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Algebraic Study of Fundamental Characteristic Classes of Sphere Bundles

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The cohomology theory of principal fibre bundles is extremely developed in these several years by many mathematicians, above all by A. Borel. Owing to this theory, the real cohomology structure of an n-universal bundle for a compact Lie group is completely determined. In particular, its base space, called n-classifying space, has the real cohomology algebra which is isomorphic in dimensions $\leqq n-1$ to the algebra of invariant polynomial functions on the Lie algebra of the structural group. Invariant polynomial functions are those which are invariant under the operations of adjoint group. As for an arbitrary given principal fibre bundle over a cell-complex with a compact Lie group, it is well-known that this bundle is induced by a mapping, say f, from its base space into an n-classifying space. The cohomology homomorphism f^* induced by f maps the cohomology algebra of the n-classifying space into that of the base space and it defines as its image the characteristic algebra whose elements are the so-called characteristic classes of the given bundle. Recall that these notions do not depend on the choices of the n-classifying space and of the mapping f. Referring to the fact mentioned above, as far as we concern real cohomology we may regard the characteristic classes as the images under a certain homomorphism of invariant polynomial functions. As a matter of fact, for a differentiable fibre bundle H. Cartan [2] and S. S. Chern [4] have availed themselves of a homomorphism explicitely given by making use of the curvature form of a connection in the bundle. This homomorphism will serve us as a foundation of the present work.

A sphere bundle is a fibre bundle whose fibre is a sphere acted by a subgroup of the orthogonal group. Its characteristic classes are those of the principal fibre bundle associated to it. As regard to the characteristic classes of a sphere bundle, there are important ones—Pontrjagin classes, Chern classes, etc.—which we call fundamental characteristic classes. These are usually defined as the f^*-images of special cohomology classes in a Grassmann manifold—a concrete n-classifying space for a compact classical group. By detailed studies of Grassmann manifolds mainly due to W. T. Wu [8], we know many formulas and duality theorems about these foundamental characteristic classes.

The purpose of the present paper will be now explained: If we restrict our attentions to differentiable sphere bundles and apply there the homomorphism of Cartan-Chern, we may regard the characteristic classes as the images by this homomorphism of invariant polynomial functions on the Lie algebra of the structural group and then we may obtain relations among the formers from algebraic relations among the latters. Resting on this principle, we shall derive properties of the fundamental characteristic classes from those of the corresponding invariant polynomial functions. Our results will thus give the formulas and duality theorems mentioned above in purely algebraic fashion. Besides, in finding relations for invariant polynomial functions, essential simplifications are provided by reducing these relations to those which exist among elementary symmetric functions. To this respect our method is previously expected by A. Borel-J. P. Serre [1], and is motivated by the questions imposed by them in connection with their proof-technique.

After we explain in §1 notations and elementary properties of invariant polynomial functions on the Lie algebra of a compact Lie group, we determine the structure of the algebras of invariant polynomial functions for the orthogonal groups (§2) and for the unitary groups (§3). The structure of these algebras is theoretically known owing to H. Cartan [2] etc. and is concretely given in lower dimensions by Chern [3] [4] to compute the cohomology of Grassmann manifolds. Resting upon the first main theorem on the vector invariants of the groups, we shall complete these results by giving algebraically independent generators of the algebras in concrete forms. In §4, we see how these generators change if we restrict them to subalgebras which belong to subgroups of special type. The relations thus obtained imply through the homomorphism of Cartan-Chern those which exist between fundamental characteristic classes. This procedure is achieved in §5 and the results furnish our main purpose. Finally we prove in §6 a theorem which may suggest a topological meaning of our algebraic method.

§ 1. Invariant polynomial functions.

Let G be a compact (not necessarily connected) Lie group and $\mathfrak{g}$ the Lie algebra of G. The inner automorphism induced by an element a of G, $x \rightarrow axa^{-1}$ $(x \in G)$, defines an automorphism of the Lie algebra $\mathfrak{g}$, which we call adjoint mapping induced by a and denote by $\mathrm{ad}(a)$.

A polynomial function F on $\mathfrak{g}$ is a function such that, if we represent an element X of $\mathfrak{g}$ by its coordinates $X_1, \cdots, X_r$ ($r=\dim \mathfrak{g}$) with respect to a coordinate system in $\mathfrak{g}$,

$$(1) \qquad F(X)=P(X_1, \cdots, X_r),$$

where P is a polynomial of r variables with real coefficients. Obviously this definition is independent of the choice of coordinate system in $\mathfrak{g}$. A polynomial P with r variables defines uniquely a polynomial function F on $\mathfrak{g}$ and this correspondence is a one-to-one homomorphism from the polynomial algebra into the algebra of functions on $\mathfrak{g}$. Therefore, we can make the algebra of polynomial functions on $\mathfrak{g}$ a graded algebra, which we denote by

$$S(G)=\sum_{k=0}^{\infty} S^k(G),$$

$S^k(G)$ being the module of polynomial functions of degree k, that is, polynomial functions F which is represented by (1) using a homogeneous polynomial P of degree k. In particular, $S^0(G)$ consists of constant polynomial functions and contains the unit 1 of $S(G)$, the polynomial function which is identically equal to 1. As is well-known, to a polynomial function of degree k we can associate a symmetric k-linear function on $\mathfrak{g}$ and vice versa. We call the k-linear function associated to a polynomial function F of degree k the polar form of F and denote it by the same letter F or, if necessary, by $F(X^1, \cdots, X^k)$ with variable vectors $X^1, \cdots, X^k$ in $\mathfrak{g}$. Then $F(X, \cdots, X)=F(X)$. F_1, F_2 being polynomial functions of degrees k and l respectively, the polar form of the product F_1F_2 is given by

$$(F_1F_2)(X^1, \cdots, X^{k+l})=\frac{1}{(k+l)!}\sum_{\pi} F_1(X^{\pi(1)}, \cdots, X^{\pi(k)})F_2(X^{\pi(k+1)}, \cdots, X^{\pi(k+l)}),$$

where π runs over all permutations of $\{1, \cdots, k+l\}$. Thus $S(G)$ may also be regarded in the following as the graded algebra spanned by symmetric multilinear functions on $\mathfrak{g}$ with this product.

By the way this notation will be used throughout this paper: $\sum_{\pi}$ shall mean the summation extending over all permutations π of the letters indicated in the summation with $\pi(\ )$. Besides ε_π will denote the signature of the permutation π.

Now a polynomial function on $\mathfrak{g}$ is called invariant if

$$F(\mathrm{ad}(a)X)=F(X), \qquad \text{for} \quad a\in G,\ X\in\mathfrak{g}.$$

A polynomial function F of degree k is invariant if and only if its polar form satisfies the condition:

$$F(\mathrm{ad}(a)X^1, \cdots, \mathrm{ad}(a)X^k) = F(X^1, \cdots, X^k), \quad \text{for} \quad a \in G,\ X^1, \cdots, X^k \in \mathfrak{g}.$$

The invariant polynomial functions on $\mathfrak{g}$ form a graded subalgebra of $S(G)$. We shall denote it by

$$I(G) = \sum_{k=0}^{\infty} I^k(G), \quad I^k(G) = I(G) \cap S^k(G).$$

$I^0(G)$ is the module of all constant functions on $\mathfrak{g}$. We shall say that some elements of positive degree in $I(G)$ are generators of $I(G)$, if they generate the algebra together with 1.

Let G and H be compact Lie groups whose Lie algebras are $\mathfrak{g}$ and $\mathfrak{h}$ respectively. A homomorphism φ of H into G induces a homomorphism, denoted also by φ, of $\mathfrak{h}$ into $\mathfrak{g}$. Since this homomorphism is a linear mapping we see at once that for a polynomial function F on $\mathfrak{g}$ $F(\varphi X)(X \in \mathfrak{h})$ is a polynomial function of $\mathfrak{h}$. The latter being denoted by φ^*F, the assignment of φ^*F to F defines a degree-preserving homomorphism φ^* of $S(G)$ into $S(H)$. Moreover, if F is invariant by G, φ^*F is invariant by H. Therefore φ^* maps $I(G)$ into $I(H)$. φ^* is called the dual mapping of φ. Note that, if H is a subgroup of G and if φ is the injection of H in G, the function φ^*F is nothing but the restriction of F on $\mathfrak{h}$.

Let G_1 and G_2 be compact Lie groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. The Lie algebra of the direct product $G_1 \times G_2$ is the direct sum $\mathfrak{g}_1 + \mathfrak{g}_2$. We know[1] that the tensor product $S(G_1) \otimes S(G_2)$ is isomorphic to $S(G_1 \times G_2)$ by the mapping which maps $F_1 \otimes F_2$ $(F_i \in S(G_i),\ i = 1, 2)$ to the polynomial function F on $\mathfrak{g}_1 + \mathfrak{g}_2$ defined by

$$F(X+Y) = F_1(X) F_2(Y), \quad X \in \mathfrak{g}_1,\ Y \in \mathfrak{g}_2.$$

In the following this function will be denoted by $F_1 \otimes F_2$. By this isomorphism the submodule $\sum_{k_1+k_2=k} S^{k_1}(G_1) \otimes S^{k_2}(G_2)$ is mapped onto $S^k(G)$, We shall now show that $I(G_1) \otimes I(G_2)$ is mapped onto $I(G_1 \times G_2)$. Obviously $I(G_1) \otimes I(G_2)$ is mapped into $I(G_1 \times G_2)$ and $\sum_{k_1+k_2=k} I^{k_1}(G_1) \otimes I^{k_2}(G_2)$ into $I^k(G_1 \times G_2)$. For an element a_i of G_i $(i=1, 2)$ the mapping $F_i(X_i) \to F_i(\mathrm{ad}(a_i)X_i)$ $(F_i \in S^{k_i}(G_i),\ X_i \in \mathfrak{g}_i)$ is a transformotion $\mathrm{ad}^{(k_i)}(a_i)$ in $S^{k_i}(G_i)$. The mapping $a_i \to \mathrm{ad}^{(k_i)}(a_i)$ is a representation of the compact group G_i on the linear space $S^{k_i}(G_i)$. Let $\chi_i^{(k_i)}$ be the character of this representa-

1) See Chevalley [5] p. 27.

tion. Since $I^{k_i}(G_i)$ is the subspace of elements fixed by this representation, we have

$$\dim I^{k_i}(G_i) = \int_{G_i} \chi_i^{(k_i)}(a_i)da_i ,$$

where the integral is taken with respect to the Haar measure on G_i with total measure 1. In the same way, we can consider the representation of $G_1 \times G_2$ on $S^k(G_1 \times G_2)$ with the character $\chi^{(k)}$ and we have

$$\dim I^k(G_1 \times G_2) = \int_{G_1 \times G_2} \chi^{(k)}(a)da .$$

Now by the isomorphism of $\sum_{k_1+k_2=k} S^{k_1}(G_1) \otimes S^{k_2}(G_2)$ onto $S^k(G_1 \times G_2)$ we see easily that

$$\chi^{(k)}(a_1 a_2) = \sum_{k_1+k_2=k} \chi_1^{(k_1)}(a_1)\chi_2^{(k_2)}(a_2) , \qquad a_i \in G_i \ (i = 1,\ 2).$$

Integrating both sides on $G_1 \times G_2$, it follows from the Fubini's theorem

$$\int_{G_1 \times G_2} \chi^{(k)}(a)da = \sum_{k_1+k_2=k} \int_{G_1} \chi_1^{(k_1)}(a_1)da_1 \int_{G_2} \chi_2^{(k_2)}(a_2)da_2 .$$

Therefore

$$\dim I^k(G_1 \times G_2) = \sum_{k_1+k_2=k} \dim I^{k_1}(G_1) \dim I^{k_2}(G_2)$$

$$\dim I^k(G_1 \times G_2) = \dim \big(\sum_{k_1+k_2=k} I^{k_1}(G_1) \otimes I^{k_2}(G_2)\big) .$$

Thus by the above isomorphism $\sum_{k_1+k_2=k} I^{k_1}(G_1) \otimes I^{k_2}(G_2)$ is mapped onto $I^k(G_1 \times G_2)$ and therefore $I(G_1) \otimes I(G_2)$ is mapped onto $I(G_1 \times G_2)$.

§ 2. Determination of $I(O(n))$ and $I(SO(n))$.

1. Reduction of the problem.

Let R^n be the real euclidean space of dimension n. We fix an orthonormal coordinate system $(e_1, \cdots, e_n)$ in R^n and the coordinates of a vector v will be designated by $(v_1, \cdots, v_n)$. The orthogonal group $O(n)$ is the group of real orthogonal matrices $a = (a_{ij})$ of degree n which operate in R^n; if $v' = av$,

$$v_i' = \sum_{j=1}^n a_{ij} v_j , \qquad i = 1, \cdots, n.$$

The rotation group $SO(n)$ is the group of orthogonal matrices of determinant 1. The group $O(n)$ being regarded as a Lie group, $SO(n)$ is the connected component of the unit element in $O(n)$. The Lie algebra $\mathfrak{o}(n)$

of $O(n)$, which is at the same time that of $SO(n)$, is represented by the Lie algebra of real skew-symmetric matrices;

$$\xi=(\xi_{ij})$$
$$\xi_{ij}+\xi_{ji}=0\,, \qquad i,\ j=1,\ \cdots,\ n,$$

and the adjoint mapping $\mathrm{ad}(a)$ induced by an element a of $O(n)$ then operates as the transformation by the matrix a;

$$(1) \qquad \mathrm{ad}(a)\xi=a\xi a^{-1}\,, \qquad \text{for } \xi\in\mathfrak{o}(n)\,.$$

Now we define, for two vectors u, v of R^n, an element $\xi(u, v)$ of $\mathfrak{o}(n)$ as follows:

$$(2) \qquad \xi(u,\ v)=\left(\frac{1}{2}(u_iv_j-v_iu_j)\right)_{i,\,j=1,\,\cdots,\,n}\,.$$

Then $\xi(u, v)$ is an $\mathfrak{o}(n)$-valued bilinear mapping on R^n and it is skew-symmetric, i.e.,

$$(3) \qquad \xi(v,\ u)=-\xi(u,\ v)\,.$$

(1) and the orthogonality of a imply

$$(4) \qquad \mathrm{ad}(a)\xi(u,\ v)=\xi(au,\ av)\,, \qquad \text{for } a\in O(n)\,.$$

On the other hand, the elements

$$(5) \qquad \xi(e_i,\ e_j)\,, \qquad 1\leqq i<j\leqq n\,,$$

form a basis of $\mathfrak{o}(n)$.

