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# ON LIFTS OF IRREDUCIBLE 2-BRAUER CHARACTERS OF SOLVABLE GROUPS

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### 1. Introduction

Let *G* be a finite group. Write  $|G| = p^a k$ , where *p* is a prime number and (k, p) = 1, and let  $\mathbf{Q}_k = \mathbf{Q}(e^{2\pi i/k})$ , the field generated by  $e^{2\pi i/k}$  over the field  $\mathbf{Q}$  of rationals. Recall that an ordinary character  $\chi$  of *G* is said to be *p*-rational if  $\chi(x) \in \mathbf{Q}_k$ , for every  $x \in G$ .

Now, let  $\varphi$  be an irreducible 2-Brauer character of *G* and denote by *n* (resp. *m*), the number of ordinary (resp. 2-rational) irreducible characters  $\xi$  of *G* such that the restriction  $\xi_{G_{\gamma'}}$  of  $\xi$  to the subset  $G_{2'}$  of 2'-elements of *G*, is equal to  $\varphi$ .

Let V be a simple FG-module affording  $\varphi$ , where F is an algebraically closed field of characteristic 2, and let Q be a vertex of V.

If Q is cyclic and |Q| > 1, the module V belongs to a 2-block B of G having Q as a defect group (see Theorem VII.15.1 in [2]). By the theory of blocks with cyclic defect groups (see, for instance, Theorem 68.1 in [1]), we have n = |Q|.

More generally, assume now that G has a normal subgroup N such that  $Q \notin N$  and that the quotient group QN/N is cyclic. Then, in case G is solvable, the main result of this paper (Theorem 1) asserts that  $n \geq |QN/N|$  and that  $m \geq 2$ .

It is worth mentioning that the statement concerning *m* above is really an exclusive feature of the prime 2. In fact, it has been shown by I.M. Isaacs that if  $\phi$  is an irreducible *p*-Brauer character of a *p*-solvable group *H*, where *p* is odd, then there exists a unique irreducible *p*-rational character  $\theta$  of *H* such that  $\theta_{H_{p'}} = \phi$  (see Theorem X.2.3 in [2]).

#### 2. Background

Although the main result of this paper (Theorem 1) concerns ordinary characters and 2-Brauer characters of solvable groups, its proof relies heavily on Isaacs' theory of partial characters developed in [6, 7]. In this section, we review few concepts of that theory needed for our purpose.

Let  $\pi$  be an arbitrary set of primes and assume throughout this section that G is a finite  $\pi$ -separable group. Recall that the  $\pi'$ -partial characters of G are just the restrictions  $\chi^0$  of ordinary characters  $\chi$  of G to the set of  $\pi'$ -elements of G. Furthermore,  $\chi^0$  is said to be irreducible if it cannot be written as a sum of two  $\pi'$ -partial characters. The set of irreducible  $\pi'$ -partial characters of G is denoted by  $I_{\pi'}(G)$ . For any  $\xi \in Irr(G)$ , there are uniquely determined nonnegative integers  $d_{\xi\psi}$ , such that  $\xi^0 = \sum_{\psi} d_{\xi\psi}\psi$ , where  $\psi$  runs through  $I_{\pi'}(G)$ .

In case  $\pi = \{p\}$ , it follows from the Fong-Swan theorem that the  $\pi'$ -partial characters of *G* are exactly the Brauer characters (at *p*) and consequently  $I_{\pi'}(G) = IBr(G)$ .

Next, assume that K is a subgroup of G and that  $\psi$  is a  $\pi'$ -partial character of G. Then, it is obvious that the restriction  $\psi_K$  is a  $\pi'$ -partial character of K. For  $\varphi \in I_{\pi'}(K)$ , we denote by  $I_{\pi'}(G | \varphi)$ , the set of all  $\omega \in I_{\pi'}(G)$  such that  $\varphi$  is a constituent of  $\omega_K$ . Induction  $\tau^G$  of a  $\pi'$ -partial character  $\tau$  of K can also be defined by using the usual formula of induced characters and applying it only to  $\pi'$ -elements. It is easy to see that  $\tau^G$  is a  $\pi'$ -partial character of G.

