Kado, J.
Osaka J. Math.
18 (1981), 55–61

UNIT-REGULAR RINGS AND SIMPLE SELF-INJECTIVE RINGS

JIRO KADO*)

(Received October 8, 1979)

Unit-regular algebras over a field are investigated from the point of view of a directed abelian group with order-unit.

In the section 1 we show that if $(K_0(R), [R])$ is an ultrasimplicial abelian group for a unit-regular algebra R over a field F, then R has a subalgebra T such that T is an ultramatricial F-algebra and R is generated as a ring by T and units of R.

In the section 2 we discuss a simple left and right self-injective ring R which is not artinian. Let F be the center of R and F_∞ be the completion of a ring which is a direct limit of $M_2(F)\rightarrow M_{2^2}(F)\rightarrow\cdots$, where homomorphisms are diagonal maps. We show that there exists a subalgebra S of R such that S is isomorphic to F_∞ as a F-algebra and that every idempotent of R is conjugate to an idempotent of S and that every element of R has the form uev, where u, v are units in R and e is an idempotent of S.

We take most of our terminologies and notations from Goodearl's recent book [3], and rely as well on this work for statements of known results.

Throughout this paper a ring is an associative ring with identity and modules are unitary

1. Unit-regular algebras

DEFINITION [2]. A ring R is *unit-regular* if for each $x\in R$ there is some unit (i.e. invertible element) $u\in R$ such that $xux=x$.

DEFINITION [3, p. 200]. For any ring R the *Grothendieck group* $K_0(R)$ is an abelian group with generators $[A]$, where A is any finitely generated projective right R-modules, and with relations $[A]+[B]=[C]$ whenever $A\oplus B\cong C$. Two generators $[A]$, $[B]$ equal in $K_0(R)$ if and only if $A\oplus R^n\cong B\oplus R^n$ for some positive integer n. Every element of $K_0(R)$ has the form $[A]$-$[B]$ for suitable modules A, B.

---

*) Supported by Grant-in-Aid for Scientific Research, No. 454017

DEFINITION [3, p. 202]. A *partially ordered abelian group* is an abelian group G equiped with a partial order $\leqslant$ which is translation invariant (i.e. $x \leqslant y$ implies $x+z \leqslant y+z$). The *positive cone* of G is the set $G^+ = \{x \in G; x \geqslant 0\}$. If the partial order on G is directed (upward or downward), then G is called a *directed abelian group*. An *order-unit* in G is an element $u > 0$ such that for any $x \in G$, there exists a positive integer n for which $x \leqslant nu$. We denote by a pair (G, u) a partially ordered abelian group G with order-unit u. We always consider a morphism f: $(G, u) \to (G', u')$ as a order-preserving group homomorphism such that $f(u) = u'$.

DEFINITION. For modules A, B, $A \lesssim B$ implies that A is isomorphic to a submodule of B.

The following Lemma is a fundamental result for a unit-regular ring.

Lemma 1 [5, Props. 2.1 and 2.2]. *Let R be a unit-regular ring and let A, B, C and D be finitely generated projective right R-modules. We define* $[A]$-$[B] \leqslant [C]$-$[D]$ *if and only if* $A \oplus D \lesssim B \oplus C$. *Then we have following results*:

(1) $(K_0(R), \leqslant)$ *is a directed abelian group.*
(2) $[A] > 0$ *for all non-zero finitely generated projective right R-modules.*
(3) $K_0(R)^+$ *consists of all elements of* $K_0(R)$ *of the form* $[A]$.
(4) *If* $R \neq 0$, *then* $[R]$ *is an order-unit in* $K_0(R)$.
(5) $[A]$-$[B] = [C]$-$[D]$ *in* $K_0(R)$ *if and only if* $A \oplus D \cong B \oplus C$.

We note that $(K_0(-), [-])$ is a functor which preserves direct limits from the category of all unit-regular rings to the category of all partially ordered abelian groups with order-unit ([3, Prop. 15.11]).

