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## ON THE COEFFICIENTS OF CERTAIN FAMILY OF MODULAR EQUATIONS

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### Abstract

The  $n$ -th modular equation for the elliptic modular function  $j(z)$  has large coefficients even for small  $n$ , and those coefficients grow rapidly as  $n \rightarrow \infty$ . The growth of these coefficients was first obtained by Cohen ([5]). And, recently Cais and Conrad ([1], §7) considered this problem for the Hauptmodul  $j_5(z)$  of the principal congruence group  $\Gamma(5)$ . They found that the ratio of logarithmic heights of  $n$ -th modular equations for  $j(z)$  and  $j_5(z)$  converges to 60 as  $n \rightarrow \infty$ , and observed that 60 is the group index  $[\overline{\Gamma(1)} : \overline{\Gamma(5)}]$ . In this paper we prove that their observation is true for Hauptmoduln of somewhat general Fuchsian groups of the first kind with genus zero.

### 1. Introduction

Let  $\mathfrak{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$  be the complex upper half plane and  $j(z) = q^{-1} + 744 + 196884q + \cdots$  be the elliptic modular function on  $SL_2(\mathbb{Z})$  with  $z \in \mathfrak{H}$  and  $q = e^{2\pi iz}$ . Further, let  $\Phi_n^j(X, Y) = 0$  be the  $n$ -th modular equation for  $j(z)$  (see [6, 10, 11]). Then  $\Phi_n^j(X, Y)$  is a polynomial with integral coefficients satisfying  $\Phi_n^j(j(z), j(nz)) = 0$ , and is irreducible as a polynomial in  $X$  over  $\mathbb{C}(Y)$ . Moreover it is known that  $\Phi_p^j(X, Y)$  satisfies the Kronecker congruences, and  $\Phi_n^j(X, Y)$  has large coefficients even for small  $n$ . For example,

$$\begin{aligned} \Phi_3^j(X, Y) = & X(X + 2^{15} \cdot 3 \cdot 5^3)^3 + Y(Y + 2^{15} \cdot 3 \cdot 5^3)^3 - X^3 Y^3 \\ & + 2^3 \cdot 3^2 \cdot 31 X^2 Y^2 (X + Y) - 2^2 \cdot 3^3 \cdot 9907 XY (X^2 + Y^2) \\ & + 2 \cdot 3^4 \cdot 13 \cdot 193 \cdot 6367 X^2 Y^2 + 2^{16} \cdot 3^5 \cdot 5^3 \cdot 17 \cdot 263 XY (X + Y) \\ & - 2^{31} \cdot 5^6 \cdot 22973 XY. \end{aligned}$$

Note that the coefficients of  $\Phi_n^j(X, Y)$  grow quite rapidly as  $n \rightarrow \infty$ , which was first estimated by Cohen ([5]) as follows.

For a nonzero polynomial  $P(X_1, \dots, X_r) \in \mathbb{C}[X_1, \dots, X_r]$ , let  $h(P(X_1, \dots, X_r))$  be the *logarithmic height* of  $P(X_1, \dots, X_r)$  defined by the logarithm of the maximum of

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the absolute values of its coefficients. And, throughout this article we use  $\mathcal{O}$ -notation which has the following meaning; let  $f$  and  $g$  be complex valued functions defined on some set  $S$  and  $h$  be a real valued positive function defined on  $S$ . Then  $f = g + \mathcal{O}(h)$  means that there exists an absolute positive constant  $A$  such that  $|f - g| \leq A \cdot h$  on  $S$ . With the aid of height and  $\mathcal{O}$ -notation Cohen showed that how rapidly  $h(\Phi_n^j(X, Y))$  grows as  $n \rightarrow \infty$ , that is, for any positive integer  $n$  we have

$$(1.1) \quad h(\Phi_n^j(X, Y)) = 6\psi(n) \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1) \right\},$$

where  $\psi(n) = n \prod_{p|n} (1 + 1/p)$ .

On the other hand Cais and Conrad recently considered the modular equations of the Hauptmodul  $j_5(z) = q^{-1/5}(1 + q - q^3 + q^5 + \cdots)$  of  $\Gamma(5)$ . For a positive integer  $n$  with  $(n, 5) = 1$  we let  $\Phi_n^{j_5}(X, Y) = 0$  be the  $n$ -th modular equation for  $j_5(z)$  defined as in [1, Definition 6.4]. Then  $\Phi_n^{j_5}(X, Y)$  is a polynomial with integral coefficients satisfying  $\Phi_n^{j_5}(j_5(z), j_5(nz)) = 0$ , and is irreducible as a polynomial in  $X$  over  $\mathbb{C}(Y)$ . In addition,  $\Phi_p^{j_5}(X, Y)$  also satisfies the Kronecker congruences ([1, Theorem 6.8]). But unlike the case of  $\Phi_n^j(X, Y)$ ,  $\Phi_n^{j_5}(X, Y)$  has much smaller coefficients, for example,

$$\Phi_3^{j_5}(X, Y) = X^4 Y^3 + X^3 - 3X^2 Y^2 - XY^4 - Y.$$

They indeed estimated the logarithmic height of  $\Phi_n^{j_5}(X, Y)$ , precisely, for any positive integer  $n$  with  $(n, 5) = 1$

$$h(\Phi_n^{j_5}(X, Y)) = \frac{1}{10} \psi(n) \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1) \right\},$$

from which they derived by comparing with  $h(\Phi_n^j(X, Y))$  that

$$\lim_{\substack{n \rightarrow \infty \\ (n, 5) = 1}} \frac{h(\Phi_n^j(X, Y))}{h(\Phi_n^{j_5}(X, Y))} = 60 = [\overline{\Gamma(1)} : \overline{\Gamma(5)}]$$

where  $\overline{\Gamma(1)}$  and  $\overline{\Gamma(5)}$  denote the images of  $\Gamma(1)$  and  $\Gamma(5)$  in  $PSL_2(\mathbb{R})$ . But Cais and Conrad did not explain why the ratio of logarithmic heights converges to the group index.

So it is natural and worthwhile to ask whether

$$\frac{h(\Phi_n^j(X, Y))}{h(\Phi_n^f(X, Y))} \rightarrow [\overline{\Gamma(1)} : \overline{\Gamma}]$$

as  $n \rightarrow \infty$  with some conditions on  $n$  for a Hauptmodul  $f(z)$  of arbitrary congruence subgroup  $\Gamma$ . In Theorem 2.1 (1) we shall prove that the answer is affirmative for clas-

sical congruence subgroups. We further consider a similar question about subgroups  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  of  $SL_2(\mathbb{R})$  which appear in “Monstrous Moonshine” phenomenon. And we will prove in Theorem 2.1 (2) that the ratio of logarithmic heights in this case is also related to a certain summand of group indices.

In what follows we fix an integer  $N$ , and define necessary congruence subgroups

$$\begin{aligned}\Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}, \\ \Gamma^0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod{N} \right\}, \\ \Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \\ \Gamma^1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \pmod{N} \right\}, \\ \Gamma(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.\end{aligned}$$

## 2. Preliminaries and statements of the results

In this section we recall the definition of modular equations for Hauptmoduln of various subgroups of  $SL_2(\mathbb{R})$ .

For a Fuchsian group  $\Gamma$  of the first kind with genus zero, we define a Hauptmodul of  $\Gamma$  by an automorphic function  $f(z)$  for  $\Gamma$  satisfying  $A_0(\Gamma) = \mathbb{C}(f(z))$ . Here by  $A_0(\Gamma)$  we mean the field of all automorphic functions for  $\Gamma$  (see [11]). In this paper we fix that  $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$  for a positive integer  $m$ , and  $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$  is a Hauptmodul of  $\Gamma$  with  $a_n \in \mathbb{R}$  for all  $n \geq 0$ . While considering this Hauptmodul  $f(z)$  of  $\Gamma$ , it is a necessary condition that the genus of  $\Gamma$  is zero, and as for the genus formula of  $\Gamma$  we refer to [9, Theorem 1.1].

For a positive integer  $n$  with  $(n, mN) = 1$  we have the following disjoint coset decomposition

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma = \bigcup_{\substack{a>0 \\ ad=n}} \bigcup_{\substack{0 \leq b < d \\ (a,b,d)=1}} \Gamma \sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where  $\sigma_a \in SL_2(\mathbb{Z})$  satisfies  $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \pmod{mN}$ . This can be proved by observing

$$\left| \Gamma \setminus \Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma \right| = n \prod_{p|n} \left( 1 + \frac{1}{p} \right) = \psi(n)$$

and using [11, Proposition 3.36].

