

Title	On finite groups with given conjugate types. II
Author(s)	Itô, Noboru
Citation	Osaka Journal of Mathematics. 1970, 7(1), p. 231-251
Version Type	VoR
URL	https://doi.org/10.18910/10804
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Osaka University

ON FINITE GROUPS WITH GIVEN CONJUGATE TYPES II*

Dedicated to Professor Keizo Asano on his sixtieth birthday

NOBORU ITO**

(Received December 11, 1969)

Let \mathfrak{G} be a finite group. Let $\{n_1, \dots, n_r\}$ be the set of integers each of which is the index of the centralizer of some element of \mathfrak{G} in \mathfrak{G} . We may assume that $n_1 > n_2 > \dots > n_r = 1$. Then the vector (n_1, \dots, n_r) is called the conjugate type vector of \mathfrak{G} . A group with the conjugate type vector (n_1, \dots, n_r) is said to be a group of type (n_1, \dots, n_r) .

In an earlier paper [5] we have proved that any group of type $(n_1, 1)$ is nilpotent. In the present paper we want to prove the following theorem.

Theorem. *Any group of type $(n_1, n_2, 1)$ is solvable.****

At few critical points the proof requires heavy group-theoretical apparatus.

NOTATION AND DEFINITION. Let \mathfrak{G} be a finite group. $Z(\mathfrak{G})$ is the center of \mathfrak{G} . $Z_2(\mathfrak{G})$ is the second center of \mathfrak{G} . $D(\mathfrak{G})$ is the commutator subgroup of \mathfrak{G} . $\Phi(\mathfrak{G})$ is the Frattini subgroup of \mathfrak{G} . Let p be a prime. $O_p(\mathfrak{G})$ is the largest normal p -subgroup of \mathfrak{G} . $F(\mathfrak{G})$ is the Fitting subgroup of \mathfrak{G} ($F(\mathfrak{G}) = \prod_p O_p(\mathfrak{G})$).

Let \mathfrak{X} be a finite set. $|\mathfrak{X}|$ is the number of elements in \mathfrak{X} . $|\mathfrak{X}|_p$ is the highest power of p dividing $|\mathfrak{X}|$. $\pi(\mathfrak{G})$ is the set of prime divisors of $|\mathfrak{G}|$. If $\mathfrak{X} \subseteq \mathfrak{G}$ and is non-empty, then $C(\mathfrak{X})$ is the centralizer of \mathfrak{X} in \mathfrak{G} . If $\mathfrak{X} = \{X\}$, $C(\mathfrak{X}) = C(X)$. $N(\mathfrak{X})$ is the normalizer of \mathfrak{X} in \mathfrak{G} . Let \mathfrak{X} be a subgroup of \mathfrak{G} and \mathfrak{Y} a subgroup of \mathfrak{X} . If $G^{-1}\mathfrak{Y}G \subseteq \mathfrak{X}$ ($G \in \mathfrak{G}$) implies that $G^{-1}\mathfrak{Y}G = \mathfrak{Y}$, we say that \mathfrak{Y} is weakly closed in \mathfrak{X} with respect to \mathfrak{G} . \mathfrak{G} is called a Frobenius group, if \mathfrak{G} is a product of a normal subgroup \mathfrak{N} and a subgroup \mathfrak{H} such that no elements ($\neq E$) of \mathfrak{N} and \mathfrak{H} commute one another. Let Σ be a group of automorphisms of \mathfrak{G} . If every element $\sigma \neq 1$ of Σ leaves no element ($\neq E$) of \mathfrak{G} fixed, Σ is called regular. If all the Sylow subgroups of \mathfrak{G} are cyclic, then \mathfrak{G} is called a Z-group. $PGL(2, q)$ and $PSL(2, q)$ denote the projective general and special linear groups of degree 2 over the field of q -elements.

* This is a continuation of [5].

** This research is partially supported by N.S.F. Grant GP 9584.

*** A part of the theorem, namely in the form of Proposition 2.2 was known at the time of [5].

A proper subgroup \mathfrak{F} of \mathfrak{G} is called fundamental, if there exists an element X of \mathfrak{G} such that $\mathfrak{F} = C(X)$. A fundamental subgroup \mathfrak{F} is called free, if \mathfrak{F} is not contained in and does not contain any other fundamental subgroup of \mathfrak{G} . \mathfrak{G} is called of type F , if all the fundamental subgroups of \mathfrak{G} are free.

Let \mathfrak{G} be a group of type $(n_1, n_2, 1)$. If \mathfrak{F}_1 and \mathfrak{F}_2 are fundamental subgroups of \mathfrak{G} such that $\mathfrak{F}_1 \cong \mathfrak{F}_2$, then \mathfrak{F}_1 and \mathfrak{F}_2 are called fundamental subgroups of \mathfrak{G} of type 1 and of type 2 respectively.

1. Preliminaries

Let \mathfrak{G} be a group of type $(n_1, n_2, 1)$ which is a counter-example of the least order against the theorem. Then \mathfrak{G} is non-solvable.

Proposition 1.1. (Burnside). $|\pi(\mathfrak{G})| \geq 3$.

Proof. ([3], p. 492).

Proposition 1.2. $Z(\mathfrak{G}) \subseteq \Phi(\mathfrak{G})$.

Proof. Otherwise, there exists a proper subgroup \mathfrak{H} of \mathfrak{G} such that $\mathfrak{G} = Z(\mathfrak{G})\mathfrak{H}$. Let X be an element of \mathfrak{H} . Since $C(X) \cong Z(\mathfrak{G})$, we have that $\mathfrak{G} : C(X) = \mathfrak{H} : \mathfrak{H} \cap C(X)$. Hence \mathfrak{H} is a group of type $(n_1, n_2, 1)$. By the choice of \mathfrak{G} \mathfrak{H} is solvable. Then \mathfrak{G} is solvable against the assumption.

Proposition 1.3. For every prime divisor p of $|\mathfrak{G}|$ there exists a p -element X such that $C(X) \neq \mathfrak{G}$.

Proof. Otherwise, a Sylow p -subgroup \mathfrak{P} of \mathfrak{G} is contained in $Z(\mathfrak{G})$. By a theorem of Zassenhaus ([3], p. 126) there exists a Sylow p -complement of \mathfrak{G} . Hence $\mathfrak{P} \not\subseteq \Phi(\mathfrak{G})$. This contradicts Proposition 1.2.

Proposition 1.4. (Cf. [5], Proposition 1.1). Let \mathfrak{F} be a free fundamental subgroup of \mathfrak{G} . Then \mathfrak{F} is either (i) abelian, or (ii) a non-abelian p -subgroup for some prime p , or (iii) a direct product of a non-abelian p -subgroup and the Sylow p -complement $\mathfrak{C}_p \neq \mathfrak{C}$ of $Z(\mathfrak{G})$.

2. Case where \mathfrak{G} is of type F

In this section we assume that \mathfrak{G} is of type F

Proposition 2.1. \mathfrak{G} contains no fundamental subgroup of prime power order.

Proof. Let \mathfrak{F} be a fundamental p -subgroup of \mathfrak{G} . Let $q (\neq p)$ be a prime divisor of $|\mathfrak{G}|$ (Cf. Proposition 1.1) and let $X (\neq E)$ be an element of the center of a Sylow q -subgroup \mathfrak{Q} of \mathfrak{G} . Then $C(X)$ contains \mathfrak{Q} . Since $Z(\mathfrak{G}) \subseteq \mathfrak{F}$,

$Z(\mathfrak{G})$ is a p -group. Now let \mathfrak{F}_1 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_1| \neq |\mathfrak{F}|$. Then \mathfrak{F}_1 contains a Sylow q -subgroup of \mathfrak{G} for every $q(\neq p)$. By Propositions 1.1 and 1.4 \mathfrak{F}_1 is abelian. Let \mathfrak{P} be a Sylow p -subgroup of \mathfrak{G} . Then $\mathfrak{G} = \mathfrak{F}_1\mathfrak{P}$, and hence \mathfrak{G} is solvable (For instance, [4]). This is a contradiction.

Proposition 2.2. *\mathfrak{G} contains a fundamental subgroup which is of the form (iii) in Proposition 1.4.*

Proof. Assume the contrary. Then by Propositions 1.4 and 2.1 all the fundamental subgroups of \mathfrak{G} are abelian. The intersection of any two distinct fundamental subgroups of \mathfrak{G} is equal to $Z(\mathfrak{G})$. Hence $\mathfrak{G}/Z(\mathfrak{G})$ admits an abelian normal partition whose components are factor groups of fundamental subgroups of \mathfrak{G} by $Z(\mathfrak{G})$. Then by a theorem of Suzuki ([6], Theorems 2 and 3) $\mathfrak{G}/Z(\mathfrak{G})$ has the following structures: If $C(XZ(\mathfrak{G}))$ is nilpotent for every involution $XZ(\mathfrak{G})$ of $G/Z(\mathfrak{G})$, then $\mathfrak{G}/Z(\mathfrak{G})$ is isomorphic to $PSL(2, q)$. If $\mathfrak{G}/Z(\mathfrak{G})$ contains an involution $XZ(\mathfrak{G})$ with non-nilpotent $C(XZ(\mathfrak{G}))$, then $\mathfrak{G}/Z(\mathfrak{G})$ is isomorphic to $PGL(2, q)$.

First assume that q is even. Then $PSL(2, q)$ ($=PGL(2, q)$) contains an involution whose centralizer is a 2-group. Hence \mathfrak{G} contains a 2-element X such that $C(X)/Z(\mathfrak{G})$ is a 2-group. Let \mathfrak{F} be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}| \neq |C(X)|$. By Proposition 1.3 $|\mathfrak{F}/Z(\mathfrak{G})|$ must be divisible by every odd prime divisor of $|\mathfrak{G}|$. But $PSL(2, q)$ contains no elements of order ab , where $a(\neq 1)$ and $b(\neq 1)$ are divisors of $q+1$ and $q-1$ respectively. This is a contradiction. So $q=p^m$ is odd. $PSL(2, q)$ and $PGL(2, q)$ contain p -elements whose centralizers are p -groups. Hence \mathfrak{G} contains a p -element X such that $C(X)/Z(\mathfrak{G})$ is a p -group. Let \mathfrak{F} be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}| \neq |C(X)|$. By Proposition 1.3 $|\mathfrak{F}/Z(\mathfrak{G})|$ must be divisible by every prime divisor of $|\mathfrak{G}|$ other than p . Let a and b be odd prime divisors of $q+1$ and $q-1$ respectively. Then $PSL(2, q)$ and $PGL(2, q)$ contains no element of order ab . This is a contradiction. Therefore $q+1$ or $q-1$ is a power of 2. But then $PSL(2, q)$ and $PGL(2, q)$ contain 2-elements whose centralizers are 2-groups. Hence \mathfrak{G} contains a 2-element Y such that $C(Y)/Z(\mathfrak{G})$ is a 2-group. Since $|C(Y)| = |\mathfrak{F}|$, this is a contradiction.

Let $\mathfrak{F}_0 = \mathfrak{P}_0 \times \mathfrak{C}_p$ be a fundamental subgroup of \mathfrak{G} which is of the form (iii) in Proposition 1.4; namely, \mathfrak{P}_0 is the non-abelian Sylow p -subgroup of \mathfrak{F}_0 and $\mathfrak{C}_p \neq \mathfrak{G}$ is the Sylow p -complement of $Z(\mathfrak{G})$. So in this section p is fixed henceforth.

Proposition 2.3. *Let \mathfrak{F}_1 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Then \mathfrak{F}_1 is abelian.*

Proof. By Propositions 2.1 and 1.3, otherwise, \mathfrak{F}_1 is a direct product of

a non-abelian q -subgroup and the Sylow q -complement $\mathfrak{C}_q \neq \mathfrak{C}$ of $Z(\mathfrak{G})$. By Propositions 1.1 and 1.3 there exist a prime divisor r of $|\mathfrak{G}|$ distinct from p and q and an r -element X of \mathfrak{G} such that $C(X) \neq \mathfrak{G}$. Then we obtain that $|C(X)|_r > |\mathfrak{G}_0|_r$ and $|C(X)|_r > |\mathfrak{F}_1|_r$. This is a contradiction.

Proposition 2.4. *Let \mathfrak{F}_1 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_1| \neq |\mathfrak{G}_0|$. Let q be a prime divisor of $|\mathfrak{G}|$ distinct from p . Let \mathfrak{Q}_1 be the Sylow q -subgroup of \mathfrak{F}_1 and \mathfrak{Q} a Sylow q -subgroup of \mathfrak{G} containing \mathfrak{Q}_1 . If \mathfrak{Q} is abelian, then $\mathfrak{Q} = \mathfrak{Q}_1$. If \mathfrak{Q} is non-abelian, then $\mathfrak{Q} : \mathfrak{Q}_1 = q$. Hence q^2 does not divide $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ and $\mathfrak{G} : N(\mathfrak{F}_1)$ is a power of p .*

Proof. If \mathfrak{Q} is abelian, then by Proposition 1.3 $\mathfrak{Q} = \mathfrak{Q}_1$. So let us assume that \mathfrak{Q} is non-abelian. Then by Proposition 2.3 $\mathfrak{Q}_1 \leq \mathfrak{Q}$, and hence $Z(\mathfrak{Q}) = \mathfrak{Q} \cap Z(\mathfrak{G})$. Let X be an element of $Z_2(\mathfrak{Q})$ such that $X \notin Z(\mathfrak{Q})$ and $X^q \in Z(\mathfrak{Q})$. Then $C(X) \supseteq D(\mathfrak{Q})$. Take any element Y of \mathfrak{Q} . Then $Y^{-1}XY = XZ$ with $Z \in Z(\mathfrak{Q})$. Therefore $C(X) = C(XZ) = C(Y^{-1}XY) = Y^{-1}C(X)Y$ and Y^q belongs to $C(X)$. Let \mathfrak{Q}_2 be the Sylow q -subgroup of $C(X)$. Then $N(C(X))$ contains \mathfrak{Q} and $\mathfrak{Q}/\mathfrak{Q}_2$ is an elementary abelian q -group.