In the sequel of this paragraph, G shall denote the group $O(n)$ or the group $SO(n)$. The Lie algebra of G is $\mathfrak{o}(n)$ in both cases, but the adjoint group on $\mathfrak{o}(n)$ may be different for two cases.

Let

$$F=F(\xi^1,\ \cdots,\ \xi^k)\,,$$

be an element of $S^k(G)$ in the polar form, that is, a real-valued symmetric k-linear function on the Lie algebra of G. We associate to F a $2k$-linear function F' on R^n by the following definition:

$$(6) \qquad F'(u^1,\ v^1,\ \cdots,\ u^k,\ v^k)=F(\xi(u^1,\ v^1),\ \cdots,\ \xi(u^k,\ v^k))\,,$$

for u^1, v^1, $\cdots$, u^k, $v^k\in R^n$. Then, by the symmetry of F,

$$(7) \qquad F'(u^{\pi(1)},\ v^{\pi(1)},\ \cdots,\ u^{\pi(k)},\ v^{\pi(k)})=F'(u^1,\ v^1,\ \cdots,\ u^k,\ v^k)\,,$$

for any permutation π of $\{1, \cdots, k\}$, and, by the property (3) of $\xi(u, v)$,

$$(8)\qquad \begin{aligned} &F'(u^1, v^1, \cdots, v^i, u^i, \cdots, u^k, v^k) \\ &= -F'(u^1, v^1, \cdots, u^i, v^i, \cdots, u^k, v^k), \qquad i=1, \cdots, k. \end{aligned}$$

We shall denote by $S_k(G)'$ the set of all $2k$-linear functions on R^n which satisfy (7) and (8). If F belongs to $I^k(G)$, then F' is G-invariant, i.e.,

$$(9)\qquad F'(au^1, av^1, \cdots, au^k, av^k) = F'(u^1, v^1, \cdots, u^k, v^k), \qquad \text{for } a\in G.$$

The set of $2k$-linear functions satisfying (7), (8) and (9) will be denoted by $I_k(G)'$. Since a basis of $\mathfrak{o}(n)$ is given by (5), we may easily see that the definition of F' for F is a one-to-one onto mapping from $S^k(G)$ to $S_k(G)'$, by which $I^k(G)$ is mapped onto $I_k(G)'$.

Now the set of all real-valued $2k$-linear functions, denoted by L_k, is a module over the real number field R by obvious definitions of scalar multiplication and addition. (L_0 is the module of constant functions). The direct sum $L=\sum_{k=0}^{\infty} L_k$ shall be a graded algebra over R by defining product of $F_1'\in L_k$ and $F_2'\in L_l$ to be the element of L_{k+l};

$$\begin{aligned} &(F_1'F_2')(u^1, v^1, \cdots, u^{k+l}, v^{k+l}) \\ &= F_1'(u^1, v^1, \cdots, u^k, v^k)\, F_2'(u^{k+1}, v^{k+1}, \cdots, u^{k+l}, v^{k+l}). \end{aligned}$$

We define next linear endomorphisms $\mathfrak{S}$ and Δ of L which map each L_k into itself: for $F'\in L_k$, put

$$(\mathfrak{S}F')(u^1, v^1, \cdots, u^k, v^k) = \frac{1}{k!}\sum_{\pi} F'(u^{\pi(1)}, v^{\pi(1)}, \cdots, u^{\pi(k)}, v^{\pi(k)}),$$

$$(\Delta F')(u^1, v^1, \cdots, u^k, v^k) = \frac{1}{2^k}\sum (-1)^{\alpha} F'(w^1, 'w^1, \cdots, w^k, 'w^k),$$

where, in the second definition, the summation extends over all $(w^1, 'w^1, \cdots, w^k, 'w^k)$ such that for each i w^i, $'w^i$ are respectively u^i, v^i or v^i, u^i and, for each term, α is the number of w^i equal to v^i. Obviously these operators on L satisfy the following relations:

$$(10)\qquad \begin{cases} \mathfrak{S}\mathfrak{S} = \mathfrak{S}, \\ \Delta\Delta = \Delta, \\ \mathfrak{S}\Delta = \Delta\mathfrak{S}, \end{cases}$$

$$(11)\qquad \begin{cases} \mathfrak{S}(F_1'F_2') = \mathfrak{S}((\mathfrak{S}F_1')(\mathfrak{S}F_2')), \\ \Delta(F_1'F_2') = (\Delta F_1')(\Delta F_2'), \qquad \text{for} \quad F_1, F_2\in L. \end{cases}$$

It follows immediately

(12) $$\mathfrak{S}\Delta(\textstyle\sum * F_1' \cdots F_s') = \sum * \mathfrak{S}((\mathfrak{S}\Delta F_1') \cdots (\mathfrak{S}\Delta F_s'))\,,$$

where $F_1', \cdots, F_s' \in L$ and $*$ denotes a constant coefficient for each term.

The conditions (8) and (9) for $F' \in L_k$ are now equivalent to the conditions

$$\mathfrak{S}F' = F'$$

and

$$\Delta F' = F'\,,$$

respectively. Moreover, by (10), F' satisfies (8) and (9) if and only if

(13) $$\mathfrak{S}\Delta F' = F'$$

and, for any $F' \in L$, $\mathfrak{S}\Delta F'$ satisfies (8) and (9). In other words, $S_k(G)'$ is the set of fixed points in L_k of $\mathfrak{S}\Delta$ and is the image of L_k under $\mathfrak{S}\Delta$. Note that $\mathfrak{S}$ and Δ and so $\mathfrak{S}\Delta$ map a G–invariant function to a G–invariant function.

The previous assignment of $F' \in L_k$ for $F \in S^k(G)$ is linearly extended and it defines a one-to-one mapping from $S(G)$ into L. The image $S(G)'$ of $S(G)$ is the direct sum of $S_k(G)'$ and the image $I(G)'$ of the subalgebra $I(G)$ is the direct sum of $I_k(G)'$. If F_1', F_2' and $(F_1F_2)'$ is the images of F_1, F_2 and F_1F_2 respectively, then obviously

$$(F_1F_2)' = \mathfrak{S}(F_1'F_2')\,.$$

In this respect, we define $\circ$–product of F_1' and F_2' in $S(G)'$ by

$$F_1' \circ F_2' = \mathfrak{S}(F_1'F_2')\,.$$

This allows us to regard $S(G)'$ as a graded algebra and $I(G)'$ as its graded subalgebra. The above formula is then written in the form

(14) $$(F_1F_2)' = F_1' \circ F_2'$$

and this means that the mapping $F \to F'$ is an isomorphism from the algebra $S(G)$ onto the algebra $S(G)'$. $I(G)$ is mapped onto $I(G)'$ by this mapping.

Thus the determination of the algebra $I(G)$ reduces to that of the algebra $I(G)'$.

2. Generators of $I(G)$.

The G–invariancy (9) of a $2k$–linear function $F'(k>0)$ means that F' is a vector invariant of G. Put

$$(u^1,\ u^2)=\sum_{i=1}^{n} u_i^1 u_i^2\,,$$
$$[u^1,\ \cdots,\ u^n]=\det|u_j^i|,$$

for $u^1, \cdots, u^n \in R^n$. Then the first main theorem[2] on the vector invariants of the group G implies that the function F' is a linear combination of terms with the following types.

i) $(u^{p_1},\ u^{p_2})\cdots(u^{p_{t-1}},\ u^{p_t})(v^{q_1},\ v^{q_2})\cdots(v^{q_{t-1}},\ v^{q_t})(u^{p_{t+1}},\ v^{q_{t+1}})\cdots(u^k,\ v^k)$,

and, in case $G=SO(n)$,

ii) $[u^{p_1},\ \cdots,\ u^{p_r},\ v^{q_1},\ \cdots,\ v^{q_s}]\ (u^{p_{r+1}},\ u^{p_{r+2}})\cdots(u^{p_{t-1}},\ u^{p_t})(v^{q_{s+1}},\ v^{q_{s+2}})$
$\cdots(v^{q_{t-1}},\ v^{q_t})(u^{p_{t+1}},\ v^{q_{t+1}})\cdots(u^{p_k},\ v^{q_k})$

with $r+s=n$. We may suppose here, by the multilinearity of F, that $\{p_1, \cdots, p_k\}$ and $\{q_1, \cdots, q_k\}$ are both permutations of $\{1, \cdots, k\}$. Therefore each of these terms itself is a $2k$-linear function with variables $u^1, v^1, \cdots, u^k, v^k$. Now, if F' belongs to $S(G)'$, then (13) implies that F' is a linear combination of the $2k$-linear functions which are obtained by applying $\mathfrak{S}\Delta$ on these $2k$-linear functions of type i) or of type ii). Since these terms are G-invariant, the resulting $2k$-linear functions are also G-invariant. Thus they belong to the module $I_k(G)'$ and span it.

Let π be a permutation of $\{1, \cdots, k\}$. It defines a mapping of L_k into itself as follows: for $F' \in L_k$,

$$(\pi F')(u^1,\ v^1,\ \cdots,\ u^k,\ v^k)=F'(u^{\pi(1)},\ v^{\pi(1)},\ \cdots,\ u^{\pi(k)},\ v^{\pi(k)})\,.$$

Then it is obvious that

$$\mathfrak{S}F'=\mathfrak{S}(\pi F')\,.$$

Suppose that an element F' of L_k $(k>0)$ is decomposable, that is, there exist $F_1' \in L_{k_1}$ $(k_1>0)$ and $F_2' \in L_{k_2}$ $(k_2>0)$ such that

$$F'=F_1'F_2'\,.$$

Then, it follows from (11)

$$\begin{aligned}\mathfrak{S}\Delta F' &= \mathfrak{S}((\mathfrak{S}\Delta F_1')(\mathfrak{S}\Delta F_2'))\\ &= (\mathfrak{S}\Delta F_1')\circ(\mathfrak{S}\Delta F_2')\,.\end{aligned}$$

The same is true if $\pi F'$ is decomoposable, that is, $\pi F'=F_1'F_2'$ for some π. Now take for F' a $2k$-linear function of type i) or type ii). If $\pi F'$ is not decomposable to a product of functions of type i) or of type ii)

2) See Weyl [7] p. 53.

for any permutation π of $\{1, \cdots, k\}$, we call F' to be irreducible. Induction on the degree of F', together with the above argument, implies that for a not irreducible function F' of type i) or of type ii), there holds

$$\mathfrak{S}\Delta F' = \mathfrak{S}\Delta F_1' \circ \cdots \circ \mathfrak{S}\Delta F_s' ,$$

where $F_1', \cdots, F_s'$ are irreducible functions of type i) or of type ii). Since 1 and the functions $\mathfrak{S}\Delta F'$ span $I(G)'$ when F' varies over all $2k$-linear functions of type i) or of type ii), we conclude that 1 and $\mathfrak{S}\Delta F'$ generate the algebra $I(G)'$ when F' runs over all irreducible $2k$-linear functions of type i) or of type ii). In the following we shall give explicit forms of these generators.

Let F' be an irreducible $2k$-linear function of type i).

If $k=1$, the function F' is of the form

$$F_1' (u^1, v^1) = (u^1, v^1) .$$

In this case put

$$K_1' = \mathfrak{S}\Delta F_1' .$$

Suppose $k>1$. We may associate to F' the following table of k pairs of numbers.

$$(15) \qquad (p_1, p_2), \cdots, (p_{t+1}, p_t), (q_1, q_2), \cdots, (q_{t-1}, q_t), (p_{t+1}, q_{t+1}), \cdots, (p_k, q_k) .$$

If $p_i=q_i$ for some $i>t$, then for the permutation π which permute p_t and 1 $\pi F'$ is decomposable with F_1' as factor. This contradicts the irreducibility of F'. We call, for convenience, two numbers are paired if they form a pair in (15). Then, this means that a number is never paired with itself. Let us define now

$$\pi(1) = p_1$$

$$\pi(2) = \begin{cases} p_2 , & \text{if } t>0 , \\ q_1 , & \text{if } t=0 . \end{cases}$$

By induction, if we define mutually distinct numbers $\pi(1), \cdots, \pi(i)$ between 1 and k so that $\pi(j)$ and $\pi(j+1)$ are paired for $j=1, \cdots, i-1$, we take for $\pi(i+1)$ a number which is paired with $\pi(i)$ and differs from $\pi(1), \cdots, \pi(i)$, as far as it exists. After a finite number of times, we arrive at the stage where $\pi(i)$ is paired only with some of $\pi(1), \cdots, \pi(i)$. Making use of the fact that a number between 1 and k appears in exactly two pairs of (15), once as p and the other time as q, we may easily show that, in this case, $\pi(i)$ is paired with $\pi(1)$ as well as with