REMARK. Since  $\sigma_a \Gamma \sigma_a^{-1} = \Gamma$  for any positive divisor  $a$  of  $n$ , we have  $\mathbb{C}(f) = A_0(\Gamma) = A_0(\sigma_a^{-1} \Gamma \sigma_a) = \mathbb{C}(f \circ \sigma_a)$ , and hence for given  $a$  we can define a rational function  $P_a(T) \in \mathbb{C}(T)$  such that  $f \circ \sigma_a = P_a(f)$ . For positive divisors  $a, b$  of  $n$  we easily see that

- (1)  $a \equiv \pm 1 \pmod{N} \Leftrightarrow P_a(T) = T$ , and  $\bar{a} = \bar{b} \in (\mathbb{Z}/N\mathbb{Z})^\times / \{\pm 1\} \Leftrightarrow P_a(T) = P_b(T)$ ,
- (2)  $P_a(P_b(T)) = P_{ab}(T) = P_b(P_a(T))$ .

If we let  $P_a(T) = A(T)/B(T) \in \mathbb{C}(T)$  with  $A(T), B(T) \in \mathbb{C}[T]$  and  $(A(T), B(T)) = 1$ , then  $\deg A(T), \deg B(T) \leq 1$  except when  $\deg A(T) = \deg B(T) = 0$  because  $\mathbb{C}(f \circ \sigma_a) = \mathbb{C}(f)$ .

We now consider the following polynomial  $\Psi_n^f(X, z)$  with the indeterminate  $X$

$$\Psi_n^f(X, z) = \prod_{\substack{a>0 \\ ad=n}} \prod_{\substack{0 \leq b < d \\ (a,b,d)=1}} \left( X - f \circ \sigma_a \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(z) \right).$$

Note that  $\deg_X \Psi_n^f(X, z) = \psi(n)$ . Since all the coefficients of  $\Psi_n^f(X, z)$  are the elementary symmetric functions of the  $f \circ \sigma_a \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ , they are invariant under  $\Gamma$ , i.e.,  $\Psi_n^f(X, z) \in \mathbb{C}(f(z))[X]$  and we may write  $\Psi_n^f(X, f(z))$  instead of  $\Psi_n^f(X, z)$ . Then as in the usual argument of modular equations, we see that  $\Psi_n^f(X, f(z))$  is irreducible over  $\mathbb{C}(f(z))$ . And we see from [8] that  $f(z)^{r_n} \Psi_n^f(X, f(z)) \in \mathbb{C}[X, f(z)]$  for  $r_n = -\sum_{s \in S_{1,\infty} \cap S_{2,0}} \text{ord}_s f(z)$ , where  $S_{1,\infty}$  (respectively,  $S_{2,0}$ ) is the set of all points of  $(\Gamma \cap \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}) \setminus \mathfrak{H}^*$  such that  $f(z)$  (respectively,  $f(nz)$ ) has poles (respectively, zeros) (see also [3, Theorem 3.3] or the proof of [4, Theorem 10]). Here we note that  $r_n \leq -\sum_{s \in S_{1,\infty}} \text{ord}_s f(z) = [\mathbb{C}(f(z), f(nz)) : \mathbb{C}(f(z))] \leq n \prod_{p|n} (1 + 1/p)$ , because  $\Psi_n^f(P_n(f(nz)), f(z)) = 0$ .

Therefore for those Hauptmoduln  $f(z)$  of  $\Gamma$  and integer  $n$  with  $(n, mN) = 1$  we define the  $n$ -th modular equation  $\Phi_n^f(X, Y) = Y^{r_n} \Psi_n^f(X, Y)$ , namely

$$\Phi_n^f(X, f(z)) = f(z)^{r_n} \cdot \prod_{\substack{a>0 \\ ad=n}} \prod_{\substack{0 \leq b < d \\ (a,b,d)=1}} \left( X - f \circ \sigma_a \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(z) \right).$$

Here we remark that if we confine ourselves to a Hauptmodul  $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$  with  $a_n \in \mathbb{Z}$ , we could justify that  $\Phi_n^f(X, Y) \in \mathbb{Z}[X, Y]$  and  $\Phi_p^f(X, Y)$  satisfies the Kronecker congruences depending on  $P_p(T)$  in the above remark. But we will not go further into this direction.

Next, unlike the case  $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$  we further consider a subgroup  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  of  $SL_2(\mathbb{R})$  which appears in “Monstrous Moonshine” phenomenon. For the details, we recommend the readers to refer [2].

Let  $N > 1$  be an integer and  $e$  be a Hall divisor of  $N$ , that is,  $e$  is a positive divisor of  $N$  such that  $(e, N/e) = 1$ . For a Hall divisor  $e$  of  $N$  we define an Atkin-Lehner involution of  $\Gamma_0(N)$  as a matrix with determinant 1 of the form

$$\begin{pmatrix} a\sqrt{e} & \frac{b}{\sqrt{e}} \\ c\frac{N}{\sqrt{e}} & d\sqrt{e} \end{pmatrix} \quad \text{where } a, b, c, d \in \mathbb{Z}.$$

Let  $W_e$  be the set of all Atkin-Lehner involutions with a fixed Hall divisor  $e$  of  $N$ . Then these sets satisfy the following multiplication rule:

$$(2.1) \quad W_e W_f = W_f W_e = W_k \quad \text{where } k = \frac{e}{(e, f)} \cdot \frac{f}{(e, f)}.$$

Notice that  $k$  is a Hall divisor of  $N$  if  $e$  and  $f$  are Hall divisors of  $N$ . Assume that  $S$  is a subset of the Hall divisors of  $N$  closed under the above multiplication rule. By  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  we mean the subgroup of  $SL_2(\mathbb{R})$  generated by all elements of  $\Gamma_0(N)$  and  $W_e$  for all  $e \in S$ . If  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  is of genus zero, then we can choose a Hauptmodul  $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$  with  $a_n \in \mathbb{Z}$ . In [2] Chen and Yui defined, for a positive integer  $n$  prime to  $N$ , the  $n$ -th modular equation  $\Phi_n^f(X, Y) = 0$  for which

$$\Phi_n^f(X, f(z)) = \prod_{\substack{a>0 \\ ad=n}} \prod_{\substack{0 \leq b < d \\ (a,b,d)=1}} \left( X - f \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(z) \right).$$

And they proved that  $\Phi_n^f(X, Y)$  is a polynomial with integral coefficients satisfying  $\Phi_n^f(f(z), f(nz)) = 0$  and it is irreducible as a polynomial in  $X$  over  $\mathbb{C}(Y)$ . But, for the purpose of this article, it is enough to assume that  $f(z)$  has only real Fourier coefficients, i.e.,  $a_n \in \mathbb{R}$  for all  $n \geq 0$ .

Now we are ready to state our main theorem.

**Theorem 2.1.** (1) Let  $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$  be a Hauptmodul of  $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$  with  $a_n \in \mathbb{R}$ . For a positive integer  $n$  with  $(n, mN) = 1$ , we get

$$h(\Phi_n^f(X, Y)) = \frac{6\psi(n)}{[\Gamma(1) : \bar{\Gamma}]} \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1) \right\}.$$

(2) Let  $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$  be a Hauptmodul of  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  with  $a_n \in \mathbb{R}$ . For a positive integer  $n$  with  $(n, N) = 1$ , we have

$$h(\Phi_n^f(X, Y)) = \sum_{e \in S} \frac{6\psi(n)}{[\Gamma(1) : \bar{\Gamma}_0(N/e)]} \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1) \right\}.$$

Combining (1.1) and Theorem 2.1, we can readily achieve the following corollary.

**Corollary 2.2.** (1) *With the notations as in Theorem 2.1 (1), we obtain*

$$\lim_{\substack{n \rightarrow \infty \\ (n, mN)=1}} \frac{h(\Phi_n^f(X, Y))}{h(\Phi_n^j(X, Y))} = \frac{1}{[\Gamma(1) : \bar{\Gamma}]}.$$

(2) *With the notations as in Theorem 2.1 (2), we get*

$$\lim_{\substack{n \rightarrow \infty \\ (n, N)=1}} \frac{h(\Phi_n^f(X, Y))}{h(\Phi_n^j(X, Y))} = \sum_{e \in S} \frac{1}{[\Gamma(1) : \bar{\Gamma}_0(N/e)]}.$$

We conclude this section with some remarks. For an arbitrary intersection of classical congruence subgroups

$$\Gamma' = \Gamma_0(N_1) \cap \Gamma^0(N_2) \cap \Gamma_1(N_3) \cap \Gamma^1(N_4) \cap \Gamma(N_5),$$

we have  $\alpha^{-1}\Gamma'\alpha = \Gamma_1(N) \cap \Gamma_0(mN)$  where  $N = \text{lcm}(N_3, N_4, N_5)$  and

$$\alpha = \begin{pmatrix} \text{lcm}(N_2, N_4, N_5) & 0 \\ 0 & 1 \end{pmatrix}, \quad m = \frac{\text{lcm}(N_1, N_3, N_5) \text{lcm}(N_2, N_4, N_5)}{N}.$$