By Propositions 1.1 and 1.3 there exist a prime divisor r of $|C(X)|$ and the Sylow r -subgroup \mathfrak{R}_2 of $C(X)$ such that $\mathfrak{R}_2 \cong Z(\mathfrak{G}) \cap \mathfrak{R}_2$. We show that $\mathfrak{Q}/\mathfrak{Q}_2$ can be considered as a regular automorphism group of $\mathfrak{R}_2/Z(\mathfrak{G}) \cap \mathfrak{R}_2$. In fact, let us assume that there exist $W \in \mathfrak{Q}$ and $V \in \mathfrak{R}_2$ such that $W \notin \mathfrak{Q}_2$, $V \notin Z(\mathfrak{G}) \cap \mathfrak{R}_2$ and $W^{-1}VW = VU$ with $U \in Z(\mathfrak{G}) \cap \mathfrak{R}_2$. Then $[V, W]^q = [V, W^q] = E = U^q$, which implies that $U = E$. Therefore W belongs to $C(V) = C(X)$, which is a contradiction. Hence $\mathfrak{Q}/\mathfrak{Q}_2$ is cyclic ([3], p. 499), and $\mathfrak{Q} : \mathfrak{Q}_2 = q$. Therefore $\mathfrak{Q} : \mathfrak{Q}_1 = q$ and \mathfrak{Q}_1 is normal in \mathfrak{Q} . Finally we show that $\mathfrak{Q} \subseteq N(\mathfrak{F}_1)$. Assume the contrary. Then there exist an element A of \mathfrak{Q} and a Sylow r -subgroup \mathfrak{R}_1 of \mathfrak{F}_1 such that $\mathfrak{R}_1 \neq A^{-1}\mathfrak{R}_1 A$. Then an abelian group $\mathfrak{F}_1 = C(\mathfrak{Q}_1)$ contains \mathfrak{R}_1 and $A^{-1}\mathfrak{R}_1 A$ as its Sylow r -subgroups. Hence $\mathfrak{R}_1 = A^{-1}\mathfrak{R}_1 A$, which is a contradiction.

Proposition 2.5. *Let \mathfrak{F}_1 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_1| \neq |\mathfrak{G}_0|$. Then $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is not a p -group and has a square-free order.*

Proof. In order to prove that $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ is square-free, it suffices to show that p^2 does not divide $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ (Proposition 2.4). Let \mathfrak{P} , \mathfrak{P}_1 , \mathfrak{P}_3 and \mathfrak{S}_p be Sylow p -subgroups of \mathfrak{G} , $N(\mathfrak{F}_1)$, \mathfrak{F}_1 and $Z(\mathfrak{G})$ such that $\mathfrak{P} \supseteq \mathfrak{P}_1 \supseteq \mathfrak{P}_3 \supseteq \mathfrak{S}_p$.

If $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is a p -group, then by Proposition 2.4 we have that $\mathfrak{G} = \mathfrak{P}\mathfrak{F}_1$. Since \mathfrak{F}_1 is abelian (Proposition 2.3), \mathfrak{G} is solvable (For instance, [4]). This is a contradiction.

Now assume that $\mathfrak{P}_1 : \mathfrak{P}_3 \geq p^2$. If $\mathfrak{P}_1 \cong \mathfrak{S}_p$, then $C(\mathfrak{P}_1) = \mathfrak{F}_1$. Thus \mathfrak{P}_1 is non-abelian. Then $Z(\mathfrak{P}_1) = Z(\mathfrak{G}) \cap \mathfrak{P}_1$, and \mathfrak{P}_1 contains an element X such that $X \in Z_2(\mathfrak{P}_1)$, $X \notin Z(\mathfrak{P}_1)$ and $X^p \in Z(\mathfrak{P}_1)$. As in the proof of Proposition 2.4 we

obtain that $\mathfrak{P}_1/\mathfrak{P}_1$ is an elementary abelian p -group. By Propositions 1.1 and 1.3 there exist a prime divisor q of $|\mathfrak{F}_1|$ distinct from p and the Sylow q -subgroup \mathfrak{Q}_1 of \mathfrak{F}_1 such that $\mathfrak{Q}_1 \neq Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. As in the proof of Proposition 2.4 we can show that $\mathfrak{P}_1/\mathfrak{P}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. Hence $\mathfrak{P}_1/\mathfrak{P}_1$ is cyclic and $\mathfrak{P}_1:\mathfrak{P}_1=p$, which is against the assumption. So we can assume that $\mathfrak{P}_1=\mathfrak{S}_p$. As above $\mathfrak{P}_1/\mathfrak{P}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. Hence if p is odd, then $\mathfrak{P}_1/\mathfrak{P}_1$ is cyclic. So $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is a Z -group. We already know that there exists a prime divisor r of $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ distinct from p . Let $\mathfrak{X}/\mathfrak{F}_1$ be a Hall $\{p, r\}$ -subgroup of $N(\mathfrak{F}_1)/\mathfrak{F}_1$. By Propositions 1.1 and 1.3 there exist a prime divisor q of $|\mathfrak{F}_1|$ distinct from p and r and a Sylow q -subgroup \mathfrak{Q}_1 of \mathfrak{F}_1 such that $\mathfrak{Q}_1 \neq Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. Then $\mathfrak{X}/\mathfrak{F}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$. So $\mathfrak{X}/\mathfrak{F}_1$ contains an element $Y\mathfrak{F}_1$ of order pr ([3], p. 499). Then $|C(Y)|_p > |\mathfrak{F}_1|_p$ and $|C(Y)|_r > |\mathfrak{F}_0|_r$. This is a contradiction. If $p=2$ and $\mathfrak{P}_1/\mathfrak{P}_1$ is cyclic, we get a contradiction as above. Thus we may assume that $\mathfrak{P}_1/\mathfrak{P}_1$ is a (generalized) quaternion group. Let $I\mathfrak{F}_1$ be an involution of $N(\mathfrak{F}_1)/\mathfrak{F}_1$. Then for every element K of \mathfrak{Q}_1 we get that $IKI \equiv K^{-1} \pmod{\mathfrak{Q}_1 \cap Z(\mathfrak{G})}$. Let $W\mathfrak{F}_1$ be an element of order r of $N(\mathfrak{F}_1)/\mathfrak{F}_1$. Then we obtain that $IW^{-1}KW I \equiv W^{-1}K^{-1}W \equiv W^{-1}IKIW \pmod{\mathfrak{Q}_1 \cap Z(\mathfrak{G})}$. Put $\mathfrak{Y}/\mathfrak{Q}_1 \cap Z(\mathfrak{G}) = C(\mathfrak{Q}_1/\mathfrak{Q}_1 \cap Z(\mathfrak{G}))$. Then $\mathfrak{Y}:\mathfrak{F}_1$ equals a power of q and \mathfrak{Y} is normal in $N(\mathfrak{F}_1)$. Since $|WI\mathfrak{Y}|=2r$, $|C(WI)/Z(\mathfrak{G})| \equiv 0 \pmod{2r}$. Then $|C(WI)|_2 > |\mathfrak{F}_1|_2$ and $|C(WI)|_r > |\mathfrak{F}_0|_r$. This is a contradiction.

Proposition 2.6. *Let \mathfrak{F}_1 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Let q be a prime divisor of $|\mathfrak{G}|$ distinct from p . Let \mathfrak{Q}_1 be the Sylow q -subgroup of \mathfrak{F}_1 and \mathfrak{Q} a Sylow q -subgroup of \mathfrak{G} containing \mathfrak{Q}_1 . If \mathfrak{Q}_1 is not weakly closed in \mathfrak{Q} with respect to \mathfrak{G} , then $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G}^2)$ is an elementary abelian q -group of order q^2 .*

Proof. This is obvious by Proposition 2.4.

Proposition 2.7. *Let \mathfrak{F}_1 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Let q be a prime divisor of $|\mathfrak{G}|$ distinct from p . Let \mathfrak{Q} , \mathfrak{Q}_1 and \mathfrak{Q}_q be Sylow q -subgroups of \mathfrak{G} , \mathfrak{F}_1 and $Z(\mathfrak{G})$ such that $\mathfrak{Q} \supseteq \mathfrak{Q}_1 \supseteq \mathfrak{Q}_q$. Now if \mathfrak{G} contains a normal subgroup \mathfrak{H} of index q , then $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ is an elementary abelian group of order q^2 .*

Proof. Since $|\mathfrak{F}_0|_q = |\mathfrak{Q}_q|$, $\mathfrak{Q}_1 \supseteq \mathfrak{Q}_q$ by Proposition 1.3. Let \mathfrak{F}_0 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_0| = |\mathfrak{F}_1|$. By Proposition 1.2 \mathfrak{H} contains $Z(\mathfrak{G})$. Thus \mathfrak{H} contains \mathfrak{F}_0 . So if for every pair of fundamental subgroups \mathfrak{F}_1 and \mathfrak{F}_1 such that $|\mathfrak{F}_1| = |\mathfrak{F}_1| \neq |\mathfrak{F}_0|$ we have that $\mathfrak{H} \cap \mathfrak{F}_1 = \mathfrak{H} \cap \mathfrak{F}_1$, then \mathfrak{H} is of type $(n'_1, n'_2, 1)$. By the minimality of \mathfrak{G} \mathfrak{H} is solvable, and hence \mathfrak{G} is solvable against the assumption. So there exists a pair of funda-

mental subgroups \mathfrak{F}_1 and $\hat{\mathfrak{F}}_1$ such that $|\mathfrak{F}_1| = |\hat{\mathfrak{F}}_1| \neq |\mathfrak{F}_0|$ and $\mathfrak{H} \supseteq \mathfrak{F}_1$ and $\hat{\mathfrak{F}}_1 : \mathfrak{H} \cap \hat{\mathfrak{F}}_1 = q$. This implies, in particular, that \mathfrak{Q} is non-abelian. Let $\hat{\mathfrak{Q}}_1$ be the Sylow q -subgroup of $\hat{\mathfrak{F}}_1$. We may assume that $\hat{\mathfrak{Q}}_1 \subseteq \mathfrak{Q}$. Then $\mathfrak{Q}_1 \neq \hat{\mathfrak{Q}}_1$ and $\mathfrak{Q}_1 \cap \hat{\mathfrak{Q}}_1 \subseteq Z(\mathfrak{Q}) = \mathfrak{Q} \cap Z(\mathfrak{G})$. Therefore $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ is an elementary abelian group of order q^2 .

Proposition 2.8. *Let \mathfrak{F}_1 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Then $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ cannot be divisible by three distinct prime numbers q, r and s which are distinct from 2 and p .*

Proof. Assume that $q > r > s$. Let $\mathfrak{X}/\mathfrak{F}_1$ be a Hall $\{r, s\}$ -subgroup of $N(\mathfrak{F}_1)/\mathfrak{F}_1$ (cf. Proposition 2.5). Let \mathfrak{Q}_1 be a Sylow q -subgroup of \mathfrak{F}_1 . Then $\mathfrak{Q}_1 \neq \mathfrak{Q}_1 \cap Z(\mathfrak{G})$ (cf. Proposition 1.3). Now $\mathfrak{X}/\mathfrak{F}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/\mathfrak{Q}_1 \cap Z(\mathfrak{G})$ (cf. Proof of Proposition 2.4). So $\mathfrak{X}/\mathfrak{F}_1$ is cyclic ([3], p. 499). The same argument holds for any Hall $\{r, t\}$ - or $\{s, t\}$ -subgroup of $N(\mathfrak{F}_1)/\mathfrak{F}_1$, where t is a prime divisor of $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ distinct from r and s . Thus $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is cyclic. Let \mathfrak{S} and \mathfrak{S}_1 be Sylow s -subgroups of $N(\mathfrak{F}_1)$ and \mathfrak{F}_1 respectively. Then \mathfrak{S} is a Sylow s -subgroup of \mathfrak{G} . If \mathfrak{S}_1 is weakly closed in \mathfrak{S} with respect to \mathfrak{G} , then since s is odd and \mathfrak{S}_1 is abelian, \mathfrak{G} contains a normal subgroup of index s ([2], p. 212). So by Propositions 2.6 and 2.7 we have that $\mathfrak{S}_1 : Z(\mathfrak{G}) \cap \mathfrak{S}_1 = s$. Let \mathfrak{Q} be a Sylow q -subgroup of $N(\mathfrak{F}_1)$. Then $\mathfrak{Q}/\mathfrak{Q}_1$ can be considered as a regular automorphism group of $\mathfrak{S}_1/Z(\mathfrak{G}) \cap \mathfrak{S}_1$. Since $q > s$, this is a contradiction.