$\pi(i-1)$. Suppose this is the case for $i<k$. We define $\pi(i+1), \cdots, \pi(k)$ to be the numbers which form the complement of $\{\pi(1), \cdots, \pi(i)\}$ in the set $\{1, \cdots, k\}$. We have then a permutation π of $\{1, \cdots, k\}$ for which

$$\begin{aligned}&\pi^{-1}F'(u^1, v^1, \cdots, u^k, v^k)\\&\quad=(w^1, {}'w^2)(w^2, {}'w^3)\cdots(w^{i-1}, {}'w^i)(w^i, {}'w^1)\,F''(u^{i+1}, v^{i+1}, \cdots, u^k, v^k),\end{aligned}$$

where w^j, ${}'w^j$ are respectively u^j, v^j or v^j, u^j and F'' is a function of type i). This contradicts the irreducibility of F'. Therefore $i=k$. Moreover, this argument shows that F' is obtained from the $2k$-linear function

$$F_k'(u^1, v^1, \cdots, u^k, v^k)=(v^1, u^2)(v^2, u^3)\cdots(v^{k-1}, u^k)(v^k, u^1),$$

by a permutation of variable vectors which permutes u^i and v^i for a number of i. Then $\Delta F'=\pm\Delta F_k'$. Therefore, $\mathfrak{S}\Delta F'$ is equal to $\pm K_k'$, where

$$K_k'=\mathfrak{S}\Delta F_k'.$$

The functions $\pm K_k'$ $(k>0)$ are generators of $I(G)'$ which are derived from the irreducible functions of type i). Besides

$$K_k'=0, \qquad \text{if } k \text{ is odd.} \tag{16}$$

For, by the property (8) of K_k',

$$\begin{aligned}&K_k'(u^1, v^1, \cdots, u^k, v^k)\\&\quad=(-1)^k K_k'(v^1, u^1, \cdots, v^k, u^k)\\&\quad=(-1)^k\mathfrak{S}\Delta((u^1, v^2)(u^2, v^3)\cdots(u^{k-1}, v^k)(u^k, v^1))\\&\quad=(-1)^k\Delta\mathfrak{S}((v^2, u^1)(v^3, u^2)\cdots(v^k, u^{k-1})(v^1, u^k))\\&\quad=(-1)^k\Delta\mathfrak{S}((v^{\pi(2)}, u^{\pi(1)})(v^{\pi(3)}, u^{\pi(2)})\cdots(v^{\pi(k)}, u^{\pi(k-1)})(v^{\pi(1)}, u^{\pi(k)}))\\&\quad=(-1)^k K_k'(u^1, v^1, \cdots, u^k, v^k),\end{aligned}$$

where π is the permutation

$$\begin{pmatrix}1, & 2, & 3, & \cdots, & k-1, & k\\ 1, & k, & k-1, & \cdots, & 3, & 2\end{pmatrix}.$$

Next let F' be an irreducible function of type ii). This function appears only if $G=SO(n)$. Moreover, since the number of variables is an even integer $2k$, it follows immediately the number of members in the bracket, that is, n, is even. Put $n=2m$. Now we can associate to it the following table of one bracket and pairs of numbers:

$$(17)\qquad [p_1, \cdots, p_r, q_1, \cdots, q_r], (p_{s+1}, p_{s+2}), \cdots, (p_{t-1}, p_t), \\ (q_{r+1}, q_{r+2}), \cdots, (q_{t-1}, q_t), (p_{t+1}, q_{t+1}), \cdots, (p_k, q_k).$$

By the same reason as before, two numbers in a pair can not be equal to one another. Let $\pi(1), \cdots, \pi(k_0)$ be the numbers which appear twice in the bracket. If $k_0=m$, we may derive from the irreducibility of F' that $k_0=k$ and $\mathfrak{S}\Delta F'$ is equal up to its sign to the function

$$W_m(u^1, v^1, \cdots, u^m, v^m)=[u^1, v^1, \cdots, u^m, v^m].$$

If $k_0>m$, there remain numbers which appear once in the bracket. One of them is defined to be $\pi(k_0+1)$. This number $\pi(k_0+1)$ appears in a pair and let $\pi(k_0+2)$ be the number paired with it by this pair. Suppose $\pi(k_0+1), \cdots, \pi(k_0+i)$ are given so that two successive numbers form a pair in (17). If $\pi(k_0+i)$ is paired in (17) with a number not equal to $\pi(k_0+1), \cdots, \pi(k_0+i)$, this number is taken as $\pi(k_0+i+1)$. Otherwise, we may show easily that $\pi(k_0+i)$ appears in the bracket. The second case occurs certainly for some $i>1$, and then we set $k_1=k_1+i$. If $k_0+1>m$, we may find a number $\pi(k_1+1)$ in the bracket which is distinct from $\pi(1), \cdots, \pi(k_0), \pi(k_0+1), \pi(k_1)$. Taking k_1 instead of k_0, we may construct in the same way $\pi(k_1+1)$, $\pi(k_1+2)$ and so on, and arrive at a number $\pi(k_2)$ in the bracket. In this manner we obtain numbers $\pi(1), \cdots, \pi(k_0), \pi(k_0+1), \cdots, \pi(k_1), \pi(k_1+1), \cdots, \pi(k_s)$ so that the bracket and some of pairs in (17) are written, up to the order of numbers in them, as follows:

$$[\pi(1), \pi(1), \cdots, \pi(k_0), \pi(k_0), \pi(k_0+1), \pi(k_1), \pi(k_1+1), \cdots, \pi(k_1)], \\ (\pi(k_0+1), \pi(k_0+2)), \cdots, (\pi(k_1-1), \pi(k_1)), \\ \cdots\cdots\cdots\cdots\cdots, \\ (\pi(k_{s-1}+1), \pi(k_{s-1}+2)), \cdots, (\pi(k_s-1), \pi(k_s)).$$

Furthermore, we may see from the irreducibility of F' that $k_s=k$ and all the pairs in (17) appear here. For later convenience, we put

$$l_0=k_0, \ l_1=k_1-k_0, \ \cdots, \ l_s=k_s-k_{s-1}$$

and consider the $2k$-linear function

$$Z'_{l_0 l_1\cdots l_s}(u^1, v^1, \cdots, u^k, v^k) \\ =\mathfrak{S}\Delta([u^1, v^1, \cdots, u^{k_0}, v^{k_0}, u^{k_0+1}, v^{k_1}, u^{k_1+1}, \cdots, v^k] \\ (v^{k_0+1}, u^{k_0+2}) \cdots (v^{k_1-1}, u^{k_1})(v^{k_1+1}, u^{k_1+2}) \cdots (v^{k-1}, u^k)).$$

Then it follows from the above argument that the function $\mathfrak{S}\Delta F'$ is equal to $\pm Z'_{l_0 l_1\cdots l_s}$.

The functions W'_m, $Z'_{l_0 l_1 \cdots l_s}$ are generators of $I(SO(2m))'$ which are derived from irreducible functions of type ii).

Now, put

$$P_k'(u^1, v^1, \cdots, u^k, v^k) = \Delta \frac{1}{k!} \sum_\pi \varepsilon_\pi (v^1, u^{\pi(1)}) \cdots (v^k, u^{\pi(k)}),$$

$$\bar{P}_k'(u^1, v^1, \cdots, u^k, v^k) = \Delta \frac{1}{k!} \sum_\pi (v^1, u^{\pi(1)}) \cdots (v^1, u^{\pi(k)}).$$

Obviously P_k' and $\bar{P}_k'$ are G-invariant $2k$-linear functions which satisfy (13). We shall show that each of $\{P_k'\}$ and $\{\bar{P}_k'\}$ generates the subalgebra of $I(G)'$ which $\{K_k'\}$ generate. Let $l_1, \cdots, l_s$ be s numbers such that $k \geqq l_1 \geqq \cdots \geqq l_s \geqq 1$ and $l_1 + \cdots + l_s = k$. For convenience we put $k_i = l_1 + \cdots + l_i (i=1, \cdots, s)$, and denote by $[l_1, \cdots, l_s]$ or briefly by $[l]$ the product of cyclic permutations $(1 \cdots k_1)(k_1+1 \cdots k_2) \cdots (k_{s-1}+1 \cdots k)$. Then, as is well-known, any permutation π of $\{1, \cdots, k\}$ is conjugate to such a permutation $[l_1, \cdots, l_s]$, i.e. $\pi = \eta [l] \eta^{-1}$ by a permutation η. In this case,

$$(v^1, u^{\pi(1)}) \cdots (v^k, u^{\pi(k)}) =$$
$$(v^{\eta(1)}, u^{\eta(2)}) \cdots (v^{\eta(k_1)}, u^{\eta(1)}) (v^{\eta(k_1+1)}, u^{\eta(k_1+2)}) \cdots (v^{\eta(k)}, u^{\eta(k_{s-1}+1)}).$$

Therefore, if $\beta_{[l]}$ is the number of permutations which commute with $[l]$,

$$P_k'(u^1, v^1, \cdots, u^k, v^k)$$
$$= \Delta \frac{1}{k!} \sum_{[l]} \varepsilon_{[l]} \frac{k!}{\beta_{[l]}} \mathfrak{S}((v^1, u^2) \cdots (v^{k_1}, u^1)(v^{k_1+1}, u^{k_1+2}) \cdots (v^k, u^{k_{s-1}+1})),$$

where the summation extends over all such $[l]$'s which represent once each conjugate class of the symmetric group. Applying (11) and (12), it follows easily

$$P_k' = \sum_{[l]} \frac{\varepsilon_{[l]}}{\beta_{[l]}} K'_{l_1} \circ \cdots \circ K'_{l_s}$$

Since, for $s=1$, $l_1 = k$, $\beta_{[l]} = k$ and $\varepsilon_{[l]} = (-1)^{k-1}$,

$$P_k' = \frac{(-1)^{k-1}}{k} K'_k + \sum_{[l], s>1} \frac{\varepsilon_{[l]}}{\beta_{[l]}} K'_{l_1} \circ \cdots \circ K'_{l_s}. \tag{18}$$

In particular, from (16),

$$P_k' = 0, \qquad \text{if } k \text{ is odd.} \tag{19}$$

Moreover, using the induction on k we may derive from (18)

$$K_k' = (-1)^{k-1} k P_k' + \sum_{[l], s>1} * P'_{l_1} \circ \cdots \circ P'_{l_s} \tag{20}$$

where $*$ denotes a constant coefficient for each term.

Just in the same way, we can prove

$$\bar{P}_k' = \frac{1}{k} K'_k + \sum_{[l], s>1} \frac{1}{\beta_{[l]}} K'_{l_1} \cdots K'_{l_s} \tag{21}$$

$$\bar{P}_k' = 0 , \qquad \text{if } k \text{ is odd} \tag{22}$$

$$K_k' = k\bar{P}_k' + \sum_{[l], s>1} * \bar{P}'_{l_1} \cdots \bar{P}'_{l_s} \tag{23}$$

The formulas (18), (20), (21), (23) imply that each of the systems of functions $\{K_k'\}$, $\{P_k'\}$, $\{\bar{P}_k'\}$ generates the same subalgebra in $I(G)'$. Moreover, since

$$\begin{aligned}\sum_\pi \varepsilon_\pi(v^1, u^{\pi(1)}) \cdots (v^k, u^{\pi(k)}) &= \det|(v^i, u^j)|_{i,j=1,\dots,k} \\ &= \det\left(|v_p^i|_{\substack{i=1,\dots,k \\ p=1,\dots,n}} \cdot |u_q^j|_{\substack{j=1,\dots,k \\ q=1,\dots,n}}\right) \\ &= 0, \qquad \text{for } k>n ,\end{aligned}$$

$$P_k' = 0 \qquad \text{for } k>n .$$

Therefore, if we denote by m the largest integer not greater than $n/2$, we see from this and (19) that P_k' are all equal to zero except for $k=2, 4, \cdots, 2m$. Thus the functions $P_2', P_4', \cdots, P'_{2m}$ generate the subalgebra which $\{K_k\}$ generate. By virtue of (18) and (23), we may see that the same holds true about the functions $K_2', K_4', \cdots, K'_{2m}$ and about the functions $\bar{P}_2', \bar{P}_4', \cdots, \bar{P}'_{2m}$.

§ 3. Structure of $I(O(n))$.

As we have seen before, the algebra $I(O(n))$ is isomorphic to the algebra $I(O(n))'$ by the mapping $F \to F'$ defined by (6) and the latter algebra is generated by the functions $\mathfrak{S}\Delta F'$ where F' are irreducible functions of type i). These functions are, up to their sign, $\{K_k'\}$. Moreover we know that each of systems of functions $\{K_2', K_4', \cdots, K'_{2m}\}$, $\{P_2', P_4', \cdots, P'_{2m}\}$, $\{\bar{P}_2', \bar{P}_4', \cdots, \bar{P}'_{2m}\}$ generates the subalgebra of $I(O(n))'$ which $\{K_k'\}$ generate, in this case $I(O(n))'$ itself. Consider the functions in $I(O(n))$ which define K_k', P_k', $\bar{P}_k'$. If k is odd, these functions are zero because $K_k' = P_k' = \bar{P}_k' = 0$. If k is even these functions are put to be K_k, P_k, $(-1)^{\frac{k}{2}}\bar{P}_k$. When K_k, P_k and $\bar{P}_k$ are considered, the suffix k will be always supposed to be even. Then, as we can easily verify,

$$K_k(\xi^1, \cdots, \xi^k) = \frac{1}{k!} \sum_\pi \sum_{\alpha_1, \cdots, \alpha_k=1}^{n} \xi^{\pi(1)}_{\alpha_1\alpha_2} \cdots \xi^{\pi(k)}_{\alpha_k\alpha_1} ,$$

$$P_k(\xi^1, \cdots, \xi^k) = \frac{1}{k!} \sum_\pi \varepsilon_\pi \sum_{\alpha_1, \cdots, \alpha_k=1}^{n} \xi^1_{\alpha_1\alpha_{\pi(1)}} \cdots \xi^k_{\alpha_k\alpha_{\pi(k)}} ,$$

$$\bar{P}_k(\xi^1, \cdots, \xi^k) = \frac{(-1)^{\frac{k}{2}}}{k!} \sum_{\pi} \sum_{\alpha_1, \cdots, \alpha_k=1}^{n} \xi^1_{\alpha_1 \alpha_{\pi(1)}} \cdots \xi^k_{\alpha_k \alpha_{\pi(k)}} .$$

The polynomial functions with these polar forms are then given by

$$
\begin{aligned}
& K_k(\xi) = \sum_{\alpha_1, \cdots, \alpha_k=1}^{n} \xi_{\alpha_1\alpha_2} \cdots \xi_{\alpha_k\alpha_1} , \\
(24) \qquad & P_k(\xi) = \frac{1}{k!} \sum \varepsilon_\pi \sum_{\alpha_1, \cdots, \alpha_k=1}^{n} \xi_{\alpha_1 \alpha_{\pi(1)}} \cdots \xi_{\alpha_k \alpha_{\pi(k)}} , \\
& \bar{P}_k(\xi) = \frac{(-1)^{\frac{k}{2}}}{k!} \sum_{\pi} \sum_{\alpha_1, \cdots, \alpha_k=1}^{n} \xi_{\alpha_1 \alpha_{\pi(1)}} \cdots \xi_{\alpha_k \alpha_{\pi(k)}} .
\end{aligned}
$$

Each of $\{K_2, K_4, \cdots, K_{2m}\}$, $\{P_2, P_4, \cdots, P_{2m}\}$, $\{\bar{P}_2, \bar{P}_4, \cdots, \bar{P}_{2m}\}$ generates the algebra $I(O(n))$.

