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## *Decompositions of a Completely Simple Semigroup*

By Takayuki TAMURA

### § 1. Introduction.

In this paper we shall study the method of finding all the decompositions of a completely simple semigroup and shall apply the result to the two special cases: an indecomposable completely simple semigroup [2] and a  $\mathfrak{S}$ -semigroup [4]. By a decomposition of a semigroup  $S$  we mean a classification of the elements of  $S$  due to a congruence relation in  $S$ . Let  $S$  be a completely simple semigroup throughout this paper. According to Rees [1], it is faithfully represented as a regular matrix semigroup whose ground group is  $G$  and whose defining matrix semigroup is  $P=(p_{\mu\lambda})$ ,  $\mu \in L$ ,  $\lambda \in M$ , that is, either  $S = \{(x; \lambda\mu) \mid x \in G, \mu \in L, \lambda \in M\}$  or  $S$  with a two-sided zero  $0$ , where the multiplication is defined as

$$(x; \lambda\mu)(y; \xi\eta) = \begin{cases} (xp_{\mu\xi}y; \lambda\eta) & \text{if } p_{\mu\xi} \neq 0 \\ 0 & \text{if } p_{\mu\xi} = 0 \text{ and hence } S \text{ has } 0. \end{cases}$$

For the sake of simplicity  $S$  is denoted as

$$\text{Simp.}(G, 0; P) \text{ or } \text{Simp.}(G; P)$$

according as  $S$  has  $0$  or not.  $L$  and  $M$  may be considered as a right-singular semigroup and a left-singular semigroup respectively [5].

### § 2. Normal Form of Defining Matrix.

We define two equivalence relations  $\overset{0}{\sim}_M$  and  $\overset{0}{\sim}_L$  in  $M$  and  $L$  respectively: we mean by  $\lambda \overset{0}{\sim}_M \sigma$  that  $p_{\eta\lambda} \neq 0$  if and only if  $p_{\eta\sigma} \neq 0$  for all  $\eta \in L$ ; by  $\mu \overset{0}{\sim}_L \tau$  that  $p_{\mu\xi} \neq 0$  if and only if  $p_{\tau\xi} \neq 0$  for all  $\xi \in M$ . Let  $L = \sum_I L_I$ , and  $M = \sum_m M_m$  be the classifications of the elements of  $L$  and  $M$  due to the relations  $\overset{0}{\sim}_L$  and  $\overset{0}{\sim}_M$  respectively.

**Lemma 1.** *A defining matrix is equivalent to one which satisfies the following two conditions. Let  $e$  be a unit of  $G$ .*

- (1) *For any  $m$ , there is  $\alpha(m) \in L$  such that  $p_{\alpha(m), \xi} = e$  for all  $\xi \in M_m$ .*
- (2) *For any  $I$ , there is  $\beta(I) \in M$  such that  $p_{\eta, \beta(I)} = e$  for all  $\eta \in L_I$ .*

Proof. First, for any  $m$ , we can easily choose  $\alpha(m) \in L$  such that

- (3)  $p_{\alpha(m), \xi} \neq 0$  for all  $\xi \in M_m$ ,
- (4) If  $\alpha(m_1) \stackrel{0}{L} \alpha(m_2)$ , then  $\alpha(m_1) = \alpha(m_2)$ .

Next, for a mapping  $m \rightarrow \alpha(m)$ ,  $\beta(I) \in M$  is determined such that the following conditions are satisfied:

- (5)  $p_{\eta, \beta(I)} \neq 0$  for all  $\eta \in L_I$
- (6) if there is  $m$  such that  $\alpha(m) \in L_I$ , then we let  $\beta(I) \in M_{m_1}$  and  $\alpha(m_1) \in L_I$  for one  $m_1$  among  $m$ .

