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Strict Convexity and Smoothness of Normed Spaces

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V. L. Klee [11][1] and M. M. Day [6] have considered various problems on strict convexity and smoothness of normed spaces. In his paper, Day [6] raised several questions. Two of these are the following:

(1) Is any L_1-space strictly convexifiable?

(2) Is there a nonreflexive nonsparable *scm* space?

In this paper, we consider these questions. In §2 we deal with spaces of bounded continuous functions and consider strict convexity and smoothness on these spaces. In §3 we give a partial answer to the first question, and in §4 we give an answer to the second, by showing an example of a nonreflexive nonseparable *scm* space.

§1. Preliminary.

Let E be a normed space. If every chord of the unit sphere has its midpoint below the surface of the unit sphere, then E is called *strictly convex* (written *SC*); if through every point of the surface of the unit sphere of E there passes a unique hyperplane of support of the unit sphere, then E is called *smooth* (written *SM*); if both occur, then E is called *SCM*. If E is isomophic to an *SM*, an *SC* and an *SCM* space, then E is called an *sm*, an *sc* and an *scm* space respectively.

If I is an index set, we define:

$m(I)$ = the space of all bounded real functions on I with $||x|| = \text{l.u.b.}_{i\in I} |x(i)|$.

$c_0(I)$ = the subspace of those x in $m(I)$ for which for each $\varepsilon > 0$ the set of i with $|x(i)| > \varepsilon$ is finite; that is, $c_0(I)$ is the set of functions vanishing at infinity on the discrete space I.

$l_p(I)$ (for $p \geqq 1$) = the set of those real functions x on I for which $||x||_{l_p} = [\sum |x(i)|^p]^{1/p} < +\infty$.

Let X be a topological space. Then $C(X)$ denotes the space of all real-valued bounded continuous functions on X such that the norm $||f|| = \sup_{x\in X} |f(x)|$.

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1) Numbers in bracket refer to the references cited at the end of the paper.

If X is a set and F is a Borel field of sets in X and if μ is a countably additive, non negative set function defined on F, then $L_p(X, \mu)$ denotes the space of all measurable functions f on X such that $\|f\|_{L_p} = [\int |f(x)|^p d\mu(x)]^{1/p}$. It is called a L_p-space.

Day [6] has proved the following theorems, which we shall frequently refer to later.

D_1. If a normed space E is isomophic with a subspace of an *sm* (or *sc*) space, then E is *sm* (or *sc*).

D_2. If E is an *sm* space and if there is a one-to-one linear continuous mapping T from E into an *scm* space F, then E is *scm*.

D_3. If E is separable then E is *scm*.

D_4. Let J be an index set and let E_j be an *sc* space for any $j \in J$. If E is a normed space of all functions f such that for any $j, f(j) \in E_j$ and $\sum_j \|f(j)\|_{E_j}^p < +\infty$ $(p \geqq 1)$ and the norm of f is $(\sum_j \|f(j)\|_{E_j}^p)^{1/p}$, then E is *sc*. E is called the l_p product of E_j.

D_5. Let J be an index set and let E_j be an *sm* space for any $j \in J$. Then the l_p product of $E_j (p > 1)$ is *sm*.

D_6. If I is infinite, then $m(I)$ is not *sm*. If I is uncountable, $m(I)$ is not *sc*.

D_7. For any index set I, $c_0(I)$ is *sm* and *sc*.

D_8. For any index set I, $l_1(I)$ is *sc*.

§ 2. Spaces of bounded continuous functions.

Throughout this paragraph, spaces are always completely regular Hausdorff spaces.

We first prove the following lemma.

Lemma 1. (i) *If R is a countably paracompact*[2] *space and if $C(R)$ is sm, then R is countably compact.*[2]

(ii) *If R is paracompact*[2] *and if $C(R)$ is sm, then R is compact.*[2]

Proof. If R is not countably compact, then there exists a countably infinite set N in R such that N has no accumulation point. Let N be a set $\{x_1, x_2, \cdots, x_n, \cdots\}$. Since R is regular, there exists a sequence of mutually disjoint open sets $\{U_n\}$ in R such that $U_n \ni x_n$ for any n. We consider an open covering $\mathfrak{U}$ consisting of $\{U_n\}$ and $R-N$. Since R is