Proposition 2.9. *Let \mathfrak{F}_1 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. If $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ is even, then $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ cannot be divisible by two distinct prime numbers q, r which are distinct from 2 and p .*

Proof. This is obvious by the proof of Proposition 2.8.

REMARK. By Propositions 2.5, 2.8 and 2.9 we have that $N(\mathfrak{F}_1) : \mathfrak{F}_1 = q$ or pq or qr or pqr , where $q \neq r$ and $q \neq p \neq r$.

Proposition 2.10. *Let \mathfrak{F}_1 be a fundamental subgroup of \mathfrak{G} such that $|\mathfrak{F}_1| \neq |\mathfrak{F}_0|$. Let \mathfrak{P} , \mathfrak{P}_1 and \mathfrak{S}_p be Sylow p -subgroups of \mathfrak{G} , \mathfrak{F}_1 and $Z(\mathfrak{G})$ respectively, such that $\mathfrak{P} \supseteq \mathfrak{P}_1 \supseteq \mathfrak{S}_p$. Then we have that $\mathfrak{P}_1 = \mathfrak{S}_p$.*

Proof. Assume that $\mathfrak{P}_1 \neq \mathfrak{S}_p$. Then since \mathfrak{P} is not abelian (Proposition 2.2), $C(\mathfrak{P}_1) = \mathfrak{F}_1$ and $N(\mathfrak{F}_1) = N(\mathfrak{P}_1) \supseteq Z_2(\mathfrak{P})$. Let \mathfrak{R} be the largest normal subgroup of \mathfrak{G} contained in $N(\mathfrak{F}_1)$. Then since $\mathfrak{G} = \mathfrak{P}N(\mathfrak{F}_1)$ (Proposition 2.4), \mathfrak{R} contains $Z_2(\mathfrak{P})$. Let X be an element of $Z_2(\mathfrak{P})$ not belonging to $Z(\mathfrak{G})$. Let \mathfrak{Q}_1 be a Sylow q -subgroup of \mathfrak{F}_1 . If X belongs to \mathfrak{F}_1 , then $\mathfrak{F}_1 = C(X)$ contains $D(\mathfrak{P})$, and $N(\mathfrak{F}_1) = N(\mathfrak{P}_1)$ contains \mathfrak{P} . This is a contradiction. Thus X does not belong to \mathfrak{F}_1 . So $XZ(\mathfrak{G})$ induces a regular automorphism on $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$.

Hence \mathfrak{R} contains \mathfrak{Q}_1 . Therefore $\mathfrak{R} \cap \mathfrak{F}_1$ is a power of p , and \mathfrak{Q}_1 is the Sylow q -subgroup of the Fitting subgroup of \mathfrak{R} . Thus \mathfrak{Q}_1 is normal in \mathfrak{G} . Since $N(\mathfrak{Q}_1) = N(\mathfrak{F}_1)$, this is a contradiction.

Proposition 2.11. *p is odd.*

Proof. Assume that $p=2$. Let $IZ(\mathfrak{G})$ be an involution of $\mathfrak{G}/Z(\mathfrak{G})$ and put $C(IZ(\mathfrak{G})) = \frac{\mathfrak{K}}{Z(\mathfrak{G})}$. Then $\mathfrak{K}:C(I)$ is a power of 2. Since $|C(I)| = |\mathfrak{F}_0|$ (Proposition 2.10), $C(I)/Z(\mathfrak{G})$ is a 2-group. So $\frac{\mathfrak{K}}{Z(\mathfrak{G})}$ is a 2-group. Therefore by a theorem of Suzuki ([9], Theorem 2) $\mathfrak{G}/Z(\mathfrak{G})$ possesses one of the following properties: (a) $\mathfrak{G}/Z(\mathfrak{G})$ contains a normal Sylow 2-subgroup. (b) $\mathfrak{P}Z(\mathfrak{G})/Z(\mathfrak{G})$ is cyclic or (generalized) quaternion and if $X^{-1}\mathfrak{P}Z(\mathfrak{G})X \neq \mathfrak{P}Z(\mathfrak{G})$, then $X^{-1}\mathfrak{P}Z(\mathfrak{G})X \cap \mathfrak{P}Z(\mathfrak{G}) = Z(\mathfrak{G})$, where \mathfrak{P} is a Sylow 2-subgroup of \mathfrak{G} . (c) $\mathfrak{G}/Z(\mathfrak{G})$ contains two normal subgroups $\mathfrak{G}_1/Z(\mathfrak{G})$ and $\mathfrak{G}_2/Z(\mathfrak{G})$ ($\mathfrak{G}_1 \supseteq \mathfrak{G}_2$) such that (i) a Sylow 2-subgroup of $\mathfrak{G}_2/Z(\mathfrak{G})$ is normal, (ii) $\mathfrak{G}:\mathfrak{G}_1$ is odd, and (iii) $\mathfrak{G}_1/\mathfrak{G}_2$ is isomorphic to $PSL(2, q)$ (q is a Fermat or a Mersenne prime) or $PSL(2, 3^2)$ or $PSL(2, 2^m)$ ($m \geq 2$) or $S(q)$ or $PSU(3, q)$ ($q > 2$) or $PSL(3, q)$ ($q > 2$) or M_q ; where $S(q)$, $PSU(3, q)$, $PSL(3, q)$ and M_q denote the Suzuki group, the 3-dimensional projective special unitary group, the 3-dimensional special linear group and the linear fractional group over the non-commutative nearfield of 9 elements respectively.

If $\mathfrak{G}/Z(\mathfrak{G})$ has Property (a), then \mathfrak{P} is normal in \mathfrak{G} . Since $\mathfrak{G} = \mathfrak{P}N(\mathfrak{F}_1)$, $\mathfrak{G}/\mathfrak{P} \cong N(\mathfrak{F}_1)/\mathfrak{P} \cap N(\mathfrak{F}_1)$. So \mathfrak{G} is solvable against the choice of \mathfrak{G} (Proposition 2.5). Suppose that $\mathfrak{G}/Z(\mathfrak{G})$ has Property (b). Since \mathfrak{P}_0 is non-abelian (Proposition 2.2), $\mathfrak{P}Z(\mathfrak{G})/Z(\mathfrak{G})$ is (generalized) quaternion. So \mathfrak{P} contains two elements A and B such that $A^{2^m} \equiv E$, $BA^{-1}B \equiv A^{-1}$, $B^2 \equiv A^{2^{m-1}} \pmod{Z(\mathfrak{P})}$. Put $BA^{-1}B = A^{-1}Z$, $Z \in Z(\mathfrak{P})$. Then since $C(B^2)$ contains A , we get that $C(B^2) \supseteq C(B)$. Since $B^2 \notin Z(\mathfrak{G})$ and \mathfrak{G} is of type F , this is a contradiction. So $\mathfrak{G}/Z(\mathfrak{G})$ has Property (c).

Suppose that $\mathfrak{G}_2 \neq Z(\mathfrak{G})$. Let \mathfrak{P}_2 be the Sylow 2-subgroup of \mathfrak{G}_2 and let \mathfrak{Q} be a Sylow q -subgroup of $N(\mathfrak{F}_1)$ not contained in \mathfrak{F}_1 (Proposition 2.5). If $\mathfrak{P}_2 \not\subseteq Z(\mathfrak{G})$, then $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ can be considered as a regular automorphism group of $\mathfrak{P}_2/\mathfrak{P}_2 \cap Z(\mathfrak{G})$ (Proposition 2.10). So $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ is cyclic ([3], p. 499), and \mathfrak{Q} is abelian. Then \mathfrak{Q} is contained in \mathfrak{F}_1 (Proposition 1.3), which is a contradiction. Thus \mathfrak{P}_2 is contained in $Z(\mathfrak{G})$ and $\mathfrak{G}_2/Z(\mathfrak{G})$ has an odd order. But then $\mathfrak{P}/\mathfrak{P} \cap Z(\mathfrak{G})$ can be considered as a regular automorphism group of $\mathfrak{G}_2/Z(\mathfrak{G})$. So $\mathfrak{P}/\mathfrak{P} \cap Z(\mathfrak{G})$ is cyclic or (generalized) quaternion. This leads to a contradiction, as above. Thus we get that $\mathfrak{G}_2 = Z(\mathfrak{G})$.

It can be easily checked that $PSU(3, q)$ and $PSL(3, q)$ ($q > 2$) contain involutions whose centralizers are not 2-groups. Thus $\mathfrak{G}_1/Z(\mathfrak{G})$ is not isomorphic to $PSU(3, q)$ nor $PSL(3, q)$ ($q > 2$). Now assume that $\mathfrak{G} \neq \mathfrak{G}_1$. Then it can

be easily checked that $\mathfrak{G}/Z(\mathfrak{G})$ contains an element of even order, which is not a power of 2. Thus we get that $\mathfrak{G}=\mathfrak{G}_1$. By the proof of Proposition 2.2 we can assume that $\mathfrak{G}/Z(\mathfrak{G})$ is isomorphic to $S(q)$ or M_9 . $S(q)$ contains no element of order ab , where a and b are prime divisors of q^2+1 and $q-1$ respectively (cf. [7]). M_9 contains no element of order 15. This contradicts Proposition 1.3.

Proposition 2.12. *A Sylow 2-subgroup \mathfrak{Q} of \mathfrak{G} is not abelian.*

Proof. If \mathfrak{Q} is abelian, then we may assume that \mathfrak{Q} is contained in \mathfrak{F}_1 . By a theorem of Feit-Thompson [1] $\mathfrak{Q} \neq \mathfrak{G}$. If $X^{-1}Z(\mathfrak{G})\mathfrak{Q}X/Z(\mathfrak{G}) \cap Z(\mathfrak{G})\mathfrak{Q}/Z(\mathfrak{G}) \neq Z(\mathfrak{G})$, then choose an element Y of $XZ^{-1}(\mathfrak{G})\mathfrak{Q}X \cap \mathfrak{Q}$ not belonging to $Z(\mathfrak{G})$. $C(Y)$ contains $X^{-1}\mathfrak{Q}X$ and \mathfrak{Q} . Since $C(Y)$ is abelian (Proposition 2.3), we get that $X^{-1}\mathfrak{Q}X = \mathfrak{Q}$. So by a theorem of Suzuki ([8], Theorem 2) $\mathfrak{G}/Z(\mathfrak{G})$ possesses one of the following properties: (a) $\mathfrak{G}/Z(\mathfrak{G})$ contains a normal Sylow 2-subgroup. (b) $\mathfrak{Q}Z(\mathfrak{G})/Z(\mathfrak{G})$ is cyclic or (generalized) quaternion. (c) $\mathfrak{G}/Z(\mathfrak{G})$ contains two normal subgroups $\mathfrak{G}_1/Z(\mathfrak{G})$ and $\mathfrak{G}_2/Z(\mathfrak{G})$ such that (i) $\mathfrak{G}/\mathfrak{G}_1$ and $\mathfrak{G}_2/Z(\mathfrak{G})$ have odd orders and (ii) $\mathfrak{G}_1/\mathfrak{G}_2$ is isomorphic to $PSL(2, q)$ ($q > 3$), $PSU(3, q)$ ($q > 2$) or $S(q)$.

If $\mathfrak{G}/Z(\mathfrak{G})$ has Property (a), then \mathfrak{Q} is normal in \mathfrak{G} . Then $N(\mathfrak{Q})=N(\mathfrak{F}_1)=\mathfrak{G}$, which implies the solvability of \mathfrak{G} (Proposition 2.5). This is a contradiction. If $\mathfrak{G}/Z(\mathfrak{G})$ has Property (b), then, since \mathfrak{Q} is abelian, $\mathfrak{Q}Z(\mathfrak{G})/Z(\mathfrak{G})$ is cyclic. Take a prime divisor r of $|N(\mathfrak{F}_1)/\mathfrak{F}_1|$ and an r -element R of $N(\mathfrak{F}_1)$ not belonging to \mathfrak{F}_1 . Then $R\mathfrak{F}_1$ induces a regular automorphism of $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$, which is a contradiction. So $\mathfrak{G}/Z(\mathfrak{G})$ has Property (c).

Suppose that $\mathfrak{G}_2 \neq Z(\mathfrak{G})$. Let \mathfrak{P}_2 be a Sylow p -subgroup of \mathfrak{G}_2 . If $\mathfrak{P}_2 \not\subseteq Z(\mathfrak{G})$, then we may assume that $N(\mathfrak{P}_2)$ contains \mathfrak{Q} . Thus $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ can be considered as a regular automorphism of $\mathfrak{P}_2/Z(\mathfrak{G}) \cap \mathfrak{P}_2$. So $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ is cyclic, which leads to a contradiction as above. Thus \mathfrak{P}_2 is contained in $N(\mathfrak{F}_1)$ and \mathfrak{G}_2 is solvable. If \mathfrak{G}_2 is contained in \mathfrak{F}_1 , then $\mathfrak{F}_1=C(\mathfrak{G}_2)$ is normal in \mathfrak{G} , which implies the solvability of \mathfrak{G} . This is a contradiction. So \mathfrak{G}_2 is not contained in \mathfrak{F}_1 . Take an element X of \mathfrak{G}_2 not belonging to \mathfrak{F}_1 . Then $X\mathfrak{F}_1$ induces a regular automorphism of $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$. Hence \mathfrak{G}_2 contains \mathfrak{Q} , which is a contradiction. Thus we get that $\mathfrak{G}_2=Z(\mathfrak{G})$.

