathfrak{P}_1 = \bar{\mathfrak{P}}_1$. If $\mathfrak{G} : \mathfrak{F}_1 \not\equiv 0 \pmod{p}$, then $\mathfrak{G} = N(\mathfrak{F}_1)$, and $\mathfrak{G}$ is solvable against the assumption. So $\mathfrak{G} : \mathfrak{F}_1 \equiv 0 \pmod{p}$, and thus $\mathfrak{P}$ is non-abelian and $Z(\mathfrak{P}) \subseteqq Z(\mathfrak{G})$. Now $\mathfrak{H}/\mathfrak{H}_1$ can be considered as a regular automorphism group of $\mathfrak{P}_1/Z(\mathfrak{G})$. Since $\mathfrak{H}/\mathfrak{H}_1$ has a square-free order, $\mathfrak{H}/\mathfrak{H}_1$ is cyclic ([3], p. 499). Then $\mathfrak{G}/\mathfrak{P}_1 C(\mathfrak{P}_1)$ is a product of a cyclic group and a p-group, and hence is solvable (For instance, [4]). On the other hand, $C(\mathfrak{P}_1) = (\mathfrak{P} \cap C(\mathfrak{P}_1))\mathfrak{H}_1$ is solvable by a theorem of Wielandt ([3], p. 680). Thus $\mathfrak{G}$ is solvable against the assumption.

Now assume that $N(\mathfrak{F}_1)$ is non-solvable. Let $\mathfrak{P}_1^*$ be a Sylow p-subgroup of $N(\mathfrak{F}_1)$. Then, since $C(\mathfrak{P}_1) \subseteqq \mathfrak{F}_1$, $\mathfrak{P}_1^*/\mathfrak{P}_1$ can be considered as a regular automorphism group of $\mathfrak{H}_1/\mathfrak{H}_1 \cap Z(\mathfrak{G})$, where $\mathfrak{H}_1$ is a Sylow p-complement of $\mathfrak{F}_1$. Hence $\mathfrak{P}_1^*/\mathfrak{P}_1$ is cyclic or (generalized) quaternion. If $\mathfrak{P}_1^*/\mathfrak{P}_1$ is cyclic, then $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is a Z-group, which implies the solvability of $N(\mathfrak{F}_1)$ against the assumption. So $\mathfrak{P}_1^*/\mathfrak{P}_1$ must be (generalized) quaternion, and, in particular, $p=2$. Let $\mathfrak{P}$ be a Sylow 2–subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_1^*$. Then $\mathfrak{P}$ is non-abelian and $Z(\mathfrak{P}) \subseteqq Z(\mathfrak{G})$. Let X be an element of $Z_2(\mathfrak{P})$ not belonging to $Z(\mathfrak{G})$ such that $X^2 \in Z(\mathfrak{G})$. Then $C(X) \supseteqq D(\mathfrak{P})$. As in the proof of Proposition 2.4 $\mathfrak{P} \subseteqq N(C(X))$ and $\mathfrak{P}/\mathfrak{P}_X$ is an elementary abelian 2-group, where $\mathfrak{P}_X$ is a Sylow 2–subgroup of $C(X)$ such that $\mathfrak{P} \supseteqq \mathfrak{P}_X \supseteqq D(\mathfrak{P})$. If $C(X)$ is not 2–singular, then by Propositions 1.4, 3.6 and 3.17 $C(X)$ is nilpotent. Let $\mathfrak{Q}_X$ be a Sylow q-subgroup of $C(X)$ with $q \neq p$. If $C(X)$ is not of type 2, $\mathfrak{P}/\mathfrak{P}_X$ can be considered as a regular automorphism group of $\mathfrak{Q}_X/\mathfrak{Q}_X \cap Z(\mathfrak{G})$. So $\mathfrak{P}/\mathfrak{P}_X$ is cyclic and $\mathfrak{P} : \mathfrak{P}_X = 2$ ([3], p. 499). Then $|\mathfrak{P}_1^*/\mathfrak{P}_1| \leq 2$, which is a contradiction. Suppose that $C(X)$ is of type 2 and that $\mathfrak{P}/\mathfrak{P}_X$ is not regular as an automorphism group of $\mathfrak{Q}_X/\mathfrak{Q}_X \cap Z(\mathfrak{G})$. Then there exist an element Y of $\mathfrak{P}$ not belonging to $\mathfrak{P}_X$ and an element Z of $\mathfrak{Q}_X$ not belonging to $Z(\mathfrak{G})$ such that $YZ = ZY$ (cf. the proof of Proposition 2.4). Then $C(Z)$ contains $\langle \mathfrak{P}_X, Y \rangle$, and is free or of type 1 and is not 2–singular. So by Propositions 3.2 and 3.17 $C(Z)$ is nilpotent. Let $\mathfrak{P}_Z$ be a Sylow 2–subgroup of $C(Z)$. Then by Proposition 3.9 we may assume that $\mathfrak{P} \supseteqq \mathfrak{P}_Z \supseteqq D(\mathfrak{P})$. Then $W^{-1}\mathfrak{P}_Z W = \mathfrak{P}_Z$ for every $W \in \mathfrak{P}$. Put $\mathfrak{Q}^* = \langle W^{-1}ZW, W \in \mathfrak{P} \rangle$. Then $\mathfrak{Q}^*$ is a $\mathfrak{P}$-invariant subgroup of $\mathfrak{Q}_X$ and $\mathfrak{P}_Z = \mathfrak{P} \cap C(\mathfrak{Q}^*)$. Now $\mathfrak{P}/\mathfrak{P}_Z$ can be considered as a regular automorphism group of $\mathfrak{Q}^*/\mathfrak{Q}^* \cap Z(\mathfrak{G})$. So $\mathfrak{P}/\mathfrak{P}_Z$ is cyclic and $\mathfrak{P} : \mathfrak{P}_Z = 2$. Then $|\mathfrak{P}_1^* \mathfrak{P}_1| \leq 2$, which is a contradiction. Hence we may assume that $C(X)$ is 2–singular.

