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DIFFEOMORPHIC EXTENSION OF BIHOLOMORPHIC MAPPINGS WITH SMOOTH MODULUS

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1. Introduction

Fefferman proved in [8] that any biholomorphic mapping between two smooth bounded strictly pseudoconvex domains D_1 and D_2 in $\boldsymbol{C}^n$ extends to a diffeomorphism of $\bar{D}_1$ onto $\bar{D}_2$. Later Fefferman's theorem was extended by Bell and Ligocka [7] and Bell [2].

Let D be a smooth bounded pseudoconvex domain in $\boldsymbol{C}^n$. Let $L^2(D)$ be the space of square-integrable functions on D. We denote by $H(D)$ the space of square-integrable holomorphic functions on D. The Bergman projection P is the orthogonal projection from $L^2(D)$ to $H(D)$. The domain D is said to satisfy condition R if P maps $C^\infty(\bar{D})$ continuously into $C^\infty(\bar{D})$. Bell's result [2] is as follows:

Let D_1 and D_2 be smooth bounded pseudoconvex domains in $\boldsymbol{C}^n$. If either D_1 or D_2 satisfies condition R, then any biholomorphic mapping between D_1 and D_2 extends to a diffeomorphism of $\bar{D}_1$ onto $\bar{D}_2$.

It is not known that any biholomorphic mapping between smooth bounded weakly pseudoconvex domains in $\boldsymbol{C}^n$ can be extended to a diffeomorphism onto the bouundary. Fornaess proved in [9] that any biholomorphic mapping f: $D_1 \rightarrow D_2$ between bounded pseudoconvex domains D_1 and D_2 in $\boldsymbol{C}^n$ with C^2-boundary extends to a C^2-diffeomorphism of $\bar{D}_1$ onto $\bar{D}_2$, if f has a C^2-extension $\hat{f}: \bar{D}_1 \rightarrow \bar{D}_2$. In this paper we shall prove the theorem of this type. Let D_1 and D_2 be smooth bounded pseudoconvex domains in $\boldsymbol{C}^n$. Using Bell's method we shall prove that any biholomorphic mapping f: $D_1 \rightarrow D_2$ extends to a C^∞-diffeomorphism of $\bar{D}_1$ onto $\bar{D}_2$, whenever $|f|^2$ is C^∞.

2. Preliminaries

Let D be a smooth bounded pseudoconvex domiain in $\boldsymbol{C}^n$. We denote by $W^s(D)$ the usual Sobolev space for $s>0$. A negaitve Sobolev space $W^{-s}(D)$ is the dual space of $W_0^s(D)$, where $W_0^s(D)$ is the closure of $C_0^\infty(D)$ in $W^s(D)$. We now consider the dual space $W^s(D)^*$ of $W^s(D)$ for $s>0$.

Let $\langle\ ,\ \rangle$ be the $L^2(D)$ inner product. For any $f \in L^2(D)$, $\langle \cdot, f\rangle$ is a continuous complex linear functional on $W^s(D)$. We set

$$|||f|||_{-s} = \sup_{\substack{\psi \in W_s(D) \\ ||\psi||_s=1}} |\langle f, \psi\rangle| .$$

Then we regard $L^2(D)$ as a subspace of $W^s(D)^*$ via $\langle\ ,\ \rangle$, and denote it by $L^{-s}(D)$. The norm of $W^{-s}(D)$ is denoted by $||\cdot||_{-s}$. By the same way as in [5], we obtain the following proposition.

Proposition 1. *Let D be a smooth bounded pseudoconvex domain in $\boldsymbol{C}^n$. Then the norms $||\cdot||_{-s}$ and $|||\cdot|||_{-s}$ are equivalent on $H(D)$.*

We set

$$|||f|||_s = \sup_{\substack{g \in L^{-s}(D) \\ |||g|||_{-s}=1}} |\langle f, g\rangle| ,$$

for $f \in L^2(D)$. If $|||f|||_s < \infty$, then we regard f as an element of the dual space $L^{-s}(D)^*$ of $L^{-s}(D)$ and we write $f \in L^{-s}(D)^*$. $H^s(D)$ is the subspace of $W^s(D)$ consisting of holomorphic functions. Note that $C^\infty(\bar{D}) = \bigcap_{s>0} W^s(D)$ and $H^\infty(\bar{D}) = \bigcap_{s>0} H^s(D)$.