In [9], a vertex of  $\psi \in I_{\pi'}(G)$  is defined to be a Hall  $\pi$ -subgroup of some subgroup J of G for which there exists  $\alpha \in I_{\pi'}(J)$  such that  $\alpha^G = \psi$  and  $\alpha(1)$  is a  $\pi'$ -number. It turns out that the set of vertices of  $\psi$  is not empty and that it forms a single conjugacy class of  $\pi$ -subgroups of G (see Theorem B in [9]). If  $\psi$  is an irreducible p-Brauer character of a p-solvable group, then it is not hard to see that the vertices of  $\psi$  defined above (when  $\pi = \{p\}$ ), are exactly the vertices of the simple module (in characteristic p) affording  $\psi$ .

It is clear from the definitions that for every  $\psi \in I_{\pi'}(G)$ , there exists  $\chi \in Irr(G)$ such that  $\chi^0 = \psi$ . However,  $\chi$  is not unique in general. Nevertheless, in [6], Isaacs has canonically defined a set  $B_{\pi'}(G)$  of irreducible characters of G such that the map  $\chi \mapsto \chi^0$  is a bijection of  $B_{\pi'}(G)$  onto  $I_{\pi'}(G)$ .

Let now  $N \triangleleft G$  and  $\mu \in B_{\pi'}(N)$ . Two characters  $\chi_1, \chi_2 \in \operatorname{Irr}(G \mid \mu)$  are said to be linked if there exists  $\psi \in I_{\pi'}(G)$  such that  $d_{\chi_1\psi} \neq 0$  and  $d_{\chi_2\psi} \neq 0$ . The equivalence classes defined by the transitive extension of this linking relation are called relative  $\pi$ blocks of *G* with respect to  $(N, \mu)$ , and the set of all these relative  $\pi$ -blocks is denoted by  $\operatorname{Bl}_{\pi}(G \mid \mu)$  (see Section 3 in [11]). In case  $(N, \mu) = (\langle 1 \rangle, 1_{\langle 1 \rangle})$ , where  $1_{\langle 1 \rangle}$  is the trivial character of  $\langle 1 \rangle$ , the relative  $\pi$ -blocks of *G* with respect to  $(N, \mu)$  are just the  $\pi$ -blocks defined by M. Slattery [12].

#### 3. The main theorem

We start this section by stating the main theorem of this paper.

**Theorem 1.** Let G be a finite solvable group and let F be an algebraically closed field of characteristic 2. Let V be a simple FG-module with vertex Q and let  $\varphi$  be the irreducible Brauer character afforded by V. Suppose that there exists a normal subgroup N of G such that  $Q \nsubseteq N$  and QN/N is cyclic. Then

(i) G has at least |QN/N| ordinary irreducible characters  $\chi$  such that the restriction  $\chi_{G_{2'}}$  of  $\chi$  to the subset  $G_{2'}$  of 2'-elements of G, is equal to  $\varphi$ .

(ii) G has at least two 2-rational irreducible characters  $\xi$  such that  $\xi_{G_{2'}} = \varphi$ .

In order to prove this theorem, we need few preliminary results. For the sake of generality, all but the last of these results are proved in the general setting of finite  $\pi$ -separable groups, where  $\pi$  is an arbitrary set of prime numbers. (Note that a solvable group is necessarily  $\pi$ -separable.)

Before stating our first preliminary result, recall that a character-triple is a triple  $(H, M, \alpha)$ , where M is a normal subgroup of the group H and  $\alpha$  is an H-invariant irreducible character of M. By definition (see Definition 11.23 in [5]), if the triple  $(H, M, \alpha)$  is isomorphic to  $(H', M', \alpha')$ , then there exists an isomorphism  $\tau : H/M \to H'/M'$ . If  $M \subseteq L \subseteq H$  and L' is the subgroup of H' containing M' such that  $\tau(L/M) = L'/M'$ , then also by the definition of character-triple isomorphism, we have a bijection  $\sigma_L$ :  $\operatorname{Irr}(L \mid \alpha) \to \operatorname{Irr}(L' \mid \alpha')$ . Let  $\sigma$  be the union of the maps  $\sigma_L$ . Then, the pair  $(\tau, \sigma)$  is the corresponding isomorphism from  $(H, M, \alpha)$  to  $(H', M', \alpha')$ .

