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AN APPROACH OF THE THEORY OF INTEGRATION BY THE MEAN OF BIASED TEST FUNCTIONS

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Introduction

In [7], O. Loos has given, in the framework of IST (the Nelson's version of non standard analysis) a definition of the Lebesgue integral of a standard function f defined on a standard compact interval of $\mathbb{R}$ using a sort of approximation of f by steps functions. The advantage of this approach is that it reduces the Lebesgue integral to the level of difficulty of the ordinary Riemann integral. Other nonstandard theories on integration exist.

The best known and universally used nonstandard formulation has been given, in the Robinson approach of nonstandard analysis, by P. Loeb in the book of Hurd and Loeb ([4]).

The theory of Pierre Cartier and Yvette Perrin [2], is in the spirit of the phenomenological approach of mathematics of Vopenka ([18]); it is very clear and use only calculus of hyper-finite sums (finite but nonstandard). We can also say that the thought process of Loos is not so far from the classical approach of Kurzweil-Henstock in [3], with gauge functions and its nonstandard improvement by J. Mawhin ([10]).

Our aim is to generalize the Loos's ideas to non standard functions defined on any intervals of $\mathbb{R}^N$ (N possibly infinitely large). The framework of this paper is Relative Set Theory (see [13],...,[17]); we describe this theory and some results in the annex at the end of the article.

Let us first give some notations and definitions.

Let N be a natural number (possibly infinitely large), we call *interval* of $\mathbb{R}^N$ a cartesian product of N intervals of $\mathbb{R}$. Let $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N)$ two points of $\mathbb{R}^N$, we denote respectively by $[a,b]$, $[a,b[$, $]a,b]$ and $]a,b[$ the intervals $\prod_{k=1}^N [a_k, b_k]$, $\prod_{k=1}^N [a_k, b_k[$, $\prod_{k=1}^N]a_k, b_k]$ and $\prod_{k=1}^N]a_k, b_k[$. Let $P = \prod_{k=1}^N I_k$ be an interval of $\mathbb{R}^N$ with $\overline{I_k} = [a_k, b_k]$ and let for all k, $\{x_1^k, \dots, x_{n_k}^k\}$ be a subdivision of $\overline{I}_k$ with $a_k = x_1^k < \dots < x_{n_k}^k = b$. We define a step function on P as a function which is constant on each open interval of $\prod_{k=1}^N]x_{j_k}^k, x_{j_k+1}^k[$, $j_k \in \{1, \dots, n_k\}$ and we denote by $\mathcal{E}(P)$ the set of all step functions on P.

<u>Topology on the sets of intervals</u>: The reader is refered to [16] for more details about

our approach of nonstandard topology. We know that the usual topology of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is defined without ambiguity from its infinitesimal proximity, $\to$, defined for any level α by,

$$\begin{aligned}
&\forall^{\alpha} x \in \mathbb{R} \forall y \in \mathbb{R} \ (y \overset{\alpha}{\to} x \iff \forall^{\alpha} \varepsilon > 0 \ |y - x| < \varepsilon), \\
&\forall y \in \overline{\mathbb{R}} \ (y \overset{\alpha}{\to} +\infty \iff \forall^{\alpha} \lambda \in \mathbb{R} \ y > \lambda), \\
&\forall y \in \overline{\mathbb{R}} \ (y \overset{\alpha}{\to} -\infty \iff \forall^{\alpha} \lambda \in \mathbb{R} \ y < \lambda).
\end{aligned}$$

We know how to extend naturally the former proximity to obtain an infinitesimal proximity, still denoted $\to$ on the cartesian product $\overline{\mathbb{R}^N}$, (even if N is nonstandard).

Let P be an interval of $\mathbb{R}^N$, let α be the level of ideality of N, let $\mathcal{P}(P)$ be the set of the intervals contained in P. We define on $\mathcal{P}(P)$ the topology, also denoted by "$\to$", defined by:

$$\text{if } F = (a, b), F' = (a', b') \text{ and } \beta \sqsupseteq \alpha, F' \overset{\beta}{\to} F \iff (a' \overset{\beta}{\to} a \text{ et } b' \overset{\beta}{\to} b).$$

We recall that, an infinitesimal proximity $\to$ being given, if $y \overset{\beta}{\to} x$ then, by the definition of the proximities, x is ${}^{\beta}$standard (see [16]).