Now consider the linear subspace $\mathfrak{h}$ of $\mathfrak{o}(n)$ defined by

$$\xi_{ij} = 0 \qquad \text{for } (i, j) \neq (2l-1, 2l) \text{ or } (2l, 2l-1), \quad l = 1, \cdots, m .$$

$\mathfrak{h}$ is then an m-dimensional commutative subalgebra of the Lie algebra $\mathfrak{o}(n)$, and it generates a maximal torus T in the group $SO(n)$. As was mentioned in § 1, we have the homomorphic mapping $t^* : I(O(n)) \to I(T)$ by taking restrictions to $\mathfrak{h}$ of invariant polynomial functions on $\mathfrak{o}(n)$. We can choose as coordinates on $\mathfrak{h}$ the restrictions $x_1, \cdots, x_m$ of $\xi_{21}, \cdots, \xi_{2m\, 2m-1}$. Then, since T is abelian, $I(T)$ is the algebra of all polynomials in $x_1, \cdots, x_m$. Now, denoting the restrictions on $\mathfrak{h}$ of the functions $K_k, P_k, \bar{P}_k$ by the corresponding small letters, we have

$$
\begin{aligned}
& k_{2l}(x) = 2(-1)^l \sum_{i=1}^{m} x_i^{2l} , \\
(25) \qquad & p_{2l}(x) = \sum_{i_1 < \cdots < i_l} x_{i_1}^2 \cdots x_{i_l}^2 \quad (1 \leqq l \leqq m) ; \ p_{2l}(x) = 0 \ (l > m) , \\
& \bar{p}_{2l}(x) = \sum_{i_1 \leqq \cdots \leqq i_l} x_{i_1}^2 \cdots x_{i_l}^2 ,
\end{aligned}
$$

for $x = (x_1, \cdots, x_m)$. The polynomials $p_2, p_4, \cdots, p_{2m}$ are elementary symmetric functions in $x_1^2, \cdots, x_m^2$, and so they are algebraically independent. It follows immediately that the functions $P_2, P_4, \cdots, P_{2m}$ are also algebraically independent as polynomials in ξ_{ij} $(1 \leqq i < j \leqq n)$. Since these functions generate the algebra $I(O(n))$ in the algebra of polynomials in ξ_{ij}, we see first that $I(O(n))$ is isomorphic to the polynomial ring with m variables over the real number field, and secondly that the mapping $t^* : I(O(n)) \to I(T)$ maps the algebra $I(O(n))$ isomorphically on the set of symmetric functions of $x_1^2, \cdots, x_m^2$. From this, we may easily deduce that $K_2, K_4, \cdots, K_{2m}$ as well as $\bar{P}_2, \bar{P}_4, \cdots, \bar{P}_{2m}$ are also algebraically independent generators of the algebra $I(O(n))$.

We thus obtain the following.

Theorm 1. *The algebra $I(O(n))$ is generated by each of the following system of functions*

$$K_2,\ K_4,\ \cdots,\ K_{2m}\,;$$
$$P_2,\ P_4,\ \cdots,\ P_{2m}\,;$$
$$\bar{P}_2,\ \bar{P}_4,\ \cdots,\ \bar{P}_{2m}\,.$$

given in (24). *The functions in each system are algebraically independent as polynormal functions.*

Note that algebraic independence of polynomial functions does not depend on their expression by coordinates of the variable vector.

Since the generators given in the above theorem are of even degree, we have

Corollary. $I^k(O(n)) = 0$, *if k is odd.*

For convenience, we put

$$P_0 = \bar{P}_0 = 1\,.$$

The it holds

$$\sum_{h=0}^{l} (-1)^h P_{2h} \bar{P}_{2(l-h)} = 0 \qquad \text{for } 1 \leqq l \leqq m\,. \tag{26}$$

Proof. Take the identity for $(m+1)$ variables $x_1, \cdots, x_m, t$;

$$\prod^{m} (1-tx_i^2) \prod^{m} (1+tx_i^2+\cdots+tx_i^{2m}) = \prod^{m} (1-(tx_i^2))^{m+1}\,.$$

The coefficients of t^l in both sides yield

$$\sum_{h=0}^{l} (-1)^h p_{2h}(x) \bar{p}_{2(l-h)}(x) = 0\,,$$

where we put $p_0 = \bar{p}_0 = 1$. Since the mapping $t^* : I(O(n)) \to I(T)$ is isomorphic into, we get the required formula (26).

4. Structure of $I(SO(n))$.

Evidently $I(O(n)) \subset I(SO(n))$, and so $I(O(n))' \subset I(SO(n))'$.

If n is odd, then irreducible functions of type ii) do not appear and the functions $\mathfrak{S}\Delta F'$ for irreducible functions F' of type i) generate the algebra $I(SO(n))'$. These generators are precisely those of $I(O(n))'$.

Therefore $I(SO(n))'=I(O(n))'$. The assertion of theorem 1 holds then for $I(SO(n))$.

Suppose that $n=2m$ is an even integer. In this case, we have as generators of $I(SO(n))'$, besides the functions $\mathfrak{S}\Delta F'$ for irreducible functions of type i), the functions $\mathfrak{S}\Delta F'$ for irreducible functions of type ii). These are given, up to their sign, by W'_m and $Z'_{l_0 l_1\cdots l_s}$. The functions in $I(SO(2m))$ which define $\frac{1}{2^m m!}W'_m$ and $\frac{1}{2^m l_0!}Z'_{l_0 l_1\cdots l_s}$ are designated by W_m and $Z_{l_0 l_1\cdots l_s}$. They are

$$W_m(\xi^1,\cdots,\xi^m)=\frac{1}{2^m m!}\sum_\pi \varepsilon_\pi \xi^1_{\pi(1)\pi(2)}\cdots\xi^m_{\pi(n-1)\pi(n)},$$

$$\begin{aligned}&Z_{l_0 l_1\cdots l_s}(\xi^1,\cdots,\xi^k)\\ &=\frac{1}{2^m l_0!}\frac{1}{k!}\sum_\eta\sum_\pi \varepsilon_\pi \sum_{\alpha_1,\cdots,\alpha_{l_1-1},\beta_1,\cdots,\gamma_{l_s-1}=1}^{n} \xi^{\eta(1)}_{\pi(1)\pi(2)}\cdots\\ &\qquad \xi^{\eta(k_0)}_{\pi(2l_0-1)\pi(2l_0)}\xi^{\eta(k_0+1)}_{\pi(2l_0+1)\alpha_1}\xi^{\eta(k_0+2)}_{\alpha_1\alpha_2}\cdots\xi^{\eta(k_1)}_{\alpha_{l_1-1}\pi(2l_0+2)}\xi^{\eta(k_1+1)}_{\pi(2l_0+3)\beta_1}\cdots\xi^{\eta(k)}_{\gamma_{l_s-1}\pi(n)}.\end{aligned}$$

and the polynomial functions with these polar forms are

$$W_m(\xi)=\frac{1}{2^m m!}\sum_\pi \xi_{\pi(1)\pi(2)}\cdots\xi_{\pi(n-1)\pi(n)} \tag{27}$$

$$\begin{aligned}Z_{l_0 l_1\cdots l_s}(\xi)&=\frac{1}{2^m l_0!}\sum_\pi \varepsilon_\pi \sum_{\alpha_1,\cdots,\alpha_{l_1-1}\beta_1,\cdots,\beta_{l_s-1}=1}^{n} \xi_{\pi(1)\pi(2)}\cdots\\ &\qquad \xi_{\pi(2l_0-1)\pi(2l_0)}\xi_{\pi(2l_0+1)\alpha_1}\xi_{\alpha_1\alpha_2}\cdots\xi_{\alpha_{l_1-1}\pi(2l_0+2)}\xi_{\pi(2l_0+3)\beta_1}\cdots\xi_{\gamma_{l_s-1}\pi(n)}.\end{aligned}$$

The restrictions of these functions on $\mathfrak{h}$ are as follows.

$$w_m(x)=(-1)^m x_1\cdots x_m, \tag{28}$$

$$z_{l_0 l_1\cdots l_s}(x)=\begin{cases}0, & \text{if some } l_h \text{ are even}, \quad h=1,\cdots,s;\\ (-1)^m x_1\cdots x_m\Big(\sum_{\{j_1\cdots j_s\}\subset\{1\cdots k\}}(-1)^{\{(l_1-1)+\cdots+(l_s-1)\}/2}x_{j_1}^{l_1-1}\cdots x_{j_s}^{l_s-1}\Big).\end{cases}$$

Now the functions W_m^2, $Z^2_{l_0 l_1\cdots l_s}$ belong obviously to $I(O(n))$. The restriction of W_m^2 on $\mathfrak{h}$ is w_m^2 which is equal to p_{2m}. Since the mapping $t^*: I(O(n))\to I(T)$ is one-to-one, it follows

$$W_m^2=P_{2m}. \tag{29}$$

As regards $Z_{l_0 l_1\cdots l_s}$, $z^2_{l_0 l_1\cdots l_s}$ reduces to 0 if some l_h is even, and by the same reason as above $Z^2_{l_0 l_1\cdots l_s}$ itself is zero. As the algebra of polynomial functions on $\mathfrak{o}(n)$ has no zero-divisor, it follows

$$Z_{l_0 l_1\cdots l_s}=0, \qquad \text{if some } l_h \text{ is even}$$

Suppose all $l_1, \cdots, l_s$ are odd. In the expression of $z_{l_0l_1\cdots l_s}$ for this case, the polynomial of $x_1, \cdots, x_m$ in the parenthesis is a symmetric function of $x_1^2, \cdots, x_m^2$, so that it is a restriction of a function F of $I(O(n))$. Therefore $z_{l_0l_1\cdots l_s}$ is a restriction of the function W_mF. The functions $Z^2_{l_0l_1\cdots l_s}$ and $(W_mF)^2$ belong to $I(O(n))$ and reduce on $\mathfrak{h}$ to the same function $z^2_{l_0l_1\cdots l_s}$. It follows by the same reason as above that $(W_mF)^2=Z^2_{l_0l_1\cdots l_s}$ and

$$Z_{l_0l_1\cdots l_s}=\pm W_mF\,.$$

This shows that $Z_{l_0l_1\cdots l_s}$ is contained in the subalgebra generated by $I(O(n))$ and by W_m in $I(SO(n))$. Together with the previous consideration, $I(SO(n))$ is then obtained by adjoining W_m to $I(O(n))$. Since $P_2, P_4, \cdots, P_{2m}$ generate $I(O(n))$ by theorem 1 and since $P_{2m}=W_m^2$ by (29), it follows that $P_2, P_4, \cdots, P_{2m-2}, W_m$ are generators of $I(SO(2m))$. Moreover, by the same argument as for $I(O(n))$, we may easily see that these generators are algebraically independent polynomial functions and furthermore that the same conculusion is true if we replace $P_2, P_4, \cdots, P_{2m-2}, W_m$ by $K_2, K_4, \cdots, K_{2m-2}, W_m$ or by $\bar{P}_2, \bar{P}_4, \cdots, \bar{P}_{2m-2}, W_m$.

The following theorem is thus proved.

Theorem 2. *The algebra $I(SO(n))$ is generated by each of the following systems of functions defined by* (24) *and* (27); *if n is odd,*

$$K_2,\ K_4,\ \cdots,\ K_{2m};$$
$$P_2,\ P_4,\ \cdots,\ P_{2m};$$
$$\bar{P}_2,\ \bar{P}_4,\ \cdots,\ \bar{P}_{2m},$$

and if $n=2m$ is even,

$$K_2,\ K_4,\ \cdots,\ K_{2m-2},\ W_m;$$
$$P_2,\ P_4,\ \cdots,\ P_{2m-2},\ W_m;$$
$$\bar{P}_2,\ \bar{P}_4,\ \cdots,\ \bar{P}_{2m-2},\ W_m.$$

We call P_k and $\bar{P}_k$ Pontrjagin- and dual Pontrjagin invariant function of degree k respectively, and W_m Euler-Poincaré invariant function.

§3. Determination of $I(U(m))$.

The structure of $I(U(m))$ is determined in a similar way as that of $I(O(n))$, so we discuss this determination briefly in this paragraph remarking differences for these two cases.

Let C^m be the complex m-dimensional euclidean space with a fixed orthonormal coordinate system $e_1, \cdots, e_m$ and the coordinates of a vector

v will be denoted by $v_1, \cdots, v_m$. The unitary group $U(m)$ is the group of unitary matrices of degree m which operate in C^m in the usual manner, and it has the Lie group structure. The Lie algebra $\mathfrak{u}(m)$ of this group is the Lie algebra of complex skew-hermitian matrices;

$$\zeta = (\zeta_{ij}), \qquad \zeta_{ij} + \bar{\zeta}_{ji} = 0, \qquad i, j = 1, \cdots, m.$$

For an element $a \in U(m)$, $\mathrm{ad}(a)$ is then the transformation

$$\mathrm{ad}(a)\zeta = a\zeta a^{-1}.$$

We define for two vectors u and v in C^m the element $\zeta(u, v)$ of $\mathfrak{u}(m)$ by the following formula,

$$\zeta(u, v) = \left(\frac{1}{2}(u_i\bar{v}_j - v_i\bar{u}_j)\right)_{i, j=1, \cdots, m}.$$

In the following, C^m is regarded as the real 2^m-dimensional vector space R^{2m} with the basis $e_1, \sqrt{-1}e_1, \cdots, e_m, \sqrt{-1}e_m$. Then $\zeta(u, v)$ is a $\mathfrak{u}(m)$-valued biliner function on R^{2m} which satisfies

$$\zeta(v, u) = -\zeta(u, v), \tag{1}$$

$$\zeta(\sqrt{-1}u, \sqrt{-1}v) = \zeta(u, v), \tag{2}$$

$$\mathrm{ad}(a)\zeta(u, v) = \zeta(au, av), \qquad \text{for} \quad a \in U(m). \tag{3}$$

The elements

$$\begin{cases} \zeta(e_i, e_j), & 1 \leqq i < j \leqq m \\ \zeta(\sqrt{-1}e_i, e_j), & 1 \leqq i \leqq j \leqq m. \end{cases} \tag{4}$$

form a basis of $\mathfrak{u}(m)$.