**Lemma 2.** Let  $\pi$  be a set of primes and let H be a  $\pi$ -separable group. Let  $M \triangleleft H$  and let  $\alpha$  be an H-invariant  $\pi'$ -special character of M. Then, there exist a central extension H' of  $\overline{H} = H/M$  by a  $\pi'$ -subgroup M' of H', a linear character  $\alpha'$  of M' and bijections  $\Psi$  of Irr( $H \mid \alpha$ ) onto Irr( $H' \mid \alpha'$ ) and  $\Psi^0$  of I $_{\pi'}(H \mid \alpha^0)$  onto I $_{\pi'}(H' \mid \alpha')$  such that

(a) For any  $\theta \in I_{\pi'}(H \mid \alpha^0)$ , if  $\xi$  is any character in  $Irr(H \mid \alpha)$  such that  $\xi^0 = \theta$ , we have  $\Psi^0(\theta) = \Psi(\xi)^0$ .

(b) For any  $\chi \in Irr(H \mid \alpha)$  and any  $\theta \in I_{\pi'}(H \mid \alpha^0)$ , we have  $d_{\chi\theta} = d_{\Psi(\chi)\Psi^0(\theta)}$ .

(c) The correspondence  $\mathcal{B} \mapsto \Psi(\mathcal{B})$  is a bijection of  $Bl_{\pi}(H \mid \alpha)$  onto the set of (Slattery)  $\pi$ -blocks of H' over  $\alpha'$ .

(d) If  $\theta \in I_{\pi'}(H \mid \alpha^0)$ , then  $\theta$  has a vertex Q such that QM/M is isomorphic to some vertex Q' of  $\Psi^0(\theta)$ .

Proof. This lemma without (d), is the invariant case of Theorem 3.1 in [10]. Recall, by the proof of that theorem, that the triple  $(H', M', \alpha')$  is chosen to be isomorphic to  $(H, M, \alpha)$ . In other words, there exists a character-triple isomorphism  $(\tau, \sigma)$ :  $(H, M, \alpha) \rightarrow (H', M', \alpha')$ . The bijection  $\Psi$  is just the map  $\sigma_H$  introduced just before the lemma. All we need now is to show (d).

Let  $\theta \in I_{\pi'}(H \mid \alpha^0)$ . Then, there exists  $\xi \in B_{\pi'}(H)$  such that  $\theta = \xi^0$ . Since the irreducible constituents of  $\xi_M$  are all in  $B_{\pi'}(M)$  (Corollary 7.5 in [6]) and  $\theta$  lies over  $\alpha^0$ , it follows that  $\xi$  lies over  $\alpha$ . Now, as  $\alpha$  is  $\pi'$ -special, Lemma 1.2 in [13] says that there exists a nucleus  $(K, \rho)$  of  $\xi$  such that  $M \subseteq K$  and  $\rho \in Irr(K \mid \alpha)$  (see Section 4 of [6], for the definition of the nucleus). In particular, we have  $\xi = \rho^H$  and hence  $\theta = \xi^0 = (\rho^0)^H$ . Moreover, since  $\xi \in B_{\pi'}(H)$ , the character  $\rho$  is  $\pi'$ -special. Therefore, a Hall  $\pi$ -subgroup Q of K is a vertex for  $\theta$ .

Next, by Lemma 11.35 in [5], we have

$$\Psi(\xi) = \Psi(\rho^H) = \sigma_H(\rho^H) = (\sigma_K(\rho))^{H'}.$$

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As  $\Psi^0(\theta) = \Psi(\xi)^0$  by (a), we get that  $\Psi^0(\theta) = (\sigma_K(\rho)^0)^{H'}$ . Let K' be the subgroup of H' containing M' such that  $\tau(K/M) = K'/M'$ . Now, since  $\Psi^0(\theta) \in I_{\pi'}(H')$ , it follows that  $\sigma_K(\rho)^0 \in I_{\pi'}(K')$ .

By Lemma 11.24 in [5], we have  $\rho(1)\alpha'(1) = \sigma_K(\rho)(1)\alpha(1)$ . Since  $\rho(1)$ ,  $\alpha(1)$  and  $\alpha'(1)$  are all  $\pi'$ -numbers, we conclude that  $\sigma_K(\rho)(1)$  is a  $\pi'$ -number. Hence, a Hall  $\pi$ -subgroup Q' of K' is a vertex of  $\Psi^0(\theta)$ .