Let $\mathfrak{F}_2$ be a fundamental subgroup of type 2 contained in $C(X)$. By Proposition 3.17 $C(X) : \mathfrak{F}_2$ is a power of a prime. If $C(X) : \mathfrak{F}_2 \not\equiv 0 \pmod{2}$, then by Proposition 3.12 $C(X) : \mathfrak{F}_2 = q$ is a prime. By Proposition 3.6 $\mathfrak{F}_2$ is nilpotent. $C(X)$ is a product of a Sylow q-subgroup of $C(X)$ and a Sylow q-complement of $\mathfrak{F}_2$. Hence by a theorem of Wielandt ([3], p. 680) $C(X)$ is solvable. Let $F(C(X)) = \mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are Sylow 2–subgroup and Sylow 2–complement of $F(C(X))$ respectively. If $\mathfrak{B} \not\subseteq Z(\mathfrak{G})$, then, since $\mathfrak{B} \subseteq \mathfrak{F}_2$, $\mathfrak{P}/\mathfrak{P}_X$ can be considered as a regular automorphism group of $\mathfrak{B}/\mathfrak{B} \cap Z(\mathfrak{G})$. So we get a contradiction as before. But if $\mathfrak{B} \subseteq Z(\mathfrak{G})$, then by a theorem of Fitting ([3], p. 277) $F(C(X)) \supseteqq C(F(C(X))) \supseteqq \mathfrak{R}_X$, where $\mathfrak{R}_X$ is a Sylow r-subgroup of $\mathfrak{F}_2$ with $r \neq q$, 2. By Propositions 1.1 and 3.5 we have that $\mathfrak{R}_X \not\subseteq Z(\mathfrak{G})$. This is a contradiction. Hence we may assume that $C(X) : \mathfrak{F}_2 =$ is a power of 2.

Let $\mathfrak{A}$ be a Sylow 2-complement of $\mathfrak{F}_2$. Suppose that a Sylow q-subgroup $\mathfrak{Q}_2$ of $\mathfrak{A}$ is non-abelian. Then by Proposition 3.15 a Sylow r-subgroup $\mathfrak{R}_2$ of $\mathfrak{A}$ is abelian. Choose an element Y of $\mathfrak{R}_2$ not belonging to $Z(\mathfrak{G})$. Then $C(XY) \supseteqq \mathfrak{Q}_2$ and $C(XY)$ is of type 2. Then $\mathfrak{Q}_2$ is abelian against the assumption. Hence $\mathfrak{A}$ is abelian (*cf.* Propositions 1.1 and 3.5). Hence, in particular, $C(X)$ is solvable (*cf.* [4]). If $C(X)$ is nilpotent, then $\mathfrak{P}/\mathfrak{P}_X$ can be considered as a regular automorphism group of $\mathfrak{A}/\mathfrak{A} \cap Z(\mathfrak{G})$, and we get a contradiction as before. Hence $C(X)$ is not nilpotent.

