Koiso, M.
Osaka J. Math.
**20** (1983), 177-183

# ON THE FINITE SOLVABILITY OF PLATEAU'S PROBLEM FOR EXTREME CURVES

MIYUKI KOISO

(Received January 22, 1981)

## 0. Introduction

For a given Jordan curve in Euclidean 3-space the existence of a solution of Plateau's problem is established with the help of well known classical methods (Douglas [3] and Radó [13]). But these methods yield no information about the number of possible solutions. Indeed it is one of the most vexing problems in connection with the classical Plateau's problem which remains unanswered today to investigate the phenomena of uniqueness and non-uniqueness as well as the isolation character of minimal surfaces. Our purpose in this paper is concerned with this problem.

Let $D$ denote the unit open disc in the $w=x+iy$-plane $(x, y \in R)$ and let $\gamma: \partial D \rightarrow R^3$ be a Jordan curve. A minimal surface $f$ spanned by the contour $\gamma$ is a map of class $C^0(\bar{D}, R^3) \cap C^2(D, R^3)$ satisfying the differential equations

$$(1) \qquad f_x \cdot f_y = 0\,, \qquad |f_x| = |f_y|$$

$$(2) \qquad \Delta f = 0$$

in $D$ together with the boundary condition

$$f \,|\, \partial D = \gamma \circ \tau$$

with some topological mapping $\tau: \partial D \rightarrow \partial D$ and the so-called "three point condition"

$$f\left(\exp\left(\frac{2}{3}k\pi i\right)\right) = \gamma\left(\exp\left(\frac{2}{3}k\pi i\right)\right), \qquad k = 1, 2, 3\,,$$

whereby no restriction in generality is imposed. We denote by $M(\gamma)$ the set of all minimal surfaces spanned by $\gamma$.

Using the complex derivative $\Phi := f_w = f_x - if_y$ of $f$, (1) and (2) can be expressed in the equivalent form

$$(1') \qquad \Phi \cdot \Phi = \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$$

and

$$(2') \qquad \Phi_{\bar{w}} = 0\,,$$

where $\Phi_1$, $\Phi_2$, and $\Phi_3$ are the components of the complex vector $\Phi$. A branch point of order $k$ of $f$ is a point $w \in \bar{D}$ where $\Phi$ vanishes in order $k$. In particular, we call a branch point of order 1 a simple branch point, also. Since $\Phi$ is holomorphic, it is clear that interior branch points $w \in D$ are isolated.

It is unknown in general, whether every solution of Plateau's problem, stable or unstable, is isolated (in a suitable topology). When this is the case, only a finite number of disc-type minimal surfaces can be spanned in a given contour. But the finite solvability of Plateau's problem is known only for certain special classes of curves. If a Jordan curve $\gamma$ satisfies one of the following conditions $1° \sim 4°$, then the solution of Plateau's problem is unique.

1° $\gamma$ admits a one-to-one parallel or central convex projection onto some plane (Radó [12]).

2° $\gamma$ is a regular analytic curve of total curvature not exceeding $4\pi$ (Nitsche [7]).

3° $\gamma$ is a $C^{3+\alpha}$-curve $(0<\alpha<1)$ with total curvature less than $4\pi$ (Gulliver and Spruck [5]).

4° $\gamma$ is a simple closed polygon of total curvature less than $4\pi$ (Nitsche [9]).

And when $\gamma$ has the following property 5°, $\gamma$ bounds only finitely many minimal surfaces (Nitsche [8]).

5° $\gamma$ is a Jordan curve of class $C^{3+\alpha}$ $(0<\alpha<1)$ whose total curvature does not exceed the value $6\pi$, and every minimal surface spanned by $\gamma$ is free of branch points in its interior and on its boundary.

To the author's knowledge, the just mentioned cases are the only ones where an explicit estimate of the number of solutions is known.

Recently F. Tomi derived another type of finiteness result. Let $\gamma$ be a regular curve of class $C^{4+\alpha}$, $0<\alpha<1$. If every surface in $M(\gamma)$ is free of boundary branch points and has at most simple branch points in its interior, then he shows the finiteness of a certain subset of $M(\gamma)$ which includes all minimal surfaces of locally minimal area (Tomi [14]). In this paper we will prove the following result using Tomi's method.

**Theorem 1′.** *A regular extreme Jordan curve of class $C^{4+\alpha}$, $0<\alpha<1$, can bound only finitely many embedded minimal surfaces of locally minimal area with respect to the $C^0$-topology.*

Here, after Gulliver and Spruck [5], we mean that a Jordan curve $\gamma$ lies on the boundary of its convex hull by saying $\gamma$ is extreme. It should be noted that soap films correspond to embedded minimal surfaces of locally minimal area, although they are not necessarily of the type of the disc. This theorem will be deduced from a more general result, Theorem 1 below.

