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SAMPLE PATH PROPERTIES OF ERGODIC SELF-SIMILAR PROCESSES

Dedicated to Professor Nobuyuki Ikeda on his 60th birthday

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1. Introduction

In this paper we shall study some sample path properties of self-similar processes with ergodic scaling transformations, in particular, a class of stable self-similar processes which includes the fractional stable processes. A large number of papers on sample path properties have been devoted to Gaussian processes and Lévy processes, i.e. stochastic processes with independent, stationary increments. In case of the Brownian motion, especially, we have Kolmogorov's test as a refinement of the law of the iterated logarithm and Chung-Erdös-Sirao's test (cf. [4]) as a refinement of Lévy's modulus of continuity.

We shall show some zero-one laws on sample path properties for general self-similar processes with ergodic scaling transformations in Sections 2 and 5. In Sections 3 and 4, we shall be concerned with a class of stable self-similar processes having stationary increments. We shall give integral tests for upper and lower functions with respect to the local growth of sample paths, which correspond to Kolmogorov's test and also to the results of Khinchin [15] for strictly stable processes. With respect to the uniform growth, in case of fractional stable processes with continuous sample paths, we shall give criteria for upper and lower functions. Furthermore, we shall show the existence of function which is neither an upper nor a lower function. This fact sharply contrasts with the Brownian motion case (cf. [4]).

Various sample path properties of self-similar processes with ergodic scaling transformations can be shown to hold with probability zero or one. Among such properties, we shall study growth properties in Section 2 and Hausdorff measure properties in Section 5. The results in Section 2 enable us to prove the above mentioned results in Section 3 by using an extension of Borel-Cantelli's lemma given in [16] rather than that of [3]. These zero-one laws on

sample path properties also have their own interest and their original version can be found in Lévy [21] (cf. also Taylor [35]), where a Hausdorff measure property of range of Brownian paths was treated.

In case of the Brownian motion, the law of the iterated logarithm means that the exponent of local growth order of sample paths is equal to 1/2 with probability one, and Lévy's modulus of continuity implies that the exponent of uniform growth order of sample paths is equal to 1/2 with probability one. The Brownian motion is, of course, a self-similar process with exponent 1/2. For the fractional Brownian motion, these three exponents are known to be equal to one another. Thus, there naturally arises the following question: do the above three exponents still coincide in case of non-Gaussian self-similar processes with dependent increments and having continuous sample paths?

In Section 2, it will be shown that the above three exponents are equal to one another for a self-similar process X with stationary increments if the tail probability of marginal distribution of X decays in an exponential order. By this fact, an affirmative answer to the above question will be given for self-similar processes represented by multiple Wiener (or Wiener-Itô) integrals (cf. Section 2, Example 2). In contrast with this, Theorem 3.4 will give a negative answer to the question for (α, β)-fractional stable processes with $\beta>0$: the exponent of uniform growth order is equal to β and strictly less than the exponent, $1/\alpha+\beta$, of self-similarity, while the exponent of local growth order is still equal to $1/\alpha+\beta$.

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2. Preliminaries and growth properties

In this section we shall first make some preparations for notions and notations on stochastic processes, and then, we shall show some zero-one laws on the local and uniform growth of sample paths for self-similar processes with ergodic scaling transformations.

Stochastic processes considered in this paper will be assumed to be real-valued and continuous in probability. Thus, we can take a separable version of such process without loss of generality. Moreover, whenever the process allows a version in $D([0, \infty)\to R)$ or $C([0, \infty)\to R)$, we shall take this version. For a stochastic process $X=\{X(t): t\geq 0\}$ and for $\kappa>0$, $a>0$, *a scaling transformation* $S_{\kappa,a}$ of X is defined by

$$(S_{\kappa,a}X)(t) = a^{-\kappa}X(at)\,, \quad t\geq 0\,.$$

DEFINITION 2.1. A stochastic process $X=\{X(t)\colon t\geq 0\}$ is called a *self-similar process with exponent* κ, $\kappa>0$ (shortly *κ-self-similar process*), if for any $a>0$, the process $\{(S_{\kappa,a}X)(t)\colon t\geq 0\}$ has the same distribution as that of X.

Throughout this paper, X will denote a self-similar process and κ will denote the exponent of X. Since, in this paper, scaling transformations will be always considered for self-similar processes, we shall write simply S_a for $S_{\kappa,a}$. By Definition 2.1, a scaling transformation S_a of X clearly preserves the distribution of X, and so the notion of ergodicity or mixing of S_a can be defined in the usual way (cf. [5]). From this point of view, we shall call X a self-similar process with *ergodic* (or *strong mixing* resp.) scaling transformations if S_a, $a>0$, $\neq 1$, is ergodic (or strong mixing resp.). For any fixed $\kappa>0$, κ-self-similar processes are characterized as follows (cf. Lamperti [19]): let $\mathcal{X}_\kappa$ be the space of κ-self-similar processes X with $X(0)=0$ *a.s.*, and $\mathcal{Y}$ be the space of strictly stationary processes $Y=\{Y(t)\colon -\infty<t<\infty\}$. Then, there is a bijective mapping $\tau_\kappa\colon \mathcal{X}_\kappa\to\mathcal{Y}$, defined by

$$\tau_\kappa(X)(t) \quad (=Y(t)) = e^{-\kappa t}X(e^t),\ -\infty<t<\infty\,.$$

A scaling transformation S_a of X corresponds to a shift transformation θ_u of Y, i.e. $\tau_\kappa\circ S_a=\theta_u\circ\tau_\kappa$, where $u=\log a$ and θ_u is defined by

$$(\theta_u Y)(t) = Y(t+u)\,, \quad -\infty<t<\infty\,.$$

Furthermore, S_a is ergodic (or strong mixing resp.) if θ_u is ergodic (or strong mixing resp.).

We shall assume, in this paper, the following:

HYPOTHESES. $X=\{X(t)\colon t\geq 0\}$ is a self-similar process with exponent $\kappa>0$, and $X(0)=0$ *a.s.*, which is separable and continuous in probability. Any scaling transformation S_a, $a>0$, $\neq 1$, is ergodic.

Next, we shall state zero-one laws on growth properties of sample paths of self-similar processes with ergodic scaling transformations. For this aim, we prepare some notions and notations on growth order properties. For a positive function g, consider the following events

$$E_g = [\text{there is } \delta>0 \text{ such that } |X(t)|\leq g(t) \quad \text{for } 0<t<\delta]\,,$$
$$E_g^\infty = [\text{there is } N>0 \text{ such that } |X(t)|\leq g(t) \quad \text{for } t>N]\,,$$
$$F_g^I = \left[\begin{array}{l}\text{there is } \delta>0 \text{ such that}\\ |X(t)-X(s)|\leq g(|t-s|) \quad \text{for } s,t\in I, |t-s|<\delta\end{array}\right],$$

where I is an interval of $[0,\infty)$. In case $I=[0,1]$, we shall write shortly F_g for

F_g^I.

DEFINITION 2.2. (i) A positive function g is called an *upper function* (or a *lower function* resp.) *with respect to the local growth at* 0, if $P(E_g)=1$ (or 0 resp.).
(ii) g is called an *upper function* (or a *lower function* resp.) *with respect to the local growth at* ∞, if $P(E_g^\infty)=1$ (or 0 resp.).
(iii) g is called an *upper function* (or a *lower function* resp,) *with respect to the uniform growth,* if $P(F_g)=1$ (or 0 resp.).

We shall denote the space of upper functions with respect to the local growth at 0 or ∞, or with respect to the uniform growth by $\mathcal{U}_l$, $\mathcal{U}_l^\infty$, or $\mathcal{U}_u$ resp., and also denote the space of lower functions with respect to the local growth at 0 or ∞, or with respect to the uniform growth by $\mathcal{L}_l$, $\mathcal{L}_l^\infty$, or $\mathcal{L}_u$ resp.

In addition, define the following functionals for $\lambda>0$ and a positive function ϕ:

$$L_{\lambda,\phi} = \limsup_{t\downarrow 0} \frac{|X(t)|}{t^\lambda \phi(t)}$$

$$L_{\lambda,\phi}^\infty = \limsup_{t\uparrow\infty} \frac{|X(t)|}{t^\lambda \phi(t)}$$

$$U_{\lambda,\phi}^I = \lim_{h\downarrow 0} \sup_{s,t\in I,|t-s|\leq h} \frac{|X(t)-X(s)|}{|t-s|^\lambda \phi(|t-s|)}$$

where I is an interval. We shall write simply $U_{\lambda,\phi}$ for $U_{\lambda,\phi}^{[0,1]}$. Note that $t^\lambda\phi(t)\in\mathcal{U}_l$ (or $\mathcal{U}_l^\infty$, $\mathcal{U}_u$ resp.) if $L_{\lambda,\phi}=0$ (or $L_{\lambda,\phi}^\infty=0$, $U_{\lambda,\phi}=0$ resp.) *a.s.*, and that $t^\lambda\phi(t)\in\mathcal{L}_l$ (or $\mathcal{L}_l^\infty$, $\mathcal{L}_u$ resp.), if $L_{\lambda,\phi}=\infty$ (or $L_{\lambda,\phi}^\infty=\infty$, $U_{\lambda,\phi}=\infty$ resp.) *a.s.*

In this and the next sections, we shall sometimes assume the stationarity of increments of processes: a stochastic process X is said to have *stationary increments* if for any $t_0\geq 0$, the process $\{X(t+t_0)-X(t_0): t\geq 0\}$ has the same distribution as that of X.

We shall now state our results on growth properties of sample paths of self-similar processes with ergodic scaling transformations, first, the following zero-one law on the local growth.

Proposition 2.1. *For a positive monotone function* ϕ,

$$P(E_{\kappa,\phi}) = 0 \text{ or } 1 \quad \text{and} \quad P(E_{\kappa,\phi}^\infty) = 0 \text{ or } 1\,,$$

where $E_{\kappa,\phi}$ *(or* $E_{\kappa,\phi}^\infty$ *resp.) denotes* E_g *(or* E_g^∞ *resp.) with* $g(t)=t^\kappa\phi(t)$.