Let $\hat{\mathfrak{F}}_2$ be a fundamental subgroup of type 2 contained in $\mathfrak{F}_1$. Since $\mathfrak{F}_1 : \hat{\mathfrak{F}}_2$ is a power of 2, every Sylow q-subgroup of $\mathfrak{Q}_1$ of $\mathfrak{F}_1$ is contained in $\hat{\mathfrak{F}}_2$ for $q \neq 2$. We show that $\mathfrak{Q}_1$ is abelian. Suppose that $\mathfrak{Q}_1$ is not abelian. Let Y be an element of $\mathfrak{Q}_1$ not belonging to $Z(\mathfrak{Q}_1)$. Then $C(Y)$ is of type 1 and contains the Sylow q-complement of $\mathfrak{F}_1$. In particular, $C(Y)$ contains $\mathfrak{R}_1$, where $\mathfrak{R}_1$ is the Sylow r-subgroup of $\mathfrak{F}_1$ (Proposition 1.1). Let Z be an element of $\mathfrak{R}_1$ not belonging to $Z(\mathfrak{G})$ (Proposition 3.5). Then $C(Z)$ contains $\mathfrak{Q}_1$ and the Sylow q-subgroup of $C(Y)$. This is a contradiction. So the Sylow 2-complement $\mathfrak{A}_1$ of $\mathfrak{F}_1$ is abelian.

Now we show that we may assume that $\mathfrak{A} = \mathfrak{A}_1$. Let $\mathfrak{Q}$, $\mathfrak{Q}_1$ and $\mathfrak{Q}_X$ be Sylow q-subgroups of $\mathfrak{G}$, $\mathfrak{F}_1$ and $C(X)$, where $q \neq 2$. We may assume that $\mathfrak{Q} \supseteqq \mathfrak{Q}_1$ and $\mathfrak{Q} \supseteqq \mathfrak{Q}_X$. Then since $\mathfrak{Q}_1$ and $\mathfrak{Q}_X$ are abelian, we have that $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ is elementary abelian of order q^2 or $\mathfrak{Q}_1 = \mathfrak{Q}_X$. If $\mathfrak{Q}_1 = \mathfrak{Q}_X$, then $C(\mathfrak{Q}_1)$ is nilpotent and contains $\mathfrak{A}_1$ and $\mathfrak{A}$ as its Sylow 2-complement. So we get that $\mathfrak{A}_1 = \mathfrak{A}$. Otherwise, let $\mathfrak{R}$, $\mathfrak{R}_1$ and $\mathfrak{R}_X$ be Sylow r-subgroups of $\mathfrak{G}$, $\mathfrak{F}_1$ and $C(X)$, where $r \neq q$, $r \neq 2$. By Propositions 1.1 and 3.5 there exists such a prime. Since we have assumed that $\mathfrak{A}_1 \neq \mathfrak{A}$, we get that $\mathfrak{R}/\mathfrak{R} \cap Z(\mathfrak{G})$ is elementary abelian of order r^2. We may assume that $r > q$. Since $\mathfrak{R}/\mathfrak{R}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$, this is a contradiction. Hence we (may) assume that $\mathfrak{A} = \mathfrak{A}_1$.

