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Osaka University
Decompositions of a Completely Simple Semigroup

By Takayuki Tamura

§ 1. Introduction.

In this paper we shall study the method of finding all the decompositions of a completely simple semigroup and shall apply the result to the two special cases: an indecomposable completely simple semigroup [2] and a ξ-semigroup [4]. By a decomposition of a semigroup S we mean a classification of the elements of S due to a congruence relation in S. Let S be a completely simple semigroup throughout this paper. According to Rees [1], it is faithfully represented as a regular matrix semigroup whose ground group is G and whose defining matrix semigroup is $P=(p_{\mu\lambda}), \mu \in L, \lambda \in M$, that is, either $S=\{(x; \lambda \mu) | x \in G, \mu \in L, \lambda \in M\}$ or $S$ with a two-sided zero 0, where the multiplication is defined as

$$(x; \lambda \mu)(y; \xi \eta) = \begin{cases} (xp_{\mu\lambda}y; \lambda \eta) & \text{if } p_{\mu\lambda} = 0 \\ 0 & \text{if } p_{\mu\lambda} = 0 \text{ and hence } S \text{ has } 0. \end{cases}$$

For the sake of simplicity S is denoted as $\text{Simp.}(G, 0; P)$ or $\text{Simp.}(G; P)$ according as $S$ has 0 or not. $L$ and $M$ may be considered as a right-singular semigroup and a left-singular semigroup respectively [5].


We define two equivalence relations $\frac{\sigma}{M}$ and $\frac{\tau}{L}$ in $M$ and $L$ respectively: we mean by $\lambda \frac{\sigma}{M} \sigma$ that $p_{\sigma \lambda} = 0$ if and only if $p_{\eta \sigma} = 0$ for all $\eta \in L$; by $\mu \frac{\tau}{L} \tau$ that $p_{\mu \xi} = 0$ if and only if $p_{\xi \eta} = 0$ for all $\xi \in M$. Let $L=\sum L_1$, and $M=\sum M_1$ be the classifications of the elements of $L$ and $M$ due to the relations $\frac{\tau}{L}$ and $\frac{\sigma}{M}$ respectively.

**Lemma 1.** A defining matrix is equivalent to one which satisfies the following two conditions. Let $e$ be a unit of $G$.

1. For any $m$, there is $\alpha(m) \in L$ such that $p_{\alpha(m), \xi} = e$ for all $\xi \in M_m$.
2. For any $l$, there is $\beta(l) \in M$ such that $p_{\eta, \beta(l)} = e$ for all $\eta \in L_l$. 

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12 (1960), 269-275.
Proof. First, for any \( \alpha(m) \in L \) such that

(3) \( p_{\alpha(m), \xi} = 0 \) for all \( \xi \in M_m \),

(4) If \( \alpha(m_1) = \alpha(m_2) \), then \( \alpha(m_1) = \alpha(m_2) \).

Next, for a mapping \( m \rightarrow \alpha(m) \), \( \beta(l) \in M \) is determined such that the following conditions are satisfied:

(5) \( p_{\eta, \beta(l)} = 0 \) for all \( \eta \in L_l \)

(6) if there is \( m \) such that \( \alpha(m) \in L_l \), then we let \( \beta(l) \in M_m \), and \( \alpha(m) \in L_l \) for one \( m \) among \( m \).

Consider the matrices

\[
Q = (q_{\lambda_1, \lambda_2}) \quad \lambda_1, \lambda_2 \in M
\]

and

\[
R = (r_{\mu_1, \mu_2}) \quad \mu_1, \mu_2 \in L
\]

where

\[
q_{\lambda_1, \lambda_2} = \begin{cases} p_{\alpha(m), \xi}^{-1} & \text{if } \lambda_1 = \lambda_2 = \xi \in M_m \\ 0 & \text{if } \lambda_1 \neq \lambda_2 \\ e & \text{if } \mu_1 = \mu_2 = \alpha(m) \text{ for some } m \end{cases}
\]

\[
r_{\mu_1, \mu_2} = \begin{cases} p_{\alpha(m'), \beta(l)}^{-1} p_{\alpha(m'), \beta(l)}^{-1} & \text{if } \alpha(m') = \mu_1 = \mu_2 = \eta \text{ for all } m', \text{ and we let } \eta \in L_l \text{ and } \beta(l) \in M_{m'} \\ 0 & \text{if } \mu_1 \neq \mu_2 \end{cases}
\]