Consider the matrices

$$Q = (q_{\lambda_1 \lambda_2}) \quad \lambda_1, \lambda_2 \in M$$

and

$$R = (r_{\mu_1 \mu_2}) \quad \mu_1, \mu_2 \in L$$

where

$$q_{\lambda_1 \lambda_2} = \begin{cases} p_{\alpha(m), \xi}^{-1} & \text{if } \lambda_1 = \lambda_2 = \xi \in M_m \\ 0 & \text{if } \lambda_1 \neq \lambda_2 \end{cases}$$

$$r_{\mu_1 \mu_2} = \begin{cases} e & \text{if } \mu_1 = \mu_2 = \alpha(m) \text{ for some } m \\ p_{\alpha(m'), \beta(I)} p_{\eta, \beta(I)}^{-1} & \text{if } \alpha(m) \neq \mu_1 = \mu_2 = \eta \text{ for all } m, \text{ and we} \\ & \text{let } \eta \in L_I \text{ and } \beta(I) \in M_{m'} \\ 0 & \text{if } \mu_1 \neq \mu_2 \end{cases}$$

Then, setting  $R(PQ) = (t_{\mu\lambda})$ , we have

$$t_{\mu\lambda} = \begin{cases} p_{\mu\lambda} p_{\alpha(m), \lambda}^{-1} & \text{if } \mu = \alpha(m'') \text{ for some } m'', \text{ and } \lambda \in M_m \\ p_{\alpha(m'), \beta(I)} p_{\mu, \beta(I)}^{-1} p_{\mu\lambda} p_{\alpha(m), \lambda}^{-1} & \text{if } \mu \neq \alpha(m'') \text{ for all } m'', \text{ we let } \lambda \in M_m, \\ & \mu \in L_I, \beta(I) \in M_{m'}, \end{cases}$$

and it is easily shown that  $RPQ$  satisfies (1) and (2). The conditions (4) and (6) are available for the proof of (2) in the case that  $\eta = \alpha(m) \in L_I$  for some  $m$ . Thus the proof of the Lemma is completed.

The form,  $RPQ$ , which satisfies (1) and (2), is called a normal form of  $P$ .

### § 3. Decompositions.

Hereafter we shall assume that  $S$  has a matrix of normal form as the defining matrix. Let  $\sim$  denote a congruence relation in  $S$ .  $\sim$  is said to be trivial if either  $x \sim y$  for all  $x, y$  or  $x \sim y$  for only  $x = y$ .

**Lemma 2.** *Let  $\sim$  be a non-trivial congruence relation.  $(x; \lambda\mu) \sim (y; \sigma\tau)$  implies  $\lambda \stackrel{0}{M} \sigma$ ,  $\mu \stackrel{0}{L} \tau$  and hence there are  $\alpha$  and  $\beta$  such that*

$p_{\alpha\lambda} = p_{\alpha\sigma} = e$ ,  $p_{\mu\beta} = p_{\tau\beta} = e$  where  $e$  is a unit of  $G$ .

Proof. Suppose  $p_{\eta_0\lambda} \neq 0$  as well as  $p_{\eta_0\sigma} = 0$  for some  $\eta_0$ , and take any element  $(u; \xi\eta)$ , then, for certain  $p_{\mu\xi_0} \neq 0$ ,

$$\begin{aligned} (u; \xi\eta) &= (ux^{-1}p_{\eta_0\lambda}^{-1}; \xi\eta_0)(x; \lambda\mu)(p_{\mu\xi_0}^{-1}; \xi_0\eta) \\ &\sim (ux^{-1}p_{\eta_0\lambda}^{-1}; \xi\eta_0)(y; \sigma\tau)(p_{\mu\xi_0}^{-1}; \xi_0\eta) = 0. \end{aligned}$$

This shows that the relation  $\sim$  is a trivial congruence relation, contradicting the assumption. The remaining part is similarly proved. The existence of  $\alpha$  and  $\beta$  is clear by a normal form of the defining matrix, q. e. d.