If  $g(z) = q_h^{-1} + \sum_{n=0}^{\infty} a_n q_h^n$  is a Hauptmodul of  $\Gamma'$  with  $h = \text{lcm}(N_2, N_4, N_5)$  and  $q_h = e^{2\pi iz/h}$ , then  $f(z) := g \circ \alpha(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$  is a Hauptmodul of  $\Gamma_1(N) \cap \Gamma_0(mN)$ . Since the  $n$ -th modular equation  $\Phi_n^g(X, Y)$  for  $g(z)$  is, essentially, irreducible as a polynomial in  $X$  over  $\mathbb{C}(Y)$  satisfying  $\Phi_n^g(g(z), g(nz)) = 0$ , we obtain  $\Phi_n^f(X, Y) = \Phi_n^g(X, Y)$  by observing  $\Phi_n^g(g(hz), g(hnz)) = 0$  and  $f(z) = g(hz)$ . Thus Theorem 2.1 (1) holds for any congruence subgroup of  $\Gamma_0(N_1)$ ,  $\Gamma^0(N_2)$ ,  $\Gamma_1(N_3)$ ,  $\Gamma^1(N_4)$ ,  $\Gamma(N_5)$  or arbitrary intersection of them. For example, since

$$\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(5) \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma_1(5) \cap \Gamma_0(25)$$

and  $f(z) := j_5(5z)$  is a Hauptmodul of  $\Gamma_1(5) \cap \Gamma_0(25)$  with the same  $n$ -th modular equation when  $(n, 5) = 1$ , we can recover the result of Cais and Conrad from Theorem 2.1 (1).

If  $S$  contains all the Hall divisors of  $N$ , we write  $\Gamma_0(N)_+$  as the group  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ . In [2, Appendix 2] Chen and Yui calculated some modular equations for Hauptmoduln of  $\Gamma_0(N)$  and  $\Gamma_0(N)_+$ . For instance,

$$\begin{aligned} \Phi_2^{\Gamma_0(3)}(X, Y) &= X^3 + (-Y^2 + 108)X^2 + (-153Y + 2268)X \\ &\quad + (Y^3 + 108Y^2 + 2268Y - 46224), \end{aligned}$$

$$\begin{aligned}\Phi_2^{\Gamma_0(3)+}(X, Y) &= X^3 + (-Y^2 + 1566)X^2 + (17343Y + 741474)X \\ &\quad + (Y^3 + 1566Y^2 + 7417474Y - 28166076),\end{aligned}$$

where  $\Phi_2^{\Gamma_0(3)}$  and  $\Phi_2^{\Gamma_0(3)+}$  stand for the second modular equations of the (normalized) Hauptmoduln of  $\Gamma_0(3)$  and  $\Gamma_0(3)+$ , respectively. We remark that Theorem 2.1 (2) also gives a reason why the logarithmic height of  $\Phi_n^{\Gamma_0(3)}$  is smaller than that of  $\Phi_n^{\Gamma_0(3)+}$  for not only  $n = 2$  but also sufficiently large  $n$ .

### 3. Proof of Theorem 2.1

To prove Theorem 2.1 it is necessary to study the behavior of Hauptmodul at each cusp of  $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$  or  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ . In this section we recall some lemmas which give us useful informations about these cusps.

First lemma provides us a criterion to determine whether or not given two cusps are equivalent under  $\Gamma$ .

**Lemma 3.1.** *Let  $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$  and*

$$\Delta = \{\pm(1 + Nk) \in (\mathbb{Z}/mN\mathbb{Z})^\times \mid k = 0, 1, \dots, m-1\}.$$

*We assume that  $a, c, a'$  and  $c'$  are integers such that  $(a, c) = (a', c') = 1$ . By  $\pm 1/0$  we mean  $\infty$ . Then the cusp  $a/c$  is equivalent to  $a'/c'$  under  $\Gamma$  if and only if there exist  $x \in \Delta$  and  $n \in \mathbb{Z}$  such that*

$$\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \begin{pmatrix} xa + nc \\ x^{-1}c \end{pmatrix} \pmod{mN}.$$

*Proof.* Suppose that  $a/c$  is equivalent to  $a'/c'$  under  $\Gamma$ , i.e., there exists  $\gamma \in \Gamma$  such that  $a'/c' = \gamma(a/c)$ . Since  $a, c, a', c'$  are integers satisfying  $(a, c) = (a', c') = 1$ , we have  $\begin{pmatrix} a' \\ c' \end{pmatrix} = \pm \gamma \begin{pmatrix} a \\ c \end{pmatrix}$ . By putting  $\gamma = \begin{pmatrix} x & n \\ z & w \end{pmatrix} \in \Gamma$  we have the desired assertion. Conversely suppose that there exist  $x \in \Delta$  and  $n \in \mathbb{Z}$  satisfying the above congruence in the hypothesis. Since the natural reduction map of  $SL_2(\mathbb{Z})$  into  $SL_2(\mathbb{Z}/mN\mathbb{Z})$  is surjective, let  $\gamma \in SL_2(\mathbb{Z})$  be a preimage of  $\begin{pmatrix} x & n \\ 0 & x^{-1} \end{pmatrix} \in SL_2(\mathbb{Z}/mN\mathbb{Z})$ . Note that  $\gamma \in \{\pm 1\} \cdot \Gamma$  and  $\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \gamma \begin{pmatrix} a \\ c \end{pmatrix} \pmod{mN}$ . Now it is an elementary fact that if  $u, v, z, w$  are integers such that  $(u, v) = (z, w) = 1$  and  $\begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} z \\ w \end{pmatrix} \pmod{N}$ , then  $u/v$  and  $z/w$  are equivalent under  $\Gamma(N)$  ([11, Lemma 1.42]). So in our case there exists  $\gamma' \in \Gamma(mN)$  such that  $a'/c' = \gamma'(\gamma(a/c))$ . This completes the proof since  $\Gamma(mN) \subset \Gamma$ .  $\square$

Let  $\phi(x)$  be the Euler function. Then it is worthy of remarking that

$$(3.1) \quad [\overline{\Gamma(1)} : \overline{\Gamma}] = [\overline{\Gamma(1)} : \overline{\Gamma_0(mN)}][\overline{\Gamma_0(mN)} : \overline{\Gamma}] = \frac{[\overline{\Gamma(1)} : \overline{\Gamma_0(mN)}]\phi(mN)}{|\Delta|},$$



which will be used in the proof of Lemma 3.10 and Lemma 3.11. From the next two lemmas we can determine whether a given cusp is equivalent to the cusp infinity under  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ .

**Lemma 3.2.** *Let  $S_{\Gamma_0(N)}$  be the set of pairs  $(c, a)$  satisfying*

- (1)  $(1, 0) \in S_{\Gamma_0(N)}$ .
  - (2)  $c > 1, c \mid N, 1 \leq a < c, (c, a) = 1$ .
  - (3) *If  $(c, a), (c, a_1) \in S_{\Gamma_0(N)}$  and  $a_1 \equiv a \pmod{c, N/c}$  then  $a = a_1$ .*
- Then the set  $\{a/c \mid (c, a) \in S_{\Gamma_0(N)}\}$  is a set of complete representatives of all inequivalent cusps of  $\Gamma_0(N)$ .*

*Proof.* This lemma is indeed well-known ([7, Proposition 1.23]). For the reader's convenience we give an alternative proof. We first observe that the cardinality of  $S_{\Gamma_0(N)}$  is  $1 + \sum_{c>1, c \mid N} \varphi((c, N/c))$  because the natural map  $(\mathbb{Z}/c\mathbb{Z})^\times \rightarrow (\mathbb{Z}/(c, N/c)\mathbb{Z})^\times$  is surjective. Since the number of inequivalent cusps of  $\Gamma_0(N)$  is  $\sum_{d \mid N} \varphi((d, N/d))$  (see [11, Proposition 1.43]), it is enough to prove that arbitrary two distinct pairs  $(c, a), (c', a') \in S_{\Gamma_0(N)}$  are inequivalent to each other. Suppose that they are equivalent under  $\Gamma_0(N)$ . By substituting  $N = 1, m = N$ , and  $\Delta = (\mathbb{Z}/N\mathbb{Z})^\times$  in Lemma 3.1, we must have that  $c = c'$  and  $x \equiv 1 \pmod{N/c}$ . Thus  $a' \equiv xa + nc \pmod{N}$  with  $x \equiv 1 \pmod{N/c}$  implies that  $a' \equiv a \pmod{c, N/c}$ . By hypothesis (3) we have  $a' = a$ .  $\square$

**Lemma 3.3.** *Let  $S$  be a subset of Hall divisors of  $N$  closed under the multiplication rule (2.1). Then the cusps*

$$\left\{ \frac{1}{N/e} \mid e \in S \right\}$$

*are all those equivalent under  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  to  $\infty$  among the set of representatives  $\{a/c \mid (c, a) \in S_{\Gamma_0(N)}\}$  described in Lemma 3.2.*

*Proof.* For given  $e \in S$  there exist  $b, d \in \mathbb{Z}$  satisfying  $de - b(N/e) = 1$ . Thus we have  $W_e = \Gamma_0(N) \begin{pmatrix} \sqrt{e} & b/\sqrt{e} \\ N/\sqrt{e} & d\sqrt{e} \end{pmatrix}$ . Since  $\begin{pmatrix} \sqrt{e} & b/\sqrt{e} \\ N/\sqrt{e} & d\sqrt{e} \end{pmatrix}(\infty) = 1/(N/e)$ , we have the assertion.  $\square$

Using the above lemmas we are able to prove Theorem 2.1 by adopting the idea of Cais and Conrad ([1]). For convenience, if  $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$  with  $a_n \in \mathbb{R}$  is a Hauptmodul of  $\Gamma$  (respectively,  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ ), then we simply write “ $f(z)$  is on  $\Gamma$ ” (respectively, on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ ).

