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DIFFEOMORPHIC EXTENSION OF BIHOLOMORPHIC MAPPINGS WITH SMOOTH MODULUS

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1. Introduction

Fefferman proved in [8] that any biholomorphic mapping between two smooth bounded strictly pseudoconvex domains D_1 and D_2 in \mathbf{C}^n extends to a diffeomorphism of \bar{D}_1 onto \bar{D}_2 . Later Fefferman's theorem was extended by Bell and Ligočka [7] and Bell [2].

Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^n . Let $L^2(D)$ be the space of square-integrable functions on D . We denote by $H(D)$ the space of square-integrable holomorphic functions on D . The Bergman projection P is the orthogonal projection from $L^2(D)$ to $H(D)$. The domain D is said to satisfy condition R if P maps $C^\infty(\bar{D})$ continuously into $C^\infty(\bar{D})$. Bell's result [2] is as follows:

Let D_1 and D_2 be smooth bounded pseudoconvex domains in \mathbf{C}^n . If either D_1 or D_2 satisfies condition R , then any biholomorphic mapping between D_1 and D_2 extends to a diffeomorphism of \bar{D}_1 onto \bar{D}_2 .

It is not known that any biholomorphic mapping between smooth bounded weakly pseudoconvex domains in \mathbf{C}^n can be extended to a diffeomorphism onto the boundary. Fornaess proved in [9] that any biholomorphic mapping $f: D_1 \rightarrow D_2$ between bounded pseudoconvex domains D_1 and D_2 in \mathbf{C}^n with C^2 -boundary extends to a C^2 -diffeomorphism of \bar{D}_1 onto \bar{D}_2 , if f has a C^2 -extension $\hat{f}: \bar{D}_1 \rightarrow \bar{D}_2$. In this paper we shall prove the theorem of this type. Let D_1 and D_2 be smooth bounded pseudoconvex domains in \mathbf{C}^n . Using Bell's method we shall prove that any biholomorphic mapping $f: D_1 \rightarrow D_2$ extends to a C^∞ -diffeomorphism of \bar{D}_1 onto \bar{D}_2 , whenever $|f|^2$ is C^∞ .

2. Preliminaries

Let D be a smooth bounded pseudoconvex domain in \mathbf{C}^n . We denote by $W^s(D)$ the usual Sobolev space for $s > 0$. A negative Sobolev space $W^{-s}(D)$ is the dual space of $W_0^s(D)$, where $W_0^s(D)$ is the closure of $C_0^\infty(D)$ in $W^s(D)$. We now consider the dual space $W^s(D)^*$ of $W^s(D)$ for $s > 0$.

Let $\langle \cdot, \cdot \rangle$ be the $L^2(D)$ inner product. For any $f \in L^2(D)$, $\langle \cdot, f \rangle$ is a

continuous complex linear functional on $W^s(D)$. We set

$$|||f|||_{-s} = \sup_{\substack{\psi \in W^s(D) \\ |||\psi|||_s=1}} |\langle f, \psi \rangle|.$$

Then we regard $L^2(D)$ as a subspace of $W^s(D)^*$ via $\langle \cdot, \cdot \rangle$, and denote it by $L^{-s}(D)$. The norm of $W^{-s}(D)$ is denoted by $\|\cdot\|_{-s}$. By the same way as in [5], we obtain the following proposition.

Proposition 1. *Let D be a smooth bounded pseudoconvex domain in \mathbb{C}^n . Then the norms $\|\cdot\|_{-s}$ and $|||\cdot|||_{-s}$ are equivalent on $H(D)$.*

We set

$$|||f|||_s = \sup_{\substack{g \in L^{-s}(D) \\ |||g|||_{-s}=1}} |\langle f, g \rangle|,$$

for $f \in L^2(D)$. If $|||f|||_s < \infty$, then we regard f as an element of the dual space $L^{-s}(D)^*$ of $L^{-s}(D)$ and we write $f \in L^{-s}(D)^*$. $H^s(D)$ is the subspace of $W^s(D)$ consisting of holomorphic functions. Note that $C^\infty(\bar{D}) = \bigcap_{s>0} W^s(D)$ and $H^\infty(\bar{D}) = \bigcap_{s>0} H^s(D)$.