Now, QM/M is a Hall  $\pi$ -subgroup of K/M and Q'M'/M' is a Hall  $\pi$ -subgroup of K'/M'. As  $K/M \cong K'/M'$ , we obtain  $QM/M \cong Q'M'/M' \cong Q'/Q' \cap M'$ . Furthermore, since Q' is a  $\pi$ -group and M' is a  $\pi'$ -group, we get  $Q' \cap M' = 1$  and it follows that  $QM/M \cong Q'$ . This proves (d) and completes the proof of the lemma.

We can now improve Theorem 3.1 of [11].

**Theorem 3.** Let N be a normal subgroup of a  $\pi$ -separable group G and let  $\mu \in B_{\pi'}(N)$  with  $T = I_G(\mu)$ . Then, there exist a central extension U of  $\overline{T} = T/N$  by a  $\pi'$ -subgroup Z of U, a linear character  $\nu$  of Z and bijections  $\Gamma$  of  $\operatorname{Irr}(G \mid \mu)$  onto  $\operatorname{Irr}(U \mid \nu)$  and  $\Gamma^0$  of  $I_{\pi'}(G \mid \mu^0)$  onto  $I_{\pi'}(U \mid \nu)$  such that the following hold.

(a) For any  $\chi \in \operatorname{Irr}(G \mid \mu)$  and any  $\phi \in I_{\pi'}(G \mid \mu^0)$ , we have  $d_{\chi\phi} = d_{\Gamma(\chi)\Gamma^0(\phi)}$ .

(b) The correspondence  $\mathcal{B} \mapsto \Gamma(\mathcal{B})$  is a bijection of  $Bl_{\pi}(G \mid \mu)$  onto the set of (Slattery)  $\pi$ -blocks of U over  $\nu$ .

(c) If  $\phi \in I_{\pi'}(G \mid \mu^0)$ , then  $\phi$  has a vertex Q such that QN/N is isomorphic to some vertex P of  $\Gamma^0(\phi)$ .

Proof. Let  $(W, \gamma)$  be a nucleus for  $\mu$  and let  $S = N_T((W, \gamma))$ , the stabilizer of  $(W, \gamma)$  in T. First, we note that this theorem without (c), is Theorem 3.1 in [11]. Recall, by its proof, that Theorem 3.1 of [11] is obtained by first applying Theorem 3.2 in [11] to the group G, the normal subgroup N and the character  $\mu \in B_{\pi'}(N)$ , and then applying the invariant form of Theorem 3.1 in [10] (this is Lemma 2 above without (d)) to the group S, the normal subgroup W and the S-invariant  $\pi'$ -special character  $\gamma$  of W.

To complete the proof, we need to show (c). Let  $\phi \in I_{\pi'}(G \mid \mu^0)$ . By Theorem 3.2 (b) in [11], there exists a partial character  $\theta \in I_{\pi'}(S \mid \gamma^0)$  such that  $\phi = \theta^G$ . Now,  $\Gamma^0(\phi)$  is the element of  $I_{\pi'}(U \mid \nu)$  corresponding to  $\theta$  via the bijection  $\Psi^0$  of Lemma 2.

By Lemma 2 (d),  $\theta$  has a vertex Q such that QW/W is isomorphic to some vertex P of  $\Gamma^0(\phi) (= \Psi^0(\theta))$ .

Now, by Lemma 3.6 (a) of [11], we have  $S \cap N = W$ . Therefore, we get  $Q \cap N = Q \cap S \cap N = Q \cap W$ . Since  $QN/N \cong Q/Q \cap N$  and  $QW/W \cong Q/Q \cap W$ , it follows that  $QN/N \cong QW/W \cong P$ . Finally, note that the subgroup Q is also a vertex of  $\phi$  as  $\phi = \theta^G$ . This proves (c) and finishes the proof of the theorem.

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Let  $(H, M, \alpha)$  be a character-triple, where H is a  $\pi$ -separable group and  $\alpha$  is  $\pi'$ -special, and let  $\mathcal{B}$  be a relative  $\pi$ -block of H with respect to  $(M, \alpha)$ .