**Lemma 3.4.**  *$\Gamma$  and  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  have no elliptic points on  $i\mathbb{R}_{>1}$ .*

**Proof.** If  $it$  ( $t > 1$ ) is a fixed point of an elliptic element  $\sigma \in SL_2(\mathbb{R})$ , then the absolute value of the trace of  $\sigma$ ,  $|\text{tr}(\sigma)|$ , is less than 2. Moreover, if  $\sigma \in SL_2(\mathbb{Z})$ , we have  $\sigma = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  which gives rise to a contradiction. If  $\sigma \in SL_2(\mathbb{R}) \setminus SL_2(\mathbb{Z})$ , then we may assume  $\sigma = \pm \begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ cN/\sqrt{e} & d\sqrt{e} \end{pmatrix}$  for  $a, b, c, d \in \mathbb{Z}$  and a Hall divisor  $e$  of  $N$ . Since  $\sigma$  fixes  $it$  and  $|\text{tr}(\sigma)| < 2$ , we have  $a = d$ ; hence  $a = 0$  and  $\sigma = \pm \begin{pmatrix} 0 & b/\sqrt{e} \\ cN/\sqrt{e} & 0 \end{pmatrix}$ . Since  $\sigma$  has determinant 1, we obtain  $-bcN = e$  and so  $bc(N/e) = -1$ , that is,  $b/c = -1$ . On the other hand  $\sigma$  fixes  $it$ , so we have  $b = -cNt$ . Thus  $Nt^2 = 1$ , which is a contradiction.  $\square$

Since  $f(z) = q^{-1} + \dots$  has real Fourier coefficients,  $f(it)$  is real and  $|f(it)| \rightarrow \infty$  as  $t \rightarrow \infty$ . Moreover  $f'(z)$  is nonvanishing on  $i\mathbb{R}_{>1}$  by Lemma 3.4, so we see that  $f(it)$  is strictly increasing for  $t \geq 1$ . Thus we can choose real numbers  $s > 1$  and  $1 \leq t_0 \leq t_1$  such that  $f(it_0) = s$ ,  $f(it_1) = 2s$ .

**Lemma 3.5.** For  $t_0 \leq t \leq t_1$ , we have

$$h(\Phi_n^f(X, f(it))) = \sum_{\substack{a>0 \\ ad=n}} S_d(t) + \mathcal{O}(\psi(n)),$$

where

$$S_d(t) = \begin{cases} \sum_{\substack{0 \leq b < d \\ (a,b,d)=1}} \log \max \left\{ 1, \left| f \circ \sigma_a \left( \frac{ait+b}{d} \right) \right| \right\} & \text{if } f(z) \text{ is on } \Gamma, \\ \sum_{\substack{0 \leq b < d \\ (a,b,d)=1}} \log \max \left\{ 1, \left| f \left( \frac{ait+b}{d} \right) \right| \right\} & \text{if } f(z) \text{ is on } \langle \Gamma_0(N), W_e \rangle_{e \in S}. \end{cases}$$

Here the implicit  $\mathcal{O}$ -constant depends only on  $f$ ,  $t_0$  and  $t_1$ .

**Proof.** It is well-known that the coefficients of a monic polynomial  $P(x) = (x - w_1) \cdots (x - w_d)$  are laid in between  $2^{-d}M$  and  $2^dM$  where  $M = \prod_{j=1}^d \max\{1, |w_j|\}$ . Taking logarithm we see that

$$(3.2) \quad h(P) = \sum_{j=1}^d \log \max\{1, |w_j|\} + \mathcal{O}(d)$$

with an implicit absolute  $\mathcal{O}$ -constant which is independent of  $d$  and  $P$ .

If  $f(z)$  is on  $\Gamma$ , then for  $t_0 \leq t \leq t_1$

$$\Phi_n^f(X, f(it)) = f(z)^{r_n} \prod_{\substack{a>0 \\ ad=n}} \prod_{\substack{0 \leq b < d \\ (a,b,d)=1}} \left( X - (f \circ \sigma_a) \left( \frac{ait+b}{d} \right) \right).$$

Applying (3.2) we have

$$\begin{aligned} h(\Phi_n^f(X, f(it))) \\ = r_n \log f(it) + \sum_{\substack{a>0 \\ ad=n}} \sum_{\substack{0 \leq b < d \\ (a,b,d)=1}} \log \max \left\{ 1, \left| f \circ \sigma_a \left( \frac{ait+b}{d} \right) \right| \right\} + \mathcal{O}(\psi(n)). \end{aligned}$$

Since  $0 \leq r_n \leq \psi(n)$  and  $s = f(it_0) \leq f(it) \leq f(it_1) = 2s$ , we get  $r_n \log f(it) = \mathcal{O}(\psi(n))$  where the implicit  $\mathcal{O}$ -constant depends only on  $f$ ,  $t_0$  and  $t_1$ .

As for the case where  $f(z)$  is on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ , the same argument can be applied, and hence we omit the detailed proof.  $\square$

Next goal is to calculate each term in the summation  $S_d(t)$ . For this purpose we are in need of the following lemma.

**Lemma 3.6.** *For  $z = \xi + i\eta \in \mathfrak{H}$ , let  $g(z) = a_{-1}q_h^{-1} + \sum_{n=0}^{\infty} a_n q_h^n$  with  $q_h = e^{2\pi iz/h}$  for a positive integer  $h$ . We assume that if  $a_{-1} = 0$  (respectively,  $a_{-1} \neq 0$ ), then  $g(z)$  (respectively,  $q_h g(z)$ ) is absolutely convergent for  $\eta > 0$ . Then for  $\eta \geq 1/2$ , we have*

$$\log \max\{1, |g(z)|\} = \begin{cases} \mathcal{O}(1) & \text{if } a_{-1} = 0, \\ \frac{2\pi i \eta}{h} + \mathcal{O}(1) & \text{if } a_{-1} \neq 0. \end{cases}$$

Here the implicit  $\mathcal{O}$ -constants depend only  $g(z)$ .

*Proof.* Since  $g(z+h) = g(z)$ , we may assume that  $-h/2 \leq \xi \leq h/2$ . Suppose first that  $a_{-1} = 0$ . Since  $|g(z)| \rightarrow |a_0|$  as  $\eta \rightarrow \infty$ , there is a real number  $\eta_0 \geq 1/2$  such that for  $\eta > \eta_0$ ,  $|a_0|/2 \leq |g(z)| \leq |a_0| + 1$ . Hence, for  $\eta > \eta_0$  we derive  $\log \max\{1, |g(z)|\} = \mathcal{O}(1)$ . Here the implicit  $\mathcal{O}$ -constant depends only on  $a_0$ , that is  $g$ . For  $1/2 \leq \eta \leq \eta_0$  we note that  $\log \max\{1, |g(z)|\}$  is a continuous function on the set

$$\left\{ \xi + i\eta \in \mathfrak{H} \mid -\frac{h}{2} \leq \xi \leq \frac{h}{2} \text{ and } \frac{1}{2} \leq \eta \leq \eta_0 \right\}$$

and hence is bounded on this set. Note that the upper bound depends only on  $g$  and is independent of the choice of  $\eta_0$ .