Now we derive the relations  $\approx$  in  $G$ ,  $\widetilde{\approx}$  in  $M$ , and  $\widetilde{\tau}$  in  $L$  from the congruence relation  $\sim$  in  $S$  as defined in the following way.

$$\begin{aligned} x \approx y &\text{ if there are } \lambda, \sigma \in M, \mu, \tau \in L \text{ such that } (x; \lambda\mu) \sim (y; \sigma\tau), \\ \lambda \widetilde{\approx} \sigma &\text{ if there are } x, y \in G, \mu, \tau \in L \text{ such that } (x; \lambda\mu) \sim (y; \sigma\tau), \\ \mu \widetilde{\tau} \tau &\text{ if there are } x, y \in G, \lambda, \sigma \in M \text{ such that } (x; \lambda\mu) \sim (y; \sigma\tau). \end{aligned}$$

**Lemma 3.** *The relations  $\approx$ ,  $\widetilde{\approx}$  and  $\widetilde{\tau}$  are all congruence relations.*

Proof. Reflexivity and symmetry are evident. Let us prove transitivity. By  $x \approx y$  and  $y \approx z$  there are  $\lambda, \mu, \sigma, \tau, \sigma', \tau', \kappa$  and  $\nu$  such that

$$(x, \lambda\mu) \sim (y; \sigma\tau), \quad (y; \sigma'\tau') \sim (z; \kappa\nu).$$

According to Lemma 2,

$$p_{\alpha\lambda} = p_{\alpha\sigma} = e, \quad p_{\mu\beta} = p_{\tau\beta} = e \quad \text{for certain } \alpha \text{ and } \beta,$$

so that we get

$$(e; \sigma'\alpha)(x; \lambda\mu)(e; \beta\tau') \sim (e; \sigma'\alpha)(y; \sigma\tau)(e; \beta\tau')$$

and hence

$$(x; \sigma'\tau') \sim (z; \kappa\nu).$$

Thus we have proved  $x \approx z$ .

Transitivity of  $\widetilde{\approx}$  is proved from  $(x; \lambda\mu) \sim (y; \sigma\tau)$ ,  $(y'; \sigma\tau') \sim (z; \kappa\nu)$  and  $(x; \lambda\mu)(y^{-1}y'; \beta\tau') \sim (y; \sigma\tau)(y^{-1}y'; \beta\tau')$  where  $p_{\mu\beta} = p_{\tau\beta} = e$ . We get transitivity of  $\widetilde{\tau}$  analogously.

Next,  $x \approx y$  implies  $xz \approx yz$  and  $zx \approx zy$  because

$$\begin{aligned} (x; \lambda\mu)(z; \beta\mu) &\sim (y; \sigma\tau)(z; \beta\mu) \\ (z; \lambda\alpha)(x; \lambda\mu) &\sim (z; \lambda\alpha)(y; \sigma\tau) \end{aligned}$$

under the assumption  $(x; \lambda\mu) \sim (y; \sigma\tau)$  where  $p_{\mu\beta} = p_{\tau\beta} = p_{\alpha\lambda} = p_{\alpha\sigma} = e$ . The proof for  $\widetilde{\approx}$  and  $\widetilde{\tau}$  is clear.

**Lemma 4.** *If  $\lambda \widetilde{\mathcal{M}} \sigma$  and  $p_{\eta\lambda} \neq 0$ , then  $p_{\eta\lambda} \approx p_{\eta\sigma} \neq 0$  for all  $\eta$ . If  $\mu \widetilde{\mathcal{L}} \tau$  and  $p_{\mu\xi} \neq 0$ , then  $p_{\mu\xi} \approx p_{\tau\xi} \neq 0$  for all  $\xi$ .*

Proof. By Lemma 2, it is evident that  $p_{\eta\sigma} \neq 0$ ,  $p_{\tau\xi} \neq 0$ . Find  $\beta$  such that  $p_{\mu\beta} = p_{\tau\beta} = e$ . Multiplying each of  $(x; \lambda\mu)$  and  $(y; \sigma\tau)$  by  $(x^{-1}; \beta\mu)$  from right, we get

$$(e; \lambda\mu) \sim (yx^{-1}; \sigma\mu) \quad \text{whence} \quad e \approx yx^{-1}.$$

Moreover, from  $(e; \lambda\eta)(e; \lambda\mu) \sim (e; \lambda\eta)(yx^{-1}; \sigma\mu)$

we have 
$$p_{\eta\lambda} \approx p_{\eta\sigma} yx^{-1} \approx p_{\eta\sigma}$$

completing the proof. Similarly  $p_{\mu\xi} \approx p_{\tau\xi}$  is proved, q. e. d.