Proof. First, we prove that $P(E_{\kappa,\phi})=0$ or 1.
Case (i) For some $a>0$, assume that $\phi(u/a)\geq\phi(u)$, $u>0$. Put $S=S_a$. If $|X(t)|\leq t^\kappa\phi(t)$, $0<t<\delta$, for some $\delta>0$, this implies that $|X(t)|\leq t^\kappa\phi(t/a)$. Put $u=t/a$. Then,

$$|(SX)(u)| = |X(au)|/a^{\kappa} \leq u^{\kappa}\phi(u), \qquad \text{for} \quad 0<u<\delta/a .$$

This means that $E_{\kappa,\phi} \subset S^{-1}E_{\kappa,\phi}$, and $P(E_{\kappa,\phi} \Delta S^{-1}E_{\kappa,\phi})=0$. Since S is ergodic, we have $P(E_{\kappa,\phi})=0$ or 1 (cf [5]).
Case (ii) For $a>0$, assume that $\phi(u/a) \leq \phi(u)$, $u>0$. If $|(SX)(t)| \leq t^{\kappa}\phi(t)$, $0<t<\delta'$, for some $\delta'>0$, this implies that

$$|X(at)| \leq (at)^{\kappa}\phi(at/a) \leq (at)^{\kappa}\phi(at) .$$

Put $u=at$. Then, $|X(u)| \leq u^{\kappa}\phi(u)$ for $0<u<a\delta'$. This means that $S^{-1}E_{\kappa,\phi} \subset E_{\kappa,\phi}$. Thus again, $P(E_{\kappa,\phi})=0$ or 1.

The proof for the second assertion goes similarly with slight modifications, such as letting N take the place of δ and the phrase "$t>N$" take the place of "$0<t<\delta$", *etc.*, and so we omit its details. *q.e.d.*

Corollary. *For a positive, monotone function ϕ, there exists a constant $c_{\kappa,\phi}$, $0 \leq c_{\kappa,\phi} \leq \infty$ (or $c^{\infty}_{\kappa,\phi}$, $0 \leq c^{\infty}_{\kappa,\phi} \leq \infty$ resp.) such that*

$$L_{\kappa,\phi} = c_{\kappa,\phi} \ (\textit{or } L^{\infty}_{\kappa,\phi} = c^{\infty}_{\kappa,\phi} \textit{ resp.}) \quad a.s.$$

Especially, for $\lambda, \neq \kappa$, and for ϕ, slowly varying at 0 (or ∞ resp.),

$$L_{\lambda,\phi} \ (\textit{or } L^{\infty}_{\lambda,\phi} \textit{ resp.}) = 0 \ a.s. \quad \textit{or} \quad \infty \ a.s.$$

Proof. Put $c_{\kappa,\phi}=\sup\{c \geq 0: P(E_{\kappa,c\phi})=0\}$. Then, it is easily verified that $L_{\kappa,\phi}=c_{\kappa,\phi}$ *a.s.* In case $\lambda \neq \kappa$, put $\psi(t)=t^{\lambda-\kappa}\phi(t)$. Although ψ is not monotone in general, it is easily shown, for example, that for some positive a, $\psi(t/a) \geq \psi(t)$ on some neighborhood of 0, if ϕ is slowly varying at 0. This is sufficient for the proof of Proposition 2.1 for $E_{\kappa,\psi}$. Thus, the last assertion can be derived from the fact that the event $[L_{\kappa,\psi}=L_{\lambda,\phi}=c_{\lambda,\phi}]$ has probability one and should be invariant with respect to any scaling transformation. The proof goes similarly for $L^{\infty}_{\kappa,\phi}$. *q.e.d.*

Next, we give a sufficient condition for the exponent of local growth order not less than κ.

Proposition 2.2. *Assume that X has stationary increments. If there exists $\gamma>0$ such that*

$$E[|X(1)|^{\gamma}]<\infty, \quad \textit{and} \quad \gamma\kappa>1 , \tag{2.1}$$

for any $\varepsilon>0$ and for a positive function ϕ, slowly varying at 0 (or ∞ resp.),

$$L_{\kappa-\varepsilon,\phi} \ (\textit{or } L^{\infty}_{\kappa+\varepsilon,\phi} \textit{ resp.}) = 0 \quad a.s.$$

Proof. By (2.1), for any s, t, $0 \leq s<t<\infty$,

$$E[|X(t)-X(s)|^{\gamma}] = E[|X(t-s)|^{\gamma}] = |t-s|^{\gamma\kappa} E[|X(1)|^{\gamma}]<\infty .$$

Because of $\gamma\kappa>1$, we can apply Theorem 1 of Móricz [28], and we obtain the following estimate for moments of $M=\max_{0\leq t\leq 1}|X(t)|$:

$$E[M^\gamma]\leq C_{\gamma\kappa}\, E[|X(1)|^\gamma]<\infty\ ,$$

where $C_{\gamma\kappa}$ is a positive constant depending only on $\gamma\kappa$. By scaling property,

$$P(|X(t)|\geq t^{\kappa-\varepsilon})=P(|X(1)|\geq t^{-\varepsilon})\leq t^{\varepsilon\gamma}\, E[|X(1)|^\gamma]$$

and

$$\begin{aligned}&P(\max_{2^{-n-1}\leq t\leq 2^{-n}}|X(t)-X(2^{-n-1})|\geq 2^{-n(\kappa-\varepsilon)})\\&\quad=P(\max_{0\leq t\leq 2^{-n}}|X(t)|\geq 2^{-n(\kappa-\varepsilon)})=P(M\geq 2^{n\varepsilon})\\&\quad\leq c\,2^{-\varepsilon\gamma n}\ ,\end{aligned}$$

where c is a positive constant independent of n. Using these estimates, we have

$$\begin{aligned}&\textstyle\sum_{n=1}^\infty P(\max_{2^{-n-1}\leq t\leq 2^{-n}}|X(t)|\geq 2^{-n(\kappa-\varepsilon)})\\&\quad\leq\textstyle\sum_{n=1}^\infty\{P(\max_{2^{-n-1}\leq t\leq 2^{-n}}|X(t)-X(2^{-n-1})|\geq 2^{-n(\kappa-\varepsilon)-1})\\&\qquad+P(|X(2^{-n-1})|\geq 2^{-n(\kappa-\varepsilon)-1})\}\\&\quad\leq c'\textstyle\sum_{n=1}^\infty 2^{-\varepsilon\gamma n}<\infty\ .\end{aligned}$$

By Borel-Cantelli's lemma, with probability one, there is a number n_0 such that

$$\max_{2^{-n-1}\leq t\leq 2^{-n}}|X(t)|\leq 2^{-n(\kappa-\varepsilon)}\qquad\text{for}\quad n\geq n_0\ .$$

This implies that

$$|X(t)|/t^{\kappa-\varepsilon}\leq 2^{\kappa-\varepsilon}\ ,\qquad\text{for}\quad 0<t\leq 2^{-n_0}\ .$$

By Corollary to Proposition 2.1, we have $L_{\kappa-\varepsilon,1}=0$ *a.s.*, where $L_{\lambda,1}$ denotes $L_{\lambda,\phi}$ with $\phi(t)\equiv 1$. Since ε is arbitrarily positive, this means that $L_{\kappa-\varepsilon,\phi}=0$ *a.s.*, for any slowly varying function ϕ.

For the second assertion, since we have $E[M^\gamma]<\infty$, the proof is completed by the result of Kôno [18]. q.e.d.

REMARK 2.1. Vervaat [36] proved that if X is *not degenerate*, i.e. $P(X(1)=0)=0$,

$$L_{\kappa,1}\ (\text{or } L_{\kappa,1}^\infty)=x_0\quad a.s.,$$

where

$$x_0=\sup\{x>0\colon P(|X(1)|\geq x)>0\}\ . \tag{2.2}$$

This implies that the exponent of local growth order of sample paths is not greater than κ. Thus, if X is not degenerated and satisfies the condition (2.1), the exponent of local growth order of sample paths of X is equal to κ. There

are two typical cases that the moment condition (2.1) is not fulfilled: the first is the case that X is a strictly stable process with exponent α. Although X does not satisfy (2.1), the exponent of the local growth of X is equal to $\kappa=1/\alpha$ (cf. [15]). On the other hand, when X is an (α, β)-fractional stable process with $0<\alpha<1$ and $-1/\alpha<\beta<1-1/\alpha$, no positive bounded function can be an upper function by the result of Maejima [23].

Next, we shall discuss the uniform growth of sample paths, and we give the following result for $U_{\kappa,\phi}$, which shows a difference between local growth properties and uniform growth ones.

Proposition 2.3. *Let ϕ be a positive function, slowly varying at* 0.
(i) *There exists a constant $c_{\kappa,\phi}$, $0\leq c_{\kappa,\phi}\leq\infty$, such that*

$$U_{\kappa,\phi}=c_{\kappa,\phi} \quad a.s.$$

(ii) *For any λ, $0<\lambda<\kappa$,*

$$P(U_{\lambda,\phi}=0),\ P(0<U_{\lambda,\phi}<\infty),\ P(U_{\lambda,\phi}=\infty)=0 \quad or \quad 1\,.$$

Moreover, assume that X has stationary increments. For any c, $0<c<\infty$,

$$P(0<U_{\lambda,\phi}\leq c),\ P(c\leq U_{\lambda,\phi}<\infty)<1\,.$$

Proof. Put $S=S_a$. First, note that for any λ,

$$U_{\lambda,\phi}^{[0,1]}\circ S=U_{\lambda,\phi}^{[0,a]} \quad a^{\lambda-\kappa},\ a>0\,, \tag{2.3}$$

where $U_{\lambda,\phi}^{[0,1]}\circ S$ denotes $U_{\lambda,\phi}^{[0,1]}$ for the process $\{(SX)(t)\colon t\geq 0\}$.
(i) By (2.3), for $a>1$ and $c>0$,

$$S^{-1}[U_{\kappa,\phi}^{[0,1]}\leq c]=[U_{\kappa,\phi}^{[0,a]}\leq c]\subset[U_{\kappa,\phi}^{[0,1]}\leq c]\,.$$

By the ergodicity of S, we have $P(U_{\kappa,\phi}\leq c)=0$ or 1. Put $c_{\kappa,\phi}=\sup\{c\geq 0\colon P(U_{\kappa,\phi}\leq c)=0\}$. Then,

$$U_{\kappa,\phi}=c_{\kappa,\phi} \quad a.s.$$

(ii) By (2.3), the events $[U_{\lambda,\phi}=0]$, $[0<U_{\lambda,\phi}<\infty]$, and $[U_{\lambda,\phi}=\infty]$ are invariant with respect to S. Thus, these events have probability zero or one.

Next, Assume that X has stationary increments and that there is $c>0$ such that $P(0<U_{\lambda,\phi}\leq c)=1$. Again by (2.3), we have

$$P(0<U_{\lambda,\phi}^{[0,a]}\leq c\ a^{\kappa-\lambda})=1\,, \qquad \text{for} \quad a>0\,.$$

On the other hand, because X has stationary increments, we have

$$P(0<U_{\lambda,\phi}^{[1-a,1]}\leq c\ a^{\kappa-\lambda})=1\,, \qquad \text{for} \quad 0<a<1\,.$$

Take a, $1/2<a<1$. Then, since

$$U_{\lambda,\phi} = \max\{U_{\lambda,\phi}^{[0,a]}, U_{\lambda,\phi}^{[1-a,1]}\},$$

we obtain

$$P(0<U_{\lambda,\phi}\leq c\, a^{\kappa-\lambda}) = 1.$$

By iterating this arguement, since $a<1$, we have

$$P(0<U_{\lambda,\phi}\leq 0) = 1.$$

This is a contradiction, and it is verified that

$$P(0<U_{\lambda,\phi}\leq c)<1, \qquad \text{for any} \quad c,\ 0<c<\infty.$$

The proof goes similarly for $P(c\leq U_{\lambda,\phi}<\infty)<1$. q.e.d.