If  $a_{-1} \neq 0$ ,  $|q_h g(z)| \rightarrow |a_{-1}|$  as  $\eta \rightarrow \infty$  so that we obtain the assertion by the same argument as above.  $\square$

Let  $M$  be a positive integer. Then it is more convenient to consider the displaced interval  $I_M = [1/(M+1), (M+2)/(M+1))$  rather than the usual interval  $[0, 1)$ . Cohen proved in [5] that  $I_M$  can be expressed as

$$I_M = \bigcup_{k=1}^M \bigcup_{\substack{h=1 \\ (h,k)=1}}^k I_M\left(\frac{h}{k}\right),$$

which is a disjoint union of sets  $I_M(h/k)$ . Here each  $I_M(h/k)$  is an interval of the form  $[\rho_1^{(h/k)}, \rho_2^{(h/k)})$  containing  $h/k$  and

$$\begin{aligned} \frac{1}{2Mk} &\leq \frac{h}{k} - \rho_1^{(h/k)} \leq \frac{1}{(M+1)k}, \\ \frac{1}{2Mk} &\leq \rho_2^{(h/k)} - \frac{h}{k} \leq \frac{1}{(M+1)k}. \end{aligned}$$

For real numbers  $h, k$  and  $x$ , we put

$$g_{h,k}(x) = \frac{2\pi nt/d^2 k^2}{(at/d)^2 + (x - h/k)^2},$$

which will be used for estimating the sum  $S_d(t)$ . Thus  $a, d$  and  $t$  are related to  $S_d(t)$ . Note that the width of the cusp  $\sigma_a(\infty)$  is 1, because  $f \circ \sigma_a = P_a(f)$  as remarked in §2. Also observe that  $\langle \Gamma_0(N), W_e \rangle_{e \in \mathcal{S}}$  contains  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Hence in any case we may reindex the sum in  $S_d(t)$  via

$$b \mapsto \begin{cases} b & \text{if } \frac{b}{d} \in \left[ \frac{1}{N+1}, 1 \right), \\ b+d & \text{if } \frac{b}{d} \in \left[ 0, \frac{1}{N+1} \right). \end{cases}$$

**Lemma 3.7.** *Let  $f$  be on  $\Gamma$ .*

(1) *If  $at/d \geq 1/2$ , then we have*

$$\log \max \left\{ 1, \left| f \circ \sigma_a \left( \frac{ait+b}{d} \right) \right| \right\} = \begin{cases} \frac{2\pi nt}{d^2} + \mathcal{O}(1) & \text{if } \bar{a} \in \Delta, \\ \mathcal{O}(1) & \text{otherwise.} \end{cases}$$

(2) *Put  $M = [d/\sqrt{nt}]$ . If  $at/d \leq 1$ , then  $M \geq 1$  and, for  $b/d \in I_M(h/k)$ , we get*

$$\begin{aligned} &\log \max \left\{ 1, \left| f \circ \sigma_a \left( \frac{ait+b}{d} \right) \right| \right\} \\ &= \begin{cases} g_{h,k}(b/d) + \mathcal{O}(1) & \text{if } k \equiv 0 \pmod{mN} \text{ and } \bar{h} \in \bar{a}\Delta, \\ \mathcal{O}(1) & \text{otherwise.} \end{cases} \end{aligned}$$

In both cases the implicit  $\mathcal{O}$ -constants depend only on  $f$ .

Proof. (1) By Lemma 3.1,  $\sigma_a(\infty)$  is equivalent to  $\infty$  under  $\Gamma$  if and only if  $\bar{a} \in \Delta$ . Using this, Lemma 3.6 gives us the assertion.

(2) Since  $(h, k) = 1$ , we can find  $\gamma_{h,k} := \begin{pmatrix} v & u \\ -k & h \end{pmatrix} \in SL_2(\mathbb{Z})$ . By routine calculation we see that

$$\operatorname{Im}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right) = \frac{nt/d^2k^2}{(at/d)^2 + (b/d - h/k)^2} = \frac{1}{2\pi} g_{h,k}\left(\frac{b}{d}\right).$$

Since  $b/d \in I_M(h/k) = [\rho_1^{(h/k)}, \rho_2^{(h/k)}]$ , we obtain

$$\left|\frac{b}{d} - \frac{h}{k}\right| \leq \frac{1}{(M+1)k} \leq \frac{\sqrt{nt}}{dk}.$$

Moreover, we achieve

$$\frac{at}{d} = \frac{nt}{d^2} \leq \frac{\sqrt{nt}}{dk}$$

which implies that

$$\operatorname{Im}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right) \geq \frac{1}{2}.$$

By Lemma 3.1,  $\sigma_a(\gamma_{h,k}^{-1}(\infty))$  is equivalent to  $\infty$  under  $\Gamma$  if and only if  $k \equiv 0 \pmod{mN}$  and  $\bar{h} \in \bar{a}\Delta$ . Taking  $g(z) = f \circ \sigma_a \circ \gamma_{h,k}^{-1}(z)$  in Lemma 3.6, we have the assertion. More precisely, if  $k \equiv 0 \pmod{mN}$  and  $\bar{h} \in \bar{a}\Delta$ , then

$$\begin{aligned} \left|f \circ \sigma_a\left(\frac{ait+b}{d}\right)\right| &= \left|f \circ \sigma_a \circ \gamma_{h,k}^{-1}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right)\right| \\ &= 2\pi \operatorname{Im}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right) = g_{h,k}\left(\frac{b}{d}\right). \end{aligned}$$

Other case corresponds to the holomorphic one in Lemma 3.6. Therefore we prove the lemma.  $\square$

**Lemma 3.8.** *Let  $f$  be on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ .*

(1) *If  $at/d \geq 1/2$ , then we have*

$$\log \max \left\{ 1, \left| f\left(\frac{ait+b}{d}\right) \right| \right\} = \frac{2\pi at}{d} + \mathcal{O}(1).$$

(2) Put  $M = [d/\sqrt{nt}]$ . If  $at/d \leq 1$ , then  $M \geq 1$  and, for  $b/d \in I_M(h/k)$ , we establish

$$\begin{aligned} & \log \max \left\{ 1, \left| f\left(\frac{ait+b}{d}\right) \right| \right\} \\ &= \begin{cases} g_{h,k}\left(\frac{b}{d}\right) + \mathcal{O}(1) & \text{for } e \in S, \quad k \equiv 0 \pmod{N/e} \quad \text{and} \quad \bar{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times, \\ \mathcal{O}(1) & \text{otherwise.} \end{cases} \end{aligned}$$

In both cases the implicit  $\mathcal{O}$ -constants depend only on  $f$ .

*Proof.* Since the first assertion can be proved in a similar way to Lemma 3.7, we only prove (2). The fact that  $\gamma_{h,k}^{-1}(\infty)$  is equivalent to  $\infty$  under  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  yields by Lemma 3.7 that  $h/k$  is equivalent to  $1/(N/e)$  under  $\Gamma_0(N)$  for some Hall divisor  $e \in S$  exactly. In other words, by Lemma 3.1 there are  $\bar{x} \in (\mathbb{Z}/N\mathbb{Z})^\times$ ,  $n \in \mathbb{Z}$  such that  $h \equiv x^{-1} + n \cdot (N/e) \pmod{N}$  and  $k \equiv x \cdot (N/e) \pmod{N}$ . This is equivalent to  $h \equiv x^{-1} \pmod{(N/e)}$  and  $k \equiv 0 \pmod{(N/e)}$ , because  $N/e$  is also a Hall divisor. Thus we have the conclusion.  $\square$

Now, we calculate  $S_d(t)$  more precisely in Lemma 3.10 and 3.10. To this end we need the following lemma in advance.

**Lemma 3.9.** *Let  $k, j$  and  $a$  be positive integers satisfying  $j \mid k$  and  $(j, a) = 1$ . We further let  $\zeta$  be a primitive  $k$ -th root of unity and let*

$$c'_k(l) = \sum_{\substack{h \in (\mathbb{Z}/k\mathbb{Z})^\times \\ h \equiv a \pmod{j}}} \zeta^{hl} \quad \text{for } l \in \mathbb{Z}.$$

Then

$$(3.3) \quad |c'_k(l)| \leq j \cdot (k, l) \quad \text{for any } l \in \mathbb{Z}.$$

*Proof.* Using a primitive  $j$ -th root of unity  $\zeta^{k/j}$  we may rewrite the sum as

$$c'_k(l) = \frac{1}{j} \sum_{i \in \mathbb{Z}/j\mathbb{Z}} \zeta^{-kia/j} \sum_{h \in (\mathbb{Z}/k\mathbb{Z})^\times} \zeta^{(l+ik/j)h}.$$

Let  $\mu(x)$  be the Möbius function. Since the Ramanujan's sum satisfies

$$\sum_{h \in (\mathbb{Z}/k\mathbb{Z})^\times} \zeta^{hx} = \mu\left(\frac{k}{(k, x)}\right) \cdot \phi(k) / \phi\left(\frac{k}{(k, x)}\right)$$

for  $x \in \mathbb{Z}$  and  $\phi(xy) \leq x\phi(y)$  for any  $x, y \in \mathbb{Z}_{>0}$ , we have

$$\begin{aligned} \left| \sum_{h \in (\mathbb{Z}/k\mathbb{Z})^\times} \zeta^{(l+ik/j)h} \right| &\leq \frac{\phi(k)}{\phi(k/(k, l+ik/j))} \leq \left( k, l + \frac{ik}{j} \right) = \left( k, l + \frac{ik}{j}, jl \right) \\ &\leq (k, jl) \leq j \cdot \left( \frac{k}{j}, l \right) \leq j \cdot (k, l), \end{aligned}$$

which implies  $|c'_k(l)| \leq j \cdot (k, l)$ .  $\square$

Here we remark that Cais and Conrad dealt with the case of a rational prime  $j$  dividing  $k$  in [1, Lemma D.3], but it seems to be not true. Indeed, we can find a counterexample when  $k = p = 3$ ,  $a = m = 1$  with the notations as in there. So we correct it and prove the expanded version. It doesn't crucially matter, however, to the results because we need just its boundedness.