It can be easily checked that Sylow 2-subgroups of $PSU(3, q)$, ($q > 2$) and $S(q)$ are non-abelian. Thus $\mathfrak{G}_1/Z(\mathfrak{G})$ is isomorphic to $PSL(2, q)$. Now if q is odd, then a Sylow 2-subgroup of $PSL(2, q)$ is dihedral and contains its own centralizer (in $PSL(2, q)$). Since \mathfrak{Q} is abelian, we get that $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ is elementary abelian of order 4. If q is even, then a Sylow 2-subgroup of $PSL(2, q)$ is an elementary abelian 2-group of order q and coincides with its own centralizer (in $PSL(2, q)$). Since $\mathfrak{F}_1=C(\mathfrak{Q})$, we get that $\mathfrak{F}_1 \cap \mathfrak{G}_1=\mathfrak{Q}Z(\mathfrak{G})$.

If r is an odd prime divisor of $(q+1)(q-1)$ distinct from p , then let R be an r -element of \mathfrak{G}_1 not belonging to $Z(\mathfrak{G})$. We may assume that $\mathfrak{F}_1 = C(R)$ and that $\mathfrak{F}_1 \supseteq \mathfrak{Q}$. Since \mathfrak{F}_1 is abelian, this is a contradiction. So we must have that $(q+1)(q-1) = 2^\alpha p^\beta$ with $\alpha, \beta \geq 0$. Since $q > 3$, if we put $q = l^m$, then $l \neq 2$ and $l \neq p$. Let L be an l -element of \mathfrak{G}_1 not belonging to $Z(\mathfrak{G})$. We may assume that $\mathfrak{F}_1 = C(L)$ and that $\mathfrak{F}_1 \supseteq \mathfrak{Q}$. Since \mathfrak{F}_1 is abelian, this is a contradiction.

REMARK. By the remark after Proposition 2.9 and by Proposition 2.12 we have that $N(\mathfrak{F}_1): \mathfrak{F}_1 = 2$ or $2p$ or $2q$ or $2pq$, where q is an odd prime distinct from p .

Proposition 2.13. *We have that $N(\mathfrak{F}_1): \mathfrak{F}_1 = 2$ or $2q$.*

Proof. If $N(\mathfrak{F}_1): \mathfrak{F}_1 = 2p$, then $N(\mathfrak{F}_1)/\mathfrak{F}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/\mathfrak{Q}_1 \cap Z(\mathfrak{G})$, where $\mathfrak{Q}_1 (\neq \mathfrak{G})$ is a Sylow q -subgroup of \mathfrak{F}_1 (By Proposition 1.1 there exists such a prime q). Thus $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is cyclic and there exists an element of order $2p$ of $\mathfrak{G}/Z(\mathfrak{G})$. This is a contradiction (Proposition 2.10). The case $N(\mathfrak{F}_1): \mathfrak{F}_1 = 2pq$ can be treated in the same way.

Proposition 2.14. *For any subgroup \mathfrak{X} of \mathfrak{G} put $\bar{\mathfrak{X}} = \mathfrak{X}Z(\mathfrak{G})/Z(\mathfrak{G})$. $N(\bar{\mathfrak{P}})$ is a Frobenius group with $\bar{\mathfrak{P}}$ as its kernel, where $\bar{\mathfrak{P}}$ is a Sylow p -subgroup of \mathfrak{G} .*

Proof. \mathfrak{G} is not p -nilpotent. In fact, if so, $N(\mathfrak{F}_1)$ is normal in \mathfrak{G} (Proposition 2.13), which implies the solvability of \mathfrak{G} against the choice of \mathfrak{G} . Hence \mathfrak{G} also is not p -nilpotent. Thus by a theorem of Frobenius ([3], p. 436) there exists a non-trivial subgroup \mathfrak{H} of $\bar{\mathfrak{P}}$ such that $N(\mathfrak{H})/C(\mathfrak{H})$ is not a p -group. We choose \mathfrak{H} so that $|\mathfrak{H}|$ is as big as possible. We show that $\mathfrak{H} = \bar{\mathfrak{P}}$. Assume that $\mathfrak{H} \subsetneq \bar{\mathfrak{P}}$. First we notice that $C(\mathfrak{H})$ is a p -group. In fact, otherwise, there exist a p -element X not belonging to $Z(\mathfrak{G})$ and an element Y which does not belong to $Z(\mathfrak{G})$ and has order prime to p , such that $XY = YX$. This contradicts Proposition 2.10. Then we get that $C(\mathfrak{H}) \subseteq \mathfrak{H}$. Otherwise, notice that $N(C(\mathfrak{H})\mathfrak{H}) \supseteq N(\mathfrak{H})$ and $C(C(\mathfrak{H})\mathfrak{H}) \subseteq C(\mathfrak{H})$, which contradicts the choice of \mathfrak{H} . Let $\bar{\mathfrak{Q}}$ be a Sylow q -subgroup of $N(\mathfrak{H})$, where $q \neq p$, and consider $\mathfrak{H}\bar{\mathfrak{Q}}$. Then the above argument shows that $\bar{\mathfrak{Q}}$ can be considered as a regular automorphism group of \mathfrak{H} . Hence $\bar{\mathfrak{Q}}$ is cyclic or (generalized) quaternion ([3], p. 499). Suppose that $\bar{\mathfrak{Q}}$ is (generalized) quaternion. Then $\bar{\mathfrak{Q}}$ contains two elements A and B such that $|AZ(\mathfrak{G})| = 4$ and $B^2 = A^2 Z_1$, $BA^{-1}B = A^{-1}Z_2$ with $Z_1, Z_2 \in Z(\mathfrak{G})$. Then $\mathfrak{G} \supseteq C(B^2) \supseteq C(B)$. Since \mathfrak{G} is of type F , this is impossible. So $\bar{\mathfrak{Q}}$ is cyclic. By a theorem of Feit-Thompson [1] $N(\mathfrak{H})$ is solvable. So let \mathfrak{H}^* and $\bar{\mathfrak{K}}$ be a Sylow p -subgroup and a Sylow p -complement of $N(\mathfrak{H})$ respectively. By assumption on \mathfrak{H} we have that $\mathfrak{H} = 0_p(N(\mathfrak{H}))$ and $\mathfrak{H}^* \supseteq \mathfrak{H}$. There exists a non-trivial cyclic subgroup $\bar{\mathfrak{J}}$ of $\bar{\mathfrak{K}}$ such that $\mathfrak{H}\bar{\mathfrak{J}}$ is normal

in $N(\mathfrak{F})$. By a theorem of Sylow we obtain that $N(\mathfrak{F}) = \mathfrak{F} \cdot N(\mathfrak{Y}) \cap N(\mathfrak{F})$. Therefore there exist an abelian subgroup \mathfrak{Y} which is not contained in $Z(\mathfrak{G})$ and has order prime to p and a p -element Z not belonging to $Z(\mathfrak{G})$ such that Z normalizes \mathfrak{Y} . Let Y be an element of \mathfrak{Y} not belonging to $Z(\mathfrak{G})$. Then $C(Y)$ and $C(ZYZ^{-1}) = Z^{-1}C(Y)Z$ contains \mathfrak{Y} . Thus we get that $C(Y) = Z^{-1}C(Y)Z$ (Proposition 2.3). This contradicts Proposition 2.13. So we must have that $\mathfrak{P} = \mathfrak{F}$.

Let $\bar{\mathfrak{X}} = \mathfrak{X}/Z(\mathfrak{G})$ be a Sylow p -complement of $N(\mathfrak{P})$. Then, as above, $\bar{\mathfrak{X}}$ can be considered as a regular automorphism group of \mathfrak{P} . Thus $N(\mathfrak{P})$ is a Frobenius group with \mathfrak{P} as its kernel.

Proposition 2.15. *Let \bar{X} be an element of $\bar{\mathfrak{G}} = \mathfrak{G}/Z(\mathfrak{G})$ whose order is divisible by p . Then \bar{X} is a p -element.*

Proof. Otherwise, put $\bar{X} = \bar{Y}\bar{Z} = \bar{Z}\bar{Y}$, where \bar{Y} is a p -element and \bar{Z} is an element whose order is prime to p . We may assume that \bar{Y} belongs to $\bar{\mathfrak{P}}$ (in Proposition 2.14). Then $\bar{Z}^{-1}\bar{\mathfrak{P}}\bar{Z} \neq \bar{\mathfrak{P}}$ (Proposition 2.14) and $\bar{Z}^{-1}\bar{\mathfrak{P}}\bar{Z} \cap \bar{\mathfrak{P}} \ni \bar{Y} \neq \bar{E}$. Let $\bar{\mathfrak{D}} = \bar{\mathfrak{P}} \cap \bar{W}^{-1}\bar{\mathfrak{P}}\bar{W}$ be a maximal intersection of $\bar{\mathfrak{P}}$ with other Sylow p -subgroups. Then $\bar{\mathfrak{D}} \neq \bar{\mathfrak{E}}$ and a Sylow p -subgroup of $N(\bar{\mathfrak{D}})$ is not normal in $N(\bar{\mathfrak{D}})$ ([10] p. 138). This leads to a contradiction as in the proof of Proposition 2.14.

Proposition 2.16. *Sylow p -subgroups of $\bar{\mathfrak{G}}$ are independent, namely if $\bar{X}^{-1}\bar{\mathfrak{P}}\bar{X} \neq \bar{\mathfrak{P}}$, then $\bar{X}^{-1}\bar{\mathfrak{P}}\bar{X} \cap \bar{\mathfrak{P}} = \bar{\mathfrak{E}}$.*

Proof. This is obvious from the proof of Proposition 2.15.

Proposition 2.17. *Let X be an element of \mathfrak{G} not belonging to $Z(\mathfrak{G})$ whose order is prime to p . Then $C(X)$ is conjugate with \mathfrak{F}_1 in \mathfrak{G} .*

Proof. If there exists a prime divisor r of $|\mathfrak{F}_1|$ distinct from 2 and q , then let \mathfrak{R} be a Sylow r -subgroup of \mathfrak{F}_1 . Then $C(\mathfrak{R}) = \mathfrak{F}_1$ and \mathfrak{R} is a Sylow r -subgroup of \mathfrak{G} (Proposition 2.13). $C(X)$ is abelian and contains $Y^{-1}\mathfrak{R}Y$ for some $Y \in \mathfrak{G}$. Thus $C(X) = C(Y^{-1}\mathfrak{R}Y) = Y^{-1}C(\mathfrak{R})Y = Y^{-1}\mathfrak{F}_1Y$. The same argument holds if \mathfrak{F}_1 contains a Sylow subgroup of \mathfrak{G} . Therefore by Proposition 2.13 we may assume that \mathfrak{F}_1 is a $\{2, q\}$ -group and that $N(\mathfrak{F}_1): \mathfrak{F}_1 = 2q$. Let $\mathfrak{S}, \mathfrak{S}_1, \mathfrak{S}_X$ be Sylow 2-subgroups of $N(\mathfrak{F}_1), \mathfrak{F}_1$ and $C(X)$ respectively. We may assume that $\mathfrak{S} \supseteq \mathfrak{S}_1$, $\mathfrak{S} \supseteq \mathfrak{S}_X$ and $\mathfrak{S}_1 \neq \mathfrak{S}_X$. Since $\mathfrak{S}_1 \cap \mathfrak{S}_X \subseteq Z(\mathfrak{G})$, we obtain that $\mathfrak{S}_1: Z(\mathfrak{G}) \cap \mathfrak{S}_1 = 2$. Now let \mathfrak{Q} and \mathfrak{Q}_1 be Sylow q -subgroups of $N(\mathfrak{F}_1)$ and \mathfrak{F}_1 respectively. Then $\mathfrak{Q}/\mathfrak{Q}_1$ can be considered as a regular automorphism group of $\mathfrak{S}_1/Z(\mathfrak{G}) \cap \mathfrak{S}_1$. This is a contradiction.

Now we count the number of elements in $\bar{\mathfrak{G}}$. Put $|\bar{\mathfrak{P}}| = p^a$, $|N(\bar{\mathfrak{F}}_1)| = x$, $|\bar{\mathfrak{F}}_1| = y$, and $|N(\mathfrak{P})| = p^a z$. By Propositions 2.15 and 2.16 there exist $\frac{x}{z}(p^a - 1)$

elements ($\neq \bar{E}$) of \mathfrak{G} whose orders are prime to p . Thus we obtain that

$$(*) \quad p^a = x \frac{x}{z} (p^a - 1) + p^a (y - 1) + 1.$$

From (*) we obtain that

$$x < \frac{x}{z} + y.$$

Since y and z are divisors of x , it is only possible when either $z=1$ or $y=x$. By Proposition 2.13 we have that $y \neq x$. By Proposition 2.14 we have that $z \neq 1$.

Thus \mathfrak{G} cannot be of type F .

3. Case where G is not of type F

In this section \mathfrak{G} is not of type F (See § 2). Let \mathfrak{F}_1 and \mathfrak{F}_2 be fundamental subgroups of \mathfrak{G} such that $\mathfrak{F}_1 \cong \mathfrak{F}_2$.