REMARK 2.2. As is shown in Proposition 2.1 and its corollary, for any positive, monotone function ϕ, $t^{\kappa}\phi(t)$ belongs to either $\mathcal{U}_l$ or $\mathcal{L}_l$, and $L_{\kappa,\phi}$ is equal to a certain constant with probability one. In contrast with this, with respect to the uniform growth, Proposition 2.3, (ii), shows the possibility of existence of a positive function ψ, slowly varying at 0, and a positive number λ, $\lambda<\kappa$, such that

$$P(0<U_{\lambda,\psi}<\infty) = 1.$$

In this case, for any $c>0$, $ct^{\lambda}\psi(t)$ can be neither an upper nor a lower function with respect to the uniform growth, because the infimum of support of distribution of $U_{\lambda,\psi}$ is equal to 0 and its supremum is equal to ∞. In Section 3, we shall show that for an (α, β)-fractional stable process with $\beta>0$,

$$P(0<U_{\beta,1}<\infty) = 1,$$

and the support of distribution of $U_{\beta,1}$ is $(0, \infty)$. This means the impossibility of general zero-one law for upper and lower functions with respect to the uniform growth, analogous to Proposition 2.1.

REMARK 2.3. For $\lambda>\kappa$, it is verified analogously that $U_{\lambda,\phi}=0$ *a.s.* or ∞ *a.s.* From the following facts, it is clear that $U_{\lambda,\phi}=\infty$ *a.s*, for non-degenerate processes: assume that X has stationary increments. Then,

(i) if X is not degenerate, $U_{\kappa,1}=x_0$ *a.s.*, where x_0 is defined by (2.2).

(ii) If the tail probability of $|X(1)|$ decays in an exponential order, i.e. there exist positive constants C_1, C_2 and γ such that

$$P(|X(1)|\geq x)\leq C_1 \exp\{-C_2 x^{\gamma}\} \qquad \text{for large } x,$$

$U_{\kappa-\varepsilon,1}=0$ *a.s.*, for $\varepsilon>0$.

Proof. (i) (This proof is due to the referee.)

By the definition, $U_{\kappa,1} \geq L_{\kappa,1}$. Thus, it is clear that $U_{\kappa,1} \geq x_0$ *a.s.*, by Remark 2.1. In case $x_0 < \infty$, put $E_{s,t} = [\,|X(t) - X(s)|/|t-s|^{\kappa} > x_0 + \delta]$ for any positive δ. Then, since X is self-similar and has stationary increments, $P(E_{s,t}) = 0$, by the definition of x_0. Thus, the union of $E_{s,t}$ over rational s, t of $[0, 1]$, has probablity zero. This means that $U_{\kappa,1} \leq x_0$ *a.s.*

(ii) By Theorem 1.1 of Bernard [1], there is a positive constant M such that $M t^{\kappa-\varepsilon} \in \mathcal{U}_u$, if

$$\int_{0+} t^{-2} P(|X(1)| \geq t^{-\varepsilon})\, dt < \infty .$$

This is easily checked by the above condition. Thus, $U_{\kappa-\varepsilon,1} \leq M$ *a.s.* This implies that $U_{\kappa-\varepsilon,1} = 0$ *a.s.*, by Proposition 2.3. q.e.d.

In case X satisfies the conditions of (i) and (ii), $t^{\kappa+\varepsilon}$ belongs to $\mathcal{U}_u$ or $\mathcal{L}_u$, according as $\varepsilon < 0$ or > 0, i.e. the exponent of the uniform growth of sample path of X is equal to κ.

In the rest of this section, we shall make some remarks on ergodic properties of scaling transformations of self-similar processes. With respect to many known self-similar processes, scaling transformations can be shown to be ergodic. We shall here show that scaling transformations are strong mixing for several typical examples, other than stable self-similar processes, with respect to which we shall discuss in the next section.

EXAMPLE 1. Let X be a Gaussian process with mean 0, and covariance

$$E[X(s)\, X(t)] = \{s^{2\kappa} + t^{2\kappa} - |t-s|^{2\kappa}\}/2,\ 0 < \kappa < 1 .$$

We can easily show the strong mixing of X by applying the criterion of Maruyama [25] on strong mixing of stationary Gaussian processes to the process $Y (= \tau_\kappa(X))$.

EXAMPLE 2. ([27]) Let m be a positive integer and consider the self-similar process X defined by

$$X(t) = \int \cdots \int_{R^m} Q_t(u_1, \cdots, u_m)\, dB(u_1) \cdots dB(u_m),\ t > 0 ,$$

where $\{B(t)\colon -\infty < t < \infty\}$ is a Brownian motion and Q_t is a square integrable function on R^m, invariant under permutations of arguments, with $Q_0 \equiv 0$ and

$$Q_{at}(au_1, \cdots, au_m) = a^{\kappa - m/2}\, Q_t(u_1, \cdots, u_m)$$
$$Q_{t+h}(u_1, \cdots, u_m) - Q_t(u_1, \cdots, u_m) = Q_h(u_1 - t, \cdots, u_m - t)$$
$$\text{for} \quad a, t > 0 \quad \text{and} \quad h \geq 0 ,$$

where $0 < \kappa < 1$. Then, X is a κ-self-similar process (cf. Mori and Oodaira

[27]). In [27], the law of the iterated logarithm is proved. Dobrushin [7] studied analogous self-similar processes represented by multiple Wiener-Itô integrals. Surgailis [33] discussed ergodicity of shift transformations of stationary random fields represented by stochastic integrals based on Poisson random measure. He introduced a notion of 'subordinated' which corresponds to the notion of 'factor' in the ergodic theroy except the necessity of taking an appropriate version of stochastic integral. In this example, it can be similarly shown that a scaling transformation of the above process X is a factor of a certain scaling transformation of the Brownian motion B. Thus, the strong mixing of scaling transformation of X is deduced from a well-known fact in the ergodic theory (cf. for example, Cornfeld, Fomin and Sinai [5], p. 230–231). It is also known that the tail probability of $|X(1)|$ decays in an exponential order (cf. [24], [27]).

EXAMPLE 3. ([14]) Let X be a process defined by

$$X(t) = \int_{-\infty}^{\infty} L_t(x)\, dZ(x),\ t>0, \quad \text{and} \quad X(0) = 0\,,$$

where $Z=\{Z(x)\colon -\infty<x<\infty\}$ is a strictly stable process with exponent α, $0<\alpha\leq 2$, and $L_t(x)$ is the local time at x of a strictly stable process Y with exponent β, $1<\beta\leq 2$, which is independent of Z. Then, X is a self-similar process with exponent $\kappa=1-1/\beta+1/(\alpha\beta)$ (cf. Kesten and Spitzer [14]). As in the above example, by taking appropriate versions of stochastic intergral and local time, we can show that a scaling transformation of X is a factor of direct product of a certain scaling transformation of Z and a transformation of $L_t(x)$ induced from a certain scaling transformation of Y. Since a direct product of strong mixing transformations is strong mixing (cf. Cornfeld, Fomin and Sinai [5], Chapter 10, Section 1, Theorem 2, for example), any scaling transformation of X is strong mixing.

EXAMPLE 4. ([12], [13]) Let $\{U(x)\colon x\geq 0\}$ and $\{M(x)\colon x\geq 0\}$ be strictly stable processes with exponents α, β, $0<\alpha$, $\beta\leq 1$, which have increasing sample paths *a.s.* Let $\{B(t)\colon t\geq 0\}$ be the standard Brownian motion. Assume that these processes are independent one another. Define a process $\{V(t)\colon t\geq 0\}$ by

$$V(t) = \int_0^{\infty} L_t(U(x))\, dM(x),\ t>0, \quad \text{and} \quad V(0) = 0\,,$$

where $L_t(x)$ is the local time at x of B. Let X be a process defined by

$$X(t) = U^{-1}(B(V^{-1}(t))),\ t>0, \quad \text{and} \quad X(0) = 0\,.$$

Then, X is a self-similar process with exponent $\kappa=\alpha\beta/(\alpha+\beta)$ (cf. Kawazu [12], Kawazu and Kesten [13]). Taking an appropriate version of the local time $L_t(x)$, we can show that a scaling transformation of X is a factor of triple direct product of scaling transformations of U, M and B. Thus, X is strong mixing

as in the previous examples.

3. Growth properties of stable self-similar processes

Pushing forward the general arguements in the previous section, we shall consider growth properties of sample paths of a class of stable self-similar processes. We shall call a stochastic process *stable* if its finite-dimensional distributions are stable distributions. We now define a class of stable self-similar processes as follows: let $0<\alpha<2$ and $\beta>-1/\alpha$. For a function f, $\not\equiv 0$, satisfying

$$0<\int_{-\infty}^{\infty}|f(s)|^{\alpha}\,ds<\infty\,, \tag{3.1}$$

put

$$f_t(s)=\begin{cases} t^{\beta}f(s/t), & t>0\,,\\ 0 & ,\ t=0.\end{cases}$$

Let $X=X_{\alpha,\beta,f}$ be a process whose finite-dimensional distribution is given by

$$\begin{aligned} &E[\exp\{i\textstyle\sum_{k=1}^{n}\theta_k X(t_k)\}]\\ &\quad=\exp\{\int_{-\infty}^{\infty}\psi(\textstyle\sum_{k=1}^{n}\theta_k f_{t_k}(s))\,ds\}\,,\\ &\qquad\text{for}\quad -\infty<\theta_1,\cdots,\theta_n<\infty,\ 0\le t_1<\cdots<t_n<\infty,\ n\ge 1\,, \end{aligned} \tag{3.2}$$

where

$$\psi(\xi)=\begin{cases}\int(e^{i\xi x}-1)\,\nu(dx) & ,\ 0<\alpha<1\,,\\ \int(e^{i\xi x}-1-i\xi x\,I_{[|x|\le 1]})\,\nu(dx), & \alpha=1\,,\\ \int(e^{i\xi x}-1-i\xi x)\,\nu(dx) & ,\ 1<\alpha<2\,.\end{cases} \tag{3.3}$$

Here $\nu(dx)$ is a Lévy measure on $R-\{0\}$, given by

$$\nu(dx)=\alpha\{C^{+}I_{[x>0]}+C^{-}I_{[x<0]}\}\,|x|^{-\alpha-1}\,dx\,,$$

where C^{+}, $C^{-}\ge 0$, $C^{+}+C^{-}>0$ (in case $\alpha=1$, $C^{+}=C^{-}$), and $I_{[\ ]}$ denotes the indicator function. Then X is a stable self-similar process with exponent $\kappa=1/\alpha+\beta$. We shall denote this class of stable self-similar processes by $\mathcal{S}(\alpha,\beta)$:

$$\mathcal{S}(\alpha,\beta)=\{X_{\alpha,\beta,f}:\ f\text{ satisfies (3.1)}\}.$$

Any X of $\mathcal{S}(\alpha,\beta)$ is continuous in probability, and can be represented as

$$X(t)=\int_{-\infty}^{\infty}f_t(s)\,dZ_{\alpha}(s)\ a.s.,\qquad\text{for}\quad t\ge 0\,,$$

where $\{Z_{\alpha}(s):\ -\infty<s<\infty\}$ is a strictly stable process with exponent α, whose

characteristic function is given by

$$E[\exp\{i\xi Z_\alpha(s)\}] = \exp\{s\,\psi(\xi)\}\ .$$

Proposition 3.1. *Any X of $\mathcal{S}(\alpha, \beta)$ is an infinitely divisible process in the sense of Maruyama* [26] (*cf. also Lee* [20]) *and is strong mixing of all order.*

This proposition will be proved in the next section.

