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INDEX OF THE EXPONENTIAL MAP OF A CENTER-FREE COMPLEX SIMPLE LIE GROUP

HENG-LUNG LAI[1)]

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0. Introduction

Let $\mathfrak{G}$ be a connected Lie group with Lie algebra G. In general, the exponential map $\exp: G \to \mathfrak{G}$ is not surjective. As in Goto [4], for an element $g \in \mathfrak{G}$, we shall define the index (of the exponential map) $\operatorname{ind}(g)$ to be the smallest positive integer q such that $g^q \in \exp G$, if it exists, otherwise, $\operatorname{ind}(g) = \infty$. The index $\operatorname{ind}(\mathfrak{G})$ of the Lie group $\mathfrak{G}$ is defined to be the least common multiple of all $\operatorname{ind}(g)$ $(g \in \mathfrak{G})$.

In Lai [6], the author proved the following theorem:

Theorem. *Let $\mathfrak{G}$ be a connected (real or complex) semisimple Lie group with finite center. Then* $\operatorname{ind}(\mathfrak{G})$ *is finite.*

More generally, M. Goto proved the following theorem:

Theorem (Goto [3]). *Let K be an algebraically closed field (of characteristic 0 or prime), and let $\mathfrak{G}$ be an algebraic group over K. Then there exists a natural number q such that for any $g \in \mathfrak{G}$, we can find a connected abelian subgroup of $\mathfrak{G}$ containing g^q.*

In case $K = \boldsymbol{C}$, this implies that $\operatorname{ind}(\mathfrak{G})$ is finite for any algebraic group $\mathfrak{G}$ over the field of complex numbers.

Theorem (Goto [4]). *Let $\mathfrak{G}$ be a semi-algebraic group over $\boldsymbol{R}$ (the field of real numbers). Then* $\operatorname{ind}(\mathfrak{G})$ *is finite.*

In Lai [6], the author also computed $\operatorname{ind}(\mathfrak{G})$ for some connected complex simple Lie groups $\mathfrak{G}$. In the case where $\mathfrak{G}$ has trivial center, which most interests us in the present paper, the results in [6] can be summarized as follows. Note that $\mathfrak{G}$ can be identified with the adjoint group $Ad(G)$ of (all inner automorphisms of) its Lie algebra G.

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(1) G is of type A_{n-1}. Then exp: $G\to Ad(G)$ is surjective. Because: $Ad(G)\cong Ad(sl(n, \boldsymbol{C}))\cong SL(n, \boldsymbol{C})/\text{center}\cong GL(n, \boldsymbol{C})/\text{center}\cong PGL(n, \boldsymbol{C})$; and in the following commutative diagram, π (the canonical projection) and Exp (the exponential map of matrices) are surjective.

$$\begin{array}{ccc} gl(n, \boldsymbol{C}) & \xrightarrow{\text{Exp}} & GL(n, \boldsymbol{C}) \\ \downarrow d\pi & & \downarrow \pi \\ sl(n, \boldsymbol{C}) & \xrightarrow{\exp} & PGL(n, \boldsymbol{C}) \end{array}$$

(2) When G is of type B, C, or D. We first considered the corresponding classical groups (the symplectic group $Sp(n, \boldsymbol{C})$ and the special orthogonal group $SO(n, \boldsymbol{C})$), and proved that the square of any element in each case lies inside the image of the exponential map. Then, in each case, we found some element in $Ad(G)$ of index exactly equal to 2.
(3) G is of type G_2. We proved that ind $(g)\in\{1, 2, 3\}$ for any $g\in Ad(G)$, and constructed elements of index equal to 2 and 3 respectively.
(4) G is of type F_4. We used a computer to compute all the determinants of the coefficient matrices of any four (linearly independent) positive roots (expressed in terms of simple root system) and we found that ind $(g)\in\{1, 2, 3, 4\}$. Again, we constructed elements of index 3 and 4 respectively.
(5) When G is of type E, we couldn't find a workable method to find ind $(Ad(G))$. We only gave some lower bounds.

For details, see [6].