Let *R* be the normal subgroup of *H* containing *M* such that  $R/M = O_{\pi'}(H/M)$ . If  $\zeta \in Irr(H \mid \alpha)$ , then by Lemma 2.3 in [6], there exists a  $\pi'$ -special character  $\delta$  of *R* such that  $\delta$  is a constituent of  $\zeta_R$ . By Lemma 3.2 in [10], if  $\beta \in B_{\pi'}(H)$  satisfies  $d_{\zeta\beta^0} \neq 0$ , we have  $\beta \in Irr(H \mid \delta)$ . Therefore, the constituents of  $\beta_R$  are precisely the constituents of  $\zeta_R$  by Clifford's theorem (Theorem 6.2 in [5]). It follows that if  $\zeta' \in Irr(H \mid \alpha)$  also satisfies  $d_{\zeta'\beta^0} \neq 0$ , then  $\zeta'$  also lies over the *H*-orbit of  $\delta$ . This implies that the characters of *B* all lie over the *H*-orbit of some  $\pi'$ -special character  $\eta$  of *R*. So, there exists a relative  $\pi$ -block  $\mathcal{B}_0$  of *H* with respect to  $(R, \eta)$  such that  $\mathcal{B} \subseteq \mathcal{B}_0$ . Assume now that  $\xi \in \mathcal{B}$  and  $\xi_0 \in \mathcal{B}_0$  satisfy  $d_{\xi\omega} \neq 0$  and  $d_{\xi_0\omega} \neq 0$  for some  $\omega \in I_{\pi'}(H)$ . Then, as  $\eta$  lies over  $\alpha$ , the character  $\xi_0$  lies over  $\alpha$  and it follows that  $\xi_0 \in \mathcal{B}$ . Consequently,  $\mathcal{B} = \mathcal{B}_0$  and thus we may view  $\mathcal{B}$  as a relative  $\pi$ -block of *H* with respect to  $(R, \eta)$ .

We now have the following "Fong reduction" type result.

**Lemma 4.** Let  $N \triangleleft G$ , where G is  $\pi$ -separable and let  $\mu \in B_{\pi'}(N)$ . If  $\mathcal{B} \in Bl_{\pi}(G \mid \mu)$ , then there exist a subgroup A of G, a normal subgroup E of A satisfying  $O_{\pi'}(A/E) = 1$  and an A-invariant  $\pi'$ -special character  $\beta$  of E such that induction defines a bijection of Irr $(A \mid \beta)$  onto  $\mathcal{B}$ . Furthermore, if D is a Hall  $\pi$ -subgroup of A, then D is a defect group of  $\mathcal{B}$  and DN/N is isomorphic to DE/E.

Proof. Let  $(W, \gamma)$  be a nucleus for  $\mu$  and let  $S = N_T((W, \gamma))$ , where  $T = I_G(\mu)$ . (Note that  $\gamma$  is  $\pi'$ -special as  $\mu \in B_{\pi'}(N)$ .) By Theorem 3.2 of [11], there exists a relative  $\pi$ -block  $\mathcal{B}_0 \in BI_{\pi}(S \mid \gamma)$  such that the induction map  $\alpha \mapsto \alpha^G$  defines a bijection of  $\mathcal{B}_0$  onto  $\mathcal{B}$ . Let now P be any defect group of  $\mathcal{B}_0$ . Then, P is also a defect group of  $\mathcal{B}$  (see Section 4 in [11]). Since  $P \subseteq S$ , we have  $P \cap N = P \cap W$ , by Proposition 4.1 in [11] and it follows that  $PN/N \cong PW/W$ .

Now, to complete the proof, we may therefore assume that  $\mu$  is a G-invariant  $\pi'$ -special character.

Let *R* be the normal subgroup of *G* containing *N* such that  $R/N = O_{\pi'}(G/N)$ . By the discussion preceding the lemma, there exists a  $\pi'$ -special character  $\eta$  of *R* lying over  $\mu$  such that  $\mathcal{B} \in Bl_{\pi}(G \mid \eta)$ . Next, let  $J = I_G(\eta)$ . Then, by Lemma 3.4 in [10], there is a unique relative  $\pi$ -block  $\widehat{\mathcal{B}}$  of *J* with respect to  $(R, \eta)$  such that the induction map  $\chi \mapsto \chi^G$  is a bijection of  $\widehat{\mathcal{B}}$  onto  $\mathcal{B}$ .