Conversely, consider congruence relations  $\widetilde{\mathcal{M}}$ ,  $\widetilde{\mathcal{L}}$ ,  $\approx$  in  $M$ ,  $L$ ,  $G$  respectively such that

$$\begin{aligned} \lambda \widetilde{\mathcal{M}} \sigma & \text{ implies } \lambda \overset{0}{\widetilde{\mathcal{M}}} \sigma, \\ \mu \widetilde{\mathcal{L}} \tau & \text{ implies } \mu \overset{0}{\widetilde{\mathcal{L}}} \tau, \\ & \approx \text{ makes Lemma 4 hold.} \end{aligned}$$

For these congruence relations, a relation  $\sim$  in  $S$  is defined as

$$(x; \lambda\mu) \sim (y; \sigma\tau) \quad \text{if} \quad x \approx y, \lambda \widetilde{\mathcal{M}} \sigma, \text{ and } \mu \widetilde{\mathcal{L}} \tau.$$

Then it is easily shown that the relation is a congruence relation.

**Theorem 1.** *We obtain, as follows, every congruence relation in a completely simple semigroup  $S$  with a ground group  $G$  and with a defining matrix  $P = (p_{\mu\lambda})$ ,  $\lambda \in M$ ,  $\mu \in L$ . First, for a pair of the congruence relations  $\widetilde{\mathcal{M}}$  and  $\widetilde{\mathcal{L}}$  taken arbitrarily, independently each other, there is at least one congruence relation  $\approx$  in  $G$  which satisfies*

$$\begin{aligned} \lambda \widetilde{\mathcal{M}} \sigma & \text{ implies } p_{\eta\lambda} \approx p_{\eta\sigma} \text{ for all } \eta, \\ \mu \widetilde{\mathcal{L}} \tau & \text{ implies } p_{\mu\xi} \approx p_{\tau\xi} \text{ for all } \xi. \end{aligned}$$

*By a triplet of the three congruence relations  $\widetilde{\mathcal{M}}$ ,  $\widetilde{\mathcal{L}}$ ,  $\approx$ , a congruence relation  $\sim$  in  $S$  is determined as*

$$(x; \lambda\mu) \sim (y; \sigma\tau) \quad \text{means that} \quad x \approx y, \lambda \widetilde{\mathcal{M}} \sigma, \text{ and } \mu \widetilde{\mathcal{L}} \tau.$$

#### § 4. Examples.

We shall arrange a few examples which follow from Theorem 1.

First, we can easily determine the structure of an indecomposable completely simple semigroup, which was obtained in [2].

**Example 1.** A completely simple semigroup  $S$  is indecomposable if and only if the following three conditions are satisfied.

- (7) The ground group is  $G = \{e\}$ .
- (8)  $\lambda \overset{0}{M} \sigma$  if and only if  $\lambda = \sigma$ .
- (9)  $\mu \overset{0}{r} \tau$  if and only if  $\mu = \tau$ .

**Example 2.** Consider a finite simple semigroup  $S$  with a ground group  $G$  and with the defining matrix  $\begin{pmatrix} e \\ e \end{pmatrix}$  or  $(e e)$ .