## 1. The precise formulation of the main theorem

In the following we shall work with the class of $C^{4+\alpha}$-curves, where $0<\alpha<1$. Every curve should be understood to be regular and of class $C^{4+\alpha}$. It has been proved by several authors that minimal surfaces have the same regularity properties as their bounding curves (see, for example, Nitsche [10]). In particular, if $\gamma$ is a $C^{4+\alpha}$-curve, $M(\gamma)$ is included in $C^{4+\alpha}(\bar{D}, R^3)$.

Whenever we say a surface $f: \bar{D}\to R^3$ is immersed, we mean $f$ is regular at every point of $\bar{D}$. As for minimal surfaces, branch points are the only possible singularities. Let $f: \bar{D}\to R^3$ be immersed. Then every surface $h$ in a sufficiently small neighborhood of $f$ and having the same boundary as $f$ can be reparametrized in the following form

$$F(f, u)=f+uN(f),$$

where $N(f)$ is the normal field of $f$ and $u$ some real function with zero boundary values (Böhme and Tomi [2]). According to Tomi [14], we shall call this special representation of $h$ to be *the normal representation of $h$ with respect to $f$.*

We denote by $J$ the open subset of $C^{4+\alpha}(\bar{D}, R^3)$ consisting of immersions, by $X$ the closed subspace of $C^{2+\alpha}(\bar{D}, R)$ of functions vanishing on $\partial D$, and by $\Omega$ the open set of all $(f, u)\in J\times X$ such that $F(f, u)=f+uN(f)$ is immersed. Let us introduce the area functional

$$A(f, u):=\int_D |F_x(f, u)\wedge F_y(f, u)|\,dxdy$$

where $\wedge$ is the vector product of $R^3$. Then $A$ is of class $C^\infty(\Omega, R)$. (As for the concept of differentiable mapping between Banach spaces the reader is referred to [4].) Define a continuous bilinear form $\beta(f)$ on $X\times X$ as follows:

$$\beta(f):=D_u^2A(f, 0).$$

Now we introduce certain important subsets of $M(\gamma)$. Let us denote by $M^*(\gamma)$ the set of all immersed surfaces in $M(\gamma)$ and by $E(\gamma)$ the set of all embedded surfaces in $M(\gamma)$, where we mean that $f\in M(\gamma)$ is immersed and one-to-one on $\bar{D}$ by saying $f$ is embedded. And furthermore, we define the subset $E_+(\gamma)$ of $E(\gamma)$ as

$$E_+(\gamma):=\{f\in E(\gamma)\,|\,\beta(f)\geq 0\}.$$

Then we see clearly the inclusions

$$M(\gamma)\supseteq M^*(\gamma)\supseteq E(\gamma)\supseteq E_+(\gamma)\neq\phi$$

hold. (As for the inequality, see Almgren and Simon [1], Meeks and Yau [6], Tomi and Tromba [15].) Our aim in this paper is to prove the finiteness of $E_{+}(\gamma)$ when $\gamma$ is extreme. Now we state our main theorem precisely:

**Theorem 1.** *Let $\gamma$ be a regular extreme curve of class $C^{4+\alpha}$, $0<\alpha<1$. Then $E_{+}(\gamma)$ is a finite set.*

## 2. The compactness of $E_{+}(\gamma)$ and a preliminary result

According to the regularity theory of minimal surfaces, there exists a constant $C$ depending only on $\gamma$ such that

$$\|f\|_{C^{4+\alpha}} \leq C, \qquad f \in M(\gamma) \tag{3}$$

when $\gamma$ is of class $C^{4+\alpha}$. So the compactness of the whole set $M(\gamma)$ with respect to every $C^{4+\delta}$-topology $(0<\delta<\alpha)$ is secured. And it follows that the all $C^{k+\beta}$-topologies $(k \in N \cup \{0\},\ 0 \leq \beta < 1)$, $0 \leq k+\beta < 4+\alpha$, coinside in $M(\gamma)$ (Böhme and Tomi [2, Lemma 2.1]). Therefore we may use any of these $C^{k+\beta}$-topologies in topological considerations about $M(\gamma)$, henceforth.

On the other hand, $E(\gamma)$ is closed in $M(\gamma)$, provided $\gamma$ is extreme. Indeed let us assume that there exists a sequence of minimal surfaces $f_n \in E(\gamma)$ converging to some $f$ in $C^0$. Then $f$ is included in $M(\gamma)$ and is one-to-one on $\bar{D}$ and has no branch points in $D$ (Gulliver and Spruck [5, Theorem 1.1]). On the other hand, it is known that every minimal surface spanned by an extreme curve has no branch points on its boundary (Nitsche [10, §366]). Consequently, $f$ is embedded, which means that $E(\gamma)$ is closed. So $E(\gamma)$ is compact, and $E_{+}(\gamma)$, which is a closed subset of $E(\gamma)$, is also compact.