2) Compactness will always mean the bicompactness of Alexandroff-Hopf [1]; a space with the property that every infinite subset has an accumulation point will be called countably compact. A Hausdorff space is paracompact (or countably paracompact) if every open covering (or countably open covering) of it can be refined by one which is locally finite, that is, every point of the space has a neighborhood meeting only a finite number of sets of the refining covering (cf. [13] and [7]).

countably paracompact, $\mathfrak{U}$ can be refined by a locally finite covering $\mathfrak{V}$. We see easily that for any n there exists a $V_n \in \mathfrak{V}$ such that $x_n \in V_n \subset U_n$. For any n we take an open set W_n and a continuous function f_n on R such that $V_n \supset \overline{W}_n$[3], $W_n \ni x_n$, $f_n(x_n)=1$, $f_n(x)$ for any $x \in R-W_n$ and $0 \leqq f_n(x) \leqq 1$ for any $x \in R$. For any $t=(t_1, t_2, \cdots, t_n, \cdots) \in m$, we put

$$f_t(x) = \sum_{n=1}^{\infty} t_n f_n(x) \qquad \text{for any } x \in R.$$

Since $\mathfrak{V}$ is locally finite and $\overline{\sum_{n=1}^{\infty} W_n} \subset \sum_{n=1}^{\infty} V_n$. We see easily that f_t is continuous for $t \in m$ and $\|f_t\| = \sup_n |t_n| = \|t\|_m$. Then by D_1 and D_6, $C(R)$ is not *sm*.

(ii) R. Arens and J. Dugungi [3] have proved that if R is paracompact, then R is compact if and only if it is countably compact. Therefore (ii) is clear by (i).

By Lemma 1. (ii) we obtain

Theorem 1. *Let R be a metric space. Then $C(R)$ is sm if and only if R is compact.*

Proof. Since a metric space is paracompact, the necessity is clear. Conversely, if R is metric and compact, then $C(R)$ is separable, therefore $C(R)$ is *sm*.

Kakutani [9] proved the following lemma.

Lemma 2. *If H is a locally separable, closed subset of a metric space R, then there is a linear isometry T of $C(H)$ into $C(R)$ such that $Tx(h) = x(h)$ for all h in H.*

Theorem 1 also follows from Lemma 2.

We obtain moreover,

Theorem 2. *Let R be a metric space. Then $C(R)$ is sc if and only if R is separable.*

Proof. Let R be separable and let $\{x_n\}$ be a countable dense set in R. Then we consider a new norm $|f|$ for any $f \in C(R)$. We define

$$|f| = \left[\|f\|^2 + \sum_{n=1}^{\infty} \frac{1}{2^{2n}} |f(x_n)|^2\right]^{\frac{1}{2}}.$$

We easily see that $C(R)$ is SC by this new norm. Conversely, if $C(R)$ is *sc*, and if N is a subset (in R) having $\aleph_1$ elements, then N has an accumulation point. For, if there is a subset N which has $\aleph_1$ elements

3) $\bar{A}$ denotes the closure of A.

and has no accumulation point, then $C(N)$ is isomorphic to $m(I)$, where I is an index set which has $\aleph_1$ elements. By D_1, D_6 and Lemma 2, $C(R)$ is not *sc*. Therefore, we may prove the following lemma.

Lemma 3. *Let R be a metric space. If every subset N (in R) having $\aleph_1$ elements has an accumulation point, then R is separable.*

Proof. Suppose the hypothesis holds. Then for any positive number ε, there is a sequence of elements, $x_1, x_2, \cdots, x_n, \cdots$ such that for any element x in R, there is an x_i with $\rho(x, x_i) < \varepsilon$, where ρ is a distance function on R. For otherwise, if x_1 is any element in R, there is an element x_2 in $R - S(x_1, \varepsilon)$, where $S(x_1, \varepsilon)$ denotes the sphere with center x_1 and radius ε. For any positive integer n, there exists x_n in $R - \bigcup_{i=1}^{n-1} S(x_i, \varepsilon)$. By repetition, for any $\alpha < \omega_1$ we can find an element x_α in R such that $x_\alpha \in R - \bigcup_{\beta<\alpha} S(x_\beta, \varepsilon)$. Therefore we have a set $N = \{x_\alpha \mid \alpha < \omega_1\}$. Since N has $\aleph_1$ elements, N has an accumulation point by hypothesis. Let x be an accumulation point of N. Then there are two distinct ordinary number α, β $(\beta < \alpha < \omega_1)$ such that $x_\alpha \in S\left(x, \frac{\varepsilon}{3}\right)$ and $x_\beta \in S\left(x, \frac{\varepsilon}{3}\right)$. Therefore $\rho(x_\alpha, x_\beta) < \frac{2}{3}\varepsilon$. This is a contradiction since $\rho(x_\alpha, x_\beta) \geqq \varepsilon$. Therefore, for any positive integer m, we can select a sequence $x_1^m, x_2^m, \cdots, x_n^m, \cdots$ such that for any x in R there is an x_i^m with $\rho(x, x_i^m) < \frac{1}{m}$. Put $D = \{x_i^m \mid m = 1, 2, \cdots . i = 1, 2, \cdots .\}$. Then D is dense in R, that is R is separable.