Proposition 3.1. *$|\mathfrak{F}_1|$ is divisible by every prime divisor p of $|\mathfrak{G}|$.*

Proof. This is obvious by Proposition 1.3.

Proposition 3.2. *If \mathfrak{F} is a free fundamental subgroup of \mathfrak{G} with $|\mathfrak{F}| = |\mathfrak{F}_1|$, then \mathfrak{F} is abelian.*

Proof. If \mathfrak{F} is of type (iii) in Proposition 1.4, then $|\mathfrak{F}_1|_q = |\mathfrak{S}_q|$, where \mathfrak{S}_q is the Sylow q -subgroup of $Z(\mathfrak{G})$ with $q \neq p$. This contradicts Proposition 1.3.

Proposition 3.3. *$|\mathfrak{F}_2|$ is divisible by every prime divisor p of $|\mathfrak{G}|$.*

Proof. Suppose that there exists a prime divisor p of $|\mathfrak{G}|$ which does not divide $|\mathfrak{F}_2|$. Since $Z(\mathfrak{G}) \subseteq \mathfrak{F}_2$, $|Z(\mathfrak{G})| \not\equiv 0 \pmod{p}$. Let $X \neq E$ be an element of $Z(\mathfrak{P})$, where \mathfrak{P} is a Sylow p -subgroup of \mathfrak{G} . Then we have that $|C(X)| = |\mathfrak{F}_1|$. If $C(X)$ is of type 1, then X belongs to a fundamental subgroup of type 2 contained in $C(X)$. Then $|\mathfrak{F}_2| \equiv 0 \pmod{p}$ against the assumption. So $C(X)$ is free, and by Proposition 3.2 $C(X)$ is abelian. Since $|C(X)| = |\mathfrak{F}_1|$ and $C(X) \cong \mathfrak{P}$, we may assume that $\mathfrak{P} \subseteq \mathfrak{F}_1$. But then $C(X) \cong Z(\mathfrak{F}_1)$ and $C(X) = \mathfrak{F}_1$. This is a contradiction.

Proposition 3.4. *We may choose \mathfrak{F}_2 so that there exist (at most) two primes p and q such that \mathfrak{F}_2 is a direct product of a $\{p, q\}$ -Hall subgroup and an abelian $\{p, q\}$ -Hall complement.*

Proof. We can find a p -element X with $C(X) = \mathfrak{F}_1$ for some prime p . Assume that for any other prime q and for any q -element Y of \mathfrak{F}_1 we have that

$C(Y) \cong \mathfrak{F}_1$. Then \mathfrak{F}_1 is a direct product of a Sylow p -subgroup and an abelian Sylow p -complement. Hence the same is true for \mathfrak{F}_2 . So we may assume that there exists a prime $q (\neq p_1)$ and a q -element Y of \mathfrak{F}_1 such that $C(Y) \cong \mathfrak{F}_1$. Then we can choose $C(XY)$ as \mathfrak{F}_2 with the claimed property.

Proposition 3.5. $|\mathfrak{F}_2/Z(\mathfrak{G})|$ is divisible by every prime divisor p of $|\mathfrak{G}|$.

Proof. Let \mathfrak{P}_2 be a Sylow p -subgroup of \mathfrak{F}_2 . Assume that \mathfrak{P}_2 is contained in $Z(\mathfrak{G})$. Let \mathfrak{P}_1 be a Sylow p -subgroup of \mathfrak{F}_1 containing \mathfrak{P}_2 . Then by Proposition 1.3 we have that $\mathfrak{P}_1 \cong \mathfrak{P}_2$. Let Y be an element of \mathfrak{P}_1 not belonging to \mathfrak{P}_2 . Then $|C(Y)| = |\mathfrak{F}_1|$ and, since $\mathfrak{F}_2 \cong Z(\mathfrak{F}_1)$, $C(Y)$ must be free. By Proposition 3.2 $C(Y)$ is abelian. Since $C(Y) \cong Z(\mathfrak{F}_1)$, $\mathfrak{F}_1 = C(Y)$. This is a contradiction.

Proposition 3.6. Every fundamental subgroup \mathfrak{F}_2 of type 2 is nilpotent.

Proof. If there exists a p -element X with $C(X) = \mathfrak{F}_2$, then \mathfrak{F}_2 is a direct product of a Sylow p -subgroup and an abelian Sylow p -complement of \mathfrak{F}_2 (cf. the proof of Proposition 3.4). So we may assume that there exists no element X of a prime power order such that $C(X) = \mathfrak{F}_2$.

Let X be a p -element of \mathfrak{F}_1 with $C(X) = \mathfrak{F}_1$, where $\mathfrak{F}_1 \cong \mathfrak{F}_2$. Let Y be an element of the least order of \mathfrak{F}_2 such that $C(Y) = \mathfrak{F}_2$. Put $\pi(|Y|) = \{q, r, \dots\}$. Then $|\pi(|Y|)| \geq 2$. Put $Y = Y_q Y_r \dots$, where $Y_q \neq E$, $Y_r \neq E \dots$ are q -, r -, \dots elements which are commutative with each other. Then by assumption $C(Y_q) \cong \mathfrak{F}_2$ for each q in $\pi(|Y|)$. If $C(Y_q) = \mathfrak{G}$, then $\mathfrak{F}_2 = C(Y) = C(r \prod_{r \neq q} Y_q)$, which contradicts the choice of Y . So we get that $|C(Y_q)| = |\mathfrak{F}_1|$. Assume that $q \neq p$. Then $\mathfrak{F}_1 \cong C(XY_q) \cong \mathfrak{F}_2$. If for every $q \neq p$ we have that $\mathfrak{F}_1 = C(XY_q) = C(Y_q)$, and if $\mathfrak{F}_1 = C(Y_p Y_q)$ provided that p belongs to $\pi(|Y|)$, then $\mathfrak{F}_1 = \mathfrak{F}_2$, which is a contradiction. So we may assume that either for some q $C(XY_q) = \mathfrak{F}_2$ or $\mathfrak{F}_2 = C(Y_p Y_q)$. Thus, in any case, \mathfrak{F}_2 is a direct product of a Hall $\{p, q\}$ -subgroup and an abelian Hall $\{p, q\}$ -complement (Proposition 3.4).

Let $r \neq p, q$ and let Z be an r -element of \mathfrak{F}_2 with $C(Z) \neq \mathfrak{G}$ (Proposition 3.5). Then we may assume that $C(Z) = \mathfrak{F}_1$. In fact, otherwise, $\mathfrak{F}_1 \cong C(XZ) \cong \mathfrak{F}_2$ and hence, $C(XZ) = \mathfrak{F}_2$. Then \mathfrak{F}_2 is a direct product of a Hall $\{p, r\}$ -subgroup and an abelian Hall $\{p, r\}$ -complement of \mathfrak{F}_2 (Proposition 3.4). Since $q \neq r$, \mathfrak{F}_2 is then nilpotent. So $C(Z) = \mathfrak{F}_1$. Then the above argument shows that there exists a prime $s \neq r$ such that \mathfrak{F}_2 is a direct product of a Hall $\{r, s\}$ -subgroup and an abelian Hall $\{r, s\}$ -complement. Since $\{p, q\} \neq \{r, s\}$, this implies that \mathfrak{F}_2 is nilpotent.

Proposition 3.7. No Sylow subgroup ($\neq \mathfrak{G}$) of \mathfrak{G} is contained in \mathfrak{F}_2 .

Proof. Let \mathfrak{P} be a Sylow p -subgroup ($\neq \mathfrak{G}$) of \mathfrak{G} . Assume that \mathfrak{P} is

contained in \mathfrak{F}_2 . Then every element of \mathfrak{G} belongs to some conjugate subgroup of $C(\mathfrak{P})$ (Proposition 3.6). This implies that $\mathfrak{G} = C(\mathfrak{P})$ and $\mathfrak{P} \subseteq Z(\mathfrak{G})$ contradicting Proposition 1.3.

REMARK. For every prime divisor p of $|\mathfrak{G}|$ we have that p^2 divides $|\mathfrak{G}|$. This is obvious by Propositions 3.5 and 3.7.

DEFINITION 3.8. Let $\mathfrak{F}_1 = C(X)$ with a p -element X . Then \mathfrak{F}_1 is called p -singular if $Z(\mathfrak{F}_1)/Z(\mathfrak{G})$ is a p -group.

Proposition 3.9. *Let \mathfrak{X} be a finite group and \mathfrak{S} a Sylow p -subgroup of \mathfrak{X} . Let \mathfrak{Y} be a p -subgroup of \mathfrak{X} such that $\mathfrak{Y} \cong D(\mathfrak{S})$. Then there exists a Sylow p -subgroup \mathfrak{T} of \mathfrak{X} such that $\mathfrak{T} \cong \mathfrak{Y} \cong D(\mathfrak{T})$.*

Proof. Let \mathfrak{T} be a Sylow p -subgroup of \mathfrak{X} such that $\mathfrak{Y} \cong D(\mathfrak{T})$ and $\mathfrak{Y} : \mathfrak{Y} \cap \mathfrak{T}$ is the least. We show that $\mathfrak{Y} = \mathfrak{Y} \cap \mathfrak{T}$. Assume that $\mathfrak{Y} : \mathfrak{Y} \cap \mathfrak{T} \neq 1$.

Since $\mathfrak{Y} \cap \mathfrak{T} \cong D(\mathfrak{T})$, $N(\mathfrak{Y} \cap \mathfrak{T})$ contains \mathfrak{T} . Put $\mathfrak{Z} = \mathfrak{Y} \cap N(\mathfrak{Y} \cap \mathfrak{T})$. Then $\mathfrak{Z} \cong \mathfrak{Y} \cap \mathfrak{T}$. If $N(\mathfrak{Y} \cap \mathfrak{T}) = \mathfrak{X}$, then $\mathfrak{Y} \cap \mathfrak{T} \cong G^{-1}D(\mathfrak{T})G$ for all $G \in \mathfrak{X}$. This contradicts the assumption $\mathfrak{Y} : \mathfrak{Y} \cap \mathfrak{T} \neq 1$. So we must have that $N(\mathfrak{Y} \cap \mathfrak{T}) \neq \mathfrak{X}$. Then by an induction argument with respect to $|\mathfrak{X}|$ we may assume that there exists a Sylow p -subgroup \mathfrak{U} of $N(\mathfrak{Y} \cap \mathfrak{T})$ such that $\mathfrak{U} \cong \mathfrak{Z} \cong D(\mathfrak{U})$. But \mathfrak{U} is a Sylow p -subgroup of \mathfrak{X} and $\mathfrak{Y} \cap \mathfrak{U} \cong \mathfrak{Z} \cong \mathfrak{Y} \cap \mathfrak{T}$. This is a contradiction.

Proposition 3.10. *Let \mathfrak{F}_1 be a fundamental subgroup of type 1. Let q be a prime divisor of $\mathfrak{G} : \mathfrak{F}_1$. If there exists no q -singular fundamental subgroup of \mathfrak{G} , then q^2 does not divide $\mathfrak{G} : \mathfrak{F}_1$.*

Proof. Let \mathfrak{Q} and \mathfrak{Q}_1 be Sylow q -subgroups of \mathfrak{G} and \mathfrak{F}_1 such that $\mathfrak{Q} \cong \mathfrak{Q}_1$. Then $Z(\mathfrak{Q}) \subseteq Z(\mathfrak{G})$ and \mathfrak{Q} is not abelian by Proposition 1.2. Let X be an element of $Z_2(\mathfrak{Q})$ not belonging to $Z(\mathfrak{Q})$ and $X^q \in Z(\mathfrak{Q})$. Let Y be an element of \mathfrak{Q} . Then $Y^{-1}XY = XZ$ with $Z \in Z(\mathfrak{G})$. Thus $C(Y^{-1}XY) = Y^{-1}C(X)Y = C(X)$ and $Y^{-q}X^{-1}Y^qX = Y^{-1}X^{-q}YX^q = E$. Therefore \mathfrak{Q} is contained in $N(C(X))$ and $\mathfrak{Q}/\mathfrak{Q}_X$ is an elementary abelian q -group, where $\mathfrak{Q}_X = \mathfrak{Q} \cap C(X)$ is a Sylow q -subgroup of $C(X)$. If $|C(X)| = |\mathfrak{F}_2|$, then by Proposition 3.6 or Proposition 1.4 $C(X)$ is nilpotent. If $\mathfrak{Q}/\mathfrak{Q}_X$ can be considered as a regular automorphism group of $\mathfrak{R}_X/Z(\mathfrak{G}) \cap \mathfrak{R}_X$, where \mathfrak{R}_X is a Sylow r -subgroup of $C(X)$ and $r \neq q$, then $\mathfrak{Q}/\mathfrak{Q}_X$ is cyclic ([3], p. 499) and $\mathfrak{Q} : \mathfrak{Q}_X = q$ (Cf. Proposition 3.5). If $\mathfrak{Q}/\mathfrak{Q}_X$ is not regular as an automorphism group of $\mathfrak{R}_X/Z(\mathfrak{G}) \cap \mathfrak{R}_X$, there exists an r -element Y in \mathfrak{R}_X not belonging to $\mathfrak{Z}(\mathfrak{G})$ such that a Sylow q -subgroup \mathfrak{Q}_Y of $C(Y)$ contains \mathfrak{Q}_X properly. By Proposition 3.9 we may assume that $\mathfrak{Q} \cong \mathfrak{Q}_Y \cong D(\mathfrak{Q})$. Let Z be an element of \mathfrak{Q} . Then $Z^{-1}\mathfrak{Q}Z = \mathfrak{Q}_Y$. Now put $\mathfrak{R}^* = \langle Z^{-1}YZ, Z \in \mathfrak{Q} \rangle$. Then \mathfrak{R}^* is a \mathfrak{Q} -invariant subgroup of \mathfrak{R}_X and $\mathfrak{Q}_Y = \mathfrak{Q} \cap C(\mathfrak{R}^*)$. Since a Sylow q -complement of $C(X)$ is abelian (cf. Proposition 3.4), $C(Y)$ is a fundamental subgroup of type 1.