CASE 1. Assume J = G. Then, the character  $\eta$  is *G*-invariant and so by Lemma 2, there exist a central extension G' of  $\overline{G} = G/R$  by a  $\pi'$ -subgroup R' of G', a linear character  $\eta'$  of R' and a bijection  $\Psi$  of  $Irr(G \mid \eta)$  onto  $Irr(G' \mid \eta')$  such that the correspondence  $b \mapsto \Psi(b)$  is a bijection of  $Bl_{\pi}(G \mid \eta)$  onto the set of (Slattery)  $\pi$ -blocks of G' over  $\eta'$ .

Since  $G/R \cong G'/R'$  and  $O_{\pi'}(G/R) = 1$ , we get that  $O_{\pi'}(G'/R') = 1$ , and thence

 $O_{\pi'}(G') = R'$ . By Theorem 2.8 in [12], G' has a single  $\pi$ -block over  $\eta'$ . It follows that  $Irr(G \mid \eta)$  consists of the single relative  $\pi$ -block  $\mathcal{B}$ . We can then take A = G, E = R and  $\beta = \eta$ .

By the definition of defect groups (see Section 4 in [10]), if D is a Hall  $\pi$ -subgroup of G, then D is a defect group for  $\mathcal{B}$ . Next, we show that  $DN/N \cong DR/R$ . Since  $DN/N \cong D/D \cap N$  and  $DR/R \cong D/D \cap R$ , it suffices to show that  $D \cap N = D \cap R$ .

First, note that  $D \cap N$  is a Hall  $\pi$ -subgroup of N. Since R/N is a  $\pi'$ -group, it follows that  $D \cap N$  is also a Hall  $\pi$ -subgroup of R. Hence,  $D \cap N = D \cap R$ , as wanted.

CASE 2. Assume J < G. Then, working by induction on the group order, we can find a subgroup A of J, a normal subgroup E of A satisfying  $O_{\pi'}(A/E) = 1$  and an A-invariant  $\pi'$ -special character  $\beta$  of E such that induction defines a bijection of  $Irr(A \mid \beta)$  onto  $\widehat{\mathcal{B}}$ . Moreover, if D is a Hall  $\pi$ -subgroup of A, then D is a defect group of  $\widehat{\mathcal{B}}$  and  $DR/R \cong DE/E$ .

Now, since induction of characters defines a bijection of  $\widehat{\mathcal{B}}$  onto  $\mathcal{B}$ , the induction map  $\theta \mapsto \theta^G$  is a bijection of  $Irr(A \mid \beta)$  onto  $\mathcal{B}$ . Next, by the definition of defect groups (Section 4 of [10]), the subgroup D is a defect group of  $\mathcal{B}$ . Furthermore, by Lemma 4.1 in [10],  $D \cap N$  is a Hall  $\pi$ -subgroup of N. Now, just as in case 1, it follows that  $DN/N \cong DR/R$ , and consequently  $DN/N \cong DE/E$ . This completes the proof of the lemma.

**Lemma 5.** Let  $\theta \in B_{2'}(G)$ , where G is solvable and let Q be a vertex of  $\varphi = \theta^0$ . Suppose that N is a normal subgroup of G such that  $Q \nsubseteq N$  and QN/N is cyclic, and let  $\mu$  be an irreducible constituent of  $\theta_N$ . Then,  $\mu \in B_{2'}(N)$  and if  $\theta \in \mathcal{B} \in Bl_2(G \mid \mu)$ , we have  $QN/N \cong DN/N$ , for any defect group D of  $\mathcal{B}$ . Furthermore,  $\mathcal{B} = \{\chi \in \operatorname{Irr}(G \mid \mu) : \chi^0 = \varphi\}$  and the number of elements of  $\mathcal{B}$  is exactly |QN/N|.

Proof. Since  $\theta \in \operatorname{Irr}(G \mid \mu)$ , the character  $\mu$  lies in  $B_{2'}(N)$  by Corollary 7.5 in [6]. Consequently, we have  $\mu^0 \in I_{2'}(N)$  and  $\varphi \in I_{2'}(G \mid \mu^0)$ .