Bell constructed a bounded linear operator $\Phi^s: H^s(D) \rightarrow W^s_0(D)$ such that $P\Phi^s h = h$ for all $h \in H^s(D)$ (see [1], [2] and [4]). This operator was extended to a bounded linear operator $\tilde{\Phi}^s: W^s(D) \rightarrow W^s_0(D)$ with $P\tilde{\Phi}^s = P$ ([11]). For $t > 0$, we denote by $L^2_t(D)$ the weighted Hilbert space of complex valued functions on D with inner product given by

$$\langle g, h \rangle_t = \int_D g(z) \overline{h(z)} e^{-t|z|^2} d\mu(z).$$

The weighted Bergman projection P_t is the orthogonal projection of $L^2_t(D)$ onto $H(D)$ with respect to the inner product $\langle \cdot, \cdot \rangle_t$. By a Kohn's result [10] it holds that for a positive integer s there exists a positive number t_0 such that P_t maps $W^s(D)$ into $W^s(D)$ continuously, if $t > t_0$. There exists a bounded linear operator $\Phi^s_t: W^s(D) \rightarrow W^s_0(D)$ such that $P_t \Phi^s_t = P_t$ (cf. [5]).

3. Holomorphic Functions with duality condition

Throughout this section we assume that D is a smooth bounded pseudoconvex domain in \mathbb{C}^n .

Proposition 2. *Let s be a positive integer and let $f \in L^2(D)$. If $f \in L^{-s}(D)^*$, then there exists a positive constant C such that*

$$|||hf|||_s \leq C |||f|||_s |||h|||_s$$

for all $h \in W^s(D)$.

Proof. For any $\psi \in L^{-s}(D)$ we have

$$|\langle hf, \psi \rangle| \leq \|f\|_s \|h\psi\|_{-s}.$$

There exists a positive constant C such that

$$\|h\varphi\|_s \leq C \|h\|_s \|\varphi\|_s$$

for all $h, \varphi \in W^s(D)$.

Then it follows that

$$\begin{aligned} \|h\psi\|_{-s} &= \sup_{\substack{\varphi \in W^s(D) \\ \|\varphi\|_s=1}} |\langle h\psi, \varphi \rangle| \\ &\leq \sup \|\psi\|_{-s} \|h\varphi\|_s \\ &\leq C \|\psi\|_{-s} \|h\|_s. \end{aligned}$$

Hence we have

$$\begin{aligned} \|hf\|_s &= \sup_{\substack{\psi \in L^{-s}(D) \\ \|\psi\|_{-s}=1}} |\langle hf, \psi \rangle| \\ &\leq C \|f\|_s \|h\|_s. \end{aligned}$$

Let $t > 0$. For $h \in H(D)$, we define

$$\|h\|_{s,t} = \sup_{\substack{g \in H(D) \\ \|g\|_{-s}=1}} |\langle h, g \rangle_t|.$$

This norm was defined in [2].

Proposition 3. *Let $f \in L^2(D)$. If $f \in L^{-s}(D)^*$, then*

$$\|P_t(hfe^{t|z|^2})\|_{s,t} \leq C \|f\|_s \|h\|_s$$

for all $h \in W^s(D)$ and all $t > 0$, where C is a constant.

Proof. By the definition we have

$$\begin{aligned} \|P_t(hfe^{t|z|^2})\|_{s,t} &= \sup_{\substack{g \in H(D) \\ \|g\|_{-s}=1}} |\langle P_t(hfe^{t|z|^2}), g \rangle_t| \\ &= \sup |\langle hf, g \rangle| \\ &\leq \sup \|hf\|_s \|g\|_{-s}. \end{aligned}$$

Then the conclusion follows from Propositions 1 and 2.

Proposition 4. *Suppose $f \in H(D)$ is contained in $L^{-s}(D)^*$. Then $f \in H^s(D)$.*

Proof. By Lemma 3 in [2], it suffices to show that $\|f\|_{s,t} < \infty$ for some $t > 0$. We prove it according to an idea of Bell [2].

First, we expand $e^{-t|z|^2}$ in a power series

$$e^{-t|z|^2} = \sum c_\alpha z^\alpha \bar{z}^\alpha.$$

Let $R > \sup \{|z|; z \in D\}$. Then we have $\|z^\alpha\|_\sigma \leq c_\sigma R^{|\alpha|}$, where c_σ is a constant depending only on the integer σ .