Therefore $\mathfrak{Q}/\mathfrak{Q}_Y$ can be considered as a regular automorphism group of $\mathfrak{R}^*/\mathfrak{R}^* \cap Z(\mathfrak{G})$. Hence $\mathfrak{Q}/\mathfrak{Q}_Y$ is cyclic ([3], p. 499) and $\mathfrak{Q}:\mathfrak{Q}_Y=q$.

If $|C(X)|=|\mathfrak{F}_1|$ and if $C(X)$ is free, then $C(X)$ is abelian by Proposition 3.2. $\mathfrak{Q}/\mathfrak{Q}_X$ can be considered as a regular automorphism group of $\mathfrak{R}_X/\mathfrak{R}_X \cap Z(\mathfrak{G})$, where \mathfrak{R}_X is a Sylow r -subgroup of $C(X)$ and $r \neq q$. Thus $\mathfrak{Q}/\mathfrak{Q}_X$ is cyclic and $\mathfrak{Q}:\mathfrak{Q}_X=q$. So we may assume that $C(X)$ is of type 1. By the assumption there exists an r -element Y such that $C(X)=C(Y)$, where $q \neq r$. Then $\mathfrak{Q}/\mathfrak{Q}_X$ can be considered as a regular automorphism group of $\mathfrak{R}_X \cap Z(C(X))/\mathfrak{R}_X \cap Z(\mathfrak{G})$, where \mathfrak{R}_X is a Sylow r -subgroup of $C(X)$. Hence $\mathfrak{Q}/\mathfrak{Q}_X$ is cyclic and $\mathfrak{Q}:\mathfrak{Q}_X=q$.

Proposition 3.11. *Let $\mathfrak{F}_1=C(X)$ be p -singular, where X is p -element. Let Y be a q -element of \mathfrak{F}_1 not belonging to $Z(\mathfrak{G})$ (Cf. Proposition 3.5). Let \mathfrak{R}_1 and \mathfrak{R}_Y be Sylow r -subgroups of \mathfrak{F}_1 and $C(XY)$ such that $\mathfrak{R}_1 \supseteq \mathfrak{R}_Y$. If $r \neq p$, then $\mathfrak{R}_1:\mathfrak{R}_Y \leq r$.*

Proof. By assumption Y does not belong to $Z(\mathfrak{F}_1)$, and thus $C(XY)$ is of type 2. Assume that $\mathfrak{R}_1 \supsetneq \mathfrak{R}_Y$. Let Z be an element of $Z(\mathfrak{R}_1)$. Then $|C(XZ)|_r > |C(XY)|_r$ and $C(X) \supsetneq C(XZ)$. Hence by assumption $C(X)=C(XZ)$ and Z belongs to $Z(\mathfrak{G})$. So $Z(\mathfrak{R}_1) \subseteq Z(\mathfrak{G})$ and \mathfrak{R}_1 is not abelian. Let W be an element of $Z_2(\mathfrak{R}_1)$ not belonging to $Z(\mathfrak{G})$ and such that $W^r \in Z(\mathfrak{G})$. Then $C(XW)$ is of type 2. Let \mathfrak{R}_W be a Sylow r -subgroup of $C(XW)$. Then as in the beginning of the proof of Proposition 3.10 we have that $\mathfrak{R}_1 \subseteq N(C(XW))$ and $\mathfrak{R}_1/\mathfrak{R}_W$ is an elementary abelian r -group. By Proposition 3.6 $C(XW)$ is nilpotent. Let \mathfrak{S}_W be a Sylow s -subgroup of $C(XW)$ with $s \neq r$. Then $\mathfrak{R}_1/\mathfrak{R}_W$ can be considered as a regular automorphism group of $\mathfrak{S}_W/\mathfrak{S}_W \cap Z(\mathfrak{G})$ (Proposition 3.5). Thus $\mathfrak{R}_1/\mathfrak{R}_W$ is cyclic ([3], p. 499) and $\mathfrak{R}_1:\mathfrak{R}_W=\mathfrak{R}_1:\mathfrak{R}_Y=r$.

Proposition 3.12. *Let \mathfrak{F}_1 be p -singular and $q \neq p$. Then q^2 does not divide $\mathfrak{F}_1:\mathfrak{F}_2$.*

Proof. This is obvious by Proposition 3.11.

Proposition 3.13. *Assume that there exist no p -singular fundamental subgroups of \mathfrak{G} for every p . If a Sylow q -subgroup \mathfrak{Q}_2 of a fundamental subgroup \mathfrak{F}_2 of type 2 is not abelian, then for every prime divisor r of $|\mathfrak{G}|$ distinct from q there exists a $\{q, r\}$ -element X such that $\mathfrak{F}_2=C(X)$. In particular, a Sylow q -complement of \mathfrak{F}_2 is abelian.*

Proof. By Proposition 3.6 \mathfrak{F}_2 is nilpotent. Let $\mathfrak{F}_1=C(Y)$ is a fundamental subgroup of type 1 containing \mathfrak{F}_2 . By assumption we may assume that Y is a p -element with $p \neq q$. If a Hall $\{p, q\}$ -complement \mathfrak{A} of \mathfrak{F}_2 contains an element Z not belonging to $Z(\mathfrak{F}_1)$, then $C(YZ)$ is of type 2 and contains \mathfrak{Q}_2 . This implies that \mathfrak{Q}_2 is abelian against the assumption (cf. the proof of Proposition 3.4). So we must have that $\mathfrak{A} \subseteq Z(\mathfrak{F}_1)$. Then for every $r \neq p, q$ there

exists an r -element W such that $\mathfrak{F}_1 = C(W)$ (Proposition 3.5). The above argument shows that a Hall $\{r, q\}$ -complement of \mathfrak{F}_2 is contained in $Z(\mathfrak{F}_1)$. By Propositions 1.1 and 3.5 a Sylow q -complement of \mathfrak{F}_2 is contained in $Z(\mathfrak{F}_1)$.

Put $\mathfrak{F}_2 = C(V)$ and $V = V_p V_q \cdots$, where $V_p, V_q \neq E, \dots$ are p -, q -, \dots elements which are commutative with each other. Let U be an r -element such that $\mathfrak{F}_1 = C(U)$. Then $\mathfrak{F}_1 \cong C(UV_q) \cong \mathfrak{F}_2$. If $\mathfrak{F}_1 = C(UV_q)$, then V_q belongs to $Z(\mathfrak{F}_1)$ and $\mathfrak{F}_1 = \mathfrak{F}_2$ which is a contradiction. So $\mathfrak{F}_2 = C(UV_q)$ as claimed.

Proposition 3.14. *Assume that there exist no p -singular fundamental subgroups of \mathfrak{G} for every prime p . Then every fundamental subgroup \mathfrak{F}_1 of type 1 is nilpotent and n_1/n_2 is a prime power. Hence all the fundamental subgroups of \mathfrak{G} are nilpotent.*

Proof. We show that for every element X of \mathfrak{F}_1 : $\mathfrak{F}_1 \cap C(X) = 1$ or n_1/n_2 . If $C(X) = \mathfrak{G}$, this is obvious. If $C(X)$ is free, then $C(X)$ is abelian (Propositions 3.5 and 1.4) and $C(X)$ contains $Z(\mathfrak{F}_1)$. This implies that \mathfrak{F}_1 contains $C(X)$, which is a contradiction. If $C(X)$ is of type 1, then we may assume that X is a p -element. By the assumption we can find a q -element Y such that $\mathfrak{F}_1 = C(Y)$ and $p \neq q$. Then $C(XY) = C(X) \cap \mathfrak{F}_1$, which implies that $\mathfrak{F}_1 : \mathfrak{F}_1 \cap C(X) = 1$ or n_1/n_2 . So we may assume that $C(X)$ is of type 2. If $C(X)$ is abelian, then $C(X)$ contains $Z(\mathfrak{F}_1)$ and $C(X)$ is contained in \mathfrak{F}_1 . Hence we may assume that a Sylow p -subgroup of $C(X)$ is not abelian for some p . Let $\mathfrak{F}_1 = C(Y)$, where Y is a q -element. By Proposition 3.13 there exists a $\{p, r\}$ -element \bar{X} such that $C(X) = C(\bar{X})$ and $q \neq p, r$. Since Y belongs to $C(X)$ and $C(X)$ is nilpotent (Proposition 3.6), $Y\bar{X} = \bar{X}Y$. Thus by Proposition 3.13 we get that $C(\bar{X}Y) = C(\bar{X})$ is contained in \mathfrak{F}_1 .

Hence by Theorem 1 of [5] \mathfrak{F}_1 is nilpotent and n_1/n_2 is a prime power.

Proposition 3.15. *There exists a p -singular fundamental subgroup of \mathfrak{G} for some p .*

Proof. Assume the contrary. Then by Proposition 3.10 $\mathfrak{G} : \mathfrak{F}_1$ is square-free, and by Proposition 3.14 \mathfrak{F}_1 is nilpotent. We show that \mathfrak{F}_1 is normal in \mathfrak{G} , whence \mathfrak{G} is solvable against the assumption. Now let \mathfrak{P}_1 and \mathfrak{P} be Sylow p -subgroups of \mathfrak{F}_1 and \mathfrak{G} such that $\mathfrak{P}_1 \subseteq \mathfrak{P}$. We show that $\mathfrak{P} \subseteq N(\mathfrak{F}_1)$. We may assume that $\mathfrak{P} : \mathfrak{P}_1 = p$. Put $\mathfrak{F}_1 = \mathfrak{P}_1 \times \bar{\mathfrak{P}}_1$, where $\bar{\mathfrak{P}}_1$ is a Sylow p -complement of \mathfrak{F}_1 . Let X be an element of \mathfrak{P} not belonging to \mathfrak{P}_1 . Then $X^{-1}\mathfrak{F}_1X = \mathfrak{P}_1 \times X^{-1}\bar{\mathfrak{P}}_1X$. Let Y be an element of \mathfrak{P}_1 not belonging to $Z(\mathfrak{G})$. Then $C(Y)$ is nilpotent (Proposition 3.14) and contains $\bar{\mathfrak{P}}_1$ and $X^{-1}\bar{\mathfrak{P}}_1X$ as Sylow p -complements. Hence $\bar{\mathfrak{P}}_1 = X^{-1}\bar{\mathfrak{P}}_1X$, and X belongs to $N(\mathfrak{F}_1)$.

Proposition 3.16. *Assume that there exists a p -singular fundamental subgroup and that there exist no q -singular fundamental subgroups of \mathfrak{G} for every*

prime q distinct from p . If a Sylow r -subgroup of a fundamental subgroup \mathfrak{F}_2 of type 2 is not abelian, then for every prime divisor s of $|\mathfrak{G}|$ distinct from r there exists a $\{r, s\}$ -element X with $\mathfrak{F}_2 = C(X)$. In particular, a Sylow r -complement of \mathfrak{F}_2 is abelian.

Proof. By Proposition 3.6 \mathfrak{F}_2 is nilpotent. Let $\mathfrak{F}_1 = C(Y)$ is a fundamental subgroup of type 1 containing \mathfrak{F}_2 . The proof of Proposition 3.13 shows that the assertion is true if we can choose Y as an s -element with $s \neq r$. Then such a choice is possible, unless $r = p$ and \mathfrak{F}_1 is p -singular. So assume that $r = p$ and \mathfrak{F}_1 is p -singular. Let \mathfrak{P}_2 be a Sylow p -subgroup of \mathfrak{F}_2 . Let $Z \neq E$ be a q -element of \mathfrak{F}_2 not belonging to $Z(\mathfrak{G})$ with $q \neq p$. Then $C(Z)$ contains \mathfrak{P}_2 . If $C(Z)$ is free or of type 2, then \mathfrak{P}_2 is abelian against the assumption. Thus $C(Z)$ is of type 1 and we may assume that $C(Z) \neq \mathfrak{F}_2$. So $C(YZ)$ is of type 2 and contains a Hall $\{p, q\}$ -complement \mathfrak{A} of \mathfrak{F}_2 . \mathfrak{A} is abelian (Proposition 3.4). Let $W \neq E$ be an s -element of \mathfrak{A} not belonging to $Z(\mathfrak{G})$ (By Proposition 3.5 such an element always exists). $C(W)$ cannot be free nor of type 2 as above. So $C(W)$ is of type 1 and contains \mathfrak{F}_2 . So we can apply the proof of Proposition 3.13.

Proposition 3.17. Assume that there exists a p -singular fundamental subgroup and that there exist no q -singular fundamental subgroups for every q distinct from p . If \mathfrak{F}_1 is not $(p-)$ singular and of type 1, then \mathfrak{F}_1 is nilpotent and n_1/n_2 is a prime power.