If X and Y are topological spaces and if there is a one-to-one continuous mapping φ from X onto Y, then Y is called a *contraction* of X, and we write $X \geqq Y$. If $X \geqq Y$ and if the inverse of φ is not continuous, then we write $X > Y$. We can assume here that X and Y are two spaces on the same set with different topologies. (Cf. [8] or [15]). If Y is metric (or locally compact), then Y is called a *metric* (or a *locally compact*) *contraction* of X.

Lemma 4. *If a completely regular space R has a metric contraction, then $C(R)$ is sm if and only if R is metric and compact.*

Proof. If R_0 is a metric contraction of R, then $C(R) \supset C(R_0)$. By D_1, if $C(R)$ is *sm*, then $C(R_0)$ is also *sm*. By Theorem 1, R_0 is (metric) compact. Now we shall prove that $R = R_0$. If $R > R_0$, then there exists an f in $C(R) - C(R_0)$ since R is completely regular. We put $d(x, y) = |f(x) - f(y)|$ for any $x, y \in R$, and put $\rho_0(x, y) = d(x, y) + \rho(x, y)$, where $\rho(x, y)$ is a distance function on R_0. Let R_1 be a metric space defined by $\rho_0(x, y)$. Then we easily see that $R \geqq R_1$ and $R_1 > R_0$. By Theorem 1, R is compact, since R_1 is metric. Since R_1 and R_2 are both compact, $R_1 = R_2$. This contradiction concludes the proof.

Similarly, we obtain

Lemma 5. *If a completely regular R has a metric contraction and if $C(R)$ is sc, then R is a least upper bound*[4] *of separable metric spaces.*

Proof. If R_0 is a metric contraction of R and if $R > R_0$, then for any open set U in R and for any $x \in U$, there is $f \in C(R)$ such that $f(x) = 1$ and $f(y) = 0$ for any $y \in R - U$. We put $d(x, y) = |f(x) - f(y)|$ and put $\rho_0(x, y) = d(x, y) + \rho(x, y)$, where $\rho(x, y)$ is a distance function on R_0. Let $R_{(U,x)}$ be a metric space defined by ρ_0. Then we easily see that R is a least upper bound of $R_{(U,x)}$.

If R is a topological space, we denote by Δ the diagonal of the topological product $R \times R$, that is, $\Delta = \{(x, x) \mid x \in R\}$.

Lemma 6. *The following two conditions are equivalent.*

a) *Δ is a G_δ set.*

b) *There exists a sequence of open coverings $\{\mathfrak{U}_n\}$ such that for any distinct two points x, y in R, no element in $\mathfrak{U}_m$ contains both x and y for some m.*

Proof. If a) holds, then $\Delta = \bigcap_{n=1}^{\infty} U_n$ for some sequence of open sets U_n in $R \times R$ containing Δ. For any $x \in R$, there is an open neighborhood $V_n(x)$ such that $V_n(x) \times V_n(x) \subset U_n$. We put $\mathfrak{U}_n = \{V_n(x) \mid x \in R\}$ for any n. Then we easily see that $\{\mathfrak{U}_n\}$ satisfies the property b).

Conversely, if b) holds and if $\mathfrak{U}_n = \{V_\alpha^n\}$, then we put $U_n = \sum_\alpha (V_\alpha^n \times V_\alpha^n)$. U_n is an open set in $R \times R$ containing Δ and $\Delta = \bigcap_{n=1}^{\infty} U_n$.

If R satisfies the equivalent condition of Lemma 6, then R will be called a *weakly metric* space. Of course, there is a weakly metric space which is not metric.