Let  $x \rightarrow f(x)$  be a homomorphism of  $G$  to certain group  $G' : G' = f(G)$ ,  $e' = f(e)$ . Then any homomorphism of  $S$  is given as either (10) or (11).

$$(10) \quad (x ; \lambda\mu) \rightarrow (f(x) ; \lambda\mu)$$

where the homomorphic image  $S'$  of  $S$  is also a completely simple semigroup in which  $G'$  is the ground group and  $P' = (f(p_{\mu\lambda}))$  is the defining matrix.

$$(11) \quad (x ; \lambda\mu) \rightarrow f(x) \quad \text{where } S' = G'.$$

**Example 3.** A finite simple semigroup  $S$  with a ground group  $G$  and with the defining matrix  $\begin{pmatrix} e & e \\ e & a \end{pmatrix}$  where  $a \neq 0$ . Any homomorphic image of  $S$  is given as one of

- $(x ; \lambda\mu) \rightarrow (f(x) ; \lambda\mu)$  where  $S' = \text{Simp.} \left( f(G) ; \begin{pmatrix} e' & e' \\ e' & f(a) \end{pmatrix} \right)$ ,
- $(x ; \lambda\mu) \rightarrow (f(x) ; \lambda 1)$  where  $S' = \text{Simp.} (f(G) ; (e' e'))$  and  $f(e) = f(a) = e'$ ,
- $(x ; \lambda\mu) \rightarrow (f(x) ; 1\mu)$  where  $S' = \text{Simp.} \left( f(G) ; \begin{pmatrix} e' \\ e' \end{pmatrix} \right)$  and  $f(e) = f(a) = e'$ .
- $(x ; \lambda\mu) \rightarrow f(x)$  where  $S' = f(G)$ .

### § 5. $\mathfrak{H}$ -Semigroups.

In this paragraph  $S$  denotes a finite simple semigroup. If a decomposition of  $S$  classifies the elements into some classes composed of equal number of elements, then the decomposition is called homogeneous. We term by  $\mathfrak{H}$ -semigroup a finite semigroup  $S$  with  $\mathfrak{H}$ -property, i.e., the property that every decomposition of  $S$  is homogeneous [4]. It goes without saying that any semigroup of order 2 and any indecomposable finite semigroup are  $\mathfrak{H}$ -semigroups. We assume that the order of  $S$  is  $> 2$ .

**Lemma 5.** *A  $\mathfrak{H}$ -semigroup is simple.*

**Proof.** If a  $\mathfrak{H}$ -semigroup  $S$  is not simple, a proper ideal  $I$  exists so

that the difference semigroup  $(S:I)$  of  $S$  modulo  $I$  would result in a non-homogeneous decomposition of  $S$ , q. e. d.

**Lemma 6.** *If a  $\mathfrak{S}$ -semigroup  $S$  has zero  $0$ , then  $S$  is indecomposable.*

Proof. Let  $\sim$  be a congruence relation in  $S$ . From Lemma 5 follows that there is nothing but the trivial decompositions, i.e., either  $0 \sim x$  for all  $x \in S$  or  $0 \sim x$  for only  $x=0$ . In the latter case, by homogeneity,  $x \neq y$  implies  $x \not\sim y$  for every  $x, y$ , q. e. d.

**Corollary 1.** *If a  $\mathfrak{S}$ -semigroup  $S$  has a non-trivial decomposition, then  $S$  is a simple semigroup without zero.*

Accordingly a  $\mathfrak{S}$ -semigroup  $S$  may be considered as a semigroup  $S = \text{Simp.}(G; (p_{ji}))$  where let  $G$  be a group of order  $g$ , let  $P = (p_{ji})$  be a matrix of  $(l, m)$  type i.e.  $i=1, \dots, m; j=1, \dots, l$ .

**Lemma 7.** *If  $S$  is a  $\mathfrak{S}$ -semigroup which has no zero, then  $m \leq 2, l \leq 2$ .*

Proof. Suppose, for example,  $m \geq 3$ . Consider a congruence relation  $\sim$  in  $S$  as follows.

$$\begin{aligned} (x; kj) &\sim (y; kj') \quad \text{for any } k > 2, \text{ any } j, \text{ and any } j', \\ (x; kj) &\not\sim (y; k'j') \quad \text{for any } k > 2, k' > 2, k \neq k' \text{ any } j, \text{ and any } j', \\ (x; 1j) &\sim (y; 2j') \quad \text{for any } j, \text{ and any } j', \end{aligned}$$

where  $x$  and  $y$  run independently throughout  $G$ .