Let $m_1\alpha_1+\cdots+m_l\alpha_l$ be the highest root of G with respect to a fixed Cartan subalgebra H expressed in terms of a simple root system $\{\alpha_1, \cdots, \alpha_l\}$. Then $I(G)=\{1, m_1, \cdots, m_l\}$ is a set of positive integers depending only on the type of G; for example, $I(A_l)=\{1\}$, $I(B_l)=I(C_l)=I(D_l)=\{1, 2\}$, $I(G_2)=\{1, 2, 3\}$, $I(F_4)=\{1, 2, 3, 4\}$. The above results suggest that ind $(Ad(G))$ may have some relationship to $I(G)$. The main purpose of this paper is to prove the following theorem.

Theorem. *Let G be a complex simple Lie algebra, $Ad(G)$ the adjoint group of G and $m_1\alpha_1+\cdots+m_l\alpha_l$ the highest root expressed in terms of a simple root system $\{\alpha_1, \cdots, \alpha_l\}$. Then $\{\text{ind}(g); g\in Ad(G)\}$ equals $I(G)=\{1, m_1, \cdots, m_l\}$.*

To prove the theorem, we use a method from Borel-Siebenthal's [1] classification of maximal subalgebras of maximal rank in a compact simple Lie algebra.

The author would like to take this opportunity to thank Professor M. Goto for his help and many useful suggestions. I would also like to thank Prof. A. Borel who pointed out some mistakes in my earlier argument.

1. Review and notation

Let G be a complex semisimple Lie algebra with a (fixed) Cartan subalgebra H. Let Δ be the root system of G with respect to H, $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ a fundamental root system of Δ, and $-\alpha_0=m_1\alpha_1+\cdots+m_l\alpha_l$ the highest root.

Let B be the Killing form on G. Then for each $\alpha\in\Delta$, we can find $h_\alpha\in H$ with $B(h, h_\alpha)=\alpha(h)$ for all $h\in H$, and $e_\alpha\in G$ such that

$$\begin{aligned} &G = H+\textstyle\sum_{\alpha\in\Delta} Ce_\alpha\,, \\ &[h, e_\alpha] = \alpha(h)e_\alpha,\ [e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta} && \text{if } \alpha+\beta\neq 0 \text{ is in } \Delta\,, \\ &[e_\alpha, e_{-\alpha}] = -h_\alpha,\ [e_\alpha, e_\beta] = 0 && \text{if } 0\neq\alpha+\beta\notin\Delta\,. \end{aligned}$$

Let $H_0\subset H$ be the real vector space spanned by $h_\alpha(\alpha\in\Delta)$, then $\beta|_{H_0}$ is real for any $\beta\in\Delta$. Since $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ is linearly independent, we can choose $h_1, \cdots, h_l\in H_0$ such that $\alpha_i(h_j)=\delta_{ij}$ $1\leq i, j\leq l$. The lattice $\Omega=\boldsymbol{Z}2\pi i h_1+\cdots+\boldsymbol{Z}2\pi i h_l\subset iH_0$ $(i=\sqrt{-1})$ is the kernel of $\exp|_H: H\to Ad(G)$. For simplicity, we identify Δ with a subset of iH_0 by the map $\alpha\mapsto\frac{1}{2\pi i}h_\alpha$, and introduce an inner product in iH_0 by $(h, h')=\frac{-1}{(2\pi)^2}B(h, h')$. Then $(\alpha, h)=\alpha(h)/2\pi i$ for $\alpha\in\Delta$, $h\in iH_0$.

Let $Ad(\Delta)$ denote the Weyl group of Δ. Any element S of $Ad(\Delta)$, regarded as a linear transformation on iH_0 can be extended to an inner automorphism of the Lie algebra G. Let $T(\Omega)$ be the group of translations of the euclidean space iH_0 induced by elements in Ω. Then, if G is simple, the group $Ad(\Delta)\cdot T(\Omega)$ acts transitively on the set of all cells, see Goto-Grosshans [5] Chapter 5. We summarize as follows:

Let G be a complex simple Lie algebra and C_0 the fundamental cell: $C_0=\{h\in iH_0; (\alpha_1, h)>0, \cdots, (\alpha_l, h)>0 \text{ and } (-\alpha_0, h)<1\}$. Let $\bar{C}_0$ denote the closure of C_0. Then for any h in iH_0, we can find $U\in Ad(\Delta)\cdot T(\Omega)$ such that $h\in U\bar{C}_0$.

In sections 2 and 3 below, we consider ind(g) for $g\in Ad(G)$ where G is a complex simple Lie algebra.

2. Upper bound for ind(g)

Theorem. *For any $g\in Ad(G)$, $\mathrm{ind}(g)\leq m_i$ for some $i=1, \cdots, l$.*

Any element g in $Ad(G)$ has a decomposition $g=g_0\cdot\exp N$ into semisimple part g_0 and unipotent part $\exp N$ such that $g_0\cdot\exp N=\exp N\cdot g_0$. Let $G(1, Ad\, g_0)$ denote the 1-eigenspace of $Ad\, g_0$ in G. Then $G(1, Ad\, g_0)$ is a subalgebra of G and $N\in G(1, Ad\, g_0)$.

By Gantmacher [2], g_0 is conjugate to some element in $\exp H$. Hence, to prove our theorem, it suffices to consider elements g whose semisimple part lies in $\exp H$, i.e. $g=\exp h_0\cdot\exp N$, $h_0\in H$, such that $N\in G(1, Ad\exp h_0)$. Let

$\Delta(h_0)=\{\alpha\in\Delta; Ad\exp h_0\cdot e_\alpha=e_\alpha\}=\{\alpha\in\Delta; \alpha(h_0)\in 2\pi i\boldsymbol{Z}\}$. Then $G(1, Ad\exp h_0)=H+\sum_{\alpha\in\Delta(h_0)}Ce_\alpha$, and $\Delta(h_0)$ satisfies (i) $-\alpha\in\Delta(h_0)$ whenever $\alpha\in\Delta(h_0)$, and (ii) if $\alpha, \beta\in\Delta(h_0)$ and $\alpha+\beta\in\Delta$, then $\alpha+\beta\in\Delta(h_0)$. Hence $\Delta(h_0)$ is a subsystem of Δ, and we can choose a simple root system $\Pi(h_0)=\{\beta_1, \cdots, \beta_r\}$ of $\Delta(h_0)$.

Lemma 1. *To find an upper bound for* ind(g) $(g\in Ad(G))$, *it suffices to consider elements with semisimple part* $\exp h_0$, *where* $h_0\in iH_0$ *and* $\Pi(h_0)$ *has cardinality* $l=\text{rank } G$.

Proof. Assume that $h_0=x_1h_1+\cdots+x_lh_l$ for some complex numbers x_i. For each $j=1, \cdots, r$, since $(\exp ad\, h_0-1)e_{\beta_j}=0$, we have $\beta_j(h_0)=2\pi ik_j$ for some $k_j\in\boldsymbol{Z}$. If k_j are all zero, then for any $N\in G(1, Ad\exp h_0)$ we have $[h_0, N]=0$, and $\exp h_0\cdot\exp N=\exp(h_0+N)$, i.e. $\text{ind}(\exp h_0\cdot\exp N)=1$. So we assume some $k_j\neq 0$, hereafter.

Since $\exp h_0=\exp(h_0+\Omega)$, if we can find a positive integer d and integers $n_1, \cdots, n_l$ such that for $h=dh_0+\sum_{j=1}^l 2\pi in_jh_j$, $[h, dN]=0$, then the index of $\exp h_0\cdot\exp N$ divides d. For this, it suffices to choose d and n_j with $\alpha(h)=0$ for all $\alpha\in\Delta(h_0)$, or equivalently for all $\alpha\in\Pi(h_0)=\{\beta_1, \cdots, \beta_r\}$. Therefore, the problem reduces to finding d so that $\beta_i(\sum_{j=1}^l n_jh_j)=-dk_i$ has integral solutions $n_1, \cdots, n_l$.

Choose $\beta_{r+1}, \cdots, \beta_l\in\Delta$ so that $\{\beta_1, \cdots, \beta_l\}$ is a maximal linearly independent subset of Δ. We write $\beta_i=\sum_{j=1}^l p_{ij}\alpha_j$ where p_{ij} are integers. Consider the following system of linear equations:

$$\begin{aligned} p_{i1}n_1+\cdots+p_{il}n_l &= -k_i \qquad i=1, \cdots, r;\\ p_{i1}n_1+\cdots+p_{il}n_l &= 0 \qquad\quad i=r+1, \cdots, l. \end{aligned}$$

Since (p_{ij}) is a nonsingular integral matrix and k_i are integers, this has a (nontrivial) rational solution, say $r_1, \cdots, r_l$.

Let $h_0'=2\pi i(r_1h_1+\cdots+r_lh_l)\in iH_0$, then $\beta_1, \cdots, \beta_l\in\Delta(h_0')$. Suppose we can find a positive integer d' and integers $n_1', \cdots, n_l'$ such that $\beta(d'h_0'+\sum_{j=1}^l 2\pi in_j'h_j)=0$ for all $\beta\in\Delta(h_0')$, then $(n_1, \cdots, n_l)=(n_1', \cdots, n_l')$ is the solution for the following system of linear equations:

$$\begin{aligned} \textstyle\sum_{j=1}^l p_{ij}n_j &= -d'k_i \qquad i=1, \cdots, r;\\ \textstyle\sum_{j=1}^l p_{ij}n_j &= 0 \qquad\quad\;\; i=r+1, \cdots, l. \end{aligned}$$

Thus we have $n_j\in\boldsymbol{Z}$ such that $\beta_i(\sum_{j=1}^l 2\pi in_jh_j)=-2\pi id'k_i$ $(i=1, \cdots, r)$. Hence for $h=d'h_0+\sum_{j=1}^l 2\pi in_jh_j$, we have $\beta_i(h)=0$ $(i=1, \cdots, r)$ and so $\beta(h)=0$ for all $\beta\in\Delta(h_0)$.

We have proved that $\text{ind}(\exp h_0\cdot\exp N)\leq\text{ind}(\exp h_0'\cdot\exp N)$. Therefore, we may replace h_0 by h_0' which satisfies Lemma 1 by our construction. ||

Given an $n\times n$ nonsingular integral matrix A, the Smith canonical form of A is a diagonal matrix $D=\mathrm{diag}(d_1, \cdots, d_n)$ such that there are $Q_1, Q_2\in GL(n, \boldsymbol{Z})$ with $A=Q_1DQ_2$ and $d_i|d_{i+1}$ (the positive integers d_i are called the elementary divisors of A). We shall denote the biggest one, d_n, by $d(A)$.

Given $h_0\in iH_0$ as in Lemma 1, the coefficient matrix $P=(p_{ij})$ of $\Pi(h_0)$ expressed in terms of a simple root system is a nonsingular $l\times l$ matrix. From the proof of Lemma 1, we see that $\mathrm{ind}(\exp h_0\cdot\exp N)\leq d(P)$, so our problem is to find $d(P)$.

Now let S be in the Weyl group $Ad(\Delta)$. Then S can be extended to an automorphism of the Lie algebra G, which can be extended to an inner automorphism σ of the Lie group $Ad(G)$. Clearly $\mathrm{ind}(g)=\mathrm{ind}(\sigma g)$ for any automorphism σ of $Ad(G)$. Therefore, to find an upper bound for $\mathrm{ind}(g)$ $(g\in Ad(G))$, we may replace g (whose semisimple part is $\exp h_0$) by an element whose semisimple part is $\exp Sh_0$ $(S\in Ad(\Delta))$.

On the other hand, $\exp h_0=\exp(h_0+\Omega)$, so we may replace h_0 by $T(\Omega)h_0$.

Combining these and the proposition we stated at end of section 1, we get

Lemma 2. *Let $-\alpha_0=m_1\alpha_1+\cdots+m_l\alpha_l$ be the highest root. To find an upper bound for* $\mathrm{ind}(g)$ $(g\in Ad(G))$, *it suffices to consider elements whose semisimple part has the form* $\exp h$ $(h\in iH_0)$ *with* $(\alpha_1, h)\geq 0, \cdots, (\alpha_l, h)\geq 0$ *and* $(-\alpha_0, h)\leq 1$.

Let $\tilde{\Pi}=\{\alpha_0, \alpha_1, \cdots, \alpha_l\}$, be the extended simple root system. To simplify our problem further, we need some discussions in the Borel-Siebenthal theory. The following two lemmas are known. For the sake of completeness, we include a proof here.

Lemma 3. *Let $h\in\bar{C}_0$ be an element satisfying Lemma* 1, *then $\Pi'=\tilde{\Pi}\cap\Delta(h)$ is a simple root system of $\Delta(h)$ with respect to a suitable ordering.*

Proof. Given any positive root $\beta=b_1\alpha_1+\cdots+b_l\alpha_l$, it suffices to prove that β can be written as a linear combination of roots in Π' with integral coefficients, all non-negative or all non-positive.

(a) $-\alpha_0\notin\Delta(h)$, i.e. $(-\alpha_0, h)\notin\boldsymbol{Z}$.

Since $0\leq(-\alpha_0, h)\leq 1$, so $0<(-\alpha_0, h)<1$. Then the inequality $0\leq b_1(\alpha_1, h)+\cdots+b_l(\alpha_l, h)=(\beta, h)\leq(-\alpha_0, h)<1$ implies that $(\beta, h)=0$ because $(\beta, h)\in\boldsymbol{Z}$. Hence $b_j(\alpha_j, h)=0$ for all j; i.e. $(\alpha_j, h)=0$ or $\alpha_j\in\Delta(h)$ whenever $b_j\neq 0$. Therefore β is a linear combination of $\alpha_j\in\Delta(h)$ with nonnegative coefficients.

(b) $-\alpha_0\in\Delta(h)$, so $(-\alpha_0, h)=0$ or 1.

If $(-\alpha_0, h)=0$, then $(\alpha_1, h)=\cdots=(\alpha_l, h)=0$ and $h=0$, which is the trivial case we have excluded (Lemma 1). Hence $(-\alpha_0, h)=1$, so $(\beta, h)=0$ or 1.

If $(\beta, h)=0$, the same argument as in (a) gives what we want.

If $(\beta, h)=1$, then $(-\alpha_0, h)=1$ and

$$0=(-\alpha_0-\beta, h)=(m_1-b_1)(\alpha_1, h)+\cdots+(m_l-b_l)(\alpha_l, h).$$

Since $m_j \geq b_j$, $(\alpha_j, h) \geq 0$, we have $\alpha_j \in \Delta(h)$ whenever $m_j-b_j \neq 0$. Hence $\beta = -\alpha_0-(m_1-b_1)\alpha_1-\cdots-(m_l-b_l)\alpha_l$ is a linear combination of roots in Π' with non-positive integral coefficients. ||

Therefore, $\Pi(h)=\tilde{\Pi} \cap \Delta(h)$ is a simple root system of $\Delta(h)$. By Lemma 1, we consider elements $h \in iH_0$ such that $\Pi(h)$ has cardinality l. If $\Pi(h)=\Pi$, then $\Delta(h)=\Delta$ and $h \in \Omega$, and in this case, $\exp h \cdot \exp N = \exp N$.

Lemma 4. *If $\Pi(h) \neq \Pi$ has cardinality l, then $h=2\pi i h_j/m_j$ for some j such that $m_j>1$.*

Proof. Since $\Pi(h)=\Delta(h) \cap \tilde{\Pi}$, we have $\Pi(h)=\tilde{\Pi}-\{\alpha_j\}$ for some $j>0$. Therefore $0<(\alpha_j, h)<1$ and $(-\alpha_0, h)=1$ because $(-\alpha_0, h) \geq m_j(\alpha_j, h)$. For $i>0$, $i \neq j$, we have $(\alpha_i, h)=0$ or 1 and the inequality

$$m_i(\alpha_i, h)<m_i(\alpha_i, h)+m_j(\alpha_j, h) \leq (-\alpha_0, h)=1$$

implies that $(\alpha_i, h)=0$ and $m_j(\alpha_j, h)=(-\alpha_0, h)=1$. So $h=2\pi i h_j/m_j$. ||

In the case $m_j=1$, we have $\Pi(2\pi i h_j/m_j)=\Pi$.

Conclusion. Let G be a complex simple Lie algebra. To find an upper bound for $\{\mathrm{ind}(g); g \in Ad(G)\}$, it suffices to consider elements $g \in Ad(G)$ whose semisimple part has the form $\exp 2\pi i h_j/m_j$ for some j, i.e. $g=\exp 2\pi i h_j/m_j \cdot \exp N$.

Clearly, $g^{m_j}=\exp m_j N$ for such g. We have proved:

Theorem. *For any $g \in Ad(G)$, there exists i such that $g^{m_i} \in \exp G$. In other words,* $\mathrm{ind}(g) \leq \max\{m_i; 1 \leq i \leq l\}$ *for all $g \in Ad(G)$. This is the same as saying that* $\mathrm{ind}(g) \in \{1, m_1, \cdots, m_l\}$.

3. Existence of elements with index m_j (in case $m_j>1$)

In [6], we have shown the existence of such elements in some cases. Here we shall give a unified short proof by using results in Steinberg [7].

We define an element x in a semisimple Lie algebra G to be regular if the centralizer $z_G(x)=\{y \in G; [x, y]=0\}$ of x (in G) has minimal dimension. By a Borel subalgebra, we mean a maximal solvable subalgebra of G. If H is a Cartan subalgebra of G with root system Δ and $U=\sum_{\alpha>0} Ce_\alpha$, then $B=H+U$ is a Borel subalgebra. Theorem 1 and its corollary in Steinberg [7] (pp. 110–112) have obviously the following Lie algebra analogues.

Theorem. *Let G be a semisimple Lie algebra with a Cartan subalgebra H, and $B=H+U$ a Borel subalgebra containing H. Let x be a nilpotent element in G. Then the following conditions are equivalent:*

(a) *x is regular.*

(b) *x belongs to a unique Borel subalgebra.*
(c) *x belongs to finitely many Borel subalgebras.*
(d) *If $U=\sum_{\alpha>0} Ce_\alpha$ and $x=\sum_{\alpha>0} c_\alpha e_\alpha (c_\alpha \in C)$, then $c_\alpha \neq 0$ for any simple root α.*

Corollary. *If $x\in U$ is regular, then $z_G(x)\subset U$. In particular, $z_G(x)$ consists of nilpotent elements.*

Retaining the notation above, consider $h_0=2\pi i h_j/m_j$. Then $\Pi=\tilde{\Pi}-\{\alpha_j\}$ is a simple root system in $\Delta(h_0)$ and $G(1, Ad \exp h_0)=H+\sum_{\alpha\in\Delta(h_0)} Ce_\alpha$. Let $N=\sum_{i=0,\cdots,l\,;\, i\neq j} e_{\alpha_i}$. Applying the above theorem, we see that N is a regular element in the semisimple subalgebra $G(1, Ad \exp h_0)$, so the above corollary implies that any element of $G(1, Ad \exp h_0)$ which commutes with N must be nilpotent.

Let $g=\exp h_0 \cdot \exp N$. If $g=\exp x$ for some $x\in G$, then x has a decomposition $x=x_0+N$, where x_0 is semisimple and $[x_0, N]=0$. Clearly $x\in G(1, Ad\, g)=G(1, Ad \exp h_0)$. Since $N\in G(1, Ad \exp h_0)$, we have $x_0\in G(1, Ad \exp h_0)$. But $[x_0, N]=0$, so the above argument implies that x_0 is nilpotent. Thus $x_0=0$ because x_0 is also semisimple. This implies that $\exp h_0=\exp x_0=1$ which is absurd ($m_j>1$). Therefore $g\notin \exp G$.

Next, let $\mathfrak{G}_1$ be the connected subgroup of $\mathfrak{G}=Ad\, G$ corresponding to the subalgebra $G_1=G(1, Ad\, g)$. Clearly, $g\in\mathfrak{G}_1$ because $\exp h_0$, $\exp N\in \exp G_1\subset\mathfrak{G}_1$. If $g^p=\exp x$ for some x in G, then x lies in G_1 because $g^p\in\mathfrak{G}_1$. (We have $G_1=\{x\in G; \exp x\in\mathfrak{G}_1\}$). But N is a regular nilpotent element in G_1, it cannot commute with any nonzero semisimple element in G_1. The same argument as above implies that the semisimple part of g^p must be 1, i.e., $\exp ph_0=1$ or $ph_0\in\Omega$. This cannot happen if $p<m_j$.

Therefore $\mathrm{ind}\,(g)=m_j$. Q.E.D.

The results in sections 2 and 3 give the following:

Theorem. *Let G be a complex simple Lie algebra and $-\alpha_0=m_1\alpha_1+\cdots+m_l\alpha_l$ the highest root expressed in terms of a simple root system. Then*

$$\{\mathrm{ind}\,(g); g\in Ad(G)\} = \{1, m_1, \cdots, m_l\},$$

which is the set of all positive integers $\leq \max\{m_i; 1\leq i\leq l\}$.

Corollary. $\mathrm{ind}(Ad(G))$ *is the least common multiple of* $\{m_1, \cdots, m_l\}$.
We can list our result in the table:

Type of G	A	B	C	D	G_2	F_4	E_6	E_7	E_8
$\max\{\mathrm{ind}(g)\}$	1	2	2	2	3	4	3	4	6
$\mathrm{ind}(Ad(G))$	1	2	2	2	6	12	6	12	60

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