Let $F = F(\zeta^1, \cdots, \zeta^k) \in S^k(U(m))$. We associate to F the $2k$-linear function F' on R^{2m} which is defined by

$$F'(u^1, v^1, \cdots, u^k, v^k) = F(\zeta(u^1, v^1), \cdots, \zeta(u^k, v^k)). \tag{5}$$

The symmetry of F and properties (1) and (2) of $\zeta(u, v)$ imply respectively

$$F'(u^{\pi(1)}, v^{\pi(1)}, \cdots, u^{\pi(k)}, v^{\pi(k)}) = F'(u^1, v^1, \cdots, u^k, v^k) \tag{6}$$

for any permutation π of $\{1, \cdots, k\}$, and

$$F'(u^1, v^1, \cdots, v^i, u^i, \cdots, u^k, v^k) = -F(u^1, v^1, \cdots, u^i, v^i, \cdots, u^k, v^k), \quad i = 1, \cdots, k, \tag{7}$$

$$F'(u^1, v^1, \cdots, \sqrt{-1}u^i, \sqrt{-1}v^i, \cdots, u^k, v^k) = F(u^1, v^1, \cdots, u^i, v^i, \cdots, u^k, v^k), \quad i = 1, \cdots, k. \tag{8}$$

From the fact that $\mathfrak{u}(m)$ has the basis (4), we see easily that F is uniquely determined by F' associated to it and that the properties (6), (7) and (8) of a $2k$-linear function F' are characteristic for F' to be associated to a function F in $S^k(U(m))$. Moreover F belongs to $I^k(U(m))$ if and only if F' is invariant, that is,

$$F'(au^1, av^1, \cdots, au^k, av^k) = F'(u^1, v^1, \cdots, u^k, v^k) \qquad \text{for } a \in U(m)\,. \tag{9}$$

We denote by $S_k(U(m))'$ the set of $2k$-linear functions which satisfy (6), (7) and (8) and by $I_k(U(m))'$ its subset consisting of invariant functions.

Now, let L_k be the module of $2k$-linear functions on R^{2m}. As is defined in §2, the direct sum $L = \sum_k L_k$ is a graded algebra over the real number field with linear endomorphisms $\mathfrak{S}$ and Δ. R^{2m} being real vector space obtained from C^m, we can define a linear endomorphism J of L by the following formula: For $F' \in L_k$,

$$\begin{aligned} JF'(u^1, v^1, \cdots, u^k, v^k) &= \frac{1}{2^k} \sum_{i_1 < \cdots < i_l} F'(u^1, v^1, \cdots, \sqrt{-1}u^{i_1}, \sqrt{-1}v^{i_1}, \cdots, \sqrt{-1}u^{i_l}, \sqrt{-1}v^{i_l}, \\ &\qquad \cdots, u^k, v^k)\,, \end{aligned}$$

where the summation is taken over all subset $\{i_1, \cdots, i_l\}$ of $\{1, \cdots, k\}$. Then

$$JJ = J, \quad \mathfrak{S}J = J\mathfrak{S}, \quad \Delta J = J\Delta\,. \tag{9}$$

$$J(F_1'F_2') = JF_1'JF_2'\,, \qquad \text{for } F_1', F_2' \in L\,. \tag{10}$$

J maps an invariant function to an invariant function. The condition (8) of a $2k$-linear function F' is now equivalent to $JF' = F'$.

The direct sums $S(U(m))' = \sum_{k=0}^{\infty} S_k(U(m))'$ and $I(U(m))' = \sum_{k=0}^{\infty} I_k(U(m))'$ are in L. Moreover, $S(U(m))'$ and $I(U(m))'$ are contained in $S(O(2m))'$, since the elements of $S(O(2m))'$ in L_k are defined by the conditions (6) and (7). $S(U(m))'$ and $I(U(m))'$ are then subalgebras of the algebra $S(O(2m))'$ with $\circ$-product, which are isomorphic to $S(U(m))$ and $I(U(m))$ respectively. While, we know that $S(O(2m))'$ is the set of elements $\mathfrak{S}\Delta F' (F' \in L)$. From this and (9), it follows that $S(U(m))'$ is the set of elements $\mathfrak{S}\Delta JF' (F' \in L)$. By the same reason, $I(U(m))'$ is the set of elements $\mathfrak{S}\Delta JF'$ where F' are invariant elements in L. Note that J is an endomorphism of the algebra $S(U(m))'$, as is easily seen from (9) and (10).

Now we extend the coefficient field of L, the real number field, to the complex number field. The algebra so obtained is denoted by

$L^c = \sum_{k=0}^{\infty} L_k^c$. An element F' of L_k^c is uniquely represented as

(11) $$F' = F_1' + \sqrt{-1}F_2'$$

with F_1', $F_2' \in L$. Therefore F' may be considered as a complex-valued $2k$-linear function on R^{2m}. If F_1' and F_2' are invariant, then F' is an invariant complex-valued $2k$-linear function, and the converse is true. The endomorphisms $\mathfrak{S}$, Δ, J are naturally extended onto L. Denote by $I_k^c(U(m))'$ the set of elements $\mathfrak{S}\Delta JF'$ where F' are invariant complex-valued $2k$-linear functions on R^{2m}. Then elements of $I_k^c(U(m))'$ are complex-valued $2k$-linear functions which satisfy (6), (7) and (8). Let $I^c(U(m))'$ be direct sum $\sum_{k=0}^{\infty} I_k^c(U(m))'$. $I^c(U(m))'$ is the complexification of $I(U(m))'$, that is, F' belongs to $I^c(U(m))'$ if and only if the components F_1', F_2' of F' in the expression (11) belong to $I(U(m))'$. According to this, we can extend the $\circ$-product in $I(U(m))'$ onto $I^c(U(m))'$ and we may also consider this product in $I^c(U(m))'$ as the product between complex-valued functions satisfying (6) and (7) which is defined in the same way as the $\circ$-product in $S(U(m))'$. From this, if we obtain a system of generators of the algebra $I^c(U(m))'$, the elements of $I(U(m))'$ which are the components in the expressions (11) of these generators generate the algebra $I(U(m))'$. Therefore, in order to obtain generators of $I(U(m))'$, it is sufficient to get those of $I^c(U(m))'$.

Let F' be a complex-valued invariant $2k$-linear function on R^{2m}. The first main theorem on the vector invariants of the group $U(m)$ implies that F' is representable as a polynomial of inner product $(\cdot, \cdot)$ between variable vectors.[3] Together with the $2k$-linearity of F', it follows that F' is a linear combination of the $2k$-linear functions of the $2k$-linear functions of the following form

(12) $$(w^1, w^2)(w^3, w^4)\cdots(w^{2k-1}, w^{2k})$$

where $\{w^1, w^2, \cdots, w^{2k-1}, w^{2k}\}$ is a permutation of $\{u^1, v^1, \cdots, u^k, v^k\}$. Since an element of $I^c(U(m))'$ is of the form $\mathfrak{S}\Delta JF'$ with an invariant $2k$-linear function F', we see from this that the functions which result by applying $\mathfrak{S}\Delta J$ to the functions (27) span the linear space $I^c(U(m))'$. Now, changing the notation, we denote the $2k$-linear function (12) by F'. Assume that, for some i, $u^i = w^{2j_1}$ and $v^i = w^{2j_2}$ $(1 \leqq j_1, j_2 \leqq k)$. Then $JF'(\cdots, \sqrt{-1}u^i, \sqrt{-1}v^i, \cdots) = -JF'(\cdots, u^i, v^i, \cdots)$, and by the property (8) of JF', $JF' = 0$, *a fortiori*, $\mathfrak{S}\Delta JF' = 0$. The same is true if $u^i = w^{2j_1-1}$,

3) See Chern [3] p. 97.

$v^i = w^{2j_2-1}(1 \leqq j_1, j_2 \leqq k)$ for some i. Therefore, we may restrict our attention to the functions F' in which $u^i = w^{2j_1-1}$, $v^i = w^{2j_2}$ or $u^i = w^{2j_1}$, $v^i = w^{2j_2-1}(1 \leqq j_1, j_2 \leqq k)$ for each i. For such a function F' it is obvious that $JF' = F'$ and so $\mathfrak{S}\Delta JF' = \mathfrak{S}\Delta F'$. If we permute u^i and v^i in the definition (12) of F', we obtain a new $2k$-linear function F_1' and $\Delta F_1' = -\Delta F'$. Therefore, as far as we require the additive generators of $I_k^c(U(m))'$, it is sufficient to consider the functions of the form

$$\mathfrak{S}\Delta(v^{p_1}, u^{q_1})(v^{p_2}, u^{q_2}) \cdots (v^{p_k}, u^{q_k}) , \tag{13}$$

where $\{p_1, \cdots, p_k\}$ and $\{q_1, \cdots, q_k\}$ are both permutations of $\{1, \cdots, k\}$.

Once we know that 1 and the functions of the form (13) (k varying also) span $I^c(U(m))'$, we can use the same argument as in §2.2 to require generators of the algebra $I(U(m))'$. The following results are then obtained: Put

$$B_k'(u^1, v^1, \cdots, u^k, v^k) = \mathfrak{S}\Delta((v^1, u^2)(v^2, u^3) \cdots (v^k, u^1)) ,$$

$$C_k'(u^1, v^1, \cdots, u^k, v^k) = \frac{1}{k!} \sum_{\pi} \varepsilon_\pi (v^1, u^{\pi(1)}) \cdots (v^k, u^{\pi(k)}) ,$$

$$\bar{C}_k'(u^1, v^1, \cdots, u^k, v^k) = \frac{1}{k!} \sum_{\pi} (v^1, u^{\pi(1)}) \cdots (v^k, u^{\pi(k)}) .$$

Then each of the systems of functions $\{B_k'\}$, $\{C_k'\}$, $\{\bar{C}_k'\}$ generates the algebra $I^c(U(m))'$. Moreover, $C_k' = 0$ for $k > m$ and $C_1', \cdots, C'_m$ generate $I^c(U(m))$. Then since $C_k'(k \leqq m)$ is expressible by $B_1', \cdots, B'_m$ and also by $\bar{C}'_1, \cdots, \bar{C}'_m$, $B_1', \cdots, B'_m$ as well as $\bar{C}_1', \cdots, \bar{C}_m'$ generate $I^c(U(m))'$.

The values of B_k' are real if k is even and are purely imaginary if k is odd. This is seen by the argument used in the proof of §2 (16). The same is true about C_k' and $\bar{C}_k'$, since they are expressible as polynomials of $B_1', \cdots, B'_m$ with real coefficients. Therefore $\frac{1}{(\sqrt{-1})^k}B_k'$, $\frac{1}{(\sqrt{-1})^k}C_k'$, $\frac{1}{(\sqrt{-1})^k}\bar{C}_k'$ belong to $I(U(m))'$ for all k. Then, according to the discussion given before and the above result, a system of generators of $I(U(m))'$ is given by each of the following systems.

$$\frac{1}{(\sqrt{-1})^k} B_k', \qquad k = 1, \cdots, m ;$$

$$\frac{1}{(\sqrt{-1})^k} C_k', \qquad k = 1, \cdots, m ;$$

$$\frac{1}{(\sqrt{-1})^k} \bar{C}_k', \qquad k = 1, \cdots, m .$$

To the functions $\frac{1}{(\sqrt{-1})^k} B_k'$, $\frac{1}{(\sqrt{-1})^k} C_k'$, $\frac{1}{(\sqrt{-1})^k} \bar{C}_k'$, there correspond the following functions in $I(U(m))$.

$$B_k(\zeta) = \frac{1}{(\sqrt{-1})^k} \sum_{\alpha_1, \cdots, \alpha_k=1}^{m} \zeta_{\alpha_1\alpha_2} \zeta_{\alpha_2\alpha_3} \cdots \zeta_{\alpha_k\alpha_1},$$

$$C_k(\zeta) = \frac{1}{(\sqrt{-1})^k} \frac{1}{k!} \sum_{\pi} \varepsilon_\pi \sum_{\alpha_1, \cdots, \alpha_k=1}^{m} \zeta_{\alpha_1\alpha_{\pi(1)}} \cdots \zeta_{\alpha_k\alpha_{\pi(k)}},$$

$$\bar{C}_k(\zeta) = \frac{1}{(\sqrt{-1})^k} \frac{1}{k!} \sum_{\pi} \sum_{\alpha_1, \cdots, \alpha_k=1}^{m} \zeta_{\alpha_1\alpha_{\pi(1)}} \cdots \zeta_{\alpha_k\alpha_{\pi(k)}}.$$

Notice that these functions are considered as polynomial functions on the real vector space $\mathfrak{u}(m)$. $I(U(m))$ is generated by each of the following systems of functions

$$B_1,\ B_2,\ \cdots,\ B_m;$$
$$C_1,\ C_2,\ \cdots,\ C_m;$$
$$\bar{C}_1,\ \bar{C}_2,\ \cdots,\ \bar{C}_m.$$

Finally we show the algebraic independence of these generators. Let T' be the maximal torus of $U(m)$ which consists of diagonal unitary matrices. The abelian subalgebra $\mathfrak{h}'$ of $\mathfrak{u}(m)$ corresponding to T' is composed of diagonal matrices $\zeta=(\zeta_{ij})$, $\zeta_{ij}=0$ if $i \neq j$. Coordinates in $\mathfrak{h}'$ are introduced by setting

$$x_1 = \frac{1}{\sqrt{-1}} \zeta_{11}, \ \cdots,\ x_m = \frac{1}{\sqrt{-1}} \zeta_{mm}.$$

$I(T')$ is the algebra of all polynomials in $x_1, \cdots, x_m$. By the mapping $t'^* : I(U(m)) \to I(T')$ dual to the injection $t' : T \to U(m)$ the functions B_k, C_k, $\bar{C}_k$ are mapped to the following functions b_k, c_k, $\bar{c}_k$.

$$\begin{aligned} & b_k(x) = \sum_{i=1}^{m} x_i^k, \\ (15) \qquad & c_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \ (k \leq m);\ c_k(x) = 0 \ (k > m), \\ & \bar{c}_k(x) = \sum_{i_1 \leqq \cdots \leqq i_k} x_{i_1} \cdots x_{i_k}, \end{aligned}$$

where $x=(x_1, \cdots, x_m)$. The polynomials $c_1, \cdots, c_m$ are elementary symmetric functions and so they are algebraically independent. It follows then that $C_1, \cdots, C_m$ are algebraically independent and by the mapping $I(U(m)) \to I(T')$, $I(U(m))$ is mapped univalently onto the algebra of elementary symmetric functions of $x_1, \cdots, x_m$. Therefore $I(U(m))$ is isomorphic to the polynomial ring with m variable over the real number field. From this, we see further that the m generators $B_1, \cdots, B_m$, and $\bar{C}_1, \cdots, \bar{C}_m$ are algebraically independent.