In what follows α is a fixed level of ideality, P is an interval of $\mathbb{R}^N$ and $f : P \to \mathbb{R}$ is a function such that (f, P, N) is ${}^{\alpha}$ standard. For any $Q \subset P$, and $g : P \to \mathbb{R}$, g_Q is the restriction of g to Q.

1. Definition of the integrability in Relative Set Theory
A- Case where P is compact

In this chapter, P will be compact. Let β be a level dominating α.

DEFINITION. A function $\psi \in \mathcal{E}(P)$ is called ${}^{\beta}$ *upper test of f on P* (respectively ${}^{\beta}$*lower test of f on P*) if and only if $\forall x \in P$, $\psi(x) \geq f({}^{\beta}x)$ (respectively $\forall x \in P$, $\psi(x) \leq f({}^{\beta}x)$).

We recall that ${}^{\beta}x$ is the unique ${}^{\beta}$standard element of P such that x is ${}^{\beta}$ infinitely close of ${}^{\beta}x$, its ${}^{\beta}$ shadow (see the annex).

We denote by ${}^{\beta}(test)^{*}(f, P)$ (respectively ${}^{\beta}(test)_{*}(f, P)$) the collection of ${}^{\beta}$ upper tests of f on P (respectively of ${}^{\beta}$ lower tests of f on P).

We say that (φ, ψ) is a ${}^{\alpha}$ *test of f on P* if and only if φ is an ${}^{\alpha}$ lower test of f on P and ψ is an ${}^{\alpha}$ upper test of f on P. We denote by ${}^{\alpha}(test)(f, P)$ the collection of the ${}^{\alpha}$ test of f on P.

Let β, γ be levels such that $\alpha \sqsubseteq \beta$ and $\alpha \sqsubseteq \gamma$, $\varphi \in {}^{\beta}test_{*}(f)$ and $\psi \in {}^{\gamma}test^{*}(f)$, we say that (φ, ψ) is ${}^{\alpha}$ *adapted* if and only if ${}^{\alpha}\int \varphi = {}^{\alpha}\int \psi \neq \pm\infty$. The ${}^{\alpha}$ shadows of the integrals are relative to the ordinary compact topology of $\overline{\mathbb{R}}$.

Now, similarly to Loos, we define the following upper and lower integrals in $\overline{\mathbb{R}}$:

$$\int_P^* f = \inf\left(st_\alpha\left\{{}^\alpha\int\psi,\ \psi\in{}^\alpha test^*(f,P)\right\}\right),$$
$$\int_{*\ P} f = \sup\left(st_\alpha\left\{{}^\alpha\int\varphi,\ \varphi\in{}^\alpha test_*(f,P)\right\}\right).$$

REMARK. The principle of transfer implies that any level dominating $[f]$ can be used to define $\int_P^* f$ and $\int_{*\ P} f$.

We can state

Theorem 1. *Let β and γ two levels such that $\alpha \sqsubset \beta$ and $\alpha \sqsubset \gamma$. Then the next three conditions are equivalent.*

i) $\qquad I = \int_P^* f = \int_{*\ P} f \neq \pm\infty,$

ii) $\qquad \forall^\alpha\ \varepsilon > 0,\ \exists(\varphi,\psi)\in{}^\alpha(test)(f,P);\ {}^\alpha\int_P \psi \neq \pm\infty$ *and* $\int_P \psi-\varphi<\varepsilon$

iii) $\qquad \exists(\varphi,\psi)\in{}^\beta(test)_*(f,P)\times{}^\gamma(test)^*(f,P),\ (\varphi,\psi)^\alpha$*adapted.*