DEFINITION [3, pp. 216 and 219]. A *simplicial directed abelian group* G is a directed abelian group whose positive cone G^+ has the form $\boldsymbol{Z}^+ x_1 + \cdots + \boldsymbol{Z}^+ x_n$ for some linearly independent elements (over $\boldsymbol{Z}$) $x_1, \cdots, x_n$. An *ultrasimplicial directed abelian group* G is isomorphic to a direct limit of a sequence $(G_1, u_1) \to (G_2, u_2) \to$ of simplicial directed abelian groups with order-unit.

DEFINITION [3, pp. 217 and 219]. Given a field F, we define a *matricial F-algebra* to be any F-algebra of the form $M_{p(1)}(F) \times \cdots \times M_{p(n)}(F)$ for any positive integers $p(1), \cdots, p(n)$. An F-algebra R is called *ultramatricial* if R is isomorphic to a direct limit (in the category of F-algebras) of a sequence $R_1 \to R_2 \to \cdots$ of matricial F-algebras.

Lemma 2 [3, Th. 15.24]. *For an ultramatricial F-algebra R, $(K_0(R), [R])$ is an ultrasimplicial directed abelian group with order-unit. Conversely for an ultrasimplicial directed abelian group with order-unit (G, u), there exists an ultramatricial F-algebra R such that $(K_0(R), [R]) \cong (G, u)$.*

The following lemma is a generalization of [3, (a) of Lemma 15.23].

Lemma 3. *Let F be a field and R be an ultramatricial F-algebra and S be a unit-regular F-algebra. If a morphism* f: $(K_0(R), [R]) \to (K_0(S), [S])$ *is given, there exists a F-algebra homomorphism* $\varphi: R \to S$ *such that* $K_0(\varphi)=f$.

Proof. Let R be the direct limit of a sequence $R_1 \xrightarrow{\pi_1} R_2 \xrightarrow{\pi_2} \cdots$ of matricial F-algebras. Let $\theta_n: R_n \to R$ be natural homomorphisms for all n. Then $(K_0(R), [R])$ is the direct limit of direct system $\{(K_0(R_n), [R_n]), K_0(\pi_n)\}$ and $K_0(\theta_n)$ are natural homomorphisms. Put $f_n = fK_0(\theta_n)$ for all n. Then by [3, Lemma 15.23], there exist F-algebra homomorphisms $\psi_n: R_n \to S$ such that $K_0(\psi_n)=f_n$ for all n. We shall construct F-algebra homomorphisms $\varphi_n: R_n \to S$ for $n=1, 2, \cdots$ such that $K_0(\varphi_n)=f_n$ and $\varphi_{n+1}\pi_n=\varphi_n$ for all n. Put $\varphi_1=\psi_1$, and assume that we have φ_k for all $k \leqslant n$. Two algebra homomorphisms φ_n, $\psi_{n+1}\pi_n$: $R_n \to S$ satisfy $K_0(\varphi_n)=K_0(\psi_{n+1}\pi_n)$. Thus we can choose an inner automorphism g of S such that $\varphi_n = g\psi_{n+1}\pi_n$ by [3, Lemma 15.23]. Put $\varphi_{n+1} = g\psi_{n+1}$. Noting that $K_0(g)$ is an identity map on $K_0(S)$, we have $K_0(\varphi_{n+1})=f_{n+1}$. For a sequence $\varphi_1, \varphi_2, \cdots$, there exists a unique F-algebra homomorphism $\varphi: R \to S$ such that $\varphi\theta_n=\varphi_n$ for all n. We have $K_0(\varphi)K_0(\theta_n)=K_0(\varphi_n)=f_n=fK_0(\theta_n)$ for all n. Then we can conclude $K_0(\varphi)=f$ by the uniqueness.

Lemma 4. *Let F be a field and R be a unit-regular F-algebra. For any ultrasimplicial directed subgroup G with order-unit* $[R]$ *of* $(K_0(R), [R])$, *there exists a subalgebra T of R such that T is an ultramatricial F-algebra and that* $K_0(i)$: $(K_0(T), [T]) \cong (G, [R])$, *where i is the inclusion map.*

Proof. By Lemma 2, there exists an ultramatricial F-algebra T' such that $(K_0(T'), [T']) \cong (G, [R])$. Let f: $(K_0(T'), [T']) \to (G, [R])$ be an order-preserving group isomorphism such that $f([T'])=[R]$. Then we can choose an F-algebra homomorphism $\varphi: T' \to R$ such that $K_0(\varphi)=f$ by Lemma 3. For any $x \in Ker\,\varphi$, $0=[\varphi(x)R]=K_0(\varphi)[xT']=f([xT'])$. Therefore $[xT']=0$, and hence we have $xT'=0$ by Lemma 1. Since φ is monomorphism, we have a desired algebra $T=\varphi(T')$.