Put $F(C(X)) = \mathfrak{C} \times \mathfrak{D}$, where $\mathfrak{C}$ and $\mathfrak{D}$ are the Sylow 2-subgroup and Sylow 2-complement of $F(C(X))$. If $\mathfrak{D} \not\subseteq Z(\mathfrak{G})$, then $C(X) \cap C(\mathfrak{D})$ is nilpotent and contains $\mathfrak{A}$ and is normal in $C(X)$. So $\mathfrak{A}$ is normal in $C(X)$. Then $\mathfrak{P} \subseteqq N(C(X)) \subseteqq N(\mathfrak{A})$. Since $C(\mathfrak{A}) = \mathfrak{F}_1$, $\mathfrak{P}_1$ is normal in $\mathfrak{P}$. Then $\mathfrak{P}$ is contained in $N(\mathfrak{F}_1)$ and $\mathfrak{G} = N(\mathfrak{F}_1)$, which implies the solvability of $\mathfrak{G}$. This is a contradiction. So we must have that $\mathfrak{D} \subseteqq Z(\mathfrak{G})$. Then $\mathfrak{C} \supsetneqq \mathfrak{P}_2$, where $\mathfrak{P}_2$ is a Sylow 2-subgroup of $\mathfrak{F}_2$. Then $\mathfrak{A}/\mathfrak{A} \cap Z(\mathfrak{G})$ can be considered as a regular automorphism of $\mathfrak{P}_X/\mathfrak{P}_2$, and hence $\mathfrak{A}/\mathfrak{A}\cap Z(\mathfrak{G})$ is cyclic. Then assume as above that $r>q$. Then since $\mathfrak{Q}_1/Z(\mathfrak{G})\cap\mathfrak{Q}_1$ is cyclic, we get a contradiction as above.

Proposition 3.19. *Assume that there exists a p-singular fundamental subgroup and that there exist no q-singular fundamental subgroups for every q distinct from p. Then there exists no free fundamental subgroup of index n_1.*

Proof. This is obvious by the proof of Proposition 3.18.

Proposition 3.20. *For at least two distinct primes p there exist p-singular fundamental subgroups of $\mathfrak{G}$.*

Proof. By Proposition 3.15 for some prime p there exists a p-singular fundamental subgroup $\mathfrak{F}_1$ of $\mathfrak{G}$. Suppose that there exists no q-singular fundamental subgroup of $\mathfrak{G}$ for every prime q distinct from p.

By Propositions 3.18 and 3.19 if X is a q-element of $\mathfrak{G}$ not belonging to $Z(\mathfrak{G})$, then $\mathfrak{G}: C(X)=n_2$. By Propositions 3.6 and 1.4 $C(X)$ is nilpotent. Furthermore by Propositions 3.18, 3.19, 3.5 and 1.1 $C(X)$ is abelian.

Let $\mathfrak{F}_2$ be a fundamental subgroup of type 2 contained in $\mathfrak{F}_1$. Let $\mathfrak{Q}$, $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ be Sylow q-subgroups of $\mathfrak{G}$, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ such that $\mathfrak{Q}\supseteqq\mathfrak{Q}_1\supseteqq\mathfrak{Q}_2$ $(q\neq p)$. By Propositions 3.10 and 3.11 we have that $\mathfrak{Q}:\mathfrak{Q}_1\leq q$ and $\mathfrak{Q}_1:\mathfrak{Q}_2\leq q$. Now we show that $\mathfrak{Q}_2$ is normal in $\mathfrak{Q}$. Assume the contrary. Then we must have that $\mathfrak{Q}:\mathfrak{Q}_1=q$, $\mathfrak{Q}_1:\mathfrak{Q}_2=q$ and $\mathfrak{Q}:\mathfrak{Q}_2=q^2$. Furthermore there exists an element Y in $\mathfrak{Q}$ such that $Y^{-1}\mathfrak{Q}_2Y\neq\mathfrak{Q}_2$. Since $\mathfrak{Q}_2$ is abelian, $Y^{-1}\mathfrak{Q}_2Y\cap\mathfrak{Q}_2=\mathfrak{Q}\cap Z(\mathfrak{G})$. So $\mathfrak{Q}_1/\mathfrak{Q}\cap Z(\mathfrak{G})$ is elementary abelian of order q^2. Let Z be an element of $\mathfrak{Q}_1$ such that $Z(\mathfrak{Q}\cap Z(\mathfrak{G}))$ is an element of $Z(\mathfrak{Q}/\mathfrak{Q}\cap Z(\mathfrak{G}))$ of order q. Let $\mathfrak{Q}_Z$ be a Sylow q-subgroup of $C(Z)$. Then $\mathfrak{Q}_Z=(\mathfrak{Q}\cap Z(\mathfrak{G}))\langle Z\rangle$ is normal in $\mathfrak{Q}$. Let $\mathfrak{R}_Z$ be a Sylow r-subgroup of $C(Z)$ with $r\neq p$, q (Proposition 3.5). Then $\mathfrak{Q}/\mathfrak{Q}_Z$ can be considered as a regular automorphism group of $\mathfrak{R}_Z/\mathfrak{R}_Z\cap Z(\mathfrak{G})$. Thus $\mathfrak{Q}/\mathfrak{Q}_Z$ is cyclic ([3], p. 499). This is a contradiction. So $\mathfrak{Q}_2$ is normal in $\mathfrak{Q}$. Let $\mathfrak{R}_2$ be a Sylow r-subgroup of $\mathfrak{F}_2$ with $r\neq p$, q. Then $\mathfrak{Q}/\mathfrak{Q}_2$ can be considered as a regular automorphism group of $\mathfrak{R}_2/\mathfrak{R}_2\cap Z(\mathfrak{G})$. Thus $\mathfrak{Q}/\mathfrak{Q}_2$ is cyclic. Since $C(\mathfrak{Q}_2)=\mathfrak{F}_2$, we get that $\mathfrak{Q}\subseteqq N(\mathfrak{F}_2)$. Therefore $\mathfrak{G}: N(\mathfrak{F}_2)$ is a power of p.