Then, setting \( R(PQ) = (t_{\mu \lambda}) \), we have

\[
t_{\mu \lambda} = \begin{cases} p_{\mu, \lambda} p_{\alpha(m), \lambda}^{-1} & \text{if } \mu = \alpha(m'') \text{ for some } m'', \text{ and } \lambda \in M_m \\ p_{\alpha(m'), \beta(l)} p_{\mu, \beta(l)}^{-1} p_{\mu, \alpha(m')} \lambda & \text{if } \mu = \alpha(m'') \text{ for all } m'', \text{ we let } \lambda \in M_m, \mu \in L_l, \beta(l) \in M_{m''}, \end{cases}
\]

and it is easily shown that \( RPQ \) satisfies (1) and (2). The conditions (4) and (6) are available for the proof of (2) in the case that \( \eta = \alpha(m) \in L_l \) for some \( m \). Thus the proof of the Lemma is completed.

The form, \( RPQ \), which satisfies (1) and (2), is called a normal form of \( P \).

\section{3. Decompositions.}

Hereafter we shall assume that \( S \) has a matrix of normal form as the defining matrix. Let \( \sim \) denote a congruence relation in \( S \). \( \sim \) is said to be trivial if either \( x \sim y \) for all \( x, y \) or \( x \sim y \) for only \( x = y \).

Lemma 2. Let \( \sim \) be a non-trivial congruence relation. \((x; \lambda \mu) \sim (y; \tau)\) implies \( \lambda \sim \sigma, \mu \sim \tau \) and hence there are \( \alpha \) and \( \beta \) such that
Proof. Suppose \( p_{\eta \lambda} \neq 0 \) as well as \( p_{\eta \sigma} = 0 \) for some \( \eta, \), and take any element \((u; \xi \eta)\), then, for certain \( p_{\mu \xi} = 0 \),

\[
(u; \xi \eta) = (ux^{-1} p_{\eta \lambda}^{-1}; \xi \eta_0) (x; \lambda \mu) (p_{\mu \xi}^{-1}; \xi \eta_0) \\
\sim (ux^{-1} p_{\eta \lambda}^{-1}; \xi \eta_0) (y; \sigma \tau) (p_{\mu \xi}^{-1}; \xi \eta_0) = 0.
\]

This shows that the relation \( \sim \) is a trivial congruence relation, contradicting the assumption. The remaining part is similarly proved. The existence of \( \alpha \) and \( \beta \) is clear by a normal form of the defining matrix, q. e. d.

Now we derive the relations \( \approx \) in \( G \), \( \equiv \) in \( M \), and \( \gamma \) in \( L \) from the congruence relation \( \sim \) in \( S \) as defined in the following way.

\[
x \approx y \text{ if there are } \lambda, \sigma M, \mu, \tau L \text{ such that } (x; \lambda \mu) \sim (y; \sigma \tau),
\]

\[
\lambda \equiv \sigma \text{ if there are } x, y G, \mu, \tau L \text{ such that } (x; \lambda \mu) \sim (y; \sigma \tau),
\]

\[
\mu \gamma \tau \text{ if there are } x, y G, \lambda, \sigma M \text{ such that } (x; \lambda \mu) \sim (y; \sigma \tau).
\]

**Lemma 3.** The relations \( \approx, \equiv \) and \( \gamma \) are all congruence relations.

Proof. Reflexivity and symmetry are evident. Let us prove transitivity. By \( x \approx y \) and \( y \approx z \) there are \( \lambda, \mu, \sigma, \tau, \sigma', \tau', \kappa \) and \( \nu \) such that

\[
(x; \lambda \mu) \sim (y; \sigma \tau), \quad (y; \sigma' \tau') \sim (z; \kappa \nu).
\]

According to Lemma 2,

\[
p_{\eta \lambda} = p_{\eta \sigma} = e, \quad p_{\mu \beta} = p_{\tau \beta} = e \quad \text{for certain } \alpha \text{ and } \beta,
\]

so that we get

\[
(e; \sigma' \alpha) (x; \lambda \mu) (e; \beta \tau') \sim (e; \sigma' \alpha) (y; \sigma \tau) (e; \beta \tau')
\]

and hence

\[
(x; \sigma' \tau') \sim (z; \kappa \nu).
\]

Thus we have proved \( x \approx z \).