Proof. It is not difficult to check that the proof of Proposition 3.14 can be applied here.

Proposition 3.18. Assume that there exists a p -singular fundamental subgroup and that there exist no q -singular fundamental subgroups for every q distinct from p . Then exists no non-singular fundamental subgroup of type 1.

Proof. Assume the contrary and let \mathfrak{F}_1 be a non-singular fundamental subgroup of type 1. By Proposition 3.10 the prime to p part of \mathfrak{G} : \mathfrak{F}_1 is square-free. By Proposition 3.17 \mathfrak{F}_1 is nilpotent. By the proof of Proposition 3.15 $\mathfrak{G} : N(\mathfrak{F}_1)$ is a power of p .

First assume that $N(\mathfrak{F}_1)$ is solvable, and let \mathfrak{H} be a Sylow p -complement of $N(\mathfrak{F}_1)$. \mathfrak{H} is a Sylow p -complement of \mathfrak{G} . Put $\mathfrak{H}_1 = \mathfrak{H} \cap \mathfrak{F}_1$. Then \mathfrak{H}_1 is a Sylow p -complement of \mathfrak{F}_1 . Let \mathfrak{P}_1 be a Sylow p -subgroup of \mathfrak{F}_1 . Then \mathfrak{P}_1 is normal in $\mathfrak{P}_1\mathfrak{H}$. Let \mathfrak{P} be a Sylow p -subgroup of \mathfrak{G} containing \mathfrak{P}_1 . Since $\mathfrak{G} = \mathfrak{P}\mathfrak{H}$, \mathfrak{P} contains a normal subgroup \mathfrak{P}_1 of \mathfrak{G} containing \mathfrak{P}_1 . \mathfrak{H}_1 is a Sylow p -complement of $C(\mathfrak{P}_1)$. Hence if $\mathfrak{P}_1 \neq \mathfrak{P}_1$, then $\mathfrak{P}_1 \cap N(\mathfrak{H}_1) \neq \mathfrak{P}_1$. But for $X \in \mathfrak{P}_1 \cap N(\mathfrak{H}_1)$ and $Y \in \mathfrak{H}_1$, we have that $X^{-1}Y^{-1}XY \in \mathfrak{P}_1 \cap \mathfrak{H}_1 = \mathfrak{E}$. Since \mathfrak{F}_1 is non-singular, we have that $C(\mathfrak{H}_1) \subseteq \mathfrak{F}_1$, $\mathfrak{P}_1 \cap N(\mathfrak{H}_1) \subseteq \mathfrak{F}_1$ and $\mathfrak{P}_1 \cap N(\mathfrak{H}_1)$

$\subseteq \mathfrak{P}_1$, which is a contradiction. Hence we get that $\mathfrak{P}_1 = \overline{\mathfrak{P}}_1$. If $\mathfrak{G} : \mathfrak{F}_1 \neq 0 \pmod{p}$, then $\mathfrak{G} = N(\mathfrak{F}_1)$, and \mathfrak{G} is solvable against the assumption. So $\mathfrak{G} : \mathfrak{F}_1 \equiv 0 \pmod{p}$, and thus \mathfrak{P} is non-abelian and $Z(\mathfrak{P}) \subseteq Z(\mathfrak{G})$. Now $\mathfrak{H}/\mathfrak{H}_1$ can be considered as a regular automorphism group of $\mathfrak{P}_1/Z(\mathfrak{G})$. Since $\mathfrak{H}/\mathfrak{H}_1$ has a square-free order, $\mathfrak{H}/\mathfrak{H}_1$ is cyclic ([3], p. 499). Then $\mathfrak{G}/\mathfrak{P}_1 C(\mathfrak{P}_1)$ is a product of a cyclic group and a p -group, and hence is solvable (For instance, [4]). On the other hand, $C(\mathfrak{P}_1) = (\mathfrak{P} \cap C(\mathfrak{P}_1))\mathfrak{H}_1$ is solvable by a theorem of Wielandt ([3], p. 680). Thus \mathfrak{G} is solvable against the assumption.

Now assume that $N(\mathfrak{F}_1)$ is non-solvable. Let \mathfrak{P}_1^* be a Sylow p -subgroup of $N(\mathfrak{F}_1)$. Then, since $C(\mathfrak{P}_1) \subseteq \mathfrak{F}_1$, $\mathfrak{P}_1^*/\mathfrak{P}_1$ can be considered as a regular automorphism group of $\mathfrak{H}_1/\mathfrak{H}_1 \cap Z(\mathfrak{G})$, where \mathfrak{H}_1 is a Sylow p -complement of \mathfrak{F}_1 . Hence $\mathfrak{P}_1^*/\mathfrak{P}_1$ is cyclic or (generalized) quaternion. If $\mathfrak{P}_1^*/\mathfrak{P}_1$ is cyclic, then $N(\mathfrak{F}_1)/\mathfrak{F}_1$ is a Z -group, which implies the solvability of $N(\mathfrak{F}_1)$ against the assumption. So $\mathfrak{P}_1^*/\mathfrak{P}_1$ must be (generalized) quaternion, and, in particular, $p=2$. Let \mathfrak{P} be a Sylow 2-subgroup of \mathfrak{G} containing \mathfrak{P}_1^* . Then \mathfrak{P} is non-abelian and $Z(\mathfrak{P}) \subseteq Z(\mathfrak{G})$. Let X be an element of $Z_2(\mathfrak{P})$ not belonging to $Z(\mathfrak{G})$ such that $X^2 \in Z(\mathfrak{G})$. Then $C(X) \cong D(\mathfrak{P})$. As in the proof of Proposition 2.4 $\mathfrak{P} \subseteq N(C(X))$ and $\mathfrak{P}/\mathfrak{P}_X$ is an elementary abelian 2-group, where \mathfrak{P}_X is a Sylow 2-subgroup of $C(X)$ such that $\mathfrak{P} \supseteq \mathfrak{P}_X \supseteq D(\mathfrak{P})$. If $C(X)$ is not 2-singular, then by Propositions 1.4, 3.6 and 3.17 $C(X)$ is nilpotent. Let \mathfrak{Q}_X be a Sylow q -subgroup of $C(X)$ with $q \neq p$. If $C(X)$ is not of type 2, $\mathfrak{P}/\mathfrak{P}_X$ can be considered as a regular automorphism group of $\mathfrak{Q}_X/\mathfrak{Q}_X \cap Z(\mathfrak{G})$. So $\mathfrak{P}/\mathfrak{P}_X$ is cyclic and $\mathfrak{P} : \mathfrak{P}_X = 2$ ([3], p. 499). Then $|\mathfrak{P}_1^*/\mathfrak{P}_1| \leq 2$, which is a contradiction. Suppose that $C(X)$ is of type 2 and that $\mathfrak{P}/\mathfrak{P}_X$ is not regular as an automorphism group of $\mathfrak{Q}_X/\mathfrak{Q}_X \cap Z(\mathfrak{G})$. Then there exist an element Y of \mathfrak{P} not belonging to \mathfrak{P}_X and an element Z of \mathfrak{Q}_X not belonging to $Z(\mathfrak{G})$ such that $YZ = ZY$ (cf. the proof of Proposition 2.4). Then $C(Z)$ contains $\langle \mathfrak{P}_X, Y \rangle$, and is free or of type 1 and is not 2-singular. So by Propositions 3.2 and 3.17 $C(Z)$ is nilpotent. Let \mathfrak{P}_Z be a Sylow 2-subgroup of $C(Z)$. Then by Proposition 3.9 we may assume that $\mathfrak{P} \supseteq \mathfrak{P}_Z \supseteq D(\mathfrak{P})$. Then $W^{-1}\mathfrak{P}_Z W = \mathfrak{P}_Z$ for every $W \in \mathfrak{P}$. Put $\mathfrak{Q}^* = \langle W^{-1}ZW, W \in \mathfrak{P} \rangle$. Then \mathfrak{Q}^* is a \mathfrak{P} -invariant subgroup of \mathfrak{Q}_X and $\mathfrak{P}_Z = \mathfrak{P} \cap C(\mathfrak{Q}^*)$. Now $\mathfrak{P}/\mathfrak{P}_Z$ can be considered as a regular automorphism group of $\mathfrak{Q}^*/\mathfrak{Q}^* \cap Z(\mathfrak{G})$. So $\mathfrak{P}/\mathfrak{P}_Z$ is cyclic and $\mathfrak{P} : \mathfrak{P}_Z = 2$. Then $|\mathfrak{P}_1^*\mathfrak{P}_1| \leq 2$, which is a contradiction. Hence we may assume that $C(X)$ is 2-singular.

Let \mathfrak{F}_2 be a fundamental subgroup of type 2 contained in $C(X)$. By Proposition 3.17 $C(X) : \mathfrak{F}_2$ is a power of a prime. If $C(X) : \mathfrak{F}_2 \neq 0 \pmod{2}$, then by Proposition 3.12 $C(X) : \mathfrak{F}_2 = q$ is a prime. By Proposition 3.6 \mathfrak{F}_2 is nilpotent. $C(X)$ is a product of a Sylow q -subgroup of $C(X)$ and a Sylow q -complement of \mathfrak{F}_2 . Hence by a theorem of Wielandt ([3], p. 680) $C(X)$ is solvable. Let $F(C(X)) = \mathfrak{A} \times \mathfrak{B}$, where \mathfrak{A} and \mathfrak{B} are Sylow 2-subgroup and Sylow 2-com-

plement of $F(C(X))$ respectively. If $\mathfrak{B} \not\subseteq Z(\mathfrak{G})$, then, since $\mathfrak{B} \subseteq \mathfrak{F}_2$, $\mathfrak{B}/\mathfrak{B}_X$ can be considered as a regular automorphism group of $\mathfrak{B}/\mathfrak{B} \cap Z(\mathfrak{G})$. So we get a contradiction as before. But if $\mathfrak{B} \subseteq Z(\mathfrak{G})$, then by a theorem of Fitting ([3], p. 277) $F(C(X)) \cong C(F(C(X))) \cong \mathfrak{R}_X$, where \mathfrak{R}_X is a Sylow r -subgroup of \mathfrak{F}_2 with $r \neq q, 2$. By Propositions 1.1 and 3.5 we have that $\mathfrak{R}_X \not\subseteq Z(\mathfrak{G})$. This is a contradiction. Hence we may assume that $C(X):\mathfrak{F}_2 =$ is a power of 2.

Let \mathfrak{A} be a Sylow 2-complement of \mathfrak{F}_2 . Suppose that a Sylow q -subgroup \mathfrak{Q}_2 of \mathfrak{A} is non-abelian. Then by Proposition 3.15 a Sylow r -subgroup \mathfrak{R}_2 of \mathfrak{A} is abelian. Choose an element Y of \mathfrak{R}_2 not belonging to $Z(\mathfrak{G})$. Then $C(XY) \cong \mathfrak{Q}_2$ and $C(XY)$ is of type 2. Then \mathfrak{Q}_2 is abelian against the assumption. Hence \mathfrak{A} is abelian (cf. Propositions 1.1 and 3.5). Hence, in particular, $C(X)$ is solvable (cf. [4]). If $C(X)$ is nilpotent, then $\mathfrak{B}/\mathfrak{B}_X$ can be considered as a regular automorphism group of $\mathfrak{A}/\mathfrak{A} \cap Z(\mathfrak{G})$, and we get a contradiction as before. Hence $C(X)$ is not nilpotent.

Let \mathfrak{F}_2 be a fundamental subgroup of type 2 contained in \mathfrak{F}_1 . Since $\mathfrak{F}_1:\mathfrak{F}_2$ is a power of 2, every Sylow q -subgroup of \mathfrak{Q}_1 of \mathfrak{F}_1 is contained in \mathfrak{F}_2 for $q \neq 2$. We show that \mathfrak{Q}_1 is abelian. Suppose that \mathfrak{Q}_1 is not abelian. Let Y be an element of \mathfrak{Q}_1 not belonging to $Z(\mathfrak{Q}_1)$. Then $C(Y)$ is of type 1 and contains the Sylow q -complement of \mathfrak{F}_1 . In particular, $C(Y)$ contains \mathfrak{R}_1 , where \mathfrak{R}_1 is the Sylow r -subgroup of \mathfrak{F}_1 (Proposition 1.1). Let Z be an element of \mathfrak{R}_1 not belonging to $Z(\mathfrak{G})$ (Proposition 3.5). Then $C(Z)$ contains \mathfrak{Q}_1 and the Sylow q -subgroup of $C(Y)$. This is a contradiction. So the Sylow 2-complement \mathfrak{A}_1 of \mathfrak{F}_1 is abelian.