Since $N(\mathfrak{F}_2)\subseteqq N(\mathfrak{P}_2)$, $\mathfrak{G}=\mathfrak{P}N(\mathfrak{P}_2)$, where $\mathfrak{P}$ is a Sylow 2-subgroup of $\mathfrak{G}$ containing $\mathfrak{P}_2$. Hence we get that $O_p(\mathfrak{G})\supseteqq\mathfrak{P}_2$.

Let $\mathfrak{F}^*$ be a free fundamental subgroup of index n_2 (*cf.* Proposition 3.19) and $\mathfrak{Q}^*$ a Sylow q-subgroup of $\mathfrak{F}^*$. We may assume that $\mathfrak{Q}^*\subseteqq\mathfrak{Q}$. We show that $\mathfrak{Q}^*$ is normal in $\mathfrak{Q}$. Assume the contrary. Then we must have that $\mathfrak{Q}:\mathfrak{Q}^*=q^2$. Since $C(\mathfrak{Q}^*\cap\mathfrak{Q}_1)$ contains $Z(\mathfrak{F}_1)$ and since $\mathfrak{F}^*$ is free and abelian, we get that $\mathfrak{Q}^*\cap\mathfrak{Q}_1\subseteqq\mathfrak{Q}\cap Z(\mathfrak{G})$. Thus $\mathfrak{Q}^*:\mathfrak{Q}\cap Z(\mathfrak{G})=\mathfrak{Q}_2:\mathfrak{Q}\cap Z(\mathfrak{G})=q$. We know already that $\mathfrak{Q}/\mathfrak{Q}_2$ is cyclic (of order q^2). Let $W\mathfrak{Q}_2$, $W\in\mathfrak{Q}$

be a generater of $\mathfrak{D}/\mathfrak{D}_2$. Then $C(W)$ has the index n_2 in $\mathfrak{G}$ and $W^q \notin Z(\mathfrak{G}) \cap \mathfrak{D}$. This is a contradiction. Now $\mathfrak{Q}/\mathfrak{Q}^*$ is cyclic; in fact, $\mathfrak{Q}/\mathfrak{Q}^*$ can be considered as a regular automorphism group of $\mathfrak{P}^*/Z(\mathfrak{G}) \cap \mathfrak{P}^*$, where $\mathfrak{P}^*$ is a Sylow p-subgroup of $\mathfrak{F}^*$ (*cf.* Proposition 3.7). Furthermore, the above argument shows that $\mathfrak{Q}: \mathfrak{Q}^* = q$ and that $\mathfrak{Q}^*: Z(\mathfrak{G}) \cap \mathfrak{Q}^* = q$. Then take a prime divisor r of $|\mathfrak{F}^*|$ distinct from p and q. Let $\mathfrak{R}$ and $\mathfrak{R}^*$ be Sylow r-subgroups of $\mathfrak{G}$ and $\mathfrak{F}^*$ such that $\mathfrak{R} \supseteqq \mathfrak{R}^*$. Then as above we obtain that $\mathfrak{R}: \mathfrak{R}^* = \mathfrak{R}^*: Z(\mathfrak{G}) \cap \mathfrak{R}^* = r$. We may assume that $r > q$. Then since $\mathfrak{R}/\mathfrak{R}^*$ can be considered as a regular automorphism group of $\mathfrak{Q}^*/Z(\mathfrak{G}) \cap \mathfrak{Q}^*$, this is a contradiction. Hence there exists no free fundamental subgroup (of index n_2).