The following lemma will be of some interest for itself.

**Lemma 1.** *Let $\gamma$ be an extreme curve of class $C^{4+\alpha}$ and let $C$ a connected component of $M^*(\gamma)$. Then, either $C$ is included in $E(\gamma)$ or does not intersect $E(\gamma)$.*

Proof. Let us consider $E(\gamma) \cap C$. From the above remarks we know $E(\gamma)$ is closed and open (Gulliver and Spruck [5, Lemma 6.1]) in $M(\gamma)$. So $E(\gamma) \cap C$ is closed and open in $C$. On the other hand, $C$ is connected by the assumption, therefore $E(\gamma) \cap C = C$ or $E(\gamma) \cap C = \phi$. This proves the lemma.

## 3. The finiteness result

In this section we shall consider only $E_{+}(\gamma)$ defined in section 1, which is a closed subset of $M(\gamma)$ as we observed in section 2. When $\gamma$ is extreme, we shall prove the isolatedness of all surfaces in $E_{+}(\gamma)$, and then finiteness of $E_{+}(\gamma)$.

**Proposition 1.** *Let $\gamma$ be an extreme Jordan curve of class $C^{4+\alpha}$. Then all surfaces in $E_{+}(\gamma)$ are isolated points of $M(\gamma)$.*

Proof. Let us assume that there exists some $f \in E_{+}(\gamma)$ which is not isolated. Then clearly $f$ is included in $M^*(\gamma)$. Since $M^*(\gamma)$ is open in $M(\gamma)$, $f$ is not an isolated point of $M^*(\gamma)$. Furthermore the form $\beta(f)$ is non-negative by the definition of $E_+(\gamma)$. So we see that there exists a $C^{2+\alpha}$-neighborhood $U$ of $f$ such that $U\cap M^*(\gamma)$, in normal representation with respect to $f$, is a one-dimensional analytic manifold (Tomi [14, Corollary 1]). From this fact it follows that the connected component $C$ of $f$ in $M^*(\gamma)$ contains more than one point. On the other hand, we know that $C$ is contained in $E(\gamma)$ by Lemma 1. And we can see that $C$ is closed in $E(\gamma)$. Indeed, since $C$ is connected, the closure of $C$, which we denote by $\bar{C}$, is also connected. By virtue of the assumption that $C$ is a connected component, $\bar{C}$ must coincide with $C$. This means that $C$ is closed in $E(\gamma)$. Since $E(\gamma)$ is a closed subset of $M(\gamma)$ as we obtained in section 2, $C$ is also closed in $M(\gamma)$, hence $C$ is compact, for $M(\gamma)$ is compact. Consequently, it follows owing to the non-negativity of $\beta(f)$ that $C$ is an isolated point (Tomi [14, Theorem 1]), which is a contradiction.

From the compactness of $E_+(\gamma)$ and Proposition 1 we immediately obtain our Theorem 1 which was mentioned in section 1.

Furthermore, we can show the local boundedness of $\#(E_+(\gamma))$. Let us denote by $\Gamma$ the set of all regular Jordan curves of class $C^{4+\alpha}$ and by $\Gamma_e$ the set of all extreme curves contained in $\Gamma$.

**Lemma 2.** *Let $\gamma_0$ be a regular extreme curve of class $C^{4+\alpha}$. Then, for every $C^{4+\delta}$-neighborhood $U$ of $E_+(\gamma_0)$, $0<\delta<\alpha$, there exists a $C^{4+\alpha}$-neighborhood $V$ of $\gamma_0$ such that $E_+(\gamma)\subseteq U$ for all $\gamma\in V$.*

Proof. Let us assume for a given $U$ the corresponding $V$ does not exist. Then there exist a sequence of curves $\gamma_n\in\Gamma$, $\gamma_n\to\gamma_0$ $(n\to\infty)$ and a sequence of minimal surfaces $f_n\in E_+(\gamma_n)$, $f_n\notin U$. Since in the estimate (3) the right hand side is locally bounded on $\Gamma$, the sequence $\{f_n\}$ is bounded in $C^{4+\alpha}$-norm. Therefore, replacing $\{f_n\}$ by a suitable subsequence if it is necessary, we may assume that $\{f_n\}$ converges in $C^{4+\delta}$ to some $f_0\in M(\gamma_0)$. Since $\gamma_0$ is extreme, we know $f_0$ is embedded (Gulliver and Spruck [5, Theorem 1.1]). And by continuity $\beta(f_0)\geq 0$. Consequently, $f_0\in E_+(\gamma_0)$, which is a contradiction.