Transitivity of \( \equiv \) is proved from \( (x; \lambda \mu) \sim (y; \sigma \tau), (y' \sigma' \tau') \sim (z; \kappa \nu) \) and \( (x; \lambda \mu) (y' \sigma' \tau') \sim (y; \sigma \tau) (y' \sigma' \tau') \) where \( p_{\mu \beta} = p_{\tau \beta} = e \). We get transitivity of \( \gamma \) analogously.

Next, \( x \approx y \) implies \( xz \approx yz \) and \( zx \approx yz \) because

\[
(x; \lambda \mu) (z; \beta \mu) \sim (y; \sigma \tau) (z; \beta \mu)
\]

\[
(z; \lambda \alpha) (x; \lambda \mu) \sim (z; \lambda \alpha) (y; \sigma \tau)
\]

under the assumption \( (x; \lambda \mu) \sim (y; \sigma \tau) \) where \( p_{\mu \beta} = p_{\tau \beta} = p_{\alpha \lambda} = p_{\alpha \sigma} = e \). The proof for \( \equiv \) and \( \gamma \) is clear.
Lemma 4. If $\lambda \mathcal{K} \sigma$ and $p_{\lambda \sigma} = 0$, then $p_{\lambda \eta} \approx p_{\eta \sigma} = 0$ for all $\eta$. If $\mu \mathcal{T} \tau$ and $p_{\mu \xi} = 0$, then $p_{\mu \eta} \approx p_{\tau \xi} = 0$ for all $\xi$.

Proof. By Lemma 2, it is evident that $p_{\sigma \eta} = 0, p_{\eta \xi} = 0$. Find $\beta$ such that $p_{\mu \beta} = p_{\beta \eta} = e$. Multiplying each of $(x; \lambda \mu)$ and $(y; \sigma \tau)$ by $(x^{-1}; \beta \mu)$ from right, we get

$$(e; \lambda \mu) \sim (yx^{-1}; \sigma \mu)$$
whence $e \approx yx^{-1}$.

Moreover, from $e; \lambda \eta)(e; \lambda \mu) \sim (e; \lambda \eta)(yx^{-1}; \sigma \mu)$
we have

$p_{\lambda \xi} \approx p_{\sigma \eta}yx^{-1} \approx p_{\eta \xi}$

completing the proof. Similarly $p_{\mu \xi} \approx p_{\tau \beta}$ is proved, q.e.d.

Conversely, consider congruence relations $\mathcal{R}, \mathcal{T}, \approx$ in $M, L, G$ respectively such that

$\lambda \mathcal{R} \sigma$ implies $\lambda \mathcal{K} \sigma$,
$\mu \mathcal{T} \tau$ implies $\mu \mathcal{K} \tau$,
$\approx$ makes Lemma 4 hold.

For these congruence relations, a relation $\sim$ in $S$ is defined as

$$(x; \lambda \mu) \sim (y; \sigma \tau)$$
if $x \approx y, \lambda \mathcal{K} \sigma$, and $\mu \mathcal{T} \tau$.

Then it is easily shown that the relation is a congruence relation.

Theorem 1. We obtain, as follows, every congruence relation in a completely simple semigroup $S$ with a ground group $G$ and with a defining matrix $P = (p_{\lambda \mu}), \lambda \in M, \mu \in L$. First, for a pair of the congruence relations $\mathcal{R}$ and $\mathcal{T}$ taken arbitrarily, independently each other, there is at least one congruence relation $\approx$ in $G$ which satisfies

$\lambda \mathcal{R} \sigma$ implies $p_{\lambda \eta} \approx p_{\eta \sigma}$ for all $\eta$,
$\mu \mathcal{T} \tau$ implies $p_{\mu \xi} \approx p_{\tau \xi}$ for all $\xi$.