If f satisfies iii), *then* ${}^\alpha\int\varphi = {}^\alpha\int\psi = I.$

Proof. i) $\Longleftrightarrow$ ii) is obvious.

ii) implies iii): Let us suppose ii). For any ${}^\alpha$standard $\varepsilon > 0$ there exists $\varphi,\psi\in\mathcal{E}(P)$ such that

$$[\varphi\in{}^\alpha(test)_*(f,P),\ \psi\in{}^\alpha(test)^*(f,P),\ {}^\alpha\int_P\psi\neq\pm\infty \text{ and } \int_P\psi-\varphi<\varepsilon],$$

which implies: for any ${}^\beta$standard $\varepsilon > 0$ there exists $\varphi,\psi\in\mathcal{E}(P)$ such that

$$[\varphi\in{}^\beta(test)_*(f,P),\ \psi\in{}^\beta(test)^*(f,P),\ {}^\beta\int_P\psi\neq\pm\infty \text{ and } \int_P\psi-\varphi<\varepsilon].$$

Using the theorem of partial transfer (see the annex), we obtain, for any ${}^\beta$standard $\varepsilon > 0$ that there exists $\varphi,\psi\in\mathcal{E}(P)$ such that

$$[\varphi\in{}^\beta(test)_*(f,P),\ \psi\in{}^\gamma(test)^*(f,P),\ {}^\beta\int_P\psi\neq\pm\infty \text{ and } \int_P\psi-\varphi<\varepsilon].$$

Let us choose $\varepsilon\ {}^\alpha\!\!\approx 0$ positive and ${}^\beta$standard. Then there exists two step functions φ,ψ such that $\varphi\in{}^\beta test_*(f,P)$, $\psi\in{}^\gamma test^*(f,P)$ and $\int\psi-\varphi\ {}^\alpha\!\!\approx 0$. This implies ${}^\alpha\int\psi = {}^\alpha\int\varphi$.

From the condition i), which is equivalent to ii), and the definition of the upper and lower integrals, we get ${}^{\beta}\int\varphi \leq \int f \leq {}^{\gamma}\int\psi$ and taking the ${}^{\alpha}$ shadows: ${}^{\alpha}\int\varphi = {}^{\alpha}\int\psi = \int^{*} f \neq \pm\infty$.

iii) $\Longrightarrow$ ii): Consider $\varphi \in {}^{\beta}(test)_{*}(f,P)$, $\psi \in {}^{\gamma}(test)^{*}(f,P)$ with ${}^{\alpha}\int\varphi = {}^{\alpha}\int\psi \neq \pm\infty$. By the definition of upper and lower integrals, we have ${}^{\beta}\int\varphi \leq \int_{*} f \leq \int^{*} f \leq {}^{\gamma}\int\psi$. If we take the ${}^{\alpha}$ shadows we obtain: ${}^{\alpha}\int\varphi = {}^{\alpha}({}^{\beta}\int\varphi) \leq \int_{*} f \leq \int^{*} f \leq {}^{\alpha}({}^{\gamma}\int\psi) = {}^{\alpha}\int\psi = {}^{\alpha}\int\varphi$. Hence $I = \int_{*} f = \int^{*} f \neq \pm\infty$. □

We can prove, by a direct adaptation of the proof of Loos's theorem and a transfer, that f satisfies i), ii) *or* iii) *if and only if it is integrable according to the Lebesgue's definition.*

REMARK.

a) If $\alpha = \sigma$ and $N = 1$, the conditions i) and ii) in the Proposition 1 are exactly Loos's criteria for integrability.

b) If we accept that $\alpha = \beta$ or $\alpha = \gamma$, then we have proved that the condition iii) suffices to have integrability.

c) If $\alpha = \beta = \gamma$ then iii) implies that f is constant. (see [1])

DEFINITION. An ${}^{\alpha}$standard function f that satisfies one of the equivalent conditions of Theorem 1 will be called *L-integrable* (the letter L beeing the initial of Loos as well as Lebesgue, or Loeb). Then the number I in i) is the *L-integral* of f.