Theorem 3. *Let R be a paracompact, weakly metric space. Then $C(R)$ is sm if and only if R is metric and compact.*

Proof. In order to prove the theorem, we may show the existence of a metric space R_0 such that $R \geqq R_0$ (cf. Lemma 4). Since R is weakly metric, there exists, by Lemma 6, a sequence of open coverings $\{\mathfrak{U}_n\}$

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4) The set of all topologies on the same set forms a lattice by the ordering $\geqq$. The least upper bound of topologies means the least upper bound on this lattice.

such that for any distinct two points x, y in R, no element in $\mathfrak{U}_m$ contains both x and y for some m. We may assume that $\mathfrak{U}_n > \mathfrak{U}_{n+1}$[5] for any n. Since R is paracompact, every covering of R is normal.[5] Therefore, there exists a sequence of coverings on R such that

$$\mathfrak{U}_n > \mathfrak{V}_n \text{ and } \mathfrak{V}_n > \mathfrak{V}_{n+1}^{*}\,^{5)} \text{ for any } n.$$

Here we have pseudo distance function[6] $\rho(x, y)$ in R such that if $y \notin S(x, \mathfrak{V}_n)$,[5] then $\rho(x, y) > 2^{-n-2}$ for any n (cf. Tukey [13]). We can prove easily that if $x \neq y$ in R, then $\rho(x, y) > 0$ by the condition of $\{\mathfrak{U}_n\}$. Therefore ρ is a distance function. If R_0 is a metric space defined by ρ, then $R \geqq R_0$.

Corollary. *If R is paracompact and if $R \times R$ is perfectly normal*[7], *then $C(R)$ is sm if and only if R is metric and compact.*

By Lemma 5 we obtain,

Theorem 4. *Let R be a paracompact, weakly metric space and let $C(R)$ be sc. Then R is a least upper bound of separable metric spaces.*

REMARK. (i) Day [6] raised the following question: Is any *sm* space an *sc* space? If R is paracompact weakly metric and if $C(R)$ is *sm*, then it is *sc* (cf. Theorem 3).

(ii) An index set J is regarded as a discrete space. Let J_0 be a non-point compactification of J (cf. [1], p. 93). Then we easily see that $C(J_0)$ is isomorphic to $c_0(J)$, therefore $C(J_0)$ is *sm* by D_7. But J_0 is not weakly metric. (J_0 is paracompact since it is compact.) Therefore, in Theorem 3, the hypothesis is necessary.

§ 3. Spaces of summable functions.

Day raised the following question: Is any L_1-space an *sc* space? We here prove that if R is paracompact, weakly metric and[8] locally compact, then $L_1(R, \mu)$ is *sc* for any positive measure[9] μ. Every L_1-space

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5) "$\mathfrak{A} > \mathfrak{B}$" denotes that $\mathfrak{B}$ is a refinement of $\mathfrak{A}$. $\mathfrak{B}^*$ is a covering consisting of $\{S(V, \mathfrak{B}) \mid V \in \mathfrak{B}\}$. $S(A, \mathfrak{B})$ denotes the sum of V ($\in \mathfrak{B}$) with $V \cap A \neq \phi$. An open covering of R will be called normal if there is an open covering of R $\mathfrak{B}$ such that $\mathfrak{A} > \mathfrak{B}^*$. A topological space will be called fully normal if every open covering is normal (cf. [14]). A. H Stone [13] has proved that paracompactness is identical with the property of "full normality" in Hausdorff spaces.

6) $\rho(x, y)$ will be called a pseudo distance function if it is continuous on $R \times R$ and if (i) $\rho(x, y) \geqq 0$, (ii) $\rho(x, y) = \rho(y, x)$ and (iii) $\rho(x, y) + \rho(y, z) \geqq \rho(x, z)$.

7) A topological space X will be called perfectly normal if any closed set in X is a G_δ set.

8) See, § 2. Lemma 6.

9) See, for example, [4].

is represented as $L_1(R, \mu)$, where R is a sum of mutually disjoint stonian spaces[10] which are both open and closed in R.[11] Therefore, by D_4, if $L_1(R, \mu)$ is *sc* when R is stonian, then every L_1-space is *sc*. But, stonian spaces are not always weakly metric. Therefore the question is yet open.