By a triplet of the three congruence relations $\mathcal{R}, \mathcal{T}, \approx$, a congruence relation $\sim$ in $S$ is determined as

$$(x; \lambda \mu) \sim (y; \sigma \tau)$$
means that $x \approx y, \lambda \mathcal{R} \sigma$, and $\mu \mathcal{T} \tau$.

§ 4. Examples.

We shall arrange a few examples which follow from Theorem 1.

First, we can easily determine the structure of an indecomposable completely simple semigroup, which was obtained in [2].
Example 1. A completely simple semigroup $S$ is indecomposable if and only if the following three conditions are satisfied.

(7) The ground group is $G = \{e\}$.

(8) $\lambda \sim^\omega_\Delta \sigma$ if and only if $\lambda = \sigma$.

(9) $\mu \sim^\omega_\tau \tau$ if and only if $\mu = \tau$.

Example 2. Consider a finite simple semigroup $S$ with a ground group $G$ and with the defining matrix $\begin{pmatrix} e \\ e \end{pmatrix}$ or $\begin{pmatrix} e & e \end{pmatrix}$.

Let $x \to f(x)$ be a homomorphism of $G$ to certain group $G' : G' = f(G)$, $e' = f(e)$. Then any homomorphism of $S$ is given as either (10) or (11).

(10) $(x ; \lambda \mu) \to (f(x) ; \lambda \mu)$

where the homomorphic image $S'$ of $S$ is also a completely simple semigroup in which $G'$ is the ground group and $P' = (f(\mu \phi))$ is the defining matrix.

(11) $(x ; \lambda \mu) \to f(x)$ where $S' = G'$.

Example 3. A finite simple semigroup $S$ with a ground group $G$ and with the defining matrix $\begin{pmatrix} e & e \\ e & a \end{pmatrix}$ where $a \neq 0$. Any homomorphic image of $S$ is given as one of

$(x ; \lambda \mu) \to (f(x) ; \lambda \mu)$ where $S' = \text{Simp.} \left( f(G) ; \begin{pmatrix} e' & e' \\ e' & f(a) \end{pmatrix} \right)$,

$(x ; \lambda \mu) \to (f(x) ; 1 \mu)$ where $S' = \text{Simp.} \left( f(G) ; \begin{pmatrix} e' \\ e' \end{pmatrix} \right)$ and $f(e) = f(a) = e'$,

$(x ; \lambda \mu) \to (f(x) ; 1 \mu)$ where $S' = \text{Simp.} \left( f(G) ; \begin{pmatrix} e' \\ e' \end{pmatrix} \right)$ and $f(e) = f(a) = e'$.

$\lambda \mu$ where $S' = f(G)$.

§ 5. $\mathfrak{S}$-Semigroups.

In this paragraph $S$ denotes a finite simple semigroup. If a decomposition of $S$ classifies the elements into some classes composed of equal number of elements, then the decomposition is called homogeneous. We term by $\mathfrak{S}$-semigroup a finite semigroup $S$ with $\mathfrak{S}$-property, i.e., the property that every decomposition of $S$ is homogeneous [4]. It goes without saying that any semigroup of order 2 and any indecomposable finite semigroup are $\mathfrak{S}$-semigroups. We assume that the order of $S$ is $\geq 2$.

Lemma 5. A $\mathfrak{S}$-semigroup is simple.

Proof. If a $\mathfrak{S}$-semigroup $S$ is not simple, a proper ideal $I$ exists so
that the difference semigroup \((S: I)\) of \(S\) modulo \(I\) would result in a
non-homogeneous decomposition of \(S\), q.e.d.

**Lemma 6.** If a \(\mathcal{D}\)-semigroup \(S\) has zero \(0\), then \(S\) is indecomposable.

Proof. Let \(\sim\) be a congruence relation in \(S\). From Lemma 5 follows that there is nothing but the trivial decompositions, i.e., either
\(0 \sim x\) for all \(x \in S\) or \(0 \sim x\) for only \(x = 0\). In the latter case, by homogeneity, \(x \equiv y\) implies \(x \sim y\) for every \(x, y\), q.e.d.

**Corollary 1.** If a \(\mathcal{D}\)-semigroup \(S\) has a non-trivial decomposition, then \(S\) is a simple semigroup without zero.