Thus we get the following

Theorem 3. *The algebra $I(U(m))$ is generated by each of the following system of functions*

$$B_1,\ B_2,\ \cdots,\ B_m\,;$$
$$C_1,\ C_2,\ \cdots,\ C_m\,;$$
$$\bar{C}_1,\ \bar{C}_2,\ \cdots,\ \bar{C}_m\,,$$

defined by (14).

The functions in each system are algebraically independent as polynomial functions.

For convenience, we put

$$C_0 = \bar{C}_0 = 1\,.$$

Then, there holds

$$\sum_{h=0}^{k}(-1)^h C_k \bar{C}_{k-h} = 0\,, \qquad \text{for} \quad 1 \leqq k \leqq m\,. \tag{16}$$

Proof is completely analogous to that of the formula §2 (20): We make use of (15) instead of §2 (25) used there.

We call C_k and $\bar{C}_k$ Chern and dual Chern invariant functions of degree k respectively.

§4. Relations.

In this paragraph we consider inclusions between some groups and study the effect of its dual mapping —restriction mapping— to the invariant polynomial functions of special types.

1. $O(n) \subset U(n)$.

The real matrices in the group $U(n)$ form its subgrup $O(n)$.

The dual mapping ι^* of the injection $O(n) \rightarrow U(n)$ maps $I(U(n))$ into $I(O(n))$. Since $I^k(O(n)) = 0$ for odd k (Corollary to theorem 1),

$$\iota^* C_k = \iota^* \bar{C}_k = 0\,, \qquad \text{if } k \text{ is odd.}$$

For even k, it is obvious from the definitions (§2. (24), §3. (14)) that

$$C_k(\xi) = (-1)^{\frac{k}{2}}\, P_k(\xi)\,,$$
$$\bar{C}_k(\xi) = \bar{P}_k(\xi)\,, \qquad \text{for} \quad \xi \in \mathfrak{o}(n)\,.$$

That is,

(1) $$\iota^* C_k = \begin{cases} (-1)^{\frac{k}{2}} P_k, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

(1)′ $$\iota^* \bar{C}_k = \begin{cases} \bar{P}_k, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

2. $U(m) \subset SO(2m)$.

The complex m-dimensional euclidean space C^m on which the group $U(m)$ operates may be regarded as the real $2m$-dimensional euclidean space R^{2m}; if $e_1, \cdots, e_m$ is the previously given orthonormal coordinate system in C^m, $e_1, \sqrt{-1}\, e_1, \cdots, e_m, \sqrt{-1}\, e_m$ give the associated orthonormal coordinate system in R^{2m}. The transformation in R^{2m} determined by an element of $U(m)$ is written by an orthogonal matrix of determinant 1 with respect to this associated coordinate system. In this way, $U(m)$ may be considered as a closed subgroup of $SO(2m)$. Let ι be the injection $U(m) \to SO(2m)$. In the notation of §§2 and 3, ι maps the maximal torus T' of $U(m)$ onto the maximal torus T of $SO(2m)$. In more detail, using concrete description of ι in matrical form, we see easily that the mapping $\mathfrak{u}(m) \to \mathfrak{o}(2m)$ induced by ι maps an element with coordinate $(x_1, \cdots, x_m)$ of $\mathfrak{h}'$ to the element with the same coordinates. Both $I(T)$ and $I(T')$ being represented as the polynomial ring with variables $x_1, \cdots, x_m$, this means that the dual mapping of the restriction ι' of ι on T', is the identity mapping of this polynomial ring. While we know in §§2 and 3 that the mappings $t^*: I(SO(2m)) \to I(T)$ and $t'^*: I(U(m)) \to I(T')$ are both univalent. Since $\iota \circ t' = t \circ \iota'$, $t'^* \circ \iota^* = \iota'^* \circ t^*$ and, after the above identification, $\iota^* = t'^{*-1} \circ t^*$.

This being said the t^*-image of P_{2l} is by definition p_{2l};

$$p_{2l}(x) = \sum_{i_1 < \cdots < i_l} x_{i_1}^2 \cdots x_{i_l}^2 \quad (l \leqq m); \; p_{2l}(x) = 0 \quad (l > m) \quad (\S 2\ (25))$$

and the t'^*-image of C_l is c_l;

$$c_l(x) = \sum_{i_1 < \cdots < i_l} x_{i_1} \cdots x_{i_l} \quad (l \leqq m); \; c_{2l}(x) = 0 \quad (l > m) \quad (\S 3\ (15)).$$

Then, in comparing the coefficients of t^{2l} in both sides of the identity

$$\prod_{i=1}^{m} (1 - (tx_i)^2) = \prod_{i=1}^{m} (1 - tx_i) \prod_{i=1}^{m} (1 + tx_i),$$

we have

$$(-1)^l p_{2l}(x) = \sum_{k=0}^{2l} (-1)^k c_k(x) c_{2l-k}(x).$$

Applying t'^{*-1}, we obtain

(2) $$(-1)^l \iota^* P_{2l} = \sum_{k=0}^{2l} (-1)^k C_k C_{2l-k}.$$

Remark that $C_k=0$ for $k>m$.

The t^*-image of $\bar{P}_{2l}$ is $\bar{p}_{2l}$;

$$\bar{p}_{2l}(x)=\sum_{i_1\leqq\cdots\leqq i_l} x_{i_1}^2\cdots x_{i_l}^2\,, \qquad (\S 2\ (25))$$

and the t'^*-image of $\bar{C}_k$ is $\bar{c}_k$;

$$\bar{c}_k(x)=\sum_{i_1\leqq\cdots\leqq i_l} x_{i_1}\cdots x_{i_l}\,, \qquad (\S 3\ (15))\,.$$

Now, it holds for an even integer N

$$\prod_{i=1}^{m}\frac{1-(tx_i)^{2(N+1)}}{1-(tx_i)^2}=\prod_{i=1}^{m}\frac{1+(tx_i)^{N+1}}{1+tx_i}\prod_{i=1}^{m}\frac{1-(tx_i)^{N+1}}{1-tx_i}\,,$$

i. e.,

$$\prod_{i=1}^{m}(1+(tx_i)^2+\cdots+(tx_i)^{2N})=\prod_{i=1}^{m}(1+(tx_i)+\cdots+(tx_i)^{N})\prod_{i=1}^{m}(1+(tx_i)+\cdots+(tx_i)^{N})\,.$$

Taking N sufficiently large, the coefficients of t^{2l} in both sides give

$$\bar{p}_{2l}(x)=\sum_{k=0}^{2l}(-1)^k\bar{c}_k(x)\bar{c}_{2l-k}x)\,.$$

Applying t'^{*-1} we have

$$(2)' \qquad \iota^*\bar{P}_{2l}=\sum_{k=0}^{2l}(-1)^k\bar{C}_k\bar{C}_{2l-k}\,.$$

Finally, since

$$w_m(x)=(-1)^m x_1\cdots x_m, \qquad (\S 2\ (28))\,,$$
$$c_m(x)=x_1\cdots x_m, \qquad (\S 3\ (15))\,,$$

we have $t^*w_m=(-1)t'^*c_m$. Therefore

$$(3) \qquad \iota^*W_m=(-1)^m C_m.$$

3. $SO(n')\times SO(n'')\subset SO(n)$ $(n'+n''=n)$.

Let n' and n'' be positive integers such that $n'+n''=n$. The elements $a=(a_{ij})$ of $SO(n)$ with $a_{ij}=\delta_{ij}$ for $n'<i\leqq n$ (resp. for $1\leqq i\leqq n'$) form a subgroup which is isomorphic to $SO(n')$ (resp. $SO(n'')$) by an obvious isomorphism. The product of these two subgroups is isomorphic to $SO(n')\times SO(n'')$. We may therefore regard the groups $SO(n')$, $SO(n'')$, $SO(n')\times SO(n'')$ and their Lie algebras $\mathfrak{o}(n')$, $\mathfrak{o}(n'')$, $\mathfrak{o}(n')+\mathfrak{o}(n'')$ as in $SO(n)$ and in $\mathfrak{o}(n)$ respectively. Let m' and m'' be the largest integers not greater than $n'/2$ and $n''/2$ respectively. By the procedure of §2, we find a maximal torus T' of $SO(n')$ and introduce coordinates

$x_1', \cdots, x'_{m'}$ in the Lie algebra $\mathfrak{h}'$ of T'. The same is done for $SO(n'')$ with two primes. Put $\mathfrak{h}_1 = \mathfrak{h}' + \mathfrak{h}''$, $\mathfrak{h}_1$ generates the subgroup $T' \times T''$ in $SO(n)$ and it is a commutative subalgebra of $\mathfrak{o}(n)$ with coordinates $(x_1', \cdots, x'_{m'}, x_1'', \cdots, x''_{m''})$. The element with coordinates $(x_1', \cdots, x'_{m'}, x_1'', \cdots, x''_{m''})$ is represented in $\mathfrak{o}(n)$ by the following matrix:

$$\begin{array}{cc}
 & \begin{array}{cccccccccccc} & & & & & * & & & & & & ** \end{array} \\
\begin{array}{c} \\ \\ \\ \\ \\ * \\ \\ \\ \\ \\ \\ ** \end{array} &
\left(\begin{array}{cccccccccccc}
0 & -x_1' & & & & \vdots & & & & & & \vdots \\
x_1' & 0 & & & & \vdots & & & & & & \vdots \\
 & & \ddots & & & \vdots & & & & & & \vdots \\
 & & & 0 & -x'_{m'} & \vdots & & & & & & \vdots \\
 & & & x'_{m'} & 0 & \vdots & & & & & & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & \vdots & 0 & -x_1'' & & & & \vdots \\
 & & & & & \vdots & x_1'' & 0 & & & & \vdots \\
 & & & & & \vdots & & & \ddots & & & \vdots \\
 & & & & & \vdots & & & & 0 & -x''_{m''} & \vdots \\
 & & & & & \vdots & & & & x''_{m''} & 0 & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{array}\right)
\end{array}$$

where the coefficients not written are all zero and the line and row indicated by $*$ (resp. by $**$) appear if and only if n' (resp. n'') is odd. The maximal torus T of $SO(n)$ and the subalgebra $\mathfrak{h}$ with coordinates $(x_1, \cdots, x_m)$ being as in § 2, we can then easily find an element $a_0 \in SO(n)$ so that $\operatorname{ad}(a_0)$ maps $\mathfrak{h}_1$ into $\mathfrak{h}$ and that the image of $(x_1', \cdots, x'_{m'}, x_1'', \cdots, x''_{m''})$ has the coordinates

$$x_1 = x_1', \cdots, x_{m'} = x'_{m'}, \ x_{m'+1} = x_1'', \cdots, x_{m'+m''} = x''_{m''},$$

and, if $m' + m'' < m$, $x_m = 0$. The last case occurs only when both n' and n'' are odd.

Now we have seen in § 1 that $I(SO(n')) \otimes I(SO(n''))$ is isomorphic to $I(SO(n') \times SO(n''))$. This isomorphism is obtained by regarding $F' \otimes F''$ ($F' \in I(SO(n'))$, $F'' \in I(SO(n''))$) as the function on $\mathfrak{o}(n') + \mathfrak{o}(n'')$

$$(F' \otimes F'')(\xi' + \xi'') = F'(\xi') F''(\xi''), \qquad \xi' \in \mathfrak{o}(n'), \ \xi'' \in \mathfrak{o}(n'').$$

Therefore, if F' (resp. F'') reduces on $\mathfrak{h}'$ (resp. on $\mathfrak{h}''$) to a function represented by a polynomial $f'(x')$ of $x_1', \cdots, x'_{m'}$ (resp. $f''(x'')$ of $x_1'', \cdots, x''_{m''}$) $F' \otimes F''$ reduces on $\mathfrak{h}_1$ to the function represented by the polynomial $f'(x') f''(x'')$ of $x_1', \cdots, x'_{m'}, x_1'', \cdots, x''_{m''}$. Since the dual mappings of the injections $T' \to SO(n')$ and $T'' \to SO(n'')$ are one-to-one, the dual mapping of the injection t_1: $T' \times T'' \to SO(n') \times SO(n'')$ is one-to-one, which means that $F' \otimes F''$ is uniquely determined by $f'(x') f''(x'')$.

On the other hand, let F be an element of $I(SO(n))$, that is, an invariant polynomial function on $\mathfrak{o}(n)$. If the restriction of F on $\mathfrak{h}$ is represented as a polynomial $f(x_1, \cdots, x_m)$ of $x_1, \cdots, x_m$, F reduces on $\mathfrak{h}_1$ to the function represented as $f(x_1', \cdots, x_{m'}', x_1'', \cdots, x_{m''}'')$ if $m'+m''=m$ and as $f(x_1', \cdots, x_{m'}', x_1'', \cdots, x_{m''}'', 0)$ if $m'+m''<m$. This is seen from the fact that $F(\mathrm{ad}(a_0)\xi)=F(\xi)$ for $\xi \in \mathfrak{o}(n)$ and from the above choice of coordinates in $\mathfrak{h}_1$. The Pontrjagin invariant function P_{2l} reducing on $\mathfrak{h}$ to the function $p_{2l}(x)$ in § 2 (25), it follows in particular that P_{2l} reduces on $\mathfrak{h}_1$ to the following function.

$$p_{2l}(x', x'')=\begin{cases} \displaystyle\sum_{\substack{1\leq i_1<\cdots<i_{l'}\leq m' \\ 1\leq j_1<\cdots<j_{l''}\leq m'',\ l'+l''=l}} x'_{i_1}\cdots x'_{i_{l'}}x''_{j_1}\cdots x''_{j_{l''}} & (1\leq l\leq m'+m''), \\ 0 \quad (l\geq m'+m''), \end{cases}$$

where $(x', x'')=(x_1', \cdots, x_{m'}', x_1'', \cdots, x_{m''}'')\in\mathfrak{h}_1$. Then, if we denote by p_k' (resp. p_k'') the restriction of the Pontrjagin invariant function P_k' (resp. P_k'') of $SO(n')$ (resp. $SO(n'')$) on $\mathfrak{h}'$ (resp. on $\mathfrak{h}''$), we have

$$p_{2l}(x', x'')=\sum_{l'+l''=l} p'_{2l'}(x')p''_{2l''}(x'').$$

Since the dual mapping of t_1: $T'\times T''\to SO(n')\times SO(n'')$ is one-to-one and since $P'_{2l'}\otimes P''_{2l''}$ is reduced to the function $p'_{2l'}(x')p''_{2l''}(x'')$ on $\mathfrak{h}_1$, we see that

$$\iota^*P_{2l}=\sum_{l'+l''=l} P'_{2l'}\otimes P''_{2l''}, \tag{4}$$

where ι^* is the dual mapping of the injection $SO(n')\times SO(n'')\to SO(n)$.