We first prove

Theorem 5. *Let R be a locally compact metric space and let μ be a positive measure on R. Then $L_1(R, \mu)$ is sc.*

Proof. Let $\mathfrak{K}(R)$ be the set of all continuous functions on R with a compact carrier. If the norm $\|f\|$ of f in $\mathfrak{K}(R)$ is $\sup_{x \in R} |f(x)|$, $\mathfrak{K}(R)$ forms a normed space. Since R is locally compact and metric, R is a sum of mutually disjoint separable locally compact metric spaces $\{R_j\}_{j \in J}$ which are both open and closed in R (cf. Alexandroff and Urysohn [2]). Index set J may be uncountable. For any f in $L_1(R, \mu)$, we denote by f_j the restriction of f on R_j and by μ_j the restriction of μ on R_j. Then we can write $f = \sum_j f_j$ and $\|f\| = \sum_j \|f_j\|$. We easily see that $\mathfrak{K}(R_j)$ is separable for any j, since R_j is separable and locally compact. Therefore for any j $L_1(R_j, \mu_j)$ is separable and *sc* (cf. [5] or [6]). By D_4 the theorem is then clear.

Moreover, we can prove the following

Theorem 6. *Let R be a paracompact, weakly metric*[8]*, locally compact space. Then $L_1(R, \mu)$ is sc for any positive measure on R.*

We first prove two lemmas.

Lemma 7. *Let R be a locally compact space and let R have a locally compact metric contraction. Then $L_1(R, \mu)$ is sc for any positive measure μ on R.*

Proof. If R_0 is a locally compact metric contraction, then R_0 is a sum of mutually disjoint separable locally compact metric spaces $\{S_j\}_{j \in J}$ which are both open and closed in R_0. Let φ be the one-to-one continuous mapping from R onto R_0, and let R_j be the inverse $\varphi^{-1}(S_j)$ for any j. Then $R = \bigcup_j R_j$ and R_j are mutually disjoint and are both open and closed in R. The mapping φ from R_j onto S_j is continuous and one-to-one. Therefore the lemma follows immediately from the next Lemma 8 and D_4.

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10) A Hausdorff space is stonian if it is compact and if $\bar{U}$ is open for any open set U.

11) Cf. for example, [10].

Lemma 8. *Let R be a locally compact space and let R have a locally compact separable metric contraction. Then $L_1(R, \mu)$ is sc for any positive measure on R.*

Proof. Let R_0 be a locally compact separable metric contraction of R and let $\mathfrak{K}(R)$ and $\mathfrak{K}(R_0)$ be the sets of all continuous functions with compact carriers of R and R_0 respectively. The norm $\|f\|$ of f in $\mathfrak{K}(R)$ (or $\mathfrak{K}(R_0)$) is $\sup_{x\in R}|f(x)|$ (or $\sup_{x\in R_0}|f(x)|$). In order to prove the lemma, we have only to prove that $L_1(R, \mu)$ is isomorphic to a subspace of $\mathfrak{K}(R_0)^*$[12], since $\mathfrak{K}(R_0)$ is separable (cf. Klee [11]). Let φ be the one-to-one continuous mapping from R onto R_0. For any f in $L_1(R, \mu)$ and for any g in $\mathfrak{K}(R_0)$, we put

$$T_f(g) = \int_R f(x)g(\varphi x)d\mu(x).$$

We are here to prove that $\|T_f\| = \sup_{\substack{g\in\mathfrak{K}(R_0)\\ g\neq 0}} \frac{|T_f(g)|}{\|g\|} = \|f\|_1$. It is clear that $\|T_f\| \leqslant \|f\|_1$. Therefore we shall prove that $\|T_f\| \geqq \|f\|_1$. For any positive number ε, there is an h in $\mathfrak{K}(R)$ such that $\int_R |f(x)-h(x)|d\mu(x) < \varepsilon/4$. We may assume that h is not identically zero. Put $U_0 = \{x \mid h(x) \neq 0\}$, $F_n = \left\{x \mid h(x) \geqq \frac{1}{n}\right\}$ and $K_n = \left\{x \mid h(x) \leqq -\frac{1}{n}\right\}$ for any natural number n. Then $U_0 = \bigcup_{n=1}^{\infty}(F_n \cup K_n)$ and therefore $\mu(U_0 - (F_n \cup K_n)) \leqslant \varepsilon/4\|h\|_\infty$ for some n. Since F_n and K_n are compact on the topology of R, φF_n and φK_n are also compact in R_0. Therefore there exists a g in $\mathfrak{K}(R_0)$ such that $g(\varphi F_n) = 1$, $g(\varphi K_n) = -1$ and $-1 \leqq g(y) \leqq 1$ for any y in R_0. For this g,