By Theorem 3, there exist a solvable group U, a 2'-subgroup  $Z \subseteq Z(U)$ , a linear character  $\nu$  of Z and bijections  $\Gamma$  of  $Irr(G \mid \mu)$  onto  $Irr(U \mid \nu)$  and  $\Gamma^0$  of  $I_{2'}(G \mid \mu^0)$  onto  $I_{2'}(U \mid \nu)$ . Furthermore,  $\omega = \Gamma^0(\varphi)$  has a vertex P such that  $P \cong QN/N$ .

Let  $b = \Gamma(\mathcal{B})$ . By Theorem 3 (b), b is a 2-block of U over  $\nu$ . Moreover,  $\omega$  is an irreducible Brauer character associated with b. As QN/N is cyclic, the vertex P is cyclic and it follows by Theorem VII.15.1 in [2], that b has P as a defect group.

Let now D be any defect group for  $\mathcal{B}$ . Then, by Theorem 4.2 in [11], we have  $D/D \cap N \cong P$ . Since  $QN/N \cong P$ , it follows that  $QN/N \cong DN/N$ .

Next, by the theory of blocks with cyclic defect groups (Theorem 68.1 in [1]),  $\omega$  is the unique irreducible Brauer character associated with *b* and there are exactly |P| (= |QN/N|) ordinary irreducible characters  $\lambda$  in *b*. Furthermore, every character  $\lambda$  lies over  $\nu$  and satisfies  $\lambda^0 = \omega$ . In particular, we have  $b = \{\eta \in \operatorname{Irr}(U \mid \nu) : \eta^0 = \omega\}$ .

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It follows from Theorem 3 (a) that  $\mathcal{B} = \Gamma^{-1}(b) = \{\chi \in \operatorname{Irr}(G \mid \mu) : \chi^0 = \varphi\}$ , and consequently, the number of elements of  $\mathcal{B}$  is equal to the number |QN/N| of elements in *b*. The proof of the lemma is now complete.

We are almost ready to start the proof of Theorem 1. All we need now are few general facts about p-rational characters.

Let *G* be any finite group and for an integer  $h \ge 1$ , denote by  $\mathbf{Q}_h$  the field  $\mathbf{Q}(e^{2\pi i/h})$  generated by  $e^{2\pi i/h}$  over the field  $\mathbf{Q}$  of rationals. Now, fix a prime *p* and write  $|G| = l = p^a k$ , where (k, p) = 1. Let  $\mathcal{G}$  denote the Galois group  $\operatorname{Gal}(\mathbf{Q}_l/\mathbf{Q}_k)$ . If  $\chi$  is a character (resp. irreducible character) of *G* and if  $\sigma \in \mathcal{G}$ , the function  $\chi^{\sigma}$  defined by  $\chi^{\sigma}(x) = \chi(x)^{\sigma}$  is also a character (resp. irreducible character) of *G*. It is clear from the definition of *p*-rational characters given in the introduction of this paper, that  $\chi$  is *p*-rational if and only if  $\chi^{\sigma} = \chi$  for all  $\sigma \in \mathcal{G}$ .

Next, let *H* be a subgroup of *G*. Then,  $\mathbf{Q}_{|H|} \subseteq \mathbf{Q}_l$  and it follows that  $\theta^{\sigma}$  is defined for every character  $\theta$  of *H* and every  $\sigma \in \mathcal{G}$ .

We can now prove our main result.

Proof of Theorem 1. Let  $\theta$  be the element of  $B_{2'}(G)$  such that  $\theta^0 = \varphi$ , and fix an irreducible constituent  $\mu$  of  $\theta_N$ . By Lemma 5,  $\mu \in B_{2'}(N)$  and if  $\theta \in \mathcal{B} \in Bl_2(G \mid \mu)$ , we have,  $\mathcal{B} = \{\chi \in Irr(G \mid \mu) : \chi^0 = \varphi\}$ , and the number of elements of  $\mathcal{B}$  is |QN/N|. This suffices to prove (i). Next, we prove (ii).

By Lemma 4, there exist subgroups  $E \triangleleft A \subseteq G$  satisfying  $O_{2'}(A/E) = 1$  and an A-invariant 2'-special character  $\beta$  of E such that induction defines a bijection of Irr( $A \mid \beta$ ) onto  $\mathcal{B}$ . Furthermore, if D is a Sylow 2-subgroup of A, then D is a defect group of  $\mathcal{B}$  and  $DE/E \cong DN/N$ . By Lemma 5, we have  $DN/N \cong QN/N$ . Therefore,  $DE/E \cong QN/N$  and it follows that DE/E is a cyclic 2-group. Moreover, as  $Q \notin N$ , we have that |DE/E| > 1.