In addition, we define a subclass $\mathcal{S}^*(\alpha, \beta)$ of $\mathcal{S}(\alpha, \beta)$ by

$$\mathcal{S}^*(\alpha, \beta) = \{X \in \mathcal{S}(\alpha, \beta):\quad X \text{ has stationary increments}\}\ .$$

If $\beta=0$ and $f(s)=I_{[0,1]}(s)$, then X is Z_α itself and belongs to $\mathcal{S}^*(\alpha, \beta)$. Next consider a function f defined by

$$f(s) = a^+\{(1-s)_+^\beta - (-s)_+^\beta\} + a^-\{(1-s)_-^\beta - (-s)_-^\beta\}\ , \tag{3.5}$$

where $x_+=\max\{x, 0\}$, $x_-=\max\{-x, 0\}$, $-1/\alpha<\beta<1-1/\alpha$, $\beta\neq 0$, $-\infty<a^+, a^-<\infty$, $|a^+|+|a^-|\neq 0$.

In this case, X is called a *fractional stable process* (cf. [11], [22], [23] and [34]). We shall call X an (α, β)-fractional stable process when we want to indicate α, β explicitly. Clearly, f_t induced from this f satisfies the following relation:

$$f_{t+h}(s) - f_t(s) = f_h(s-t) \quad a.e. \text{ in } s, \qquad \text{for any fixed} \quad t, h \geq 0\ . \tag{3.6}$$

Thus, (α, β)-fractional stable processes belong to $\mathcal{S}^*(\alpha, \beta)$.

We shall now discuss growth properties of sample paths of these stable self-similar processes. First, we give integral tests for upper and lower functions with respect to the local growth.

Theorem 3.1. *Let $1<\alpha<2$, $\beta>0$, and ϕ be a positive monotone function. With respect to any X of $\mathcal{S}^*(\alpha, \beta)$,*

$$t^\kappa \phi(t) \in \mathcal{U}_l \quad (\textit{or } \mathcal{U}_l^\infty \textit{ resp.})$$

if the integral

$$I_0 = \int_{0+} t^{-1}\,\phi(t)^{-\alpha}\,dt \ (\textit{or } I^\infty = \int_\infty t^{-1}\phi(t)^{-\alpha}\,dt \quad \textit{resp.})$$

converges, where $\kappa=1/\alpha+\beta$. In this case,

$$L_{\kappa,\phi} = 0 \ (\textit{or } L_{\kappa,\phi}^\infty = 0 \textit{ resp.}) \quad a.s.$$

Therem 3.2. *Let $0<\alpha<2$, $\beta>-1/\alpha$ and ϕ be a positive monotone function. Assume that there exists $\delta_0>0$ such that*

$$\int_{-\infty}^{\infty} |s|^{\delta_0} |f(s)|^{\alpha} \, ds < \infty \,, \tag{3.7}$$

$$\int_{-\infty}^{\infty} |f(s)|^{\alpha+\delta_0} \, ds < \infty \,. \tag{3.8}$$

Then, with respect to any $X_{\alpha,\beta,f}$ *of* $\mathcal{S}(\alpha, \beta)$,

$$t^{\kappa}\phi(t) \in \mathcal{L}_l \ (\textit{or } \mathcal{L}_l^{\infty} \textit{ resp.})$$

if the integral I_0 *(or* I_∞ *resp.) diverges.*

In this case, $L_{\kappa,\phi} = \infty$ *(or* $L_{\kappa,\phi}^{\infty} = \infty$ *resp.) a.s.*

REMARK 3.1. (i) If X is a fractional stable process, i.e. f is given by (3.5), the conditions (3.7) and (3.8) are satisfied.

(ii) Put

$$\phi(t) = |\log t|^{1/\alpha} |\log_{(2)} t|^{1/\alpha} \cdots |\log_{(n)} t|^{1/\alpha+\varepsilon} \,,$$

where $n \geq 1$ and $\log_{(k)}$ is the logarithm function iterated k times. Then, $t^{\kappa}\phi(t) \in \mathcal{U}_l$ and $\mathcal{U}_l^{\infty}$ if $\varepsilon > 0$, $1 < \alpha < 2$ and $\beta > 0$, $t^{\kappa}\phi(t) \in \mathcal{L}_l$ and $\mathcal{L}_l^{\infty}$ if $\varepsilon \leq 0$, $0 < \alpha < 2$ and $\beta > -1/\alpha$.

(iii) In case X is an (α, β)-fractional stable process with $0 < \alpha < 1$, any positive bounded function becomes a lower function and the assertion of Theorem 3.2 is meaningless, since any version of X is nowhere bounded (cf. Maejima [23]).

Next we shall discuss the uniform growth properties of sample paths of fractional stable processes. From Proposition 2.3, it can be expected that there are some aspects of problems of the uniform growth, different from those of the local growth. In the rest of this section, we shall investigate behavior of sample paths with respect to the fractional stable processes with continuous sample paths and show a remarkable difference of their uniform growth properties from their local growth properties. These results also contrast sharply with known results on Gaussian self-similar processes (cf. for example, Chung, Erdös and Sirao [4]).

Let $1 < \alpha < 2$, $0 < \beta < 1 - 1/\alpha$ and let f_t be a function induced from f defined by (3.5), i.e.

$$f_t(s) = a^+ \{(t-s)_+^{\beta} - (-s)_+^{\beta}\} + a^- \{(t-s)_-^{\beta} - (-s)_-^{\beta}\} \,.$$

Condider

$$X(t) = X_{\alpha,\beta,f}(t) = \int_{-\infty}^{\infty} f_t(s) \, dZ_{\alpha}(s) \,.$$

We take a version of Z_{α} whose sample paths are right-continuous and have left-

limit with probability one. We also take a version of X having continuous sample paths with probability one.

We denote jumps of Z_α at time t by $\Delta_Z(t)$, i.e.

$$\Delta_Z(t) = Z_\alpha(t) - Z_\alpha(t-0), \quad -\infty < t < \infty .$$

Of course, X satisfies the conditions of Theorem 3.1 and Theorem 3.2. Therefore, upper and lower functions with respect to the local growth are determined by the integral tests and the exponent of local growth order is equal to $\kappa = 1/\alpha + \beta$. The following theorem, however, shows that almost all sample paths of X have certain times at which they behave like β-Hölder continuous functions.

Theorem 3.3. *Let $1<\alpha<2$, $0<\beta<1-1/\alpha$ and $X=X_{\alpha,\beta,f}$ with f defined by* (3.5). *Then,*

$$\lim_{h\downarrow 0}(X(t+h)-X(t))\, h^{-\beta} = a^+\Delta_Z(t),\ 0\leq t\leq 1,\ a.s.$$

$$\lim_{h\downarrow 0}(X(t)-X(t-h))\, h^{-\beta} = -a^-\Delta_Z(t),\ 0<t\leq 1,\ a.s.$$

Corollary. *Let f' be also defined by* (3.5) *with certain a'^+ and a'^-. Then, for some constant $c\neq 0$, $X_{\alpha,\beta,f}$ and $cX_{\alpha,\beta,f'}$ have the same distribution if and only if $f=cf'$.*

REMARK 3.2. Cambanis and Maejima [2] proved the result of this corollary for the case that $1<\alpha<2$, $-1/\alpha<\beta<1-1/\alpha$ and Z_α is a symmetric stable process. We obtain their result only for $0<\beta<1-1/\alpha$ but without the symmetry of distribution of Z_α.

We next consider the functional $U_{\beta,1}$ and show that $U_{\beta,1}$ behaves a little differently than as it is expected from the last theorem.

THEOREM 3.4. *Under the same assumptions in Theorem* 3.3,

$$U_{\beta,1} = \max_{-\infty<s<\infty}|f(s)|\sup_{0\leq t\leq 1}|\Delta_Z(t)|,\ a.s.$$

REMARK 3.3. In case $a^+a^-\geq 0$,

$$\max_{-\infty<s<\infty}|f(s)| = \max\{|a^+|,\ |a^-|\} .$$

On the other hand, in case $a^+a^-<0$,

$$\begin{aligned}\max_{-\infty<s<\infty}|f(s)| &= |a^+|\{|a^-/a^+|^{1/(1-\beta)}+1\}^{1-\beta}\\ &>\max\{|a^+|,\ |a^-|\} .\end{aligned}$$

Corollary. *Under the same assumptions in Theorem* 3.4, *let $\phi(t)$ be a positive function defined for $t>0$.*

(i) *Assume that* $\lim_{t\downarrow 0}\phi(t)=\infty$. *Then, $t^\beta\phi(t)\in\mathcal{U}_u$ and $U_{\beta,\phi}=0$ a.s.*

(ii) *Assume that* $\lim_{t\downarrow 0}\phi(t)=0$. *Then,* $t^{\beta}\phi(t)\in\mathcal{L}_u$ *and* $U_{\beta,\phi}=\infty$ *a.s.*
(iii) *Assume that*

$$0<\liminf_{t\downarrow 0}\phi(t)\leq\limsup_{t\downarrow 0}\phi(t)<\infty\ .$$

Then, $t^{\beta}\phi(t)$ *belongs to neither* $\mathcal{U}_u$ *nor* $\mathcal{L}_u$, *and the support of the distribution of* $U_{\beta,\phi}$ *is* $(0,\infty)$.

The first version of the results on the uniform growth was unsatisfactory and improved by referee's suggestions.

4. Proofs for results of Section 3

Before giving proofs for main theorems, we first verify the strong mixing of all order of stable self-similar processes.