Then we have a non-homogeneous decomposition

$$S = \bar{S} \cup S_3 \cup S_4 \cup \dots$$

where 
$$\begin{aligned} \bar{S} &= \{(x; ij) \mid x \in G, i = 1, 2; 1 \leq j \leq l\}, \\ S_k &= \{(x; kj) \mid x \in G, 1 \leq j \leq l\}, \quad k = 3, 4, \dots \end{aligned}$$

and the order of  $\bar{S}$  is  $2gl$ , that of  $S_k$  is  $gl$ . This contradicts the assumption of  $\mathfrak{S}$ . Hence  $m \leq 2$ . Similarly  $l \leq 2$  is proved.

Therefore a  $\mathfrak{S}$ -semigroup which has no zero must have a structure of the following four.

$$\begin{aligned} &\text{Simp.} \left( G; \begin{pmatrix} e \\ e \end{pmatrix} \right), \\ &\text{Simp.} (G; (e \ e)), \\ &\text{Simp.} \left( G; \begin{pmatrix} e & e \\ e & a \end{pmatrix} \right), \\ &\text{Group.} \end{aligned}$$

On the other hand, Examples 2, 2' and 3 show that every decomposition of them is homogeneous. At last we have arrived at

**Theorem 2.** *A finite semigroup is a  $\mathfrak{S}$ -semigroup of order  $\geq 2$  if and only if it is one of the following six cases.*

- (C<sub>1</sub>) *a z-semigroup of order 2 or a semilattice of order 2*  
 (C<sub>2</sub>) *a finite group of order  $\geq 2$*   
 (C<sub>3</sub>) *an indecomposable finite semigroup of order  $> 1$*   
 (C<sub>4</sub>) *Simp.  $(G; \begin{pmatrix} e \\ e \end{pmatrix})$*   
 (C<sub>5</sub>) *Simp.  $(G; (e e))$*   
 (C<sub>6</sub>) *Simp.  $(G; \begin{pmatrix} e & e \\ e & a \end{pmatrix})$*  } *where  $G$  is a finite group of order  $\geq 1$ ,  
 $a \neq 0$*

### § 6. Relations between $\mathfrak{S}$ -property and $\mathfrak{S}$ -property.

In the paper [3, 4] we defined  $\mathfrak{S}$ -property of a finite semigroup and proved that an  $\mathfrak{S}$ -semigroup is one of the above cases except (C<sub>3</sub>).

Immediately we have

**Theorem 3.**  *$\mathfrak{S}$ -property implies  $\mathfrak{S}$ -property. Though the converse is not true, it is true that a  $\mathfrak{S}$ -semigroup which has a proper decomposition is an  $\mathfrak{S}$ -semigroup.*

By the way we give a few theorems.

**Theorem 4.** *A unipotent  $\mathfrak{S}$ -semigroup of order  $> 2$  is a group. A unipotent  $\mathfrak{S}$ -semigroup of order  $> 2$  is so also.*

**Theorem 5.** *A subsemigroup of an  $\mathfrak{S}$ -semigroup is an  $\mathfrak{S}$ -semigroup. A subsemigroup of an indecomposable semigroup is not always a  $\mathfrak{S}$ -semigroup, but  $\mathfrak{S}$ -property, in the other cases, is preserved in a subsemigroup.  $\mathfrak{S}$ -property and  $\mathfrak{S}$ -property are both preserved in a homomorphic image.*

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 [4] ———: Note on finite semigroups which satisfy certain group-like condition, Proc. of Jap. Acad. **36**, 62-64 (1960).  
 [5] I have called  $L$  a right-singular semigroup if  $xy=y$  for all  $x, y \in L$ . In p. 62, [4], I find a misprint: for "left-singular", read "right-singular". But this is not essential for discussion.