Bell constructed a bounded linear operator Φ^s: $H^s(D) \to W_0^s(D)$ such that $P\Phi^s h = h$ for all $h \in H^s(D)$ (see [1], [2] and [4]). This operator was extended to a bounded linear operator $\tilde{\Phi}^s$: $W^s(D) \to W_0^s(D)$ with $P\tilde{\Phi}^s = P$ ([11]). For $t>0$, we denote by $L_t^2(D)$ the weighted Hilbert space of complex valued functions on D with inner product given by

$$\langle g, h\rangle_t = \int_D g(z)\overline{h(z)} e^{-t|z|^2} d\mu(z) .$$

The weighted Bergamn projection P_t is the orthogonal projection of $L_t^2(D)$ onto $H(D)$ with respect to the inner product $\langle\ ,\ \rangle_t$. By a Kohn's result [10] it holds that for a positive integer s there exists a positive number t_0 such that P_t maps $W^s(D)$ into $W^s(D)$ continuously, if $t > t_0$. There exists a bounded linear operator Φ_t^s: $W^s(D) \to W_0^s(D)$ such that $P_t \Phi_t^s = P_t$ (cf. [5]).

3. Holomorphic Functions with duality condition

Throughout this section we assume that D is a smooth bounded pseudoconvex domain in $\boldsymbol{C}^n$.

Proposition 2. *Let s be a positive integer and let $f \in L^2(D)$. If $f \in L^{-s}(D)^*$, then there exists a positive constant C such that*

$$|||hf|||_s \leq C |||f|||_s ||h||_s$$

for all $h \in W^s(D)$.

Proof. For any $\psi\in L^{-s}(D)$ we have

$$|\langle hf, \psi\rangle| \leq |||f|||_s|||\bar{h}\psi|||_{-s}.$$

There exists a positive constant C such that

$$||h\varphi||_s \leq C||h||_s||\varphi||_s$$

for all $h, \varphi\in W^s(D)$.
Then it follows that

$$\begin{aligned}|||\bar{h}\psi|||_{-s} &= \sup_{\substack{\varphi\in W^s(D)\\ ||\varphi||_s=1}} |\langle \bar{h}\psi, \varphi\rangle| \\ &\leq \sup |||\psi|||_{-s}||h\varphi||_s \\ &\leq C|||\psi|||_{-s}||h||_s.\end{aligned}$$

Hence we have

$$\begin{aligned}|||hf|||_s &= \sup_{\substack{\psi\in L^{-s}(D)\\ |||\psi|||_{-s}=1}} |\langle hf, \psi\rangle| \\ &\leq C|||f|||_s||h||_s.\end{aligned}$$

Let $t>0$. For $h\in H(D)$, we define

$$|||h|||_{s,t} = \sup_{\substack{g\in H(D)\\ ||g||_{-s}=1}} |\langle h, g\rangle_t|.$$

This norm was defined in [2].

Proposition 3. *Let $f\in L^2(D)$. If $f\in L^{-s}(D)^*$, then*

$$|||P_t(hfe^{t|z|^2})|||_{s,t} \leq C|||f|||_s||h||_s$$

for all $h\in W^s(D)$ and all $t>0$, where C is a constant.

Proof. By the definition we have

$$\begin{aligned}|||P_t(hfe^{t|z|^2})|||_{s,t} &= \sup_{\substack{g\in H(D)\\ ||g||_{-s}=1}} |\langle P_t(hfe^{t|z|^2}), g\rangle_t| \\ &= \sup|\langle hf, g\rangle| \\ &\leq \sup|||hf|||_s||g|||_{-s}.\end{aligned}$$

Then the conclusion follows from Propositions 1 and 2.

Proposition 4. *Suppose $f\in H(D)$ is contained in $L^{-s}(D)^*$. Then $f\in H^s(D)$.*

Proof. By Lemma 3 in [2], it suffices to show that $|||f|||_{s,t}<\infty$ *for* some $t>0$. We prove it according to an idea of Bell [2].

First, we expand $e^{-t|z|^2}$ in a power series

$$e^{-t|z|^2} = \sum c_\alpha z^\alpha \bar{z}^\alpha .$$

Let $R > \sup\{|z|;\ z \in D\}$. Then we have $\|z^\alpha\|_\sigma \le c_\sigma R^{|\alpha|}$, where c_σ is a constant depending only on the integer σ.
For $g \in H(D)$ we have

$$\begin{aligned} |\langle f, g\rangle_t| &= \left| \int_D f \bar{g} e^{-t|z|^2} \right| \\ &\le \sum |c_\alpha| \, |\langle z^\alpha f, z^\alpha g\rangle| . \end{aligned}$$