In a completely similar manner, we obtain

$$\iota^*\bar{P}_{2l}=\sum_{l'+l''=l} \bar{P}'_{2l'}\otimes\bar{P}''_{2l''}, \tag{4$'$}$$

where $\bar{P}_{2l}$, $\bar{P}'_{2l'}$ and $\bar{P}''_{2l''}$ are the dual Pontrjagin invariant functions of $SO(n)$, $SO(n')$ and $SO(n'')$ respectively.

If $n=2m$, the Eular-Poincaré invariant function W_m is considered. Analogous argument implies that

$$W_m=\begin{cases} W_{m'}\otimes W_{m''} & \text{if } n'=2m' \text{ and } n''=2m'' \\ 0 & \text{if } n' \text{ and } n'' \text{ are odd.} \end{cases} \tag{5}$$

4. $U(m')\times U(m'')\subset U(m)$ $(m'+m''=m)$.

Let m' and m'' be positive integers such that $m'+m''=m$. The elements $a=(a_{ij})$ of $U(m)$ with $a_{ij}=\delta_{ij}$ for $m'<i\leq m$ (resp. $1\leq i\leq m'$) form a subgroup isomorphic to $U(m')$ (resp. $U(m'')$). The product of these subgroups is isomorphic to $U(m')\times U(m'')$. The injection of $U(m')\times U(m'')$ is denoted by ι. T' and T'' being maximal tori of $U(m')$ and $U(m'')$ respectively which are chosen in § 3, the product $T'\times T''$ coin-

cides with the maximal torus T of $U(m)$ given in § 3. Therefore the subalgebra $\mathfrak{h}$ of $\mathfrak{u}(m)$ which belongs to T is the direct sum of the subalgebras $\mathfrak{h}'$ and $\mathfrak{h}''$ which belong to T' and to T'' respectively. Moreover the coordinates in $\mathfrak{h}'$ and in $\mathfrak{h}''$ give those in $\mathfrak{h}$.

This being settled, we can use the same argument as in the previous section to find the forms of $\iota^* C_k$ and of $\iota^* \bar{C}_k$. Considering them of $\mathfrak{h}$ and using the fact that the dual mapping of the injection $T' \times T'' \to U(m') \times U(m'')$ are one-to-one, we obtain the following formulas.

$$(6) \qquad \iota^* C_k = \sum_{k'+k''=k} C_{k'} \otimes C_{k''} ,$$

$$(6)' \qquad \iota^* \bar{C}_k = \sum_{k'+k''=k} \bar{C}_{k'} \otimes \bar{C}_{k''} .$$

§ 5. Fundamental characteristic classes

Let (E, M, G) be a differentiable principal fibre bundle: E is the bundle space and M is the base space; E and M are differentiable manifolds and G is a Lie group. All mappings which appear in the definition of principal fibre boundle are supposed to be differentiable.[4] We assume further that E, M and G are all compact and call such a bundle a G-bundle. $\mathfrak{g}$ will denote the Lie algebra of G.

In the following cohomology is considered over the real number field if the contrary is not stated. Owing to Cartan [2] and Chern [4][5] the following results about the characteristic algebra of a G-bundle (E, M, G) are known. Define a connection in the bundle and let Ω be the curvature form of the connection. Ω is a $\mathfrak{g}$-valued differential form of degree 2 on E. For an element F of $I^k(G)$, that is, an invariant polynomial function of degree k on $\mathfrak{g}$, we can consider the real-valued differential form $F(\Omega)$ of degree $2k$ on E. Then $F(\Omega)$ may be regarded as a differential form on M and it is closed on M. By a theorem of de Rham, $F(\Omega)$ represents a cohomology class $[F(\Omega)]$ of M. This definition being linearly extended over $I(G)$, the mapping $F \to [F(\Omega)]$ is a homomorphism of the algebra $I(G)$ into the cohomology algebra $H(M)$ of M. Denote this homomorphism by χ. A theorem of Weil asserts that χ is independent of the choice of the connection in the bundle. We shall define the dimension of a polynomial function of degree k to be $2k$ and 0 shall have all dimensions $\geqq 0$. A main theorem states that if the G-bundle is n-universal in the sense of Steenrod [6] the

4) About the definitions of principal fibre bundles and of related notions we reffer to Steenrod [6].

5) The assumption on the connectedness of structural group imposed in Chern [4] Chap. III can be excluded when we generalize the definition of adojoint mapping as in §1..

homomorphism χ is isomorphic onto in dimensions $<n$. An arbitrary G–bundle (E, M, G) is induced from an n–universal bundle $(\tilde{E}, \tilde{M}, G)$ with $n>\dim M$ by a mapping f of M into $\tilde{M}$. Then the composition of the homomorphism χ for $(\tilde{E}, \tilde{M}, G)$ and of the cohomology homomorphism f^* induced by f coincides with the homomorphism χ for (E, M, G). Since f^* is the characteristic homomorphism of the G–bundle and since $H(M)=0$ in dimensions $\geqq n$, this allows us to regard the image of χ as the characteristic algebra of the bundle.

We apply the above general theory first to $O(n)$–bundles and $SO(n)$–bundles. In such a bundle, we put

$$(1)\qquad \begin{aligned} P^{4l}&=\frac{1}{(2\pi)^{2l}}P_{2l}\,, \\ \bar{P}^{4l}&=\frac{1}{(2\pi)^{2l}}\bar{P}_{2l}\,, \qquad l=0, 1, 2, \cdots \end{aligned}$$

and, if the bundle is an $SO(2m)$–bundle,

$$(2)\qquad W^n=\frac{1}{(2\pi)^m}\chi W_m\,.$$

The members in the left-hand sides are characteristic classes of dimension indicated by the superscript. P^{4l}, $\bar{P}^{4l}$ and W^n are called the l–th Pontrjagin class, the l–th dual Pontrjagin class and the Euler-Poincaré class of the given bundle. Since χ is a homomorphism the following theorems follow immediately from theorems 1, 2 and § 2 (26), (29) respectively. (Recall that m is the largest integer $\leqq\frac{n}{2}$).

Therem 4. *The characteristic algebra of an $O(n)$–bundle is generated by the Pontrjagin classes P^0, $P^4, \cdots, P^{4m}$ and also by the dual Pontrjagin classes $\bar{P}^0$, $\bar{P}^4, \cdots, \bar{P}^{4m}$.*

Theorem 5. *The characteristic algebra of an $SO(n)$–bundle is generated by the Pontrjagin classes P^0, $P^4, \cdots, P^{4m}$ and also by the dual Pontrjagin classes $\bar{P}^0$, $\bar{P}^4, \cdots, \bar{P}^{4m}$ if n is odd. It is generated by the classes P^0, $P^4, \cdots, P^{4m-4}$ and the Euler-Poincaré class W^n and also by the classes $\bar{P}^0$, $\bar{P}^4, \cdots, \bar{P}^{4m-4}$ and W^n if $n=2m$ is even.*

Theorem 6. *In an $O(n)$– or $SO(n)$–bundle,*

$$\sum_{h=0}^{l}(-1)^h P^{4h}\cup\bar{P}^{4(l-h)}=0\,, \qquad for\ 1\leqq l\leqq m\,.$$

Therem 7. *In an $SO(2m)$–bundle,*

$$W^n \cup W^n = P^{4m}.$$

Next consider an $U(m)$-bundle. We put

$$(3) \qquad \begin{aligned} C^{2k} &= \frac{1}{(2\pi)^k} \chi C_k , \\ \bar{C}^{2k} &= \frac{1}{(2\pi)^k} \chi \bar{C}_k , \qquad k=0,1,2,\cdots \end{aligned}$$

C^{2k} and $\bar{C}^{2k}$ are the k-th Chern class and the k-th dual Chern class respectively of the $U(m)$-bundle. The following theorems follow from Theorem 3 and § 3 (6).

Theorem 8. *The characteristic algebra of an $U(m)$-bundle is generated by the Chern classes $C^0, C^2, \cdots, C^{2m}$ and also by the dual Chern classes $\bar{C}^0, \bar{C}^2, \cdots, \bar{C}^{2m}$.*

Theorem 9. *In an $U(m)$-bundle,*

$$\sum_{h=0}^{k} (-1)^h C^{2h} \cup \bar{C}^{2(k-h)} = 0 , \qquad \textit{for } 1 \leqq k \leqq m .$$

Now let G' be a closed subgroup of a compact Lie group G. A G'-bundle (E', M, G') defines in a natural way a G-bundle (E, M, G). We say in this case that the G-bundle is equivaledt to the G'-bundle and also that the G'-bundle is a G'-subbundle or simply a subbundle of the G-bundle. A connection in the G'-subbundle may be extended to a connection in the G-bundle, and then the curvature form of the latter connection reduces on the bundle space E' of the subbundle to the curvature form of the former connection. Considering the homomorphisms χ and χ' for the G- and G'-bundle by these connections, we can easily obtain the following formula. Denoting by ι^* the dual mapping of the injection of G' into G,

$$(4) \qquad \chi = \chi' \circ \iota^* .$$

The real matrices in the group $U(n)$ form its subgroup $O(n)$. Suppose that an $U(n)$-bundle is equivalent to an $O(n)$-bundle. Applying the homomorphism χ' of the $O(n)$-bundle to both sides of § 4 (1) and (1)′ and using the relation (4), we obtain by the definitions (1) and (3) the following

Theorem 10. *If an $U(n)$-bundle is equivalent to an $O(n)$-bundle, there hold among the Chern classes and the dual Chern classes of the $U(n)$-bundle and the Pontrjagin classes and the dual Pontrjagin classes of the $O(n)$-bundle the following relations*

$$C^{2k}=\begin{cases}(-1)^{\frac{k}{2}}P^{2k}, & \text{if } k \text{ is even,}\\ 0\quad , & \text{if } k \text{ is odd;}\end{cases}$$

$$\bar{C}^{2k}=\begin{cases}P^{2k}, & \text{if } k \text{ is even,}\\ 0\ , & \text{if } k \text{ is odd.}\end{cases}$$

Let the inclusion $U(m)\subset SO(2m)$ be as in § 4. 2. We obtain from the formulas § 4 (2), (2)′ and (3) the following theorem in the similar way as above.

Theorem 11.[6] *Let the inclusion $U(m)\subset SO(2m)$ be as in* § 4. 2. *If an $SO(2m)$-bundle is equivalent to an $U(m)$-bundle, the Pontrjagin classes, the dual Pontrjagin classes and the Euler-Poincaré classes of the $SO(2m)$-bundle are expressed by the Chern classes and the dual Chern classes of the $U(m)$-subbundle as follows*:

$$(-1)^l P^{4l}=\sum_{h=0}^{2l}(-1)^h C^{2h}\cup C^{4l-2h},$$
$$\bar{P}^{4l}=\sum_{h=0}^{2l}(-1)^h \bar{C}^{2h}\cup \bar{C}^{4l-2h},$$
$$W^m=(-1)^m C^m.$$

Suppose that the direct product $G_1\times G_2$ of two groups G_1 and G_2 is a closed subgroup of a compact Lie group G. If a G-bundle (E, M, G) equivalent to a $G_1\times G_2$-bundle, there are weakly associated to the $G_1\times G_2$-subbundle a G_1-bundle (E_1, M, G_1) and a G_2-bundle (E_2, M, G_2). We say then that the G-bundle is the Whitney product of these two bundles. The $G_1\times G_2$-subbundle is induced from the product bundle $(E_1\times E_2, M\times M, G_1\times G_2)$ with the structural group $G_1\times G_2$ by the disgonal mapping d of the space M into $M\times M$: $d(x)=(x, x)$ for $x\in M$. In order to study the homomorphism χ' of the $G_1\times G_2$-subbundle, introduce connections in the G_1-bundle and in the G_2-bundle. Obviously these connections define a connection in the product bundle. Using these connections we consider the homomorphisms χ_1, χ_2 and $\tilde{\chi}$ for the G_1-bundle, for the G_2-bundle and for the product bundle respectively. Then, from the definitions of these homomorphisms we may easily deduce the following relations: Denote by φ_1 and φ_2 the projections of $M\times M$ onto the first factor M and onto the second factor M respectively, and the cohomology homomorphisms induced by them are indicated by adjoining asterisk. $I(G_1\times G_2)$ being identified with $I(G_1)\otimes I(G_2)$ as before, there hold for $F_1\in I(G_1)$ and $F_2\in I(G_2)$

6) This theorem is originally obtained by Wu for the corresponding integral classes. See Wu [8] Theorem 9.

$$\tilde{\chi}(F_1 \otimes 1) = \varphi_1{}^*\chi_1 F_1\,,$$
$$\tilde{\chi}(1 \otimes F_2) = \varphi_2{}^*\chi_2 F_2\,.$$

Since $\tilde{\chi}$ is a homomorphism, it follows

$$\tilde{\chi}(F_1 \otimes F_2) = \varphi_1{}^*\chi_1 F_1 \cup \varphi_2{}^*\chi_2 F_2\,.$$