Theorem 1. *Let R be a unit-regular algebra over a field F, and assume that* $(K_0(R), [R])$ *is ultrasimplicial. Then there exists a subalgebra T of R such that*

(*a*) *T is an ultramatricial F-algebra*

(*b*) *every idempotent of R is cojuagte to an idempotet of T.*

(*c*) *every element of R is a product of a unit and a conjugate of an idempotent of T, i.e. every element of R has the form uev, where u, v, are units of R and e is an idempotent of T.*

Proof. By Lemma 4, there exists an ultramatricial F-algebra T of R such that $K_0(i)$: $(K_0(T), [T]\cong(K_0(R), [R])$, where i: $T\rightarrow R$ is the inclusion. Let e be any idempotent of R. There exist finitely generated projective right T-modules A, B such that $[eR]=K_0(i)[A]$, $[(1-e)R]=K_0(i)[B]$. Since $K_0(i)([T])=K_0(i)([A\oplus B])$, then $[T]=[A\oplus B]$. We have $T\cong A\oplus B$ by Lemma 1. We choose an idempotent f of T such that $fT\cong A$. Since $[eR]=K_0(i)([fT])=[fR]$, then $eR\cong fR$ by Lemma 1. Therefore we have $e=u^{-1}fu$ for some unit u of R by [7, Th. 2]. For every $x\in R$, there exists a unit u of R such that xu is an idempotent. Then (c) is a immediate consequence by (b).

REMARK. The example R given in [3, Example 15.28] is a unit-regular algebra and $(K_0(R), [R])$ is an ultrasimplicial directed abelian group but is not ultramatricial.

2. Simple self-injective rings

DEFINITION [3, p. 80]. A regular ring R satisfies the *comparability axiom* if we have either $J\lesssim K$ or $K\lesssim J$ for any two principal right ideals J, K. A ring R is *directly finite* if $xy=1$ *implies* $yx=1$ for $x, y\in R$.

DEFINITION [3, p. 226]. A *rank function of a regular ring* R is a map N: $R\rightarrow [0, 1]$ such that

(a) $N(1)=1$
(b) $N(xy)\leqslant min\ \{N(x), N(y)\}$ for all $x, y, \in R$
(c) $N(e+f)=N(e)+N(f)$ for all orthogonal idempotents $e, f\in R$
(d) $N(x)>0$ for all non-zero $x\in R$.

Let R be any simple unit-regular ring which satisfies the comparability axiom. By [3, Cor. 18.12 and Th. 18.17], R has a unique rank function N which is determined by the rule: $N(x)=sup\ \{kn^{-1};\ k, n\in \boldsymbol{Z}, k\geqslant 0, n>0, (R_R)^k\lesssim(xR)^n\}$. If a regular ring R has a rank function N, N induces a metric δ on R by the rule $\delta(x, y)=N(x-y)$ for all $x, y\in R$ ([3, p. 282]). We call this metric a *rank-metric induced by* N or N-*metric*. Moreover the completion $\bar{R}$ of R with respect to N-metric is a unit-regular left and right self-injective ring ([3, Th. 19.7]). We call $\bar{R}$ the N-completion of R.

DEFINITION [1]. A *factor sequence* $\mu=(p(1), p(2), \cdots)$ is a infinite sequence of positive integers such that $p(n)|p(n+1)$ for all n and $p(n)\rightarrow\infty$ when $n\rightarrow\infty$.