Now we show that we may assume that $\mathfrak{A} = \mathfrak{A}_1$. Let \mathfrak{Q} , \mathfrak{Q}_1 and \mathfrak{Q}_X be Sylow q -subgroups of \mathfrak{G} , \mathfrak{F}_1 and $C(X)$, where $q \neq 2$. We may assume that $\mathfrak{Q} \supseteq \mathfrak{Q}_1$ and $\mathfrak{Q} \supseteq \mathfrak{Q}_X$. Then since \mathfrak{Q}_1 and \mathfrak{Q}_X are abelian, we have that $\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G})$ is elementary abelian of order q^2 or $\mathfrak{Q}_1 = \mathfrak{Q}_X$. If $\mathfrak{Q}_1 = \mathfrak{Q}_X$, then $C(\mathfrak{Q}_1)$ is nilpotent and contains \mathfrak{A}_1 and \mathfrak{A} as its Sylow 2-complement. So we get that $\mathfrak{A}_1 = \mathfrak{A}$. Otherwise, let \mathfrak{R} , \mathfrak{R}_1 and \mathfrak{R}_X be Sylow r -subgroups of \mathfrak{G} , \mathfrak{F}_1 and $C(X)$, where $r \neq q, r \neq 2$. By Propositions 1.1 and 3.5 there exists such a prime. Since we have assumed that $\mathfrak{A}_1 \neq \mathfrak{A}$, we get that $\mathfrak{R}/\mathfrak{R} \cap Z(\mathfrak{G})$ is elementary abelian of order r^2 . We may assume that $r > q$. Since $\mathfrak{R}/\mathfrak{R}_1$ can be considered as a regular automorphism group of $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$, this is a contradiction. Hence we (may) assume that $\mathfrak{A} = \mathfrak{A}_1$.

Put $F(C(X)) = \mathfrak{C} \times \mathfrak{D}$, where \mathfrak{C} and \mathfrak{D} are the Sylow 2-subgroup and Sylow 2-complement of $F(C(X))$. If $\mathfrak{D} \not\subseteq Z(\mathfrak{G})$, then $C(X) \cap C(\mathfrak{D})$ is nilpotent and contains \mathfrak{A} and is normal in $C(X)$. So \mathfrak{A} is normal in $C(X)$. Then $\mathfrak{B} \subseteq N(C(X)) \subseteq N(\mathfrak{A})$. Since $C(\mathfrak{A}) = \mathfrak{F}_1$, \mathfrak{B}_1 is normal in \mathfrak{B} . Then \mathfrak{B} is contained in $N(\mathfrak{F}_1)$ and $\mathfrak{G} = N(\mathfrak{F}_1)$, which implies the solvability of \mathfrak{G} . This is a contradiction. So we must have that $\mathfrak{D} \subseteq Z(\mathfrak{G})$. Then $\mathfrak{C} \cong \mathfrak{B}_2$, where \mathfrak{B}_2 is a Sylow 2-subgroup of \mathfrak{F}_2 . Then $\mathfrak{A}/\mathfrak{A} \cap Z(\mathfrak{G})$ can be considered as a

regular automorphism of $\mathfrak{P}_X/\mathfrak{P}_2$, and hence $\mathfrak{X}/\mathfrak{X} \cap Z(\mathfrak{G})$ is cyclic. Then assume as above that $r > q$. Then since $\mathfrak{Q}_1/Z(\mathfrak{G}) \cap \mathfrak{Q}_1$ is cyclic, we get a contradiction as above.

Proposition 3.19. *Assume that there exists a p -singular fundamental subgroup and that there exist no q -singular fundamental subgroups for every q distinct from p . Then there exists no free fundamental subgroup of index n_1 .*

Proof. This is obvious by the proof of Proposition 3.18.

Proposition 3.20. *For at least two distinct primes p there exist p -singular fundamental subgroups of \mathfrak{G} .*

Proof. By Proposition 3.15 for some prime p there exists a p -singular fundamental subgroup \mathfrak{F}_1 of \mathfrak{G} . Suppose that there exists no q -singular fundamental subgroup of \mathfrak{G} for every prime q distinct from p .

By Propositions 3.18 and 3.19 if X is a q -element of \mathfrak{G} not belonging to $Z(\mathfrak{G})$, then $\mathfrak{G}:C(X) = n_2$. By Propositions 3.6 and 1.4 $C(X)$ is nilpotent. Furthermore by Propositions 3.18, 3.19, 3.5 and 1.1 $C(X)$ is abelian.

Let \mathfrak{F}_2 be a fundamental subgroup of type 2 contained in \mathfrak{F}_1 . Let \mathfrak{Q} , \mathfrak{Q}_1 and \mathfrak{Q}_2 be Sylow q -subgroups of \mathfrak{G} , \mathfrak{F}_1 and \mathfrak{F}_2 such that $\mathfrak{Q} \supseteq \mathfrak{Q}_1 \supseteq \mathfrak{Q}_2$ ($q \neq p$). By Propositions 3.10 and 3.11 we have that $\mathfrak{Q}:\mathfrak{Q}_1 \leq q$ and $\mathfrak{Q}_1:\mathfrak{Q}_2 \leq q$. Now we show that \mathfrak{Q}_2 is normal in \mathfrak{Q} . Assume the contrary. Then we must have that $\mathfrak{Q}:\mathfrak{Q}_1 = q$, $\mathfrak{Q}_1:\mathfrak{Q}_2 = q$ and $\mathfrak{Q}:\mathfrak{Q}_2 = q^2$. Furthermore there exists an element Y in \mathfrak{Q} such that $Y^{-1}\mathfrak{Q}_2Y \neq \mathfrak{Q}_2$. Since \mathfrak{Q}_2 is abelian, $Y^{-1}\mathfrak{Q}_2Y \cap \mathfrak{Q}_2 = \mathfrak{Q} \cap Z(\mathfrak{G})$. So $\mathfrak{Q}_1/\mathfrak{Q} \cap Z(\mathfrak{G})$ is elementary abelian of order q^2 . Let Z be an element of \mathfrak{Q}_1 such that $Z(\mathfrak{Q} \cap Z(\mathfrak{G}))$ is an element of $Z(\mathfrak{Q}/\mathfrak{Q} \cap Z(\mathfrak{G}))$ of order q . Let \mathfrak{Q}_Z be a Sylow q -subgroup of $C(Z)$. Then $\mathfrak{Q}_Z = (\mathfrak{Q} \cap Z(\mathfrak{G}))\langle Z \rangle$ is normal in \mathfrak{Q} . Let \mathfrak{R}_Z be a Sylow r -subgroup of $C(Z)$ with $r \neq p, q$ (Proposition 3.5). Then $\mathfrak{Q}/\mathfrak{Q}_Z$ can be considered as a regular automorphism group of $\mathfrak{R}_Z/\mathfrak{R}_Z \cap Z(\mathfrak{G})$. Thus $\mathfrak{Q}/\mathfrak{Q}_Z$ is cyclic ([3], p. 499). This is a contradiction. So \mathfrak{Q}_2 is normal in \mathfrak{Q} . Let \mathfrak{R}_2 be a Sylow r -subgroup of \mathfrak{F}_2 with $r \neq p, q$. Then $\mathfrak{Q}/\mathfrak{Q}_2$ can be considered as a regular automorphism group of $\mathfrak{R}_2/\mathfrak{R}_2 \cap Z(\mathfrak{G})$. Thus $\mathfrak{Q}/\mathfrak{Q}_2$ is cyclic. Since $C(\mathfrak{Q}_2) = \mathfrak{F}_2$, we get that $\mathfrak{Q} \subseteq N(\mathfrak{F}_2)$. Therefore $\mathfrak{G}:N(\mathfrak{F}_2)$ is a power of p .

Since $N(\mathfrak{F}_2) \subseteq N(\mathfrak{P}_2)$, $\mathfrak{G} = \mathfrak{P}N(\mathfrak{P}_2)$, where \mathfrak{P} is a Sylow 2-subgroup of \mathfrak{G} containing \mathfrak{P}_2 . Hence we get that $O_p(\mathfrak{G}) \supseteq \mathfrak{P}_2$.

Let \mathfrak{F}^* be a free fundamental subgroup of index n_2 (cf. Proposition 3.19) and \mathfrak{Q}^* a Sylow q -subgroup of \mathfrak{F}^* . We may assume that $\mathfrak{Q}^* \subseteq \mathfrak{Q}$. We show that \mathfrak{Q}^* is normal in \mathfrak{Q} . Assume the contrary. Then we must have that $\mathfrak{Q}:\mathfrak{Q}^* = q^2$. Since $C(\mathfrak{Q}^* \cap \mathfrak{Q}_1)$ contains $Z(\mathfrak{F}_1)$ and since \mathfrak{F}^* is free and abelian, we get that $\mathfrak{Q}^* \cap \mathfrak{Q}_1 \subseteq \mathfrak{Q} \cap Z(\mathfrak{G})$. Thus $\mathfrak{Q}^*:\mathfrak{Q} \cap Z(\mathfrak{G}) = \mathfrak{Q}_2:\mathfrak{Q} \cap Z(\mathfrak{G}) = q$. We know already that $\mathfrak{Q}/\mathfrak{Q}_2$ is cyclic (of order q^2). Let $W \in \mathfrak{Q}_2$, $W \in \mathfrak{Q}$

be a generator of $\mathfrak{D}/\mathfrak{D}_2$. Then $C(W)$ has the index n_2 in \mathfrak{G} and $W^q \notin Z(\mathfrak{G}) \cap \mathfrak{D}$. This is a contradiction. Now $\mathfrak{D}/\mathfrak{D}^*$ is cyclic; in fact, $\mathfrak{D}/\mathfrak{D}^*$ can be considered as a regular automorphism group of $\mathfrak{P}^*/Z(\mathfrak{G}) \cap \mathfrak{P}^*$, where \mathfrak{P}^* is a Sylow p -subgroup of \mathfrak{F}^* (cf. Proposition 3.7). Furthermore, the above argument shows that $\mathfrak{D}:\mathfrak{D}^*=q$ and that $\mathfrak{D}^*:Z(\mathfrak{G}) \cap \mathfrak{D}^*=q$. Then take a prime divisor r of $|\mathfrak{F}^*|$ distinct from p and q . Let \mathfrak{R} and \mathfrak{R}^* be Sylow r -subgroups of \mathfrak{G} and \mathfrak{F}^* such that $\mathfrak{R} \supseteq \mathfrak{R}^*$. Then as above we obtain that $\mathfrak{R}:\mathfrak{R}^*=\mathfrak{R}^*:Z(\mathfrak{G}) \cap \mathfrak{R}^*=r$. We may assume that $r>q$. Then since $\mathfrak{R}/\mathfrak{R}^*$ can be considered as a regular automorphism group of $\mathfrak{D}^*/Z(\mathfrak{G}) \cap \mathfrak{D}^*$, this is a contradiction. Hence there exists no free fundamental subgroup (of index n_2).

Now every p -element is contained in some fundamental subgroup of type 2. Hence we get that $O_p(\mathfrak{G})=\mathfrak{P}$. Since \mathfrak{H} is solvable, \mathfrak{G} is solvable against the assumption.

Proposition 3.21. *Let \mathfrak{F}_1 and \mathfrak{F}_2 be fundamental subgroups of type 1 and 2 such that $\mathfrak{F}_1 \supset \mathfrak{F}_2$. Then $\mathfrak{F}_1:\mathfrak{F}_2$ is square-free.*

Proof. This is obvious by Propositions 3.12 and 3.20.

Now let \mathfrak{F}_1 be p -singular and \mathfrak{F}_2 be q -singular, where $p \neq q$. Let \mathfrak{F}_2 be a fundamental subgroup of type 2 contained in \mathfrak{F}_1 . By Proposition 3.6 \mathfrak{F}_2 is nilpotent. Since \mathfrak{F}_1 is p -singular, $\mathfrak{F}_1:N(\mathfrak{F}_2) \cap \mathfrak{F}_1=p$ or 1. Next let \mathfrak{F}_2 be a fundamental subgroup of type 2 contained in \mathfrak{F}_1 . By Proposition 3.6 \mathfrak{F}_2 is nilpotent. Since \mathfrak{F}_1 is q -singular, $\mathfrak{F}_1:N(\mathfrak{F}_2) \cap \mathfrak{F}_1=q$ or 1. Assume that $p>q$. Let \mathfrak{P}_2 be a Sylow p -subgroup of \mathfrak{F}_2 . Since $N(\mathfrak{P}_2) \cap \mathfrak{F}_1=N(\mathfrak{F}_2) \cap \mathfrak{F}_1$, we see that \mathfrak{P}_2 is normal in \mathfrak{F}_1 . Hence \mathfrak{F}_2 is normal in \mathfrak{F}_1 . Let X be an element of \mathfrak{F}_1 not belonging to \mathfrak{F}_2 such that $|X|$ is prime to q . Assume that $\mathfrak{F}_1=C(Y)$, where Y is a q -element. Then $C(XY)$ is a fundamental subgroup of type 2 contained in \mathfrak{F}_1 . The above argument shows that $C(XY)$ is a nilpotent normal subgroup of \mathfrak{F}_1 . Then $\mathfrak{F}_2 C(XY)$ is nilpotent. This is a contradiction. This implies that $\mathfrak{F}_1:\mathfrak{F}_2=q$. But then $\mathfrak{F}_1:\mathfrak{F}_2=q$ and \mathfrak{F}_2 is normal in \mathfrak{F}_1 . There exists an element Z of \mathfrak{F}_1 not belonging to \mathfrak{F}_2 such that $|Z|$ is prime to p . Assume that $\mathfrak{F}_1=C(W)$, where W is a p -element. Then $C(ZW)$ is a fundamental subgroup of type 2 contained in \mathfrak{F}_1 . The above argument shows that $C(ZW)$ is a nilpotent normal subgroup of \mathfrak{F}_1 . Then $\mathfrak{F}_2 C(ZW)$ is nilpotent. This is a contradiction (cf. Proposition 1.1).

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