On the other hand, since the diagonal mapping induces the bundle mapping from the $G_1 \times G_2$-subbundle to the product bundle, it follows from a general theorem[7] that $\chi' = d^* \circ \chi$. Besides, $\varphi_1 \circ d$ and $\varphi_2 \circ d$ being the identity mapping of M, $d^* \circ \varphi_1{}^*$ and $d^* \circ \varphi_2{}^*$ are the identity mapping of $H(M)$. Therefore, application of the cohomology homomorphism d^* to both sides of the above formula implies

$$(5) \qquad \chi'(F_1 \otimes F_2) = \chi_1 F_1 \cup \chi_2 F_2\,.$$

Now we apply these considerations to the inclusion $SO(n') \times SO(n'') \subset SO(n)$ defined in §4. 4. If an $SO(n)$-bundle is the Whitney product of an $SO(n')$-bundle and an $SO(n'')$-bundle, we can consider the Pontrjagin classes and the dual Pontrjagin classes of the $SO(n)$-bundle, of the $SO(n')$-bundle and of the $SO(n'')$-bundle simultaneously in the cohomology algebra of the common base space. They are denoted by P^{4l}, $\bar{P}^{4l}$; $'P^{4l}$, $'\bar{P}^{4l}$ and $''P^{4l}$, $''\bar{P}^{4l}$ respectively. Then, χ, χ', χ_1 and χ_2 being the homomorphisms for the $SO(n)$-bundle, for the $SO(n') \times SO(n'')$-subbundle, for the $SO(n')$-bundle and for the $SO(n'')$-bundle respectively,

$$\begin{aligned}
P^{4l} &= \frac{1}{(2\pi)^{2l}} \chi P_{2l} \\
&= \frac{1}{(2\pi)^{2l}} \chi'(\iota^* P_{2l}) \qquad \text{(by (4))} \\
&= \frac{1}{(2\pi)^{2l}} \chi'\Big(\sum_{l'+l''=l} P'_{2l'} \otimes P''_{2l''}\Big) \qquad \text{(by §4 (4))} \\
&= \sum_{l'+l''=l} \frac{1}{(2\pi)^{2l'}} \chi_1(P'_{2l'}) \cup \frac{1}{(2\pi)^{2l''}} \chi_2(P''_{2l''}) \qquad \text{(by (5))} \\
&= \sum_{l'+l''=l} {}'P^{4l'} \cup {}''P^{4l''}\,.
\end{aligned}$$

Similary §4 (4)′ implies

$$\bar{P}^{4l} = \sum_{l'+l''=l} {}'\bar{P}^{4l'} \cup {}''\bar{P}^{4l''}\,.$$

If $n = 2m$ then §4 (5) implies

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7) See Chern [4] p. 65.

$$W^n=\begin{cases} 'W^{n'}\cup''W^{n''}, & \text{if } n' \text{ and } n'' \text{ are even,} \\ 0 \quad , & \text{if } n' \text{ and } n'' \text{ are odd,} \end{cases}$$

where $'W^{n'}$ and $''W^{n''}$ are the Euler-Poincaré classes of $SO(n')$-bundle and of $SO(n'')$-bundle respectively.

Thus we get the following

Theorem 12. *If an $SO(n)$-bundle is the Whitney product of an $SO(n')$-bundle and an $SO(n'')$-bundle, the Pontrjagin classes and the dual Pontrjagin classes and, if $n=2m$, the Euler-Poincaré class of the $SO(n)$-bundle are expressed by those of the $SO(n'')$-bundle and of the $SO(n'')$-bundle as follows*

$$P^{4l}=\sum_{l'+l''=l}{}'P^{4l'}\cup{}''P^{4l''},$$

$$\bar{P}^{4l}=\sum_{l'+l''=l}{}'\bar{P}^{4l'}\cup{}''\bar{P}^{4l''},$$

$$W^n=\begin{cases} 'W^{n'}\cup''W^{n''}, & \textit{if } n' \textit{ and } n'' \textit{ are even,} \\ 0 \quad , & \textit{if } n' \textit{ and } n'' \textit{ are odd.} \end{cases}$$

In a similar way we can deduce from § 4 (6) and (6)′ the following

Theorem 13.[8] *If an $U(m)$-bundle is the Whitney product of an $U(m')$-bundle and an $U(m'')$-bundle, the Chern classes and the dual Chern classes of $U(m)$-bundle is expressed by those of the $U(m')$-bundle and of the $U(m'')$-bundle as follows*

$$C^{2k}=\sum_{k'+k''=k}{}'C^{2k'}\cup{}''C^{2k''},$$

$$\bar{C}^{2k}=\sum_{k'+k''=k}{}'\bar{C}^{2k'}\cup{}''\bar{C}^{2k''}.$$

REMARK. In the above argument we restrict our consideration to the real cohomology. If we denote by $H(M, Z)$ the integral cohomology ring and by $H(M)$ the real cohomology algebra of a manifold M, there is a natural ring homomorphism from $H(M, Z)$ into $H(M)$. This homomorphism is one-to-one if $H(M, Z)$ has no torsion. Now we know that the characteristic classes of a G-bundle considered in this paragraph are the images of the corresponding integral characteristic classes by this homomorphism.[9] Therefore the relations among the formers obtained in this paragraph hold still true among the latters in the integral cohomology ring if $H(M, Z)$ has no torsion. Moreover the relations given in theorems 9 and 11 hold true in integral cohomology ring. For, an $(2N-1)$-universal bundle for the group $U(m)$ is constructed over

8) Cf. Wu [8] Theorem 15.
9) See Chern [4] Chap. IV.

the complex Grassmann manifold $G(m, N)$ which is composed of m-planes in a complex $(m+N)$-dimensional vector space. This Grassmann manifold has the cellular decomposition by even-dimensional cells, and so $H(G(m, N), Z)$ has no torsion. Therefore the relations hold in it. Since any $U(m)$-bundle $(E, M, U(m))$ is induced by a mapping f: $M \rightarrow G(m, N)$ for a sufficiently large N and since f^* induces a ring-homomorphism of $H(G(m, N), Z)$ into $H(M, Z)$ which defines integral Chern classes and integral Pontrjagin classes etc., we see immediately that the relations hold in $H(M, Z)$. A universal bundle for the group $U(m') \times U(m'')$ is provided by a bundle over the product of two Grassmann manifolds $G(m', N') \times G(m'', N'')$, and the cohomology ring of this product manifold has no torsion. Since a Whitney product of an $U(m')$-bundle and an $U(m')$-bundle has a bundle mapping to this bundle with sufficiently large N' and N'', we may see by an analogous argument that the relations given in theorem 13 hold true for integral Chern classes and for integral dual Chern classes.

§ 6. The dual mapping of injection

Consider a compact Lie group G and a closed subgroup K of G, whose Lie algebras are respectively $\mathfrak{g}$ and $\mathfrak{k}$. We shall give in this paragraph a theorem which might provide topological signification for the dual mapping of the injection $\iota : K \rightarrow G$.

Let (E, M_G, G) be a G-bundle in the sense of § 5 and π_G its projection. We construct the weakly associsted fibre bundle over M_G with the fibre G/K. Let M_K be its bundle space and ρ its projection. ρ induces a mapping of the cohomology algebra $H(M_G)$ into $H(M_K)$. On the other hand there is a natural mapping π_K of E onto M_K and it defines a K-bundle (E, M_K, K). And there holds obviously

$$\pi_G = \rho \circ \pi_K . \tag{1}$$

As we have explained in § 5 the homomorphisms

$$\chi_G : I(G) \rightarrow H(M_G) ,$$
$$\chi_K : I(K) \rightarrow H(M_K)$$

are defined using connections in the bundles. Besides there is the dual mapping

$$\iota^* : I(G) \rightarrow I(K) ,$$

which is defined by taking the restrictions of invariant polynomial functions on $\mathfrak{g}$ to $\mathfrak{k}$.

Theorem 14. *Notations being as above,*

$$\rho^* \circ \chi_G = \chi_K \circ \iota^*$$

Proof.[10] Take a connection in the G-bundle (E, M_G, G). This is given by a $\mathfrak{g}$-valued linear differential form ω on E with the following properties. (All differential forms considered are supposed to be differentiable): $R_a^*\omega$ being the mapping for differential forms induced by the right translation of E by an element a of G,

$$R_a^*\omega = \mathrm{ad}\,(a^{-1})\omega \qquad \text{for} \quad a \in G; \tag{2}$$

To an element X of $\mathfrak{g}$, i.e., to a left-invariant vector field X on G, let X^* be the well-defined vector field on E which is transformed to X by any admissible mapping of G onto a fibre. Then the value of ω for X^* is constant and it is equal to X, that is,

$$\omega(X^*) = X \qquad \text{for} \quad X \in \mathfrak{g}. \tag{3}$$

The curvature form Ω_G of this connection being the covariant differential of ω, it is a $\mathfrak{g}$-valued differential form of degree 2 on E. One knows the following structure equation.

$$\Omega_G = d\omega + \frac{1}{2}[\omega, \omega]. \tag{4}$$

Now, since $\mathfrak{k}$ is stable under the adjoint mappings $\mathrm{ad}\,(a)$ for $a \in K$ and since $a \to \mathrm{ad}\,(a)$ $(a \in K)$ is a represention on $\mathfrak{g}$ of the compact group K, there is a subspace $\mathfrak{m}$ of $\mathfrak{g}$ so that it is stable under $\mathrm{ad}\,(a)$ for $a \in K$ and that $\mathfrak{g}$ is the direct sum of $\mathfrak{k}$ and $\mathfrak{m}$. Decomposition of the values of ω according to this decomposition of ω yields a $\mathfrak{k}$-valued linear differential form ω_1 on E and an $\mathfrak{m}$-valued one ω_2 such that

$$\omega = \omega_1 + \omega_2. \tag{5}$$

If we denote by $\mathrm{ad}_{\mathfrak{k}}(a)$ and $\mathrm{ad}_{\mathfrak{m}}(a)$ the linear transformations of $\mathfrak{k}$ and of $\mathfrak{m}$ respectively which is induced by $\mathrm{ad}\,(a)$ for $a \in K$, it follows from (2) and (3) the following relations:

$$\begin{cases} R_a^*\omega_1 = \mathrm{ad}_{\mathfrak{k}}(a^{-1})\omega_1 & \text{for} \quad a \in K, \\ \omega_1(X^*) = X, & \text{for} \quad X \in \mathfrak{k}; \end{cases} \tag{6}$$

$$\begin{cases} R_a^*\omega_2 = \mathrm{ad}_{\mathfrak{m}}(a^{-1})\omega_2 & \text{for} \quad a \in K, \\ \omega_2(X^*) = 0 & \text{for} \quad X \in \mathfrak{k}. \end{cases} \tag{7}$$

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10) In this proof, we use notions and elementary results in the theory of connections. See Chern [4] Chap. III or Nomizu; Lie groups and differential geometry, Tokyo, 1956.

(6) means that ω_1 defines a connection in the K-bundle (E, M_K, K). Let D be the covariant differentiation and Ω_K the curvature form of this connection. Then, the following formulas are known

$$\Omega_K = d\omega_1 + \frac{1}{2}[\omega_1, \omega_1], \tag{8}$$

and

$$D\Omega_K = 0. \tag{9}$$

From the properties (7) of ω_2, we can prove the following relations just as (8) and (9)

$$D\omega_2 = d\omega_2 + [\omega_1, \omega_2], \tag{10}$$

$$D^2\omega_2 = [\Omega_K, \omega_2]. \tag{11}$$

By virtue of (5) and (10), (4) is transformed into the form

$$\Omega_G = \Omega_K + D\omega_2 + \frac{1}{2}[\omega_2, \omega_2]. \tag{12}$$

If we consider the homomorphisms χ_G and χ_K using these connections in the bundles, we have by definition,

$$\chi_G(F) = [F(\Omega_G)], \quad \text{for} \quad F \in I^k(G),$$
$$\chi_K(F') = [F'(\Omega_K)], \quad \text{for} \quad F' \in I^k(K), \ k = 0, 1, \cdots,$$

where $F(\Omega_G)$ and $F'(\Omega_K)$ are considered as differential forms on M_G and on M_K respectively, i.e. those which induce $F(\Omega_G)$ and $F'(\Omega_K)$ by the dual mapping π_G^* and π_K^* of the projections, and [ ] denotes the cohomology class represented by the closed differential form in it. Therefore, the assertion of the theorem states

$$\rho^*[F(\Omega_G)] = [(\iota^* F)(\Omega_K)], \quad \text{for} \quad F \in I^k(G), \ k = 0, 1, \cdots.$$

This holds obviously for $k=0$. Suppose $k>0$. The dual mapping π_K^* for differential forms induced by projection π_K is one-to-one and it commutes with the differentiation. Then, together with the definition of ρ^* and ι^*, it follows from (1) that the above equation is equivalent to the following: There exists a differential form Φ on M_K of degree $2k-1$ such that

$$F(\Omega_G) - F(\Omega_K) = d\pi_K^*\Phi. \tag{13}$$

Such a differential form Φ is obtained by a similar procedure as in the proof of a theorem of Weil (Chern [4] p. 58). In fact, F being considered in the polar form, put

$$Q(X, Y) = F(X, Y, \cdots, Y), \qquad \text{for} \quad X, Y \in \mathfrak{g}.$$

Then

$$F(W+Y+Z) - F(W) = k \int_0^1 Q(Y+2tz, W+tX+t^2Z)dt.$$

Therefore, (12) implies

$$F(\Omega_G) - F(\Omega_K) = k \int_2^1 Q(D\omega_2 + t[\omega_2, \omega_2], \Omega_K + tD\omega_2 + \frac{t^2}{2}[\omega_2, \omega_2])dt.$$

Now, by (9) and (11) and $[[\omega_2, \omega_2], \omega_2] = 0$,

$$\begin{aligned} dQ(\omega_2, \Omega_K + tD\omega_2 + \frac{t^2}{2}[\omega_2, \omega_2]) & \\ &= DQ(\omega_2, \Omega_K + tD\omega_2 + \frac{t^2}{2}[\omega_2, \omega_2]) \\ &= Q(D\omega_2 + t[\omega_2, \omega_2], \Omega_K + tD\omega_2 + \frac{t^2}{2}[\omega_2, \omega_2]). \end{aligned}$$

Therefore

$$F(\Omega_G) - F(\Omega_K) = d(\int_0^1 kQ(\omega_2, \Omega_K + tD\omega_2 + \frac{t^2}{2}[\omega_2, \omega_2])dt.$$

The differential form in the parenthesis can be expressed in the form $\pi^*\Phi$, as is seen from the properties (7) of ω_2 and and analogous properties of Ω_K, $D\omega_2$ and $[\omega_2, \omega_2]$. The theorem is thus proved.

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