Let F be a field. For a factor sequence μ, F_μ denotes the ultramatricial F-algebra determined by a direct system $\{M_{p(n)}(F), f_n\}$, where f_n: $M_{p(n)}(F)\rightarrow M_{p(n+1)}(F)$ is a block diagonal homomorphism. F_μ is a simple unit regular ring which satisfies the comparability axiom. Then F_μ has a unique rank function N. The N-completion $\bar{F}_\mu$ of F_μ is a simple unit-regular left and right self-injective ring which is not artinian and its center is isomorphic to F ([4, Th. 2.8]). Since $\bar{F}_\mu \cong \bar{F}_\nu$ as F-algebra by [10] for any two factor sequences μ, ν, then we denote the completion of F_μ by F_∞ instead of $\bar{F}_\mu$.

Let R be a simple left and right self-injective ring which is not artinian. R has a unique rank function N and it is complete with respect to N-metric ([3, Cor. 21.14]). The center F of R is a field ([3, Cor. 1.15]). Now we investigate the relation between R, F_μ and F_∞. We note that the additive group $\boldsymbol{R}$ of all real numbers is considered as partially ordered abelian group with normal order.

Theorem 2. *Let R be a simple left and right self-injective ring which is not artinian and let F be the center of R. Then the following results hold*:

(1) *For any factor sequence μ, there exist a subalgebra T of R which is isomorphic to F_μ as F-algebra.*

(2) *For every non-zero idempotent e of R, there exists a (single or infinite) sequence $\{e_1, e_2, \cdots\}$ of orthogonal non-zero idempotents in a subalgebra conjugate to T such that $\oplus_n Re_n$ is essential in Re and also $\oplus_n e_nR$ is essential in eR.*

(3) *There exists a subalgebra S of R such that*

(*a*) *S is isomorphic to F_∞ as F-algebra*

(*b*) *every idempotent of R is conjugate to an idempotent of S.*

(*c*) *every element of R is a product of a unit of R and a conjugate of an idempotent of S.*

Proof. (1) Since R is a simple (unit-) regular right self-injective ring of "Type II_f" (See [3, pp. 102, 113 and 120]), then $(K_0(R), [R]) \cong (\boldsymbol{R}, 1)$ by [3, Th. 15.8]. Put $\mu = (p(1), p(2), \cdots)$ and N be the unique rank function of F_μ. For $x \in F_\mu$, we know $N(x) = p(n)^{-1} rank(x)$, where $x \in M_{p(n)}(F)$. A morphism $\theta: K_0(F_\mu) \to \boldsymbol{R}$ induced by the rule $\theta([xF_\mu]) = N(x)$ for all $x \in F_\mu$ is a order-preserving group isomorphism such that $\theta([F_\mu]) = 1$ by [3, Cor. 16, 15]. We have $Image\,\theta = \cup_n \boldsymbol{Z} \cdot p(n)^{-1}$ by easy caluculation. This is an ultrasimplicial directed abelian group. Then there exists an ultramatricial F-algebra T of R such that $(K_0(T), [T]) \cong (Image\,\theta, 1)$ by Lemma 3. By [3, Th. 15.26], we have $T \cong F_\mu$ as F-algebra.

(2) Again N denote the rank function of T. We define $D(xT) = N(x)$ for all $x \in T$. Then D is a unique dimension function on the set of all finitely generated projective right T-modules ([3, Prop. 16.8]). Moreover D is extended to a function on all projective right T-modules ([9]). Then we may assume that D is this extended function. A morphism induced by the rule: $[A] \to D(A)$ is equal to θ. Let N' be the unique rank function of R. We note that the resrtictive map of N' to T is equal to N by the uniqueness. Now let

e be any non-zero idempotent of R. First we assume that $N'(e) \in Image\,\theta$. We choose a finitely generated projective T-module A such that $N'(e)=\theta([A])$. Since $\theta([A])=D(A)\leqslant 1$, then $A \lesssim T_T$ by [9, Lemma 2.2]. There exists an idempotent f in T such that $N'(e)=N'(f)$. Since $eR \cong fR$ by [3, Cor. 16.15], then there exists a unit $u \in R$ such that $e=u^{-1}fu$ by [7, Th. 2]. Next we assume that $N'(e) \notin Image\,\theta$. Since $Image\,\theta=\cup_n Zp(n)^{-1}$ is dense in the space of all real numbers with respect to the normal metric, then there exists an infinite sequence $x_1, x_2, \cdots$ of positive real numbers in $Image\,\theta$ such that $N'(e)=\sum_n x_n$. We choose finitely generated right T-modules A_n such that $D(A_n)=x_n$ for each n. Let $A=\oplus_n A_n$ be an outer direct sum. Since $D(A)=\sum_n D(A_n)=N'(e)\leqslant 1$ ([9, p. 406]), then $A \lesssim T_T$ by [9, Lemma 2.3]. Hence we may assume that there exists a countably generated right ideal A of T such that $D(A)=N'(e)$. By [3, Prop. 2.14], there exists an infinite sequence $g_1, g_2, \cdots$ of orthogonal non-zero idempotents in T such that $A=\oplus_n g_nT$. Since R is left and right self-injective, then there exists an idempotent $g \in R$ such that $\oplus_n Rg_n$ is essential in Rg and also $\oplus_n g_nR$ is essential in gR by [11, Th. 6.4]. Hence we have $N'(g)=\sum_n N'(g_n)$ by [3, Th. 21.11 and 21.13]. We see that a directed sequence $\{g_1+\cdots+g_n;\ n=1, 2, \cdots\}$ of idempotents is convergent to g with respect to N'-metric. We write $g=\sum_n g_n$. Let S be the closure of T in R. Since $(T, N$-metric) is a subspace of (R, N'-metric), then we have $S \cong F_\infty$. We have $N'(g)=\sum_n N(g_n)=\sum_n D(g_nT)=D(A)=N'(e)$. Since $gR \cong eR$ by [3, Cor. 16.15], then we have $e=u^{-1}gu$ for some unit $u \in R$ by [7, Th. 2]. A sequence $\{u^{-1}g_nu;\ n=1, 2, \cdots\}$ is a family of orthogonal nonzero idempotents in a subalgebra $u^{-1}Tu$. Put $e_n=u^{-1}g_nu$ for all n. Since $g=\sum_n g_n$, then $e=u^{-1}gu=\sum_n e_n$. Therefore we conclude $\oplus_n Re_n$ is essential in Re and also $\oplus_n e_nR$ in eR by [8, Prop. 3].

(3) We have already shown that the closure S of T is a desired algebra which satisfies (a) and (b). For every element $x \in R$, there exists a unit $u \in R$ such that xu is an idempotent. Applying (2) to xu, we can conclude (c).

REMARK. The (a) of Theorem 2 was proved in the case $\mu=(2, 2^2, 2^3, \cdots)$ by D. Handelman ([6, Remark (1) of Prof. 9]).

Acknowledgment. I would like to thank Prof. K.R. Goodearl for his useful suggestions.

References

[1] B.P. Dawkins and I. Halperin: *The isomorphism of certain continuous rings*, Canad. J. Math. **18** (1966), 1333–1344.

[2] G. Ehrlich: *Unit-regular rings*, Portugal Math. **27** (1968), 209-212.

[3] K.R. Goodearl: Von Neumann regular rings, Pitman, 1979.

[4] K.R. Goodearl: *Centers of regular self-injective rings*, Pacific J. Math. **76** (1978),

381–389.

[5] K.R. Goodearl and D. Handelman: *Rank functions and K_0 of regular rings*, J. Pure Appl. Algebra **7** (1976), 195–216.

[6] D. Handelman: *Simple regular rings with a uniquerank function*, J. Algebra **42** (1976), 60–80.

[7] D. Handelman: *Perspectivity and cancellation in regular rings*, J. Algebra **48** (1977), 1–16.

[8] D. Handelman and R. Raphael: *Regular Schur rings*, Arch. Math. **31** (1978), 332–338.

[9] J. Kado: *Projective modules over simple regular rings*, Osaka J. Math. **16** (1979), 405–412.

[10] Von Neumann: *Independence of F_∞ from the sequence ν*, Collected Works of John Von Neumann, Vol. IV.

[11] Y. Utumi: *On continuous rings and self-injective rings*, Trans. Amer. Math. Soc. **118** (1965), 158–173.

Departmeut of Mathematics
Osaka City University
Osaka 558, Japan