Now every p-element is contained in some fundamental subgroup of type 2. Hence we get that $O_p(\mathfrak{G}) = \mathfrak{P}$. Since $\mathfrak{H}$ is solvable, $\mathfrak{G}$ is solvable against the assumption.

Proposition 3.21. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be fundamental subgroups of type* 1 *and* 2 *such that $\mathfrak{F}_1 \supset \mathfrak{F}_2$. Then $\mathfrak{F}_1: \mathfrak{F}_2$ is square-free.*

Proof. This is obvious by Propositions 3.12 and 3.20.

Now let $\mathfrak{F}_1$ be p-singular and $\hat{\mathfrak{F}}_1$ be q-singular, where $p \neq q$. Let $\mathfrak{F}_2$ be a fundamental subgroup of type 2 contained in $\mathfrak{F}_1$. By Proposition 3.6 $\mathfrak{F}_2$ is nilpotent. Since $\mathfrak{F}_1$ is p-singular, $\mathfrak{F}_1: N(\mathfrak{F}_2) \cap \mathfrak{F}_1 = p$ or 1. Next let $\hat{\mathfrak{F}}_2$ be a fundamental subgroup of type 2 contained in $\hat{\mathfrak{F}}_1$. By Proposition 3.6 $\hat{\mathfrak{F}}_2$ is nilpotent. Since $\hat{\mathfrak{F}}_1$ is q-singular, $\hat{\mathfrak{F}}_1: N(\hat{\mathfrak{F}}_2) \cap \hat{\mathfrak{F}}_1 = q$ or 1. Assume that $p > q$. Let $\hat{\mathfrak{P}}_2$ be a Sylow p-subgroup of $\hat{\mathfrak{F}}_2$. Since $N(\hat{\mathfrak{P}}_2) \cap \hat{\mathfrak{F}}_1 = N(\hat{\mathfrak{F}}_2) \cap \hat{\mathfrak{F}}_1$, we see that $\hat{\mathfrak{P}}_2$ is normal in $\hat{\mathfrak{F}}_1$. Hence $\hat{\mathfrak{F}}_2$ is normal in $\hat{\mathfrak{F}}_1$. Let X be an element of $\hat{\mathfrak{F}}_1$ not belonging to $\hat{\mathfrak{F}}_2$ such that $|X|$ is prime to q. Assume that $\hat{\mathfrak{F}}_1 = C(Y)$, where Y is a q-element. Then $C(XY)$ is a fundamental subgroup of type 2 contained in $\hat{\mathfrak{F}}_1$. The above argument shows that $C(XY)$ is a nilpotent normal subgroup of $\hat{\mathfrak{F}}_1$. Then $\hat{\mathfrak{F}}_2 C(XY)$ is nilpotent. This is a contradiction. This implies that $\hat{\mathfrak{F}}_1: \hat{\mathfrak{F}}_2 = q$. But then $\mathfrak{F}_1: \mathfrak{F}_2 = q$ and $\mathfrak{F}_2$ is normal in $\mathfrak{F}_1$. There exists an element Z of $\mathfrak{F}_1$ not belonging to $\mathfrak{F}_2$ such that $|Z|$ is prime to p. Assume that $\mathfrak{F}_1 = C(W)$, where W is a p-element. Then $C(ZW)$ is a fundamental subgroup of type 2 contained in $\mathfrak{F}_1$. The above argument shows that $C(ZW)$ is a nilpotent normal subgroup of $\mathfrak{F}_1$. Then $\mathfrak{F}_2\, C(ZW)$ is nilpotent. This is a contradiction (cf. Proposition 1.1).

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