**Lemma 3.10.** *Let  $f$  be on  $\Gamma$ .*

- (1) *If  $d < \sqrt{nt}$ , then  $S_d(t) = \mathcal{O}(n/d)$ . Here the implicit  $\mathcal{O}$ -constant depends only upon  $f$ ,  $t_0$  and  $t_1$ .*
- (2) *If  $d \geq \sqrt{nt}$ , then*

$$S_d(t) = \frac{1}{[\Gamma(1) : \bar{\Gamma}]} \cdot \frac{6d}{(a, d)} \phi((a, d)) \log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{(a, d)}\right)\right) + \mathcal{O}\left(\frac{d\sigma_1((a, d))}{(a, d)}\right),$$

where  $\phi(x)$  is the Euler function and  $\sigma_1(x)$  is the sum of positive divisors of  $x$ . Here the implicit  $\mathcal{O}$ -constant depends only upon  $\Gamma$ ,  $f$ ,  $t_0$  and  $t_1$ .

*Proof.* (1) Since the number of elements in  $\{b \mid 0 \leq b < d, (a, b, d) = 1\}$  is  $d\phi((a, d))/(a, d)$ , by Lemma 3.6 and the fact that  $\phi((a, d))/(a, d) \leq 1$  we have

$$\begin{aligned} |S_d(t)| &\leq \sum_{\substack{0 \leq b < d \\ (a, b, d) = 1}} \log \max \left\{ 1, \left| f \circ \sigma_a \left( \frac{ait + b}{d} \right) \right| \right\} \\ &= \begin{cases} \frac{d\phi((a, d))}{(a, d)} \frac{2\pi nt}{d^2} + \mathcal{O}\left(\frac{d\phi((a, d))}{(a, d)}\right) & \text{if } \bar{a} \in \Delta, \\ \mathcal{O}\left(\frac{d\phi((a, d))}{(a, d)}\right) & \text{otherwise} \end{cases} \\ &\leq \begin{cases} \frac{2\pi nt}{d} + C \cdot d & \text{if } \bar{a} \in \Delta \\ C'd & \text{otherwise.} \end{cases} \end{aligned}$$

Using the fact that  $d < nt/d \leq nt_1/d$  we conclude the first assertion.

(2) Note that the assumption  $d \geq \sqrt{nt}$  implies  $at/d \leq 1$ . Put  $M = [d/\sqrt{nt}] \geq 1$ . Then we have by Lemma 3.7

$$\begin{aligned}
 S_d(t) &= \sum_{k=1}^M \sum_{\substack{h=1 \\ (h,k)=1}}^k \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} \log \max \left\{ 1, \left| f \circ \sigma_a \left( \frac{ait+b}{d} \right) \right| \right\} \\
 &= \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{mN} \\ h \in \bar{a}\Delta}} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} \left( g_{h,k} \left( \frac{b}{d} \right) + \mathcal{O}(1) \right) + \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ \text{otherwise}}} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} \mathcal{O}(1) \\
 &= \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{mN} \\ h \in \bar{a}\Delta}} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} g_{h,k} \left( \frac{b}{d} \right) + \mathcal{O}(d).
 \end{aligned}$$

Since the total number for error terms  $\mathcal{O}(1)$  is less than  $d$  and so  $\mathcal{O}(d)$  lies outside of the summation, we can get the last expression in the above summation.

Meanwhile, we see from [5, Lemma 6] that

$$\sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d, (a,b,d)=1}} g_{h,k} \left( \frac{b}{d} \right) = k^{-2} \sum_{f|(a,d)} \mu(f) F_f \left( \frac{dh}{fk} \right) + \mathcal{O} \left( \frac{\sqrt{n} \sigma_1((a, d))}{k(a, d)} \right),$$

where  $F_f(\theta) = (2\pi^2 d/f) \sum_{v \in \mathbb{Z}} e^{-2\pi |v|nt/df} e^{2\pi i v \theta}$  and  $\mu(x)$  is the Möbius function.

Since we have as in [1]

$$\begin{aligned}
 (3.4) \quad C \frac{\sqrt{n} \sigma_1((a, d))}{(a, d)} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1}} \frac{1}{k} &= C \frac{\sqrt{n} \sigma_1((a, d))}{(a, d)} \sum_{1 \leq k \leq M} \frac{\phi(k)}{k} \leq C \cdot M \frac{\sqrt{n} \sigma_1((a, d))}{(q, d)} \\
 &\leq C \frac{\sqrt{n} \sigma_1((a, d))}{(a, d)} \frac{d}{\sqrt{nt}} \leq C \frac{d \sigma_1((a, d))}{(a, d)},
 \end{aligned}$$

we establish that

$$\begin{aligned}
 S_d(t) &= \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{mN} \\ h \in \bar{a}\Delta}} \left\{ k^{-2} \sum_{f|(a,d)} \mu(f) F_f \left( \frac{dh}{fk} \right) + \mathcal{O} \left( \frac{\sqrt{n} \sigma_1((a, d))}{k(a, d)} \right) \right\} + \mathcal{O}(d) \\
 &= \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{mN} \\ h \in \bar{a}\Delta}} \left\{ k^{-2} \sum_{f|(a,d)} \mu(f) F_f \left( \frac{dh}{fk} \right) \right\} + \mathcal{O} \left( \frac{d \sigma_1((a, d))}{(a, d)} \right)
 \end{aligned}$$



$$= \sum_{f|(a,d)} \mu(f) \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{mN} \\ h \in \bar{a}\Delta}} k^{-2} F_f \left( \frac{dh}{fk} \right) + \mathcal{O} \left( \frac{d\sigma_1((a,d))}{(a,d)} \right).$$

We now consider the sum

$$(3.5) \quad \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{mN} \\ h \in \bar{a}\Delta}} k^{-2} F_f \left( \frac{dh}{fk} \right) = \frac{2\pi^2 d}{f} \sum_{v \in \mathbb{Z}} C_M \left( \frac{dv}{f} \right) e^{-2\pi |v|nt/df},$$

where

$$C_M(l) = \sum_{\substack{1 \leq k \leq M \\ k \equiv 0 \pmod{mN}}} k^{-2} c_k(l)$$

and

$$c_k(l) = \sum_{\substack{1 \leq h \leq k \\ (h,k)=1 \\ h \in \bar{a}\Delta}} e^{2\pi i h l / k} \quad \text{for any } l \in \mathbb{Z}.$$

We have to calculate  $C_M(l)$  and  $c_k(l)$  to know the upper bound of the sum of (3.5). By Lemma 3.9 we know that  $|c_k(l)| \leq |\Delta| mN(k, l)$  for  $l \in \mathbb{Z} - \{0\}$ . So when  $l \neq 0$ , we have

$$\begin{aligned} |C_M(l)| &\leq |\Delta| mN \sum_{k=1}^{\infty} k^{-2}(k, l) \leq |\Delta| mN \sum_{d|l} d \sum_{j=1}^{\infty} \frac{1}{j^2 d^2} \\ &= |\Delta| mN \frac{\pi^2}{6} \frac{1}{|l|} \sum_{d|l} \frac{|l|}{d} = |\Delta| mN \frac{\pi^2}{6} \frac{\sigma_1(|l|)}{|l|}; \end{aligned}$$

hence

$$|C_M(l)| = \mathcal{O} \left( \frac{\sigma_1(|l|)}{|l|} \right)$$

for  $l \neq 0$ , where the implicit  $\mathcal{O}$ -constant depends only on  $\Gamma$ . In case of  $l = 0$  we consider the natural surjective homomorphism  $\pi : (\mathbb{Z}/k\mathbb{Z})^\times \rightarrow (\mathbb{Z}/mN\mathbb{Z})^\times$  which gives us

$$c_k(0) = |\pi^{-1}(\Delta)| = |\Delta| |\ker \pi| = |\Delta| \frac{\phi(k)}{\phi(mN)}.$$

Hence by [1, Lemma D.1] and (3.1) we obtain

$$\begin{aligned} C_M(0) &= \sum_{\substack{1 \leq k \leq M \\ k \equiv 0 \pmod{mN}}} k^{-2} \frac{|\Delta|}{\phi(mN)} \phi(k) \\ &= \frac{6}{\pi^2} \frac{|\Delta|}{\phi(mN)[\Gamma(1) : \Gamma_0(mN)]} \log M + \mathcal{O}(1) \\ &= \frac{6}{\pi^2 [\overline{\Gamma}(1) : \overline{\Gamma}]} \log M + \mathcal{O}(1), \end{aligned}$$

where the implicit  $\mathcal{O}$ -constant is absolute, i.e., it is independent of  $\Gamma$  and  $M$ .