Next we obtain

$$\begin{aligned} |\langle z^\alpha f, z^\alpha g\rangle| &= |\langle z^\alpha f e^{t|z|^2}, z^\alpha g\rangle_t| \\ &= |\langle P_t(z^\alpha f e^{t|z|^2\alpha}), z^\alpha g\rangle_t| \\ &\le |||P_t(z^\alpha f e^{t|z|^2})|||_{s,t} \|z^\alpha g\|_{-s} . \end{aligned}$$

It follows from Proposition 3 that

$$|||P_t(z^\alpha f e^{t|z|^2})|||_{s,t} \le C_1 |||f|||_s \|z^\alpha\|_s .$$

And also we have

$$\begin{aligned} \|z^\alpha g\|_{-s} &= \sup_{\substack{\varphi \in C_0^\infty(D) \\ \|\varphi\|_s = 1}} |\langle z^\alpha g, \varphi\rangle| \\ &\le \sup \|g\|_{-s} \|\bar{z}^\alpha \varphi\|_s \\ &\le C_2 \|g\|_{-s} \|z^\alpha\|_s . \end{aligned}$$

Since $\|z^\alpha\|_s \le c_s R^{|\alpha|}$, we finally obtain

$$\begin{aligned} |||f|||_{s,t} &= \sup_{\substack{g \in H(D) \\ \|g\|_{-s} = 1}} |\langle f, g\rangle_t| \\ &\le C_3 \sum |c_\alpha| \, |||f|||_s \|z^\alpha\|_s^2 \\ &\le C_4 \, |||f|||_s \sum |c_\alpha| R^{2|\alpha|} \\ &\le C_4 |||f|||_s e^{tnR^2} . \end{aligned}$$

Hence the proof finishes.

4. Theorem

Theorem. *Let D_1 and D_2 be smooth bounded pseudoconvex domains in $\boldsymbol{C}^n$, and let $f\colon D_1 \to D_2$ be a biholomorphic mapping. If $|f|^2$ is C^∞, then f extends to a diffeomorphism of $\bar{D}_1$ onto $\bar{D}_2$.*

Proof. First we show that $U\cdot h\circ F\in L^{-s}(D_2)^*$ for all $s>0$ and all $h\in H^\infty(\bar{D}_1)$, where $F=f^{-1}$ and $U=\mathrm{Det}[F']$.

Let $s>0$ and let $h\in H^\infty(\bar{D}_1)$. Take a $t>0$ such that P_t: $W^s(D_2)\to H^s(D_2)$ is bounded. There exists an integer M such that the operator $\varphi\mapsto U\cdot(\varphi\circ F)$ is bounded from $W_0^{s+M}(D_1)$ to $W_0^s(D_2)$ (Lemma 4 in [2].). For $\psi\in L^{-s}(D_2)$ we have

$$|\langle U\cdot h\circ F,\ \psi\rangle|=|\langle U\cdot h\circ F,\ P_t(\psi e^{t|w|^2})\rangle_t|.$$

Letting $g=P_t(\psi e^{t|w|^2})$, we obtain

$$|\langle U\cdot h\circ F,\ g\rangle_t|=|\langle h,\ u\cdot g\circ fe^{-t|f|^2}\rangle|,$$

where $u=\mathrm{Det}[f']$. Using a bounded operator Φ^{s+M}: $W^{s+M}(D_1)\to W_0^{s+M}(D_1)$ with $P\Phi^{s+M}=P$, we get

$$\begin{aligned}|\langle h,\ u\cdot g\circ fe^{-t|f|^2}\rangle|&=|\langle\Phi^{s+M}(he^{-t|f|^2}),\ u\cdot g\circ f\rangle|\\&\leq C_1\|h\|_{s+M}\|u\cdot g\circ f\|_{-s-M}\,.\end{aligned}$$

Now we estimate the norm $\|u\cdot g\circ f\|_{-s-M}$. By the definition we have

$$\begin{aligned}\|u\cdot g\circ f\|_{-s-M}&=\sup_{\substack{\varphi\in C_0^\infty(D_1)\\ \|\varphi\|_{s+M}=1}}|\langle u\cdot g\circ f,\ \varphi\rangle|\\&=\sup|\langle g,\ U\cdot\varphi\circ F\rangle|\\&=\sup|\langle P_t(\psi e^{t|w|^2}),\ U\cdot\varphi\circ Fe^{t|w|^2}\rangle_t|\\&=\sup|\langle\psi,\ P_t(U\cdot\varphi\circ Fe^{t|w|^2})\rangle|\\&\leq\sup|||\psi|||_{-s}|||P_t(U\cdot\varphi\circ Fe^{t|w|^2})|||_s\,.\end{aligned}$$