Proof of Proposition 3.1. Since every finite-dimensional distribution of X is stable by the definition, X is clearly an infinitely divisible process in the sense of [26].

From (3.2), the characteristic function of $(Y(0), Y(t))$, $Y=\tau_\kappa(X)$, is given by

$$\begin{aligned}&E[\exp\{i(\theta Y(0)+\eta Y(t))\}]\\&=\exp\{\iint[\exp\{ix(\theta f(u)+\eta g(u;t))\}-1\\&\quad -ia(x)(\theta f(u)+\eta g(u;t))]\,\nu(dx)\,du\}\ ,\end{aligned}$$

where $g(u;t)=f(ue^{-t})\,e^{-t/\alpha}$ and

$$a(x)=\begin{cases}0\ , & \text{if}\ \ 0<\alpha<1\,,\\ xI_{[|x|\leq 1]}(x)\,, & \text{if}\ \ \alpha=1\,,\\ x\ , & \text{if}\ \ 1<\alpha<2\,.\end{cases}$$

Since Y has no Gaussian component, it is enough for verification of the conditions of Theorem 6 of [26], to show the following:

(A) $$\lim_{t\to\infty}\iint_{F(t,\delta)}\nu(dx)\,du=0\,,\quad\text{for}\quad\delta>0\,,$$

(B) $$\lim_{t\to\infty}\iint_{F(t)}|f(u)\,g(u;t)|\,x^2\,\nu(dx)\,du=0\,,$$

where

$$\begin{aligned}F(t,\delta)&=\{(x,u)\colon|f(u)\,g(u;t)|\,x^2\geq\delta\}\,,\quad\text{and}\\F(t)&=\{(x,u)\colon 0<x<(f(u)^2+g(u;t)^2)^{-1/2}\}\,.\end{aligned}$$

(A) is derived from the fact that for any fixed $\delta>0$,

$$\iint_{F(t,\delta)}|x|^{-\alpha-1}\,dx\,du\leq c\int|f(u)\,g(u;t)|^{\alpha/2}\,du$$

and (B) is derived from the fact that

$$\iint_{F(t)} |f(u)\,g(u;t)|\,x^2\,\nu(dx)\,du \leq c' \int |f(u)\,g(u;t)|^{\alpha/2}\,du\,,$$

where c and c' are positive constants, independent of t, because $|f(u)|^{\alpha/2}$ and $|g(u;t)|^{\alpha/2}$ are square integrable and so

$$\int |f(u)\,g(u;t)|^{\alpha/2}\,du \to 0 \quad \text{as} \quad t \to \infty\,. \qquad \text{q.e.d.}$$

We shall now turn to proofs of theorems in Section 3. Let X be a process of $\mathcal{S}^*(\alpha, \beta)$. First, note the following fact: there exist positive constants K_1, K_2 such that

$$K_1\,x^{-\alpha} \leq P(|X(1)| \geq x) \leq K_2\,x^{-\alpha}\,,$$

and

$$K_1\,|t-s|^{\alpha\kappa}\,x^{-\alpha} \leq P(|X(t)-X(s)| \geq x) \leq K_2\,|t-s|^{\alpha\kappa}\,x^{-\alpha}\,, \tag{4.1}$$
$$\text{for} \quad 0 \leq s, t < \infty, \quad \text{and for} \quad x \geq 1\,.$$

This is derived from a well-known estimate for tail probability of stable distributions (cf. Gnegenko and Kolmogorov [8], p. 182). In case $\alpha\kappa > 1$, i.e. $\beta > 0$, by using Theorem 3.2 of Móricz, Serfling and Stout [29], we can obtain the next estimate for maximum of X, which plays an important role in the proof of Theorem 3.1.

Lemma 4.1. *For a process X of $\mathcal{S}^*(\alpha, \beta)$, there exists a positive constant K_3, depending on α, β and K_2 such that*

$$P(\max_{0 \leq t \leq 1} |X(t)| \geq x) \leq K_3\,x^{-\alpha} \qquad \text{for} \quad x \geq 1\,.$$

Outline of proof. Fix $n \geq 1$. Using the notations of [29], put $X_k = X(k/n) - X((k-1)/n)$, and $g(i,j) = K_2\,|(j-i)/n|^{\alpha\kappa}$ and $\phi(t) = t^{\alpha}$, for $1 \leq k \leq n$, $1 \leq i, j \leq n$ and $t > 0$. Let $\alpha\kappa$ take the place of α in [29]. Then, it is easily verified that each condition in [29] can be satisfied. Thus,

$$P(\max_{1 \leq k \leq n} |X(k/n)| \geq x) \leq K_3\,x^{-\alpha}\,.$$

By Theorem 3.2 of [29], K_3 depends not on n, but only on α, β, K_2, and so we have the assertion of lemma by letting $n \to \infty$. q.e.d.

Using this lemma, we prove Theorem 3.1.

Proof of Theorem 3.1. We start from the proof for $\mathcal{U}_l$. It is enough to prove the theorem for a decreasing function ϕ, by Remark 2.1. For $c > 0$, define events

$$E_n = [\,|X(2^{-n})|\,2^{n\kappa} \geq c\,\phi(2^{-n})]\,,$$
$$F_n = [\max_{2^{-n-1}\leq t\leq 2^{-n}} |X(t)-X(2^{-n-1})| \geq c\,2^{-n\kappa}\,\phi(2^{-n})]\,.$$

By (4.1) and Lemma 4.1,

$$\textstyle\sum_{n=1}^{\infty} P(E_n) \leq c_1 \sum_{n=1}^{\infty} \phi(2^{-n})^{-\alpha} \leq c_2 \int_{0+} t^{-1}\,\phi(t)^{-\alpha}\,dt < \infty\,,$$
$$\textstyle\sum_{n=1}^{\infty} P(F_n) \leq \sum_{n=1}^{\infty} P(M \geq c\,2^{\kappa}\phi(2^{-n}))$$
$$\textstyle\leq c_3 \sum_{n=1}^{\infty} \phi(2^{-n})^{-\alpha} \leq c_4 \int_{0+} t^{-1}\,\phi(t)^{-\alpha}\,dt < \infty\,,$$

where $M=\max_{0\leq t\leq 1}|X(t)|$, and c_1, c_2, c_3, c_4 are positive constants independent of n. By Borel-Cantelli's lemma, with probability one, there is a number n_0 such that

$$\begin{aligned} |X(t)| &\leq |X(2^{-n-1})| + |X(t)-X(2^{-n-1})| \\ &\leq 2\,c\,2^{-n\kappa}\,\phi(2^{-n}) \leq 2^{1+\kappa}\,c\,t^{\kappa}\,\phi(t)\,, \\ \text{for}\quad & 2^{-n-1}\leq t\leq 2^{-n},\ n\geq n_0\,. \end{aligned}$$

This implies that $L_{\kappa,\phi}\leq 2^{1+\kappa}\,c$ a.s. Since c is arbitrarily positive, this means that $L_{\kappa,\phi}=0$ a.s., and that $\phi\in\mathcal{U}_l$. For the local growth at ∞, the proof goes similarly with trivial modifications. q.e.d.

We now proceed to the proof of Theorem 3.2. For this purpose, we prepare the following estimate for tail probability of two-dimensional distribution of X.

Lemma 4.2. *Let X be a process of $\mathcal{S}(\alpha, \beta)$. There exists a positive constant K_4 such that for $a, b>0$ and for $0<h\leq 1$,*

$$P(|X(h)|\,h^{-\kappa}\geq a,\ |X(1)|\geq b) \leq K_4\{h^{\delta}(ab)^{-\alpha/2}+(ab)^{-\alpha}\}\,,$$

where $\delta=\min\{\delta_0, \delta_0/(\alpha+\delta_0)\}/4$.

Proof. Let $\rho(x)$, $-\infty<x<\infty$, be a non-negative, infinitely differentiable, even function such that $|\rho(x)|\leq 1$ and

$$\rho(x) = \begin{cases} 1 & \text{if} \quad |x|\leq 1/2\,, \\ 0 & \text{if} \quad |x|\geq 1\,, \end{cases}$$

and $\tilde{\rho}(\xi)$ be its image of Fourier transformation, i.e.

$$\rho(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{ix\xi}\,\tilde{\rho}(\xi)\,d\xi\,.$$

Put $X'=X(h)\,h^{-\kappa}$, $X''=X(1)$ and

$$\Phi_{X'}(\xi) = E[\exp(i\xi X')]\,,\quad \Phi_{X''}(\eta) = E[\exp(i\eta X'')]\,,$$
$$\Phi_{X',X''}(\xi,\eta) = E[\exp(i\xi X'+i\eta X'')]\,.$$

Note that

$$(2\pi)^{-1}\int_{-\infty}^{\infty}\tilde{\rho}(\xi)\,d\xi=\rho(0)=1\,.$$

Using this relation, we have the following estimate:

$$\begin{aligned}
&P(|X(h)|h^{-\kappa}\geq a,\ |X(1)|\geq b)\\
&\quad\leq E[(1-\rho(X'/a))\,(1-\rho(X''/b))]\\
&\quad=(2\pi)^{-2}\,E[\{\int(1-e^{i\xi X'/a})\,\tilde{\rho}(\xi)\,d\xi\}\ \{\int(1-e^{i\eta X''/b})\,\tilde{\rho}(\eta)\,d\eta\}]\\
&\quad=(2\pi)^{-2}\iint\{1-\Phi_{X'}(\xi/a)-\Phi_{X''}(\eta/b)+\Phi_{X',X''}(\xi/a,\,\eta/b)\}\ \tilde{\rho}(\xi)\,\tilde{\rho}(\eta)\,d\xi\,d\eta\\
&\quad=(2\pi)^{-2}\iint\{\Phi_{X',X''}(\xi/a,\,\eta/b)-\Phi_{X'}(\xi/a)\,\Phi_{X''}(\eta/b)\}\ \tilde{\rho}(\xi)\,\tilde{\rho}(\eta)\,d\xi\,d\eta\\
&\qquad+(2\pi)^{-2}\iint\{1-\Phi_{X'}(\xi/a)\}\,\{1-\Phi_{X''}(\eta/b)\}\ \tilde{\rho}(\xi)\,\tilde{\rho}(\eta)\,d\xi\,d\eta\\
&\quad=(2\pi)^{-2}\iint\Phi_{X',X''}(\xi/a,\,\eta/b)\,\{1-\frac{\Phi_{X'}(\xi/a)\,\Phi_{X''}(\eta/b)}{\Phi_{X',X''}(\xi/a,\,\eta/b)}\}\ \tilde{\rho}(\xi)\,\tilde{\rho}(\eta)\,d\xi\,d\eta\\
&\qquad+E[1-\rho(X'/a)]\,E[1-\rho(X''/b)]\,.
\end{aligned}$$

It is enough to estimate the first term in the last side, because we have the following estimates for the second term:

$$\begin{aligned}
E[1-\rho(X'/a)]&\leq P(|X'|\geq a/2)\leq c_1\,a^{-\alpha}\,,\\
E[1-\rho(X''/b)]&\leq P(|X''|\geq b/2)\leq c_1\,b^{-\alpha}\,,
\end{aligned}$$

where c_1 is a positive constant independent on a, b and h. By (3.2),

$$\begin{aligned}
\Phi_{X'}(\xi)&=\exp\{\int\psi(\xi f(u/h)\,h^{-1/\alpha})\,du\}\,,\\
\Phi_{X''}(\eta)&=\exp\{\int\psi(\eta f(u))\,du\}\,,\\
\Phi_{X',X''}(\xi,\,\eta)&=\exp\{\int\psi(\xi f(u/h)\,h^{-1/\alpha}+\eta f(u))\,du\}\,.
\end{aligned}$$

Here ψ is given by (3.3). Thus,

$$\begin{aligned}
\frac{\Phi_{X'}(\xi)\,\Phi_{X''}(\eta)}{\Phi_{X',X''}(\xi,\eta)}=\exp\{\int\{&\psi(\xi f(u/h)\,h^{-1/\alpha})+\psi(\eta f(u))\\
&-\psi(\xi f(u/h)\,h^{-1/\alpha}+\eta f(u))\}\,du\}\,.
\end{aligned}$$

It is easily derived from (3.3) that

$$\begin{aligned}
|\psi(\xi)+\psi(\eta)-\psi(\xi+\eta)|&\leq c_2\,|\int(e^{i\xi x}-1)(e^{i\eta x}-1)\,|x|^{-\alpha-1}\,dx|\\
&\leq c_3\,|\xi|^{\alpha/2}\,|\eta|^{\alpha/2}
\end{aligned}$$

where c_2 and c_3 depend only on C^+, C^- and . Therefore, by Schwarz' inequality,

we have

$$\begin{aligned}&\int |\psi(\xi f(u/h)\, h^{-1/\alpha})+\psi(\eta f(u))-\psi(\xi f(u/h)\, h^{-1/\alpha}+\eta f(u))|\, du\\ &\leq c_4 |\xi\eta|^{\alpha/2} \int h^{-1/2} |f(u/h) f(u)|^{\alpha/2}\, du\\ &\leq c_5 |\xi\eta|^{\alpha/2}\,.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\iint \left|1-\frac{\Phi_{X'}(\xi/a)\, \Phi_{X''}(\eta/b)}{\Phi_{X',X''}(\xi/a,\, \eta/b)}\right| |\tilde{\rho}(\xi)\, \tilde{\rho}(\eta)|\, d\xi\, d\eta\\ &\leq c_6 \{(ab)^{-\alpha/2} \iint \{\int h^{-1/2} |f(u/h) f(u)|^{\alpha/2}\, du\} |\xi\eta|^{\alpha/2} |\tilde{\rho}(\xi)\, \tilde{\rho}(\eta)|\, d\xi\, d\eta\\ &\quad+\iint_{|\xi\eta|>ab} |\tilde{\rho}(\xi)\, \tilde{\rho}(\eta)|\, d\xi\, d\eta\}\,.\end{aligned}$$

Because ρ is infinitely differentiable, $\tilde{\rho}$ has moments of all order and we have

$$\iint_{|\xi\eta|>ab} |\tilde{\rho}(\xi)\, \tilde{\rho}(\eta)|\, d\xi\, d\eta \leq c_7 (ab)^{-2}\,.$$

Getting these estimates together, we have

$$\begin{aligned}&P(|X(h)|\, h^{-\kappa} \geq a,\ |X(1)| \geq b)\\ &\leq c_8 \{(ab)^{-\alpha/2} \int h^{-1/2} |f(u/h) f(u)|^{\alpha/2}\, du+(ab)^{-\alpha}\}\,,\end{aligned}$$

where c_4, c_5, c_6, c_7 and c_8 are positive constants independent of a, b, h, ξ and η.

Therefore, by the following lemma, the proof is completed.

Lemma 4.3. *Under the same assumptions and by the same notations as in Lemma* 4.2, *there exists* $c'>0$ *such that*

$$\int h^{-1/2} |f(u/h) f(u)|^{\alpha/2}\, du \leq c'\, h^{\delta}, \qquad \textit{for} \quad 0<h\leq 1\,.$$

Proof. Put

$$F(x) = \int_{[|u|>x]} |f(u)|^{\alpha}\, du \qquad \text{for} \quad x>0\,.$$

Divide the integral into three parts:

$$\begin{aligned}&\int h^{-1/2} |f(u/h) f(u)|^{\alpha/2}\, du = \int_{-\infty}^{-\sqrt{h}} + \int_{-\sqrt{h}}^{\sqrt{h}} + \int_{\sqrt{h}}^{\infty},\\ &= I_1+I_2+I_3\,.\end{aligned}$$

We give estimates for these three integrals by using Schwarz' inequality. For I_1,

$$I_1 \leq \{\int_{-\infty}^{-\sqrt{h}} |f(u)|^{\alpha} du\}^{1/2} \{\int_{-\infty}^{-\sqrt{h}} |f(u/h)|^{\alpha} du/h\}^{1/2}$$
$$\leq \{\int_{-\infty}^{\infty} |f(u)|^{\alpha} du\}^{1/2} \{\int_{-\infty}^{-1/\sqrt{h}} |f(u)|^{\alpha} du\}^{1/2}$$
$$\leq c_9 F(1/\sqrt{h})^{1/2} .$$

For I_2 and I_3,

$$I_2 \leq c_{10}(F(0)-F(\sqrt{h}))^{1/2}, \quad I_3 \leq c_{11} F(1/\sqrt{h})^{1/2} ,$$

where c_9, c_{10} and c_{11} are positive constants independent of h.
From the conditions (3.7) and (3.8), we obtain the following estimates for $F(x)$ by Hölder's and Chebyshev's inequalities:

$$F(x) \leq c_{12} x^{-\delta_0} , \quad F(0)-F(x) \leq c_{13} x^{\delta'} , \qquad \text{for} \quad x>0 ,$$

where c_{12} and c_{13} are positive constants independent of x, and $\delta'=\delta_0/(\alpha+\delta_0)$.

Getting these estimates together, we verify the estimate of the lemma and we complete the proof of Lemma 4.2. q.e.d.

Using Lemma 4.2, we now prove Theorem 3.2. In proving a function to be a lower function, we usually use an extension of Borel-Cantelli's lemma by Chung and Erdös [3]. In this paper, we apply the following lemma by Kochen and Stone [16], which we state here in a form convenient for our use.

Lemma 4.4. ([16]) *Let E_n be events and assume that*

$$\sum_{n\geq 1} P(E_n) = \infty ,$$
$$\liminf_{N\to\infty} \sum_{m,n=1}^{N} P(E_m \cap E_n)/\{\sum_{n=1}^{N} P(E_n)\}^2 < \infty .$$

Then, $P(E_n \text{ i.o.})>0$.

Zero-one laws in Section 2 enable us to prove results by this lemma, whose conditions are much more easily cheked than those of [3].

Proof of Theorem 3.2. First, we prove the result with respect to the local growth at 0. It is enough to consider the problem for decreasing function ϕ, because of Remark 2.1. Define events

$$E_n = [\,|X(2^{-n})|\,2^{n\kappa} \geq c\,\phi(2^{-n})] , \qquad \text{for} \quad c>0,\ n\geq 1 .$$

Then,

$$\sum_{n=1}^{\infty} P(E_n) \geq c_1 \sum_{n=1}^{\infty} \phi(2^{-n})^{-\alpha} \geq c_2 \int_{0+} t^{-1} \phi(t)^{-\alpha} dt = \infty .$$

By Lemma 4.2, we have for $n>m$,

$$P(E_m \cap E_n)$$
$$\leq c_3\{2^{-\delta(n-m)}\{\phi(2^{-m})\,\phi(2^{-n})\}^{-\alpha/2}+\{\phi(2^{-m})\,\phi(2^{-n})\}^{-\alpha}\}\ ,$$

where c_1, c_2 and c_3 are positive constants independent of m and n. Therefore,

$$\sum_{m,n=1}^{N} P(E_m \cap E_n) \leq 2\sum_{m=1}^{N}\sum_{n=m}^{N} P(E_m \cap E_n)$$
$$\leq 2\,c_3 \sum_{m=1}^{N}\sum_{n=m}^{N}\{2^{-\delta(n-m)}\,\phi(2^{-m})^{-\alpha}+\{\phi(2^{-m})\,\phi(2^{-n})\}^{-\alpha}\}$$
$$\leq c_4\{\sum_{n=1}^{N} P(E_n)\}^2\,.$$

Thus, by Lemma 4.4, we have $P(E_n \text{ i.o.})>0$. This implies that $L_{\kappa,\phi}\geq c$ a.s. Since c is arbitrarily positive, we have

$$L_{\kappa,\phi}=\infty \text{ a.s.,} \quad \text{and} \quad t^{\kappa}\phi(t)\in\mathcal{L}_l\,.$$

The proof goes similarly for the growth at ∞, with slight modifications. q.e.d.

Next we shall turn to the proofs for Theorem 3.3 and Theorem 3.4. The following lemma is suggested by the referee and plays an important role in the proofs.

Put

$$g_t(s)=-\frac{\partial f_t(s)}{\partial s},\ s\neq 0,\ t,\ t>0\,,$$

and

$$Y(t)=\int_{-\infty}^{\infty} Z_\alpha(s)\,g_t(s)\,ds,\ t>0\,,$$
$$Y(0)=0\,.$$

Note here that $Y(t)$ can be defined for almost all sample paths of Z_α, because by the results of Khinchin [15], for any $\varepsilon>0$,

$$\text{(4.2)} \qquad \limsup_{s\to\pm\infty}|Z_\alpha(s)|\,|s|^{-1/\alpha}(\log|s|)^{-1/\alpha-\varepsilon}=0 \quad \text{a.s.,}$$

and $|g_t(s)|=O(|s|^{\beta-2})$ as $s\to\pm\infty$.

Lemma 4.5. *The process $Y=\{Y(t);\ t\geq 0\}$ has the same distribution as that of $X_{\alpha,\beta,f}$, i.e. Y is a version of fractional stable process $X_{\alpha,\beta,f}$.*

Outline of Proof. For $N>0$, using approximations of stochastic integral

$$\int_{-N}^{N} f_t(s)\,dZ_\alpha(s)$$

and integral

$$\int_{-N}^{N} Z_\alpha(s)\,g_t(s)\,ds$$

by Riemann's sums, we can easily show that for any $t>0$,

$$\int_{-N}^{N} f_t(s)\, dZ_\alpha(s)$$

and

$$\int_{-N}^{N} Z_\alpha(s)\, g_t(s)\, ds + f_t(N)\, Z_\alpha(N) - f_t(-N)\, Z_\alpha(-N)$$

have the same distribution. Since $|f_t(s)| = O(|s|^{\beta-1})$ as $s \to \pm\infty$, we have by (4.2)

$$|f_t(N)\, Z_\alpha(N)|,\ |f_t(-N)\, Z_\alpha(-N)| \to 0 \quad \text{as} \quad N \uparrow \infty\,.$$

This implies that $X(t)$ and $Y(t)$ have the same distribution. To show that X and Y have the same finite-dimensional distributions, it is needed only to replace $f_t(s)$ by

$$\textstyle\sum_k \theta_k f_{t_k}(s),\ -\infty < \theta_k < \infty,\ 0 \le t_k < \infty\,,$$

in the above arguments. q.e.d.

Proof of Theorem 3.3. Since the functionals under consideration are measurable, it is enough to show the following

Lemma 4.6.

$$\lim_{h\downarrow 0}(Y(t+h) - Y(t))\, h^{-\beta} = a^+ \Delta_Z(t),\ 0 \le t \le 1\,, \quad a.s.$$
$$\lim_{h\downarrow 0}(Y(t) - Y(t-h))\, h^{-\beta} = -a^- \Delta_Z(t),\ 0 < t \le 1\,, \quad a.s.$$

Proof. First, note that for $h > 0$,

$$g_{t+h}(s) - g_t(s) = g_h(s-t) \quad \text{and} \quad g_h(s) = h^{\beta-1} g(s/h)$$

where $g(s) = -\dfrac{df(s)}{ds},\ s \neq 0,\ 1\,.$

This is derived from the fact that f_t satisfies (3.6). Using this relation, we have

$$\begin{aligned} Y(t+h) - Y(t) &= \int_{-\infty}^{\infty} Z_\alpha(s)\, (g_{t+h}(s) - g_t(s))\, ds \\ &= h^\beta \int_{-\infty}^{\infty} Z_\alpha(t+hv)\, g(v)\, dv\,. \end{aligned}$$

By (4.2), there is $M > 0$ such that

$$|Z_\alpha(t+hv)| \le \max\{M,\ |v|^{1/\alpha}(\log|v|)^{1/\alpha+1}\}$$
$$\text{for} \quad -\infty < v < \infty \quad \text{and} \quad 0 \le t \le 1\,.$$

Therefore,

$$|Z_\alpha(t+hv)\, g(v)| \le \max\{M,\ |v|^{1/\alpha}(\log|v|)^{1/\alpha+1}\}\, |g(v)|\,.$$

Since the right hand side is an integrable function of v, it is derived from Lebesgue's convergence theorem that

$$\lim_{h\downarrow 0}(Y(t+h)-Y(t))\,h^{-\beta}=\int_{-\infty}^{\infty}\lim_{h\downarrow 0}Z_\alpha(t+hv)\,g(v)\,dv\,.$$

Here note that

$$\lim_{h\downarrow 0}Z_\alpha(t+hv)=\begin{cases}Z_\alpha(t) & \text{if} \quad v\geq 0\,,\\ Z_\alpha(t-0) & \text{if} \quad v<0\,,\end{cases}$$

$$\int_{-\infty}^{\infty}g(v)\,dv=f(-\infty)-f(\infty)=0\,,$$

and

$$\int_{0}^{\infty}g(v)\,dv=f(0)-f(\infty)=a^{+}\,.$$

Then, we obtain

$$\lim_{h\downarrow 0}(Y(t+h)-Y(t))\,h^{-\beta}=a^{+}\Delta_Z(t),\ 0\leq t\leq 1\,,\quad \text{a.s.}$$

Next we prove the second assertion. For $h>0$ and $t-h>0$,

$$\begin{aligned}Y(t)-Y(t-h)&=\int_{-\infty}^{\infty}Z_\alpha(s)\,(g_t(s)-g_{t-h}(s))\,ds\\&=\int_{-\infty}^{\infty}Z_\alpha(s)\,g_h^*(s-t)\,ds\\&=h^{\beta}\int_{-\infty}^{\infty}Z_\alpha(t+hv)\,g^*(v)\,dv\,,\end{aligned}$$

where $g^*(s)=-\dfrac{df^*(s)}{ds}$,

$$f^*(s)=-a^{+}\{(1+s)_-^{\beta}-(s)_-^{\beta}\}-a^{-}\{(1+s)_+^{\beta}-(s)_+^{\beta}\}\,,$$

$$g_t^*(s)=-\frac{\partial f_t^*(s)}{\partial s}\quad\text{and}\quad f_t^*(s)=t^{\beta}f^*(s/t)\,,$$

and in the above we use the following relations:

$$g_t(s)-g_{t-h}(s)=g_h^*(s-t)\quad\text{and}\quad g_h^*(s)=h^{\beta-1}g^*(s)\,.$$

Since $|g^*(v)|=O(|v|^{\beta-2})$ as $v\to\pm\infty$, Lebesgue's convergence theorem can be again applied and we obtain

$$\begin{aligned}\lim_{h\downarrow 0}(Y(t)-Y(t-h))\,h^{-\beta}&=\int_{-\infty}^{\infty}\lim_{h\downarrow 0}Z_\alpha(t+hv)\,g^*(v)\,dv\\&=-a^{-}\Delta_Z(t),\ 0<t\leq 1\,,\quad\text{a.s.}\end{aligned}$$

q.e.d.

Proof of Theorem 3.4. As in the proof of Theorem 3.3, it is enough to show the following

Lemma 4.7.

$$\lim_{h\downarrow 0} \sup_{0<s<t\leq 1,\, t-s\leq h} |Y(t)-Y(s)|h^{-\beta}$$
$$= \max_{-\infty<s<\infty} |f(s)| \sup_{0\leq t\leq 1} |\Delta_Z(t)| \quad a.s.$$

Proof. Since sample paths of Z_α are right-continuous and have left-limit, for any $\varepsilon>0$ and $0\leq t\leq 1$ there is $\eta=\eta(t,\varepsilon)>0$ such that

$$|Z_\alpha(t-0)-Z_\alpha(s)|\leq\varepsilon \qquad \text{for} \quad t-\eta\leq s<t\,, \quad \text{and}$$
$$|Z_\alpha(t)-Z_\alpha(s)|\leq\varepsilon \qquad \text{for} \quad t<s\leq t+\eta\,.$$

Let ε be fixed. Because $[0,1]$ is compact, there are $t_1, \cdots, t_m$ of $[0,1]$ such that $[0,1]\subset\bigcup_{k=1}^m (t_k-\eta_k/2,\, t_k+\eta_k/2)$, where η_k is η corresponding to t_k. Here note that if $0\leq t\leq 1$ and $|\Delta_Z(t)|>2\varepsilon$, then $t=t_k$ for some $1\leq k\leq m$. Put

$$B_k = [t_k-\eta_k,\, t_k+\eta_k]$$

and

$$B_k^* = (t_k-\eta_k/2,\, t_k+\eta_k/2)\,.$$

Put $\rho=(t-s)/h$ for $0<h<1$ and $0\leq s<t\leq 1$, $t-s\leq h$. Then,

$$Y(t)-Y(s) = \int_{-\infty}^{\infty} Z_\alpha(u)\,(g_t(u)-g_s(u))\,du$$
$$= h^\beta \int_{-\infty}^{\infty} Z_\alpha(s+hv)\,g_\rho(v)\,dv\,.$$

Since $|g_\rho(v)|\leq|g(v)|$ for $|v|\geq 2$, there is $N>0$ such that

$$\sup_{0\leq s\leq 1} \int_{|v|>N} |Z_\alpha(s+hv)\,g_\rho(v)|\,dv$$
$$\leq \sup_{0\leq s\leq 1} \int_{|v|>N} |Z_\alpha(s+hv)\,g(v)|\,dv\leq\varepsilon\,,$$

and

$$\int_{|v|>N} |g_\rho(v)|\,dv\leq\int_{|v|>N} |g(v)|\,dv\leq\varepsilon\,.$$

Now we have for s, t of B_k^*,

$$(Y(t)-Y(s))\,h^{-\beta}$$
$$= \int_{-N}^{N} Z_\alpha(t_k-0)\,g_\rho(v)\,dv + \int_{-N}^{N} \Delta_Z(t_k)\,H(s+hv;\,t_k)\,g_\rho(v)\,dv$$
$$+ \int_{-N}^{N} (Z_\alpha(s+hv)-Z_k(t_k-0))\,(1-H(s+hv;\,t_k))\,g_\rho(v)\,dv$$
$$+ \int_{-N}^{N} (Z_\alpha(s+hv)-Z_\alpha(t_k))\,H(s+hv;\,t_k)\,g_\rho(v)\,dv$$
$$+ \int_{|v|>N} Z_\rho(s+hv)\,g_\rho(v)\,dv\,,$$

where $H(x;x_0)=\begin{cases}1 & \text{if} \quad x\geq x_0\\ 0 & \text{if} \quad x<x_0\,.\end{cases}$

Because $\int_{-\infty}^{\infty} g_\rho(v)\,dv=0$ and $\int_{|v|>N}|g_\rho(v)|\,dv<\varepsilon$, we have

$$\left|\int_{-N}^{N} g_\rho(v)\,dv\right|\leq\varepsilon \quad \text{and} \quad \left|\int_{-N}^{N} Z_\alpha(t_k-0)\,g_\rho(v)\,dv\right|\leq|Z_\alpha(t_k-0)|\,\varepsilon\,.$$

Let $\eta_0=\min_{1\leq k\leq m}\eta_k>0$ and $h<\eta_0/(2N)$. Then, $s+hv$ belongs to B_k for s of B_k^* and $|v|\leq N$. Therefore,

$$\begin{aligned}&\left|\int_{-N}^{N}\{(Z_\alpha(s+hv)-Z_\alpha(t_k-0))\,(1-H(s+hv;t_k))\right.\\&\quad\left.+(Z_\alpha(s+hv)-Z_\alpha(t_k))\,H(s+hv;t_k)\}\,g_\rho(v)\,dv\right|\\&\leq\varepsilon\int_{-\infty}^{\infty}|g_\rho(v)|\,dv\leq 2\varepsilon(|a^+|+|a^-|)\,\rho^\beta\\&\leq 2\varepsilon(|a^+|+|a^-|)\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{-N}^{N}\Delta_Z(t_k)\,H(s+hv;t_k)\,g_\rho(v)\,dv\\&=\Delta_Z(t_k)f_\rho(v_k)-\int_{|v|>N}\Delta_Z(t_k)\,H(s+hv;t_k)\,g_\rho(v)\,dv\,,\end{aligned}$$

where $v_k=(t_k-s)/h$.