Therefore we get

$$\begin{aligned} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ h \equiv 0 \pmod{mN} \\ h \in \bar{a}\Delta}} k^{-2} F_f \left( \frac{dh}{fk} \right) &= \frac{12d}{f[\Gamma(1) : \overline{\Gamma}]} \log M + \mathcal{O} \left( \frac{d}{f} \right) \\ &\quad + \mathcal{O} \left( \sum_{v \in \mathbb{Z} - \{0\}} \frac{\sigma_1(d|v|/f)}{|v|} e^{-2\pi|v|nt/df} \right), \end{aligned}$$

where the implicit  $\mathcal{O}$ -constants depend only on  $\Gamma$  and  $t$ . Since  $f \mid (a, d)$  and  $(a, d) \mid a = n/d$ , we have  $df \leq n$ ; hence  $1 \leq t_0 \leq t$  implies that

$$e^{-2\pi(|v|-1)nt/df} \leq e^{-2\pi(|v|-1)t} \leq e^{-2\pi(|v|-1)}.$$

By putting

$$C_1 = \sum_{v \in \mathbb{Z} - \{0\}} \frac{\sigma_1(|v|)}{|v|} e^{-2\pi(|v|-1)}$$

and using the fact

$$\sigma_1 \left( \frac{d}{f} |v| \right) \leq \sigma_1 \left( \frac{d}{f} \right) \sigma_1(|v|),$$

we obtain

$$\sum_{v \in \mathbb{Z} - \{0\}} \frac{\sigma_1(d|v|/f)}{|v|} e^{-2\pi|v|nt/df} \leq C_1 \sigma_1 \left( \frac{d}{f} \right) e^{-2\pi nt/df} \leq C_1 \sigma_1 \left( \frac{d}{f} \right) e^{-2\pi n/df}.$$

Thus we deduce

$$\mathcal{O} \left( \sum_{v \in \mathbb{Z} - \{0\}} \frac{\sigma_1(d|v|/f)}{|v|} e^{-2\pi|v|nt/df} \right) = \mathcal{O} \left( \sigma_1 \left( \frac{d}{f} \right) e^{-2\pi n/df} \right),$$

where the implicit  $\mathcal{O}$ -constant depends only on  $\Gamma$ .

Since  $M = [\sqrt{d^2/(nt)}]$  and  $1 \leq t_0 \leq t \leq t_1$ , we see that  $\log M = (1/2) \log(d^2/n) + \mathcal{O}(1)$  where the implicit  $\mathcal{O}$ -constant depends only on  $t_0$  and  $t_1$ .

Consequently, we have

$$\sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{mN} \\ h \in \bar{a}\Delta}} k^{-2} F_f \left( \frac{dh}{fk} \right) = \frac{6d}{f[\Gamma(1) : \bar{\Gamma}]} \log \left( \frac{d^2}{n} \right) + \mathcal{O} \left( \frac{d}{f} \right) + \mathcal{O} \left( \sigma_1 \left( \frac{d}{f} \right) e^{-2\pi n/df} \right)$$

where the implicit  $\mathcal{O}$ -constants depend only on  $\Gamma$ ,  $t_0$  and  $t_1$ . By substituting this for the sum of  $S_d(t)$  we obtain

$$S_d(t) = \sum_{f|(a,d)} \mu(f) \left( \frac{6d}{f[\Gamma(1) : \bar{\Gamma}]} \log \left( \frac{d^2}{n} \right) + \mathcal{O} \left( \frac{d}{f} \right) + \mathcal{O} \left( \sigma_1 \left( \frac{d}{f} \right) e^{-2\pi n/df} \right) \right) + \mathcal{O} \left( \frac{d\sigma_1((a, d))}{(a, d)} \right),$$

where the implicit  $\mathcal{O}$ -constants depend only on  $\Gamma$ ,  $f$ ,  $t_0$  and  $t_1$ . Since

$$\sum_{f|(a,d)} \left| \mu(f) \frac{d}{f} \right| \leq \sum_{f|(a,d)} \frac{d}{f} = \frac{d\sigma_1((a, d))}{(a, d)},$$

the first error term contributes  $\mathcal{O}(d\sigma_1((a, d))/(a, d))$ .

Similarly, since  $\sigma_1(df/(a, d)) \leq \sigma_1(d/(a, d))\sigma_1(f)$  and  $e^{-2\pi nf/d(a,d)} \leq e^{-2\pi f}$ , we derive

$$\begin{aligned} \sum_{f|(a,d)} \left| \mu(f) \sigma_1 \left( \frac{d}{f} \right) e^{-2\pi n/df} \right| &\leq \sum_{f|(a,d)} \sigma_1 \left( \frac{d}{f} \right) e^{-2\pi n/df} = \sum_{f|(a,d)} \sigma_1 \left( \frac{df}{(a, d)} \right) e^{-2\pi nf/d(a,d)} \\ &\leq \sigma_1 \left( \frac{d}{(a, d)} \right) \sum_{f|(a,d)} \sigma_1(f) e^{-2\pi f}, \end{aligned}$$

and so the second error term contributes  $\mathcal{O}(\sigma_1(d/(a, d)))$ . From the fact  $\phi((a, d)) = \sum_{f|(a,d)} \mu(f)(a, d)/f$  we finally obtain

$$S_d(t) = \frac{6}{[\Gamma(1) : \bar{\Gamma}]} \frac{d}{(a, d)} \phi((a, d)) \log \left( \frac{d^2}{n} \right) + \mathcal{O} \left( \sigma_1 \left( \frac{d}{(a, d)} \right) \right) + \mathcal{O} \left( \frac{d\sigma_1((a, d))}{(a, d)} \right),$$

where the implicit  $\mathcal{O}$ -constants depend only on  $\Gamma$ ,  $f$ ,  $t_0$  and  $t_1$ . This completes the proof.  $\square$

**Lemma 3.11.** *Let  $f$  be on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ .*

- (1) *If  $d < \sqrt{nt}$ , then  $S_d = \mathcal{O}(n/d)$ . Here the implicit  $\mathcal{O}$ -constant depends only upon  $f$ ,  $t_0$  and  $t_1$ .*  
 (2) *If  $d \geq \sqrt{nt}$ , then*

$$S_d = \sum_{e \in S} \frac{1}{[\Gamma(1) : \Gamma_0(N/e)]} \frac{6d}{(a, d)} \phi((a, d)) \log \left( \frac{d^2}{n} \right) \\ + \mathcal{O} \left( \sigma_1 \left( \frac{d}{(a, d)} \right) \right) + \mathcal{O} \left( \frac{d\sigma_1((a, d))}{(a, d)} \right).$$

Here the implicit  $\mathcal{O}$ -constant depends only upon  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ ,  $f$ ,  $t_0$  and  $t_1$ .

Proof. It is possible for us to prove (1) with similar arguments as in Lemma 3.10, so we omit the detail. We only give a proof of (2). By using Lemma 3.8 we have

$$S_d(t) = \sum_{k=1}^M \sum_{\substack{h=1 \\ (h,k)=1}}^k \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} \log \max \left\{ 1, \left| f \left( \frac{ait+b}{d} \right) \right| \right\} \\ = \sum_{e \in S} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{N/e} \\ \bar{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} g_{h,k} \left( \frac{b}{d} \right) + \mathcal{O}(d \cdot s),$$

where  $M = [d/\sqrt{nt}]$  and  $s$  is the number of Hall divisors in  $S$ . From [5, Lemma 6] and (3.4) we can see that

$$S_d(t) = \sum_{e \in S} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{N/e} \\ \bar{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} \left\{ k^{-2} \sum_{f|(a,d)} \mu(f) F_f \left( \frac{dh}{fk} \right) + \mathcal{O} \left( \frac{\sqrt{n}\sigma_1((a, d))}{k(a, d)} \right) \right\} + \mathcal{O}(d \cdot s) \\ = \sum_{e \in S} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{N/e} \\ \bar{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} \left\{ k^{-2} \sum_{f|(a,d)} \mu(f) F_f \left( \frac{dh}{fk} \right) + \mathcal{O} \left( \frac{\sqrt{n}\sigma_1((a, d))}{k(a, d)} \right) \right\} + \mathcal{O}(d \cdot s) \\ = \sum_{e \in S} \sum_{f|(a,d)} \mu(f) \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{N/e} \\ \bar{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2} F_f \left( \frac{dh}{fk} \right) + \mathcal{O} \left( \frac{d\sigma_1((a, d))}{(a, d)} \cdot s \right).$$