Next, write  $|G| = 2^r k$  and  $|A| = h = 2^s m$ , where both k and m are odd integers. Assume that a character  $\zeta \in Irr(A | \beta)$  is 2-rational. Then, the values of  $\zeta$  are in  $\mathbf{Q}_m$ . Since m divides k, the values of  $\zeta$  lie in  $\mathbf{Q}_k$  and it follows that the values of the character  $\zeta^G$  all lie in  $\mathbf{Q}_k$ . In other words,  $\zeta^G$  is 2-rational. Therefore, to show (ii), it suffices to find two 2-rational characters in  $Irr(A | \beta)$ .

Recall that *D* is a Sylow 2-subgroup of *A*. Then, DE/E is a Sylow 2-subgroup of A/E. Since  $O_{2'}(A/E) = 1$  and DE/E is cyclic, it follows from Theorem 6.3.3 in [4], that  $DE/E \triangleleft A/E$ . Now, let *R* be the subgroup of *A* such that R/E is the unique subgroup of DE/E of index 2. As  $DE/E \triangleleft A/E$ , it is clear that  $R/E \triangleleft A/E$ , and so  $R \triangleleft A$ .

Since  $\beta$  is A-invariant, Corollary 4.8 in [3] implies that there exists a 2'-special character  $\zeta_0 \in \operatorname{Irr}(A \mid \beta)$ . Set  $(\zeta_0)^0 = \omega$ . Next, fix an irreducible constituent  $\eta$  of  $(\zeta_0)_R$  and write  $S = \{\lambda \in \operatorname{Irr}(A \mid \eta) : \lambda^0 = \omega\}$ . As  $\zeta_0$  is 2'-special, D is a vertex of  $\omega$ . Moreover, since DR/R is cyclic of order 2, Lemma 5 says that S contains exactly 2

elements, the character  $\zeta_0$ , of course, and another character  $\zeta_1$ .

As  $\zeta_0 \in \operatorname{Irr}(A \mid \beta)$  and  $\beta$  is A-invariant, we have  $\eta \in \operatorname{Irr}(R \mid \beta)$ , and consequently  $S \subseteq \operatorname{Irr}(A \mid \beta)$ . So now, to complete the proof, it suffices to show that the characters  $\zeta_0$  and  $\zeta_1$  are 2-rational. First, note that  $\zeta_0$  is 2-rational by Lemma 3.1 in [8]. Next, we prove that  $\zeta_1$  is 2-rational.

Write  $(\zeta_1)_R = \sum_{i=1}^n \eta_i$ , where  $\eta_i \in \operatorname{Irr}(R)$  and  $\eta_1 = \eta$ . Now, let  $\sigma \in \operatorname{Gal}(\mathbf{Q}_h/\mathbf{Q}_m)$ . Then, for each i,  $\eta_i^{\sigma}$  is well defined (see the remarks preceding the proof) and  $(\zeta_1^{\sigma})_R = \sum_{i=1}^n \eta_i^{\sigma}$ . Since  $\zeta_0$  is 2'-special, then so is  $\eta$  by Lemma 2.2 in [6]. Hence,  $\eta$  is 2-rational by Lemma 3.1 in [8]. In other words, the values of  $\eta$  are in  $\mathbf{Q}_l$ , where l is the order of a Hall 2'-subgroup of R. As l divides m, we have  $\mathbf{Q}_l \subseteq \mathbf{Q}_m$  and it follows that  $\eta^{\sigma} = \eta$ . This shows that  $\zeta_1^{\sigma} \in \operatorname{Irr}(A \mid \eta)$ . Next, we have  $(\zeta_1)^0 = \omega$ . So, clearly  $(\zeta_1^{\sigma})^0 = \omega$  and we conclude that  $\zeta_1^{\sigma} \in S$ . Now, as  $S = \{\zeta_0, \zeta_1\}$  and  $\zeta_0$  is 2-rational, we have  $\zeta_1^{\sigma} = \zeta_1$ , necessarily. Hence,  $\zeta_1$  is 2-rational, as wanted.

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