We also have

$$\begin{aligned}&\left|\int_{|v|>N}\Delta_Z(t_k)\,H(s+hv;t_k)\,g_\rho(v)\,dv\right|\\&\leq|\Delta_Z(t_k)|\int_{|v|>N}|g_\rho(v)|\,dv\leq|\Delta_Z(t_k)|\,\varepsilon\,.\end{aligned}$$

Getting the above estimates together, we obtain

$$\begin{aligned}&|(Y(t)-Y(s))\,h^{-\beta}-\Delta_Z(t_k)f_\rho(v_k)|\\&\leq\varepsilon\{|Z_\alpha(t_k-0)|+2(|a^+|+|a^-|)+|\Delta_Z(t_k)|\}\end{aligned}$$

for $0\leq s<t\leq 1$, $s,\,t\in B_k^*$, $0<t-s\leq h$ and $h<\eta_0/(2N)$. Note here that

$$\sup_{0<t-s<h,\,s,\,t\in B_k^*}|f_\rho(v_k)|=\max_{-\infty<u<\infty}|f(u)|\,.$$

Then, we have

$$\begin{aligned}&\left|\sup_{0<t-s\leq h,\,s,\,t\in B_k^*}|Y(t)-Y(s)|\,h^{-\beta}-\max_{-\infty<u<\infty}|f(u)|\,|\Delta_Z(t_k)|\right|\\&\leq\varepsilon\,\{|Z_\alpha(t_k-0)|+2(|a^+|+|a^-|)+|\Delta_Z(t_k)|\}\,.\end{aligned}$$

Since $\{B_k^*,\,k=1,\cdots,m\}$ is a covering of $[0,1]$, this means

$$|\lim_{h\downarrow 0} \sup_{\substack{0\leq s<t\leq 1\\ t-s\leq h}} |Y(t)-Y(s)|h^{-\beta} - \max_{-\infty<v<\infty} |f(v)| \sup_{0\leq t\leq 1} |\Delta_Z(t)| \ |$$

$$\leq \varepsilon \{\sup_{1\leq k\leq m} |Z_\rho(t_k-0)| + 2(|a^+|+|a^-|) + \sup_{0\leq t\leq 1} |\Delta_Z(t)|\} .$$

Thus, letting $\varepsilon \downarrow 0$, we obtain the desired result. q.e.d.

5. Other sample path properties

In this section we shall consider sample path properties other than growth order properties dealt in Section 2, especially, we shall discuss some properties related to Hausdorff measures. First, we recall the definition of Hausdorff measures. Let ϕ be a positive, continuous function and A be a subset of R^d, $d\geq 1$. Denote the *Hausdorff measure* of A *with repsect to measure function* $x^\gamma\phi(x)$ by $m_{\gamma,\phi}(A)$, defined as follows:

$$m_{\gamma,\phi}(A) = \lim_{\delta\downarrow 0} \inf \sum_{U\in C_\delta} (d(U))^\gamma \phi(d(U)) ,$$

where inf denotes the infimum over all coverings C_δ of A with balls U, $d(U)<\delta$, and $d(U)$ denotes the diameter of U, and $\gamma>0$. In case $\phi\equiv 1$, m_γ stands for $m_{\gamma,\phi}$.

(1) Hausdorff measure of range of sample paths

In considering this problem, we assume that X takes values in R^d, $d\geq 2$, because in one-dimensional case this problem becomes trivial. Denote the range of path by R_t:

$$R_t = \{X(s): 0\leq s\leq t\} , \qquad \text{for} \quad t>0 .$$

P. Lévy made a comment on Hausdorff measure of range of Brownian paths in R^d in the introduction of [21]: for a positive, continuous function ϕ, slowly varying at 0,

$$m_{2,\phi}(R_t) = C\,t \qquad \text{for} \quad t>0 , \quad \text{a.s.},$$

where C is a constant, $0\leq C\leq\infty$. He gave only an idea of proof (cf. footnote (2), he reduced arguments to Kolmogorov's zero-one law). We shall derive this fact from ergodicity of scaling transformations for general self-similar processes.

Proposition 5.1. *Let ϕ be a positive, continuous, monotone function, slowly varying at* 0. *There exists a constant C, $0\leq C\leq\infty$, such that*

$$m_{1/\kappa,\phi}(R_t) = C\,t \qquad \textit{for} \quad t>0 , \quad \textit{a.s.}$$

Proof. For $\lambda>0$, define an event E_λ by

$$E_\lambda = [m_{1/\kappa,\phi}(R_t)\geq\lambda\, t,\, t>0]\,.$$

Note that

$$R_t\circ S = \{a^{-\kappa}\, X(as)\colon 0\leq s\leq t\} = a^{-\kappa}\, R_{at}\,,$$

where $S=S_a$, $R_t\circ S$ denotes R_t with respect to the process $\{(SX)(s);\, s\geq 0\}$ and $c\,A=\{cx\colon x\in A\}$. Since ϕ is slowly varying,

$$m_{1/\kappa,\phi}(a^{-\kappa}\, R_{at}) = m_{1/\kappa,\phi}(R_{at})/a\,.$$

Thus, we have $S^{-1}\,E_\lambda\subset E_\lambda$. Therefore, $P(E_\lambda)=0$ or 1. Put

$$C = \sup\,\{\lambda\colon P(E_\lambda) = 1\}\,.$$

Then, the assertion of the proposition can be proved. q.e.d.

(2) Hausdorff dimension of zero points

Put $Z_t=\{s\colon 0\leq s\leq t,\, X(s)=0\}$. Taylor [35] made another approach to zero-one law for Hausdorff dimension of Z_t in case X is the Brownian motion (cf. [35], Lemma 1). He reduced arguments, in contrast with [21], to strong Markovian properties of the Brownian motion. We give an extension of Lemma 1 of [35] to general self-similar processes with ergodic scaling transformations. The *Hausdorff dimension* of a subset A is defined by

$$\dim A = \inf\,\{\gamma>0\colon m_\gamma(A) = 0\}\,.$$

Proposition 5.2. *For any* $\gamma>0$,

$$P(m_\gamma(Z_1)>0) = 0 \quad or \quad 1\,.$$

Furthermore, there exists γ_0, $0\leq\gamma_0\leq 1$, *such that*

$$\dim Z_1 = \gamma_0 \quad a.s.$$

Proof. Consider an event $F=[m_\gamma(Z_1)>0]$. Let $0<a<1$, and $S=S_a$. Then, we have $S^{-1}\,F\subset F$, since $Z_1\circ S=Z_a$, where $Z_1\circ S$ denotes Z_1 for the process $\{(SX)(s)\colon s\geq 0\}$. This implies the first assertion. For the second assertion, the proof goes similarly as in [35]. q.e.d.

(3) Hausdorff dimension of graphs of sample paths

Denote the graph of path by G_t:

$$G_t = \{(s,\, X(s))\colon 0\leq s\leq t\}\,.$$

Proposition 5.3. *There exists* γ_0, $0\leq\gamma_0\leq 2$, *such that*

$$\dim G_1 = \gamma_0 \quad a.s.$$

Proof. It is enough to show that for any $\gamma > 0$,

$$P(m_\gamma(G_1) > 0) = 0 \quad \text{or} \quad 1 .$$

Note that

$$\begin{aligned} G_1 \circ S &= \{(s, a^{-\kappa} X(as)) \colon 0 \leq s \leq 1\} \\ &= \{(u/a, a^{-\kappa} X(u)) \colon 0 \leq u \leq a\} , \end{aligned}$$

where $G_1 \circ S$ denotes G_1 for the process $\{SX(s) \colon s \geq 0\}$. This implies that $m_\gamma(G_1 \circ S) \leq c m_\gamma(G_a)$ for some constant c. Take a, $0 < a < 1$. Then, $S^{-1}[m_\gamma(G_1) > 0] \subset [m_\gamma(G_1) > 0]$, and so we have the above assertion. q.e.d.

Kôno [17] gave some estimates for the Hausdorff dimension of graph and range of sample paths. Checking his conditions by using well-known fact about the density of stable distribution (cf. Ibragimov and Linnik [9]), we have that the Hausdorff dimension of graph of (α, β)-fractional stable process is equal to $2-\kappa$ for $1 < \alpha < 2$ and $0 < \beta < 1 - 1/\alpha$. From this fact, we expect that the Hausdorff measure properties of sample paths are closely related to the local growth properties rather than the uniform growth property.

(4) Slow points

Kahane [10] showed the existence of slow points for Brownian paths and recently this problem attracts interests of probabilists (cf. [6], [31]). A point T is called a *slow point* of a sample path if

$$\limsup_{\delta \downarrow 0} | X(T+\delta) - X(T) | \delta^{-\kappa} < \infty .$$

Proposition 5.4.

$$P(\textit{ there exist slow points }) = 0 \quad \textit{or} \quad 1 .$$

Furthermore, there exists γ_0, $0 \leq \gamma_0 \leq 1$, *such that*

$$\dim \{ \textit{slow points} \} = \gamma_0 \quad a.s.$$

Sketch of proof. If T is a slow point of a path, aT is a slow point of SX. Thus, the first assertion is easily derived from the ergodicity of S. The second assertion is derived similarly as in the proof of Proposition 5.3 with slight modifications such as letting the set of slow points take the place of G_t, and so we omit its details. q.e.d.

(5) Irregularity points

Orey and Taylor [30] studied the Hausdorff measure properties of ir-

regularity points of Brownian paths. This problem will be formulated for self-similar processes as follows: put

$$L_\phi(t) = \limsup_{h\downarrow 0} \frac{X(t+h)-X(t)}{h^\kappa\, \phi(h)}$$

$$E_{\lambda,t} = \{s\colon 0\leq s\leq t,\ L_\phi(s)\geq\lambda\}\ ,$$

for a positive, slowly varying function ϕ and $\lambda>0$.

Proposition 5.5. *There exists a constant C_λ, $0\leq C_\lambda\leq 1$, such that*

$$\dim E_{\lambda,1} = C_\lambda \quad a.s.$$

Proof. Since ϕ is slowly varying, $L_\phi(t)\circ S=L_\phi(at)$ and $E_{\lambda,t}\circ S=E_{\lambda,at}$, where $S=S_a$ and $L_\phi(t)\circ S$ and $E_{\lambda,t}\circ S$ denote $L_\phi(t)$ and $E_{\lambda,t}$ for the process $\{SX(s)\colon s\geq 0\}$. This means that

$$S^{-1}\,[m_\gamma(E_{\lambda,1})>0]\subset[m_\gamma(E_{\lambda,1})>0]\ ,$$

for any $\gamma>0$ and $0<a<1$. Therefore,

$$P(m_\gamma(E_{\lambda,1})>0) = 0 \text{ or } 1\ .$$

The uniqueness of the dimension is proved similarly as in the proofs of the previous propositions. q.e.d.

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