Using this lemma and the method of Tomi [14], we can obtain the following fact:

**Theorem 2.** *The number of elements of $E_+(\gamma)$ is locally bounded on $\Gamma_e$.*

## 4. Concluding remarks

Tomi [14] calls a curve $\gamma\in\Gamma$ to be *proper* if every minimal surface in $M(\gamma)$ is free of boundary branch points and has at most simple branch points in its interior. Let us denote by $\Gamma_p$ the set of all proper curves. And let us define a certain subset of $M^*(\gamma)$ as follows:

$$M_+^*(\gamma):=\{f\in M^*(\gamma)\,|\,\beta(f)\geq 0\}\ .$$

Then, Tomi [14] proved that the number of $M_+^*(\gamma)$ is locally bounded on $\Gamma_p$.

If $\gamma\in\Gamma$, then

$$M_+^*(\gamma)\supseteq E_+(\gamma)$$

is valid. Now we compare $\Gamma_p$ with $\Gamma_e$. The definition of proper curves is not geometric, and it is not trivial whether or not there exist proper curves which don't satisfy any conditions $1°\sim 5°$ in the introduction. So Tomi gives a special geometric situation which implies that the curve considered is proper: Namely, let a regular Jordan curve $\gamma$ of class $C^{4+\alpha}$ satisfy the following condition

(♯) $\gamma$ satisfies an interior 4-point Radó condition and a boundary 0-point Radó condition simultaneously.

Then $\gamma$ is a proper curve. By virtue of this geometric sufficient condition, we know the existence of proper curves satisfying none of the conditions $1°\sim 5°$. Therefore Tomi's result is significant. On the other hand, we can easily see that for every Jordan curve extremeness is equivalent to the boundary 0-point Radó condition. Consequently every regular Jordan curve of class $C^{4+\alpha}$ which satisfies the condition (♯) is contained in $\Gamma_e$. It is evident that there exist many Jordan curves in $\Gamma_e$ which don't satisfy (♯). So it is meaningful to prove our Theorem 1 and Theorem 2.

---

## References


[1] F.J. Almgren, Jr. and L. Simon: *Existence of embedded solutions of Plateau's problem,* Ann. Scuola Norm. Sup. Pisa (4) **6** (1979), 447–495.

[2] R. Böhme and F. Tomi: *Zur Struktur der Lösungsmenge des Plateauproblems,* Math. Z. **133** (1973), 1–29.

[3] J. Douglas: *Solution of the problem of Plateau,* Trans Amer. Math. Soc. **33** (1931), 263–321.

[4] S. Fucik, J. Necas, J. Soucek, and V. Soucek: Special analysis of nonlinearoperators, Lecture Notes in Mathematics 346, Springer, Berlin-Heidelberg-New York, 1973.

[5] R. Gulliver and J. Spruck: *On embedded minimal surfaces,* Ann. of Math. **103** (1976), 331–347.

[6] W.H. Meeks, III and S.-T. Yau: *The classical Plateau problem and the topology of three-dimensional manifolds,* Arch. Rational Mech. Anal. (to appear).

[7] J.C.C. Nitsche: *A new uniqueness theorem for minimal surfaces,* Arch. Rational Mech. Anal. **52** (1973), 319–329.

[8] J.C.C. Nitsche: *Contours bounding at most finitely many solutions of Plateau's*

*problem,* Complex analysis and its applications (Russian), Nauka, Moscow, 1978, 438–446.

[9] J.C.C. Nitsche: *Uniqueness and non-uniqueness for Plateau's problem — one of the last major questions,* Minimal Submanifolds and Geodesics, Kaigai Publications, Tokyo, 1978, 143–161.

[10] J.C.C. Nitsche: Vorlesungen über Minimalflächen, Springer, Berlin-Heidelberg-New York, 1975.

[11] R. Osserman: *A proof of the regularity everywhere of the classical solution to Plateau's problem,* Ann. of Math. **91** (1970), 550–569.

[12] T. Radó: On the problem of Plateau, Julius Springer, Berlin, 1933.

[13] T. Radó: *The problem of the least area and the problem of Plateau,* Math. Z. **32** (1930), 763–796.

[14] F. Tomi: *On the finite solvability of Plateau's problem,* Springer Lect. Notes Math. 597 (1977), 679–695.

[15] F. Tomi and A. Tromba: *Extreme curves bound embedded minimal surfaces of the type of the disc,* Math. Z. **158** (1978), 137–145.

[16] M. Beeson: *Some results on finiteness in Plateau's problem,* part I, Math. Z. **175** (1980), 103–123.

After the author had completed the manuscript of this paper, a recent work [16] by M. Beeson drew her attention.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan