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Remarks on the Structure of Maximally Almost Periodic Groups

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Introduction

The purpose of this note is to give two remarks on the structure of maximally almost periodic (abbreviated to m. a. p.) locally compact groups.

In Part I, we shall determine the structure of m. a. p. locally compact groups, which are not necessarily connected but satisfy the following condition: the factor group of the group by the connected component of the identity is compact. We shall call a group which satisfies this condition to be of *type* (A). The results are similar to those of H. Freudenthal [1],[1] who treated the same subject for connected groups and whose main theorem [2] states that a connected m. a. p. locally compact group decomposes into the direct product of a vector group and a compact group.[3] It will be needed to obtain our results this Freudenthal's theorem and a theorem which was proved as a lemma by K. Iwasawa [2]. The proof of our Theorem 1 will be reduced namely in the first half to these two theorems, and in the second half to a simple lemma (Lemma 4). These both reductions of the subject with some results was found previously by M. Kuranishi [3] in the case when the groups under consideration are Lie groups. Our results are thus only the generalizations of these Kuranishi's results to the case of locally compact groups, but much or less simpler methods will be used in proving the applicability of the above reductions.

Part II will be devoted to construct concretely a m. a. p. group with the following significance. For convenience, we call that a group has a *two-sidedly invariant uniform structure,* if the group satisfies the

1) Numbers in bracket refer to the Bibliography at the end of this note.

2) See [1] p. 68, Hauptsatz 1.

3) In [1], the groups in question are always supposed to be separable. But this assumption can be excluded. Cf. A. Weil: L'intégration dans les groupes topologiques et ses applications.

following condition: there exists in the group a complete system of neighbourhoods of the identity which are invariant under all inner automorphisms in the group. For a group satisfying the first axiom of countability, this condition is equivalent to that the group can be topologized by a both right- and left-invariant metric. It was proved also by H. Freudenthal [1] that a connected locally compact group is m. a. p. if and only if the group has a two-sidedly invariant uniform structure.[3] [4] When we use our results in Part I, it may be easily seen that this theorem is still valid if the group is not connected but of type (A). In more general cases, as far as the author knows, it has been believed to be plausible that m. a. p. group has in any case a two-sidedly invariant uniform structure even though the converse is obviously false.[5] The group constructed in Part II is a counter-example to this fact.

Parts I and II are independent of one another except for the use of Lemma 4, which will play an essential rôle in both Parts.

We note that the author is suggested to use the above reductions of the proof of Theorem 1 from the paper [3] which was fortunately communicated to him before its publication. For this and for his kind encouragements to write this note, the present author is grateful to Mr. M. Kuranishi.

Part I. On the structure of m. a. p. locally compact groups of type (A)

1. Throughout this part we shall use the following

NOTATIONS. The letters V, T and K (with a prime or a suffix if necessary) are reserved for a (finite-dimensional) vector group, a toroidal group and a compact group respectively. Furthermore e denotes always the identity of the group in question. When we denote by G a group, we indicate by G^0 the connected component of e in G.

As we have defined in Introduction, a group G is called of type (A) if the factor group G/G^0 is compact.

Our main object of this part is to prove the following

Theorem 1. *A locally compact group G of type (A) is m. a. p. if and only if G contains a maximal compact subgroup K and a normal sub-*

4) See [1] p. 74. Hauptsatz VIII.

5) This was stated by H. Freudenthal, [1] p. 75 § 30.

6) X denotes the direct product of groups.

7) This follows from the facts that $C/T=V$ and that the only one compact subgroup of V is the trivial subgroup. The same reasoning will be repeatedly used hereafter.

group V isomorphic with a vector group so that $G=KV, K \frown V=e$ and that every element of V commutes with any element of the connected component K^0 of e in K.