$$\begin{aligned}\left| T_f(g) - \int_R |h(x)|d\mu(x)\right| &\leqslant \left|\int_R f(x)g(\varphi x)d\mu(x) - \int_R h(x)g(\varphi x)d\mu(x)\right| \\ &+ \left|\int_R |h(x)|d\mu(x) - \int_R h(x)g(\varphi x)d\mu(x)\right| < \frac{\varepsilon}{4} \\ &+ 2\int_{U_0-(F_n\cup G_n)} |h(x)|d\mu(x) < \frac{\varepsilon}{4} + 2\|h\|_\infty \cdot \varepsilon/4\|h\|_\infty = \frac{3}{4}\varepsilon.\end{aligned}$$

Therefore $|T_f(g)| > \int_R |h(x)|d\mu(x) - \frac{3}{4}\varepsilon \geqq \int_R |f(x)|d\mu(x) - \varepsilon = \|f\|_1 - \varepsilon$. Since $\|g\|_\infty = 1$, $\|T_f\| \geqq \|f\|_1$.

Proof of Theorem 5. Since R is weakly metric and locally compact, we can assume, in the proof of Lemma 6, that $V_1(x)$ is relatively compact for any $x \in R$. In the proof of Theorem 3, we easily see that R

12) For any normed space E, E^* denotes the conjugate space of E.

has a locally compact metric contraction R_0. Then by Lemma 7 the theorem is clear.

Theorem 7. *If an L_1-space E is lattice-isomophic and isometric to a conjugate space of an AM space and if E is sm, then E is lattice-isomorphic and isometric to l_1.*

Proof. If an AL space E with an F-unit[11] is lattice-isomorphic and isometric to a conjugate space of an AM space, then E is lattice-isomorphic and isometric to l_1 (cf. [16]). The theorem is clear since if E is *sm*, then E has an F-unit (cf. [6]).

§4. scm spaces.

Day [6] has proved that if a normed space E is separable, then it is *scm*. He raised the following question: Is there a nonreflexive, nonseparable *scm* space? We give an example of nonreflexive, nonseparable *scm* spaces. If E and F are two normed spaces, we mean by the l_p product of E and F a normed space of all pairs $z=(x, y)$, $x \in E$, $y \in F$ with the norm $(\|x\|^p+\|y\|^p)^{1/p}$, $(p>1)$.

We first prove the following.

Theorem 8. *If E is a separable normed space, then the l_p product of E and $l_p(I)$, $E \overset{l_p}{\times} l_p(I)$, is scm $(p>1)$.*

Proof. Since E is separable, there exists a one-to-one linear continuous mapping V from E into l_p. V is as follows: let $\{f_j\}$ be a bounded sequence of elements of E^* total over E. Then $V(x)=\{f_j(x)/2^j\}$ for any $x \in E$. For any $z=(x, y) \in M=E \overset{l_p}{\times} l_p(I)$, we put

$$W(z)=(V(x), y)$$

$(V(x), y)$ is in $l_p \overset{l_p}{\times} l_p(I)$. Since $l_p \overset{l_p}{\times} l_p(I)$ is isomorphic to an $l_p(J)$ for an index set J, W is a one-to-one linear continuous mapping from M into $l_p(J)$. By D_5 M is *sm*. Since $l_p(J)$ is *scm*, by D_2, M is *scm*.

Example. Let E be the space c_0, that is, the set of all sequences of real numbers which converge to zero. Then E is separable and nonreflexive. If the index set I is uncountable, $l_p(I)$ is nonseparable. Therefore, if $p>1$ and if I is uncountable, $M=c_0 \overset{l_p}{\times} l_p(I)$ is an example of a nonreflexive, nonseparable *scm* space. For, M is nonreflexive since any closed subspace of a reflexive space is also reflexive (cf. Pettis [12]), and is *scm* by Theorem 8.

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Added in proof: The question (1) is already solved (cf. M. M. Day: Every L-space is isomorphic to a strictly convex space, Proc. Amer. Math. Soc. **8**, 415–417 (1957).