Since $|||\cdot|||_s\leq\|\cdot\|_s$, we obtain

$$\|u\cdot g\circ f\|_{-s-M}\leq C_2|||\psi|||_{-s}\,.$$

Thereofre we get

$$\begin{aligned}|||U\cdot h\circ F|||_s&=\sup_{\substack{\psi\in L^{-s}(D_2)\\ |||\psi|||_{-s}=1}}|\langle U\cdot h\circ F,\ \psi\rangle|\\&\leq C_2\|h\|_{s+M}\,.\end{aligned}$$

Then it follows from Proposition 4 that $U\cdot h\circ F\in H^\infty(\bar{D}_2)$ for all $h\in H^\infty(\bar{D}_1)$. Putting $h=1$ and $h=z^\alpha$, we obtain $U\in H^\infty(\bar{D}_2)$ and $U\cdot F^\alpha\in H^\infty(\bar{D}_2)$. Then F extends smoothly to the boundary (see the proof of Theorem 1 in [6]). Of course it implies $|F|^2\in C^\infty(\bar{D}_2)$. Similarly, we obtain that $u\cdot H\circ f\in L^{-s}(D_1)^*$ for all $s>0$ and all $H\in H^\infty(\bar{D}_2)$. By the same reason we obtain that f extends smoothly to the boundary. Since $U=1/u$, f is a diffeomorphism of $\bar{D}_1$ onto $\bar{D}_2$.

5. Remarks

By the proof of Theorem we obtain that a biholomorphic mapping $f\colon D_1 \to D_2$ between smooth bounded pseudoconvex domains D_1 and D_2 extends to a diffeomorphism of $\bar{D}_1$ onto $\bar{D}_2$, if f has the following property

$$u \cdot H \circ f \in L^{-s}(D_1)^*$$

for all $s>0$ and all $H \in H^\infty(\bar{D}_2)$.
The relation between the above property and condition R is as follows.

Proposition 5. *Let D_1 and D_2 be smooth bounded pseudoconvex domains in $\boldsymbol{C}^n$. Suppose that D_1 satisfies condition R. Then any proper holomorphic mapping $f\colon D_1 \to D_2$ has the property*

$$u \cdot H \circ f \in L^{-s}(D_1)^*$$

for all $s>0$ and all $H \in H^\infty(\bar{D}_2)$.

Proof. Let P_i be the Bergman projection associated to D_i. Bell [3] proved a transformation formula

$$P_1(u \cdot \varphi \circ f) = u \cdot (P_2 \varphi) \circ f$$

for $\varphi \in L^2(D_2)$. For a given $s>0$, there exists an integer M such that $P_1\colon W^{s+M}(D_1) \to H^s(D_1)$ is bounded. We have an integer N depending on s and M such that the operator $\varphi \mapsto u \cdot \varphi \circ f$ is bounded from $W_0^N(D_2)$ to $W_0^{s+M}(D_1)$. There exists a bounded operator $\Phi^N\colon W^N(D_2) \to W_0^N(D_1)$ such that $P_2 \Phi^N = P_2$. For $\psi \in L^{-s}(D_1)$ it holds that

$$\begin{aligned} |\langle u \cdot H \circ f, \psi \rangle| &= |\langle u \cdot (\Phi^N H) \circ f, P_1 \psi \rangle| \\ &\leq \|u \cdot (\Phi^N H) \circ f\|_{s+M} \|P_1 \psi\|_{-s-M} . \end{aligned}$$

It is easily seen that

$$\|u \cdot (\Phi^N H) \circ f\|_{s+M} \leq C_1 \|H\|_N .$$

By the boundedness of P_1 we have

$$\begin{aligned} \|P_1 \psi\|_{-s-M} &= \sup_{\substack{\varphi \in C_0^\infty(D_1) \\ \|\varphi\|_{s+M}=1}} |\langle P_1 \psi, \varphi | \rangle \\ &= \sup |\langle \psi_1, P_1 \varphi \rangle| \\ &\leq \sup \||\psi|\|_{-s} \|P_1 \varphi\|_s \\ &\leq C_2 \||\psi|\|_{-s} . \end{aligned}$$

Therefore we obtain

$$|||u\cdot H\circ f|||_s = \sup_{\substack{\psi\in L^{-s}(D_1)\\ |||\psi|||_{-s}=1}} |\langle u\cdot H\circ f,\ \psi\rangle| \\ \leq C_3||H||_N\,.$$

This completes the proof.

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