For $g \in H(D)$ we have

$$\begin{aligned} |\langle f, g \rangle_t| &= \left| \int_D f \bar{g} e^{-t|z|^2} \right| \\ &\leq \sum |c_\alpha| |\langle z^\alpha f, z^\alpha g \rangle|. \end{aligned}$$

Next we obtain

$$\begin{aligned} |\langle z^\alpha f, z^\alpha g \rangle| &= |\langle z^\alpha f e^{t|z|^2}, z^\alpha g \rangle_t| \\ &= |\langle P_t(z^\alpha f e^{t|z|^2}), z^\alpha g \rangle_t| \\ &\leq \|P_t(z^\alpha f e^{t|z|^2})\|_{s,t} \|z^\alpha g\|_{-s}. \end{aligned}$$

It follows from Proposition 3 that

$$\|P_t(z^\alpha f e^{t|z|^2})\|_{s,t} \leq C_1 \|f\|_s \|z^\alpha\|_s.$$

And also we have

$$\begin{aligned} \|z^\alpha g\|_{-s} &= \sup_{\substack{\varphi \in C_0^\infty(D) \\ \|\varphi\|_s = 1}} |\langle z^\alpha g, \varphi \rangle| \\ &\leq \sup \|g\|_{-s} \|z^\alpha \varphi\|_s \\ &\leq C_2 \|g\|_{-s} \|z^\alpha\|_s. \end{aligned}$$

Since $\|z^\alpha\|_s \leq c_s R^{|\alpha|}$, we finally obtain

$$\begin{aligned} \|f\|_{s,t} &= \sup_{\substack{g \in H(D) \\ \|g\|_{-s} = 1}} |\langle f, g \rangle_t| \\ &\leq C_3 \sum |c_\alpha| \|f\|_s \|z^\alpha\|_s^2 \\ &\leq C_4 \|f\|_s \sum |c_\alpha| R^{2|\alpha|} \\ &\leq C_4 \|f\|_s e^{tnR^2}. \end{aligned}$$

Hence the proof finishes.

4. Theorem

Theorem. *Let D_1 and D_2 be smooth bounded pseudoconvex domains in \mathbb{C}^n , and let $f: D_1 \rightarrow D_2$ be a biholomorphic mapping. If $|f|^2$ is C^∞ , then f extends to a diffeomorphism of \bar{D}_1 onto \bar{D}_2 .*

Proof. First we show that $U \cdot h \circ F \in L^{-s}(D_2)^*$ for all $s > 0$ and all $h \in H^\infty(\bar{D}_1)$, where $F = f^{-1}$ and $U = \text{Det}[F']$.

Let $s > 0$ and let $h \in H^\infty(\bar{D}_1)$. Take a $t > 0$ such that $P_t: W^s(D_2) \rightarrow H^s(D_2)$ is bounded. There exists an integer M such that the operator $\varphi \mapsto U \cdot (\varphi \circ F)$ is bounded from $W_0^{s+M}(D_1)$ to $W_0^s(D_2)$ (Lemma 4 in [2]). For $\psi \in L^{-s}(D_2)$ we have

$$|\langle U \cdot h \circ F, \psi \rangle| = |\langle U \cdot h \circ F, P_t(\psi e^{t|w|^2}) \rangle_t|.$$

Letting $g = P_t(\psi e^{t|w|^2})$, we obtain

$$|\langle U \cdot h \circ F, g \rangle_t| = |\langle h, u \cdot g \circ f e^{-t|f|^2} \rangle|,$$

where $u = \text{Det}[f']$. Using a bounded operator $\Phi^{s+M}: W^{s+M}(D_1) \rightarrow W_0^{s+M}(D_1)$ with $P\Phi^{s+M} = P$, we get

$$\begin{aligned} |\langle h, u \cdot g \circ f e^{-t|f|^2} \rangle| &= |\langle \Phi^{s+M}(h e^{-t|f|^2}), u \cdot g \circ f \rangle| \\ &\leq C_1 \|h\|_{s+M} \|u \cdot g \circ f\|_{-s-M}. \end{aligned}$$

Now we estimate the norm $\|u \cdot g \circ f\|_{-s-M}$. By the definition we have

$$\begin{aligned} \|u \cdot g \circ f\|_{-s-M} &= \sup_{\substack{\varphi \in C_0^\infty(D_1) \\ \|\varphi\|_{s+M} = 1}} |\langle u \cdot g \circ f, \varphi \rangle| \\ &= \sup |\langle g, U \cdot \varphi \circ F \rangle| \\ &= \sup |\langle P_t(\psi e^{t|w|^2}), U \cdot \varphi \circ F e^{t|w|^2} \rangle_t| \\ &= \sup |\langle \psi, P_t(U \cdot \varphi \circ F e^{t|w|^2}) \rangle| \\ &\leq \sup \|\psi\|_{-s} \|P_t(U \cdot \varphi \circ F e^{t|w|^2})\|_s. \end{aligned}$$