2. Before we prove Theorem 1, we shall establish some lemmas.

Lemma 1. *Let C be a connected abelian Lie group, and let Φ be a finite group of automorphisms in C. Then C decomposes into the form $V\times T$,*[6] *where V and T are invariant under (all automorphisms of) Φ.*

PROOF. Since C is a connected abelian Lie group, C has the form $V'\times T$, in which T is characteristic since it is the uniquely determined maximal compact subgroup of C,[7] while V' may not be invariant under Φ. Consider the character group C^* of C, and let V^* and A' be the annihilators of T and V' respectively. Then, as may be readily seen, C^* decomposes into the form $V^*\times A'$, where V^* is isomorphic with V', and A' is a free abelian group of rank equal to the dimension of T.

The dual automorphisms of thcse belonging to Φ obviously form a finite group, which we shall denote by Ψ. It follows immediately from the definitions of V^* and of dual automorphisms that V^* is invariant under Ψ.

Now, since $C^*=V^*\times A'$, we may suppose that C^* is a subgroup of a vector group W whose dimension is equal to the sum of the dimension of V^* and the rank of A', and we can introduce coordinate systems both in C^* and in W by a system of linearly independent elements belonging either to V^* or to A'. Then with respect to the coordinate system in C^* every automorphism in C^* is represented by a non-singular matrix and so it can be extended to the automorphism defined by the same matrix with respect to the coordinate system in W. For this reason, we can regard Ψ as a finite group of automorphisms acting on W, which leaves invariant the subgroups C^* and V^*.

Since a finite linear group is completely reducible, we can take such a subgroup V_1 in W that $V_1 \frown V^*=e$, and $V_1V^*=W$, and that V_1 is invariant under Ψ. Set $A=V_1 \frown C^*$. It is obvious that V_1 is invariant under Ψ and that $C^*=V^*\times A$. Denoting by V the annihilator of A in C, it is also easy to see that V is invariant under Φ, that V is isomorphic with V' and that C is written in the form $C=V\times T$, which gives the required decomposition of C.

Lemma 2. *Let G be a locally compact group of type (A), and assume that G^0 decomposes into the form: $G^0=K\times V$. Then, for an arbitrary neighbourhood U of e in G, there exists in $K\frown U$ such a normal subgroup N of G that K/N is a Lie group.*

PROOF. We first remark that for a compact group to be a Lie group it is necessary and sufficient that there exists a neighbourhood of the identity such that no normal subgroups exist in it.

Since K is compact, we can find in $K \cap U$ a subgroup M so that it is normal in K and that K/M is a Lie group. We shall prove that the number of conjugate subgroups of M in G (which are also contained and normal in K) is finite, and then that their intersection has the required properties.

From the above remark applied to K/M follows the existence of such a neighbourhood U_1 of e in G that MU_1 contains no conjugate subgroups of M different from M. Indeed, this is satisfied by the neighbourhood U_1 for which the image of $K \cap U_1$ under the natural homomorphism on K to K/M contains no normal subgroups.

Now let us denote by H the normalizer of M in G. H is an open subgroup of G; in fact, by the compactness of M we can easily find a neighbourhood U_2 of e so that $U_2MU_2^{-1} \subset MU_1$, and then we see from the choice of U_1 that $U_2 \subset H$, which shows the openness of H. Obviously $G^0 \subset H$. From these facts and the compactness of G/G^0, it follows easily that the index of H is finite, which means that the number of conjugate subgroups of M is finite. Let these subroups be $M = M_1, M_2, \dots, M_n$. We notice that all the groups K/M_i $(i=1, 2, \dots, n)$ are isomorphic with the Lie group K/M.

We proceed to prove that the intersection N of all M_i $(i=1, 2, \dots, n)$ has the required properties. It is obvious that this subgroup N is a normal subgroup contained in $K \cap U$. Setting $N_j = M_1 \cap M_2 \cap \dots \cap M_j$ for $j=1, 2, \dots, n$, we shall prove by induction for K/N_j to be Lie groups. The case $j=1$ is trivial by assumption. Suppose that K/N_j is a Lie group. Then, since N_j/N_{j+1} is isomorphic with the subgroup $(M_{j+1}N_j)/M_{j+1}$ of the Lie group K/M_{j+1}, that K/N_{j+1} is a Lie group follows from the following principle: if a compact group K contains a normal subgroup N such that both N and K/N are Lie groups, then K itself is a Lie group.[8] This may be easily proved by means of the remark mentioned at the top of this proof. Thus, for $j=n$, the proof of Lemma 2 is completed.

Lemma 3. *Suppose the same assumptions as in Lemma 2 be valid for G and G^0. Then, we can find in G^0 a normal subgroup V' of G isomorphic with V such that $G^0 = K \times V'$.*

REMARK. In the assumptions of Lemma 2, we supposed that G^0 of a locally compact group G of type (A) has the form $K \times V$, where K is a characteristic subgroup of G since it is the uniquely determined

8) This is a particular case of Theorem 7 in [2], p. 537.

maximal connected compact subgroup of G, but V is not necessarily a normal subgroup of G. This Lemma 3 states that we may assume here without loss of generality that V is a normal subgroup of G as well as K.

PROOF. We first note a simple proposition:

(1) *If a locally compact group G has a compact normal subgroup K such that G/K is connected, then G is of type (A).*

This is observed from the fact that the factor group $G/(G^0K)$ becomes at the same time connected and 0-dimensional.

We now consider a family $\mathfrak{H}$ of subgroups H satisfying the following conditions:

a) H is an abelian normal subgroup of G contained in the center of G^0,

b) $G^0=KH$, and

c) H is of type (A).

For example, the center of G^0 (which is a normal subgroup of G containing V) belongs to $\mathfrak{H}$, and hence $\mathfrak{H}$ is not empty. Now, after introducing into $\mathfrak{H}$ an order by the relation of set-inclusion, we see that the assumption of the Zorn's lemma on the existence of the minimal elements is satisfied; let $\{H_\alpha\}$ be a totally ordered subset of $\mathfrak{H}$, then $H=\bigcap_\alpha H_\alpha$ gives the infimum of this subset belonging to $\mathfrak{H}$. In fact, for H, the condition *a*) is obvious, *b*) follows from the compactness of K and *c*) can be verified by means of property *b*) and of Proposition (1). Thus by the Zorn's lemma the existence of at least one minimal element, say V', of $\mathfrak{H}$ is established.

Since V' belongs to $\mathfrak{H}$, $G^0=KV'$ and V' is a normal subgroup of G. Therefore, the proof of this lemma will be completed for this subgroup V', if we show that $K\cap V'=e$, which follows immediately by the minimality of V' from the following proposition.

(2) *Let H be a subgroup belonging to $\mathfrak{H}$ and assume that $K\cap H$ has more elements than e. Then there exists a proper subgroup H' of H which also belongs to $\mathfrak{H}$.*

Proof of (2). Set $P=K\cap H$. By assumption, $P\neq e$ and hence there exists a neighbourhood U of e which does not contain P, that is, $P\cap U\neq P$. Applying Lemma 2 for this neighbourhood U, we can find a normal subgroup N contained in $K\cap U$ such that G^0/N is a Lie group. Set $Q=P\cap N$. Clearly $Q=H\cap N$ and so Q is a normal subgroup of G. Since HN is a closed subgroup according to the compactness of $N, H/Q$ is a group isomorphic with the subgroup $(HN)/N$ of the Lie group G^0/N. While H is abelian and of type (A), so is

H/Q, too. The H/Q is an abelian Lie group of type (A), and therefore decomposes into the form $C \times E$, where C is the connected component of the identity and E is a finite group.

Now, we set $\bar{G}=G/Q$, $\bar{G}^0=G^0/Q$ and $\bar{H}=H/Q$. Since C is the connected component of e in the normal subgroup $\bar{H}$ of $\bar{G}$, C is also a normal subgroup of G. Therefore each inner automorphism in $\bar{G}$ defined by an element $\bar{x}$ induces in C an automorphism $\alpha^{\bar{x}}$; $\alpha^{\bar{x}}(\bar{z})=\bar{x}^{-1}\bar{z}\bar{x}$, where $\bar{z} \in C$. The mapping: $\bar{x} \to \alpha^{\bar{x}}$ is obviously a continuous homomorphism from $\bar{G}$ into the group $A(C)$ of all automorphisms in C, where $A(C)$ is topologized as usual. Since H is central in G^0, $\bar{H}$ is a central subgroup of $\bar{G}^0$ and hence the kernel of this homomorphism contains $\bar{G}^0$. Then, $\bar{G}/\bar{G}_0$ being compact and 0-dimensional as G/G^0, the image of $\bar{G}$ in $A(C)$ under this homomorphism is also compact and 0-dimensional. On the other hand, $A(C)$ is a Lie group because C is a connected Lie group. Thus the above image must be a finite group. In other words, the inner automorphisms in $\bar{G}$ induces in C a finite group of automorphisms. By Lemma 1, we have then a decomposition of C: $C=V_1 \times T$, where V_1 and T are invariant under all inner automorphisms in $\bar{G}$, that is, they are normal subgroup in $\bar{G}$.

We set H' and K' for the complete inverse image of V_1 and $T \times E$ respectively under the natural homomorphism on G to $\bar{G}$, and we shall show that H' is one of the required proper subgroups of H which belong to $\mathfrak{H}$. Since $H'/Q=V_1$, Q is the uniquely determined maximal compact subgroup of H', and therefore $K \frown H' \subset Q \subset U$. Whence $K \frown H' \neq P = K \frown H$, which shows that H' is a proper subgroup of H. Now we verify the conditions *a*), *b*) and *c*) for H' to belong to $\mathfrak{H}$. *a*) is obvious except for the normality of H' in G, which follows from that of V_1 in $\bar{G}$. *b*) follows by (1) since $H'/Q=V_1$. Finally, *c*) is verified as follows: since both $K'/Q=T \times E$ and Q are compact, K' is compact, and therefore $K' \subset K$. Then the relation $H=K'H'$ implies that $KH'=KH=G^0$, q. e. d.

Lemma 4. *Let G be a group and N a normal subgroup of G with finite index. Then, if N is m. a. p., G is also m. a. p.*

In other words, every finite extension of a m. a. p. group is also m. a. p.

Proof. To prove this lemma, it is sufficient to show that, for every continuous almost periodic function $f(x)$ defined on N, there is a continuous almost periodic function $F(x)$ defined on G which coincides with $f(x)$ for $x \in N$. This function $F(x)$ is given by the following formula:

$$F(x)=\begin{cases} f(x), & \text{if } x\in N, \\ 0, & \text{if } x\notin N. \end{cases}$$

The continuity of this function is obvious since G/N is discrete. We shall prove that $F(x)$ is an almost periodic function, i. e., every sequence of functions of the form

$$F_n(x)=F(a_nx), \text{ where } a_n\in G, n=1, 2, \dots ,$$

contains a subsequence which uniformly converges on G. Since the number of cosets modulo N is finite, we first extract from the original sequence an infinite number of functions for which a_n's belong to one and the same coset a_0N modulo N, where a_0 is a representative of this coset. According to the definition of $F(x)$, all these functions vanish on any coset modulo N except on the coset of the form $a_0^{-1}N$, and on the latter one they are expressed in the form $f(a_na_0^{-1}x)$ where $x\in N$. Hence from the almost periodicity of $f(x)$ on N, it follows immediately that we can select again from these functions an infinite number of functions which form the required uniformly convergent subsequence, q. e. d.

The following lemma was obtained recently by K. Iwasawa [2] [9].

Lemma 5. *Let G be a locally compact group and V be a normal subgroup of G. If V is isomorphic with a vector group, and if G/V is compact, then G contains a compact subgroup K such that $G=KV$, $K\frown V=e$. Moreover, K is the maximal compact subgroup unique up to its conjugate subgroups.*

3. PROOF OF THEOREM 1. Suppose G be a m. a. p. locally compact group of type (A). Then G^0 is obviously m. a. p. and by the Freudenthal's theorem on the structure of connected locally compact m. a. p. groups (cited in Introduction), G^0 has the form $K_1\times V$, where we may assume by Lemma 3 that both K_1 and V are normal subgroups of G. Then, since G is of type (A), G/V is compact. Thus, by Lemma 5, we have the decomposition of G: $G=KV, K\frown V=e$. The maximality of K as a compact subgroup is involved in Lemma 5. Since it is easily seen that the connected component of K^0 of e in K coincides with K_1, K^0 and V are elementwise commutative.

Conversely, let G be a group with the above decomposition and assume that every element of V commutes with any element of K^0. To each element k of K there corresponds the automorphism σ^k in V

9) See [2] p. 524, Lemma 3.8.

defined by $\sigma^k(v)=k^{-1}vk$ for $q\in V$. The mapping: $k\to\sigma^k$ obviously gives a continuous homomorphism on K into the group of all automorphisms in V, which is the full linear group of some degree. By the second assumption, this homomorphism induces an isomorphism between a factor group of K/K^0 and a subgroup of the full linear group. The former being a compact 0-dimensional group and the latter being a Lie group, these groups must be finite. Hence the kernel K' of the homomorphism: $k\to\sigma^k$ is a subgroup of finite index in K. It follows then that the centralizer of G (which is a normal subgroup of G) decomposes into the direct product of K' and V and that it has a finite index in G. Thus by Lemma 4 we see immediately that G is a m. a. p. group. This completes the proof of Theorem 1.

In the first half of this proof, we have only used that G^0 is m. a. p. Therefore by the second half of it, we have proved incidentally the following

Theorem 2. *Let G be a locally compact group of type (A), G is m. a. p. when (and obviously only when) G^0 is m. a. p.*

4. As a consequence of Theorems 1 and 2, we note the following.

Proposition. *Let G be a locally compact group and assume that G has a two-sidedly invariant uniform structure. Then G contains an open normal subgroup N which decomposes into the form $K\times G$*[10].

PROOF. In this case G^0 has the two-sidedly invariant uniform structure induced from that of G, and hence, by the second theorem of Freudenthal cited in Introduction, G^0 is m. a. p. while as is easily seen, G contains an open subgroup L of type (A) which contains G^0. By Theorem 2, L is m. a. p., and by Theorem 1 it is of the form K_1V, where K_1 is a compact subgroup. Now as was shown in the proof of Theorem 1, the centralizer M of V in L is an open subgroup of L and so of G, and it takes the form $K'\times V$. Let N be the intersection of all the conjugate subgroups of M. Then N is the required open normal subgroup of G, since it contains G^0 and so V, and since its openness is observed from the fact that M contains a neighbourhood of e which is invariant under all inner automorphisms in G, q. e. d.

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10) This structural condition is not sufficient for a group to have a two-sidedly invariant uniform structure. For example, the group composed of all matrices of the form $\begin{pmatrix} r & x \\ 0 & 1 \end{pmatrix}$, where r runs over the discrete multiplicative group of positive rational numbers and x runs over the (usual) additive group of real numbers.

REMARKS. This proposition would be the best possible answer for the conjecture of E. R. van Kampen (which reads that a locally compact group with a two-sidedly invariant uniform structure may be represented as a direct product of a vector group and a group in which an open compact normal subgroup exists), since this conjecture was disproved by M. Kuranishi [3].

Part II. Example of a m. a. p. group which has no two-sidedly invariant uniform structures

1. Consider a 0-dimensional locally compact group G which has a two-sidedly invariant uniform structure. Then there exists in G an open compact normal subgroup N. In fact, since G is 0-dimensional and locally compact, we can find an open subgroup O in a conditionally compact neighbourhood of e. The intersection of all its conjugate subgroups gives the required normal subgroup N, since its openness follows from the fact that the former subgroup O contains a neighbourhood of e which is invariant under all inner automorphisms in G.

Thus we can state as follows:

An example of a locally compact group G which is m. a. p. but has no two-sidedly uniform structures, can be given by a 0-dimensional m. a. p. locally compact group which admits no open compact subgroups.

In the next section, we shall construct such a group concretely.

2. CONSTRUCTION OF THE EXAMPLE. First we define two kinds of groups.

Let D be the weak direct product of countable number of groups, each of which is isomorphic with the additive group of integers. Namely, an element of D is a sequence $\{x_n\}$ of integers, where $x_n(n=1, 2, \dots)$ are equal to zero except for a finite number of n's, and the product of two elements $\{x_n\}, \{y_n\}$ in D is defined to be the element $\{x_n+y_n\}$. Denote by A_r and B_r two subgroups of D which consist of those elements $\{x_n\}$ that satisfy the condition $x_n=0$ for $n>r$, and the condition $x_n=0$ for $n\leq r$ respectively. In the following D will be regarded as a topological group with discrete topology.

Secondly, let K be the direct product (in the sense of a topological group) of countable number of groups, each of which is isomorphic with the multiplicative group of two numbers ± 1. That is to say, an element of K is expressed in the form $\{\varepsilon_n\}$, where $\varepsilon_n=\pm 1$, and the product of two elements $\{\varepsilon_n\}, \{\delta_n\}$ in K is $\{\varepsilon_n\delta_n\}$. The subgroups L_r and M_r are defined as those formed by such elements $\{\varepsilon_n\}$ that satisfy the condition $\varepsilon_n=1$ for $n>r$, and the condition $\varepsilon_n=1$ for $n\leqq r$ respectively. K is a 0-dimensional compact group.

Now the required group G is the group of all pairs $(\{\varepsilon_n\}, \{x_n\})$ of elements from K and D, in which the product of two elements is defined by the formula:

$$(\{\varepsilon_n\}, \{x_n\})(\{\delta_n\}, \{y_n\})=(\{\varepsilon_n\delta_n\}, \{\varepsilon_n y_n+x_n\}).$$

It is easy to verify that G forms an algebraic group, and that the set of all the elements of the form $(\{\varepsilon_n\}, \{0\})$, and the set of those of the form $(\{1\}, \{x_n\})$ constitute respectively a subgroup isomorphic with K and a normal subgroup isomorphic with D. Denoting such corresponding groups by the same letters, we can see easily that B_r and M_rD are normal subgroups in G.

Moreover the following relations hold in G:

$$G=KD, K\frown D=e.$$

Hence we can assign to G the direct product topology of K and D. Obviously by this topologization G becomes a 0-dimensional locally compact group.

3. We are to show that G is m. a. p. Since G/D is isomorphic with the compact group K, we have only to prove that every element $(\neq e)$ of G can be separated from e by a suitable bounded representation of G. For this purpose it is enough to show that G/B_r is m. a. p., since each element $(\neq e)$ of D is contained in some A_r and then has its natural homomorphic image different from the identity into G/B_r. Now a simple calculation shows that the normal subgroup $(M_rD)/B_r$ in G/B_r is abelian and hence is m. a. p., while the factor group of G/B_r by $(M_rD)/B_r$ is isomorphic with the finite group L_r. Thus, by Lemma 4 in the foregoing Part we conclude that G/B_r is m. a. p.

Next we shall prove that G admits no open compact normal subgroups. Suppose there be such a normal subgroup N in G. Then, by the definition of the topology in G, there would exist a suitable r such that $N\supset M_r$, and therefore an element $k=(\{\varepsilon_n\}, \{0\})$ in which $\varepsilon_n=1$ for $n\leqq r, \varepsilon_n=-1$ for $n>r$, must belong to N. Let $d=(\{1\}, \{x_n\})$ be an element of D, in which $x_{r+1}\neq 0$. By virtue of the normality of N and of D, the element $dkd^{-1}k^{-1}=(\{1\}, \{x_n-\varepsilon_n x_n\})$ which is not equal to e because $x_{r+1}-\varepsilon_{r+1}x_{r+1}=2\ x_{r+1}\neq 0$, must be contained both in N and in D. This implies that D should have non-trivial compact (i. e. finite, since D is discrete) subgroup $N\frown D$, which contradicts the algebraic structure of D.

Thus the construction of the example mentioned in 1 has been finished.

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BIBLIOGRAPHY

[1] H. Freudenthal: *Topologische Gruppen mit genügend vielen fastperiodischen Funktionen,* Ann. of Math. Vol. 37 (1936) 57–75.

[2] K. Iwasawa: *On some types of topological groups,* Ann. of Math. Vol. 50 (1949) 507–558.

[3] M. Kuranishi: *On non-connected maximally almost periodic groups,* to appear in Tôhoku Math. Journal.