As we did in (3.5) we change the inner summand as

$$\sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{N/e} \\ \bar{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2} F_f\left(\frac{dh}{fk}\right) = \frac{2\pi^2 d}{f} \sum_{v \in \mathbb{Z}} C_M\left(\frac{dv}{f}\right) e^{-2\pi|v|nt/df},$$

where

$$C_M(l) = \sum_{\substack{1 \leq k \leq M \\ k \equiv 0 \pmod{N/e}}} k^{-2} c_k(l)$$

and

$$c_k(l) = \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ \bar{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} e^{2\pi i h l / k} \quad \text{for any } l \in \mathbb{Z}.$$

Then, by Lemma 3.9 we know that  $|c_k(l)| \leq \phi(N/e) \cdot (N/e) \cdot (k, l)$  for  $l \in \mathbb{Z} - \{0\}$ . So when  $l \neq 0$ , we have

$$|C_M(l)| = \mathcal{O}\left(\frac{\sigma_1(|l|)}{|l|}\right),$$

where the implicit  $\mathcal{O}$ -constant depends only on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ . When  $l = 0$ , we obtain  $c_k(0) = \phi(k)$ . Hence it follows from [1, Lemma D.1] that

$$\begin{aligned} C_M(0) &= \sum_{\substack{1 \leq k \leq M \\ k \equiv 0 \pmod{N/e}}} k^{-2} \phi(k) \\ &= \frac{6}{\pi^2} \frac{1}{[\Gamma(1) : \Gamma_0(N/e)]} \log M + \mathcal{O}(1), \end{aligned}$$

where the implicit  $\mathcal{O}$ -constant is absolute, namely it is independent of  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  and  $M$ .

Therefore we get

$$\begin{aligned} &\sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{N/e} \\ \bar{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2} F_f\left(\frac{dh}{fk}\right) \\ &= \frac{12d}{f[\Gamma(1) : \Gamma_0(N/e)]} \log M + \mathcal{O}\left(\frac{d}{f}\right) + \mathcal{O}\left(\sum_{v \in \mathbb{Z} - \{0\}} \frac{\sigma_1(d|v|/f)}{|v|} e^{-2\pi|v|nt/df}\right), \end{aligned}$$

where the implicit  $\mathcal{O}$ -constants depend only on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$  and  $t$ . Applying the same estimates as in the proof of Lemma 3.10 we have

$$\begin{aligned} & \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \pmod{N/e} \\ \bar{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2} F_f \left( \frac{dh}{fk} \right) \\ &= \frac{6d}{f[\Gamma(1) : \Gamma_0(N/e)]} \log \left( \frac{d^2}{n} \right) + \mathcal{O} \left( \frac{d}{f} \right) + \mathcal{O} \left( \sigma_1 \left( \frac{d}{f} \right) e^{-2\pi n/df} \right), \end{aligned}$$

where the implicit  $\mathcal{O}$ -constants depend only on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ ,  $t_0$  and  $t_1$ . By plugging this into the sum of  $S_d(t)$  we achieve

$$\begin{aligned} S_d(t) &= \sum_{e \in S} \sum_{f|(a,d)} \mu(f) \left( \frac{6d}{f[\Gamma(1) : \Gamma_0(N/e)]} \log \left( \frac{d^2}{n} \right) + \mathcal{O} \left( \frac{d}{f} \right) + \mathcal{O} \left( \sigma_1 \left( \frac{d}{f} \right) e^{-2\pi n/df} \right) \right) \\ &\quad + \mathcal{O} \left( \frac{d\sigma_1((a,d))}{(a,d)} \cdot s \right), \end{aligned}$$

where the implicit  $\mathcal{O}$ -constants depend only on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ ,  $f$ ,  $t_0$  and  $t_1$ . Thus, in like manner as in the proof of Lemma 3.10 we finally conclude

$$\begin{aligned} S_d(t) &= \sum_{e \in S} \frac{6}{[\Gamma(1) : \Gamma_0(N/e)]} \frac{d}{(a,d)} \phi((a,d)) \log \left( \frac{d^2}{n} \right) \\ &\quad + \mathcal{O} \left( \sigma_1 \left( \frac{d}{(a,d)} \right) \cdot s \right) + \mathcal{O} \left( \frac{d\sigma_1((a,d))}{(a,d)} \cdot s \right), \end{aligned}$$

where the implicit  $\mathcal{O}$ -constants depend only on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ ,  $f$ ,  $t_0$  and  $t_1$ . The number  $s$  depends only on the group  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ , and so we have the assertion.  $\square$

**Lemma 3.12.** For  $1 \leq t_0 \leq t \leq t_1$ ,

(1) if  $f(z)$  is on  $\Gamma$ , then we have

$$h(\Phi_n^f(X, f(it))) = \frac{6\psi(n)}{[\Gamma(1) : \bar{\Gamma}]} \left( \log n - 2 \sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1) \right),$$

(2) if  $f(z)$  is on  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ , then we achieve

$$h(\Phi_n^f(X, f(it))) = \sum_{e \in S} \frac{6\psi(n)}{[\Gamma(1) : \Gamma_0(N/e)]} \left( \log n - 2 \sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1) \right).$$

In case (1) (resp., (2)) the implicit  $\mathcal{O}$ -constant depends only on  $\Gamma$  (resp.,  $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ ),  $f$ ,  $t_0$  and  $t_1$ .

Proof. In each case we are able to use the same method. Thus we put down only the first case. From Lemma 3.10 we know that

$$h(\Phi_n^f(X, f(it))) = \sum_{\substack{a>0 \\ ad=n}} S_d(t) + \mathcal{O}(\psi(n)) = H_1 + H_2 + \mathcal{O}(\psi(n)),$$

where

$$H_1 = \sum_{\substack{a>0, ad=n \\ d<\sqrt{nt}}} S_d(t) = \mathcal{O}(\psi(n))$$

and

$$H_2 = \sum_{\substack{a>0, ad=n \\ d\geq\sqrt{nt}}} S_d(t) = \frac{6\psi(n)}{[\Gamma(1) : \overline{\Gamma}]} \left( \log n - 2 \sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1) \right)$$

by means of Cohen's results in [5, §4].  $\square$

**Lemma 3.13.** *Let  $P(X) \in \mathbb{C}[X]$  be any nonzero polynomial of degree  $\leq D$ . Then for any  $\theta > 0$ , there exists an absolute constant  $c_\theta > 0$ , depending only on  $\theta$ , such that*

$$\left| h(P(X)) - \log \sup_{\theta \leq x \leq 2\theta} |P(x)| \right| \leq c_\theta D.$$

Proof. We refer to [1] or [5].  $\square$

Now we are ready to prove our main theorem.

Proof of Theorem 2.1. To avoid troublesome, we define  $h(0) = -\infty$ . Let  $D = \psi(n)$  and we write

$$\Phi_n^f(X, Y) = P_0(Y)X^D + P_1(Y)X^{D-1} + \cdots + P_D(Y)$$

with  $P_j(Y) \in \mathbb{C}[Y]$  and  $P_0(Y) \neq 0$ . Certainly,  $h(\Phi_n^f(X, Y)) = \max_{0 \leq j \leq D} h(P_j(Y))$ . Since  $\deg P_j(Y) \leq D$ , Lemma 3.13 yields that

$$\begin{aligned} h(\Phi_n^f(X, Y)) &= \max_{0 \leq j \leq D} \log \sup_{s \leq y \leq 2s} |P_j(y)| + \mathcal{O}(D) \\ &= \sup_{s \leq y \leq 2s} \max_{0 \leq j \leq D} \log |P_j(y)| + \mathcal{O}(D) \end{aligned}$$

where the implicit  $\mathcal{O}$ -constant depends only on  $s$ . Since  $\max_{0 \leq j \leq D} \log |P_j(y)| = h(\Phi_n^f(X, y))$ , we obtain

$$h(\Phi_n^f(X, Y)) = \sup_{s \leq y \leq 2s} h(\Phi_n^f(X, y)) + \mathcal{O}(D).$$

Here we note that the interval  $[t_0, t_1]$  corresponds bijectively to the interval  $[s, 2s]$ , and so we have

$$h(\Phi_n^f(X, Y)) = \sup_{t_0 \leq t \leq t_1} h(\Phi_n^f(X, f(it))) + \mathcal{O}(D).$$

Therefore, we get the conclusion by Lemma 3.12.  $\square$

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