Since $\|\cdot\|_s \leq \|\cdot\|_s$, we obtain

$$\|u \cdot g \circ f\|_{-s-M} \leq C_2 \|\psi\|_{-s}.$$

Therefore we get

$$\begin{aligned} \|U \cdot h \circ F\|_s &= \sup_{\substack{\psi \in L^{-s}(D_2) \\ \|\psi\|_{-s} = 1}} |\langle U \cdot h \circ F, \psi \rangle| \\ &\leq C_2 \|h\|_{s+M}. \end{aligned}$$

Then it follows from Proposition 4 that $U \cdot h \circ F \in H^\infty(\bar{D}_2)$ for all $h \in H^\infty(\bar{D}_1)$. Putting $h=1$ and $h=z^\alpha$, we obtain $U \in H^\infty(\bar{D}_2)$ and $U \cdot F^\alpha \in H^\infty(\bar{D}_2)$. Then F extends smoothly to the boundary (see the proof of Theorem 1 in [6]). Of course it implies $|F|^2 \in C^\infty(\bar{D}_2)$. Similarly, we obtain that $u \cdot H \circ f \in L^{-s}(D_1)^*$ for all $s > 0$ and all $H \in H^\infty(\bar{D}_2)$. By the same reason we obtain that f extends smoothly to the boundary. Since $U=1/u$, f is a diffeomorphism of \bar{D}_1 onto \bar{D}_2 .

5. Remarks

By the proof of Theorem we obtain that a biholomorphic mapping $f: D_1 \rightarrow D_2$ between smooth bounded pseudoconvex domains D_1 and D_2 extends to a diffeomorphism of \bar{D}_1 onto \bar{D}_2 , if f has the following property

$$u \cdot H \circ f \in L^{-s}(D_1)^*$$

for all $s > 0$ and all $H \in H^\infty(\bar{D}_2)$.

The relation between the above property and condition R is as follows.

Proposition 5. *Let D_1 and D_2 be smooth bounded pseudoconvex domains in \mathbb{C}^n . Suppose that D_1 satisfies condition R . Then any proper holomorphic mapping $f: D_1 \rightarrow D_2$ has the property*

$$u \cdot H \circ f \in L^{-s}(D_1)^*$$

for all $s > 0$ and all $H \in H^\infty(\bar{D}_2)$.

Proof. Let P_i be the Bergman projection associated to D_i . Bell [3] proved a transformation formula

$$P_1(u \cdot \varphi \circ f) = u \cdot (P_2 \varphi) \circ f$$

for $\varphi \in L^2(D_2)$. For a given $s > 0$, there exists an integer M such that $P_1: W^{s+M}(D_1) \rightarrow H^s(D_1)$ is bounded. We have an integer N depending on s and M such that the operator $\varphi \mapsto u \cdot \varphi \circ f$ is bounded from $W_0^N(D_2)$ to $W_0^{s+M}(D_1)$. There exists a bounded operator $\Phi^N: W^N(D_2) \rightarrow W_0^N(D_1)$ such that $P_2 \Phi^N = P_1$. For $\psi \in L^{-s}(D_1)$ it holds that

$$\begin{aligned} |\langle u \cdot H \circ f, \psi \rangle| &= |\langle u \cdot (\Phi^N H) \circ f, P_1 \psi \rangle| \\ &\leq \|u \cdot (\Phi^N H) \circ f\|_{s+M} \|P_1 \psi\|_{-s-M}. \end{aligned}$$

It is easily seen that

$$\|u \cdot (\Phi^N H) \circ f\|_{s+M} \leq C_1 \|H\|_N.$$

By the boundedness of P_1 we have

$$\begin{aligned} \|P_1 \psi\|_{-s-M} &= \sup_{\substack{\varphi \in C_0^\infty(D_1) \\ \|\varphi\|_{s+M} = 1}} |\langle P_1 \psi, \varphi \rangle| \\ &= \sup |\langle \psi, P_1 \varphi \rangle| \\ &\leq \sup \|\psi\|_{-s} \|P_1 \varphi\|_s \\ &\leq C_2 \|\psi\|_{-s}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \|u \cdot H \circ f\|_s &= \sup_{\substack{\psi \in L^2_s(D_1) \\ \|\psi\|_{-s} = 1}} |\langle u \cdot H \circ f, \psi \rangle| \\ &\leq C_3 \|H\|_N. \end{aligned}$$

This completes the proof.

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