Accordingly a \(\mathcal{D}\)-semigroup \(S\) may be considered as a semigroup
\(S = \text{Simp}(G; (p_{ji}))\) where let \(G\) be a group of order \(g\), let \(P = (p_{ji})\) be
a matrix of \((l, m)\) type i.e. \(i = 1, \ldots, m; j = 1, \ldots, l\).

**Lemma 7.** If \(S\) is a \(\mathcal{D}\)-semigroup which has no zero, then \(m \leq 2, l \leq 2\).

Proof. Suppose, for example, \(m \geq 3\). Consider a congruence relation
\(\sim\) in \(S\) as follows.

\[(x; kj) \sim (y; kj')\quad \text{for any } k > 2, \text{ any } j, \text{ and any } j',\]
\[(x; kj) \sim (y; kj')\quad \text{for any } k > 2, k' > 2, k \equiv k' \text{ any } j, \text{ and any } j',\]
\[(x; 1j) \sim (y; 2j')\quad \text{for any } j, \text{ and any } j',\]

where \(x\) and \(y\) run independently throughout \(G\).

Then we have a non-homogeneous decomposition
\[S = \bar{S} \cup S_3 \cup S_4 \cup \ldots\]
where
\[\bar{S} = \{(x; ij) | x \in G, i = 1, 2; 1 \leq j \leq l\},\]
\[S_k = \{(x; kj) | x \in G, 1 \leq j \leq l\}, k = 3, 4, \ldots\]
and the order of \(\bar{S}\) is \(2gl\), that of \(S_k\) is \(gl\). This contradicts the assumption of \(\mathcal{D}\). Hence \(m \leq 2\). Similarly \(l \leq 2\) is proved.

Therefore a \(\mathcal{D}\)-semigroup which has no zero must have a structure
of the following four.

\[
\text{Simp.}\left(G; \begin{pmatrix} e \\ e \end{pmatrix}\right),
\text{Simp.}\left(G; \begin{pmatrix} e \\ e \end{pmatrix}\right),
\text{Simp.}\left(G; \begin{pmatrix} e \\ e \end{pmatrix}\right),
\text{Group.}
\]

On the other hand, Examples 2, 2' and 3 show that every decom-
position of them is homogeneous. At last we have arrived at
Theorem 2. A finite semigroup is a $\Theta$-semigroup of order $\geq 2$ if and only if it is one of the following six cases.

(C) a $z$-semigroup of order 2 or a semilattice of order 2
(C) a finite group of order $\geq 2$
(C) an indecomposable finite semigroup of order $> 1$

\[
\begin{align*}
\text{Simp.} (G; \begin{pmatrix} e \\ e \end{pmatrix}) & \quad \text{where } G \text{ is a finite group of order } \geq 1, \\
\text{Simp.} (G; \begin{pmatrix} e & e \\ e & a \end{pmatrix}) & \quad a \neq 0
\end{align*}
\]

§ 6. Relations between $\Theta$-property and $\Phi$-property.

In the paper [3, 4] we defined $\Theta$-property of a finite semigroup and proved that an $\Theta$-semigroup is one of the above cases except (C).

Immediately we have

Theorem 3. $\Theta$-property implies $\Phi$-property. Though the converse is not true, it is true that a $\Phi$-semigroup which has a proper decomposition is an $\Theta$-semigroup.

By the way we give a few theorems.

Theorem 4. A unipotent $\Theta$-semigroup of order $> 2$ is a group. A unipotent $\Phi$-semigroup of order $> 2$ is so also.

Theorem 5. A subsemigroup of an $\Theta$-semigroup is an $\Theta$-semigroup. A subsemigroup of an indecomposable semigroup is not always a $\Phi$-semigroup, but $\Phi$-property, in the other cases, is preserved in a subsemigroup. $\Theta$-property and $\Phi$-property are both preserved in a homomorphic image.

(Received July 8, 1960)

References

[5] I have called $L$ a right-singular semigroup if $xy = y$ for all $x, y \in L$. In p. 62, [4], I find a misprint: for “left-singular”, read “right-singular”. But this is not essential for discussion.