A subset X of P will be called L-integrable if the characteristic function $\mathbb{I}_X$ of X is L-integrable. If f is defined only on an integrable subset P' of P, then we extend the definition of the integrability by: f is L-integrable if the function g such that $g_{P'} = f$ and $g_{P-P'} = 0$ is L-integrable.

APPLICATION. Let K be a compact subset of P. We know that $\mathbb{I}_K$, is L-intégrable (classic) but how do the tests for such a function look like?

For the sake of simplicity, we suppose that $N = 2$, and that P and K are standard.

Consider $P = [a,b] \times [a',b']$ and $F = \{x_1, x_2, \ldots, x_n\}$ (resp $F' = \{x'_1, x'_2, \ldots, x'_m\}$) be finite set such that $a = x_1 < x_2 < \ldots < x_n = b$ and for any $i = 1 \ldots n-1$, $x_i{}^{\sigma} \approx x_{i+1}$, (resp $a' = x'_1 < x'_2 < \ldots < x'_n = b'$ and for any $i = 1 \ldots m-1$, $x'^{\sigma}_i \approx x'_{i+1}$). Put $\beta = [(F,F')]$. We shall call a *cell* any of the sets: $C_{ij} = [x_i, x_{i+1}[\times[x'_j, x'_{j+1}[$, $\partial_i = [x_i, x_{i+1}[\times\{b'\}$, $\partial'_i = \{b\} \times [x'_i, x'_{i+1}[$ and $B = \{(b,b')\}$.

Let ψ be the function defined as follows, $\psi(x,y) = 1$ if (x,y) is in the cell C and $\overline{C} \cap K \neq \phi$, 0 otherwise.

Consider $(x,y) \in C$ such that $\psi(x,y) = 1$, there exists (z,t) such that $(z,t) \in$

$\overline{C} \cap K$; Hence, as K is a standard compact, $({}^\sigma z, {}^\sigma t) = ({}^\sigma x, {}^\sigma y) \in K$, and $\mathbb{I}_K({}^\sigma x, {}^\sigma y) = 1$. So $\psi \in {}^\sigma test_*(f)$.

Let us suppose now that $\mathbb{I}_K({}^\beta x, {}^\beta y) = 1$. Then $({}^\beta x, {}^\beta y) \in K$. If $(x, y) \in C$, then, as C is ${}^\beta$standard, $({}^\beta x, {}^\beta y) \in \overline{C}$. This implies that $\overline{C} \cap K \neq \phi$ and $\psi(x, y) = 1$. So $\psi \in {}^\beta test^*(f)$

The test (ψ, ψ) is clearly ${}^\sigma$ adapted.

REMARK. In this particular situation, there are two surprising facts:
(a) the test function ψ is both a lower test and an upper test,
(b) ψ is ${}^\beta$standard, and this is not true in general.

So we state.

Problem. Describe the class of all the L-integrable functions such that (a) or (b) holds. The next proposition solves the point (a).

Theorem 2. *If $\alpha \sqsubset \beta$, then there exists $\psi \in {}^\alpha(test)_*(f, P) \cap {}^\beta(test)^*(f, P)$ (resp. $\psi \in {}^\beta(test)_*(f, P) \cap {}^\alpha(test)^*(f, P)$) if and only if any point of P is a local maximum (resp. minimum) of f.*

Proof. Let us suppose that $\varphi \in {}^\alpha test_*(f) \cap {}^\beta test^*(f)$ exists, then for any $x \in P$, we have $f({}^\beta x) \leq \varphi(x) \leq f({}^\alpha x)$, which give $f({}^\beta x) \leq f({}^\alpha x)$. Let x_α be a fixed ${}^\alpha$standard point of P. For any ${}^\beta$standard element x_β of P such that $x_\beta {}^\alpha\!\approx x_\alpha$, we have $f(x_\beta) = f({}^\beta x_\beta) \leq f({}^\alpha x_\beta) = f(x_a)$. This can be expressed by $\forall^\alpha x \forall^\beta y\ (y {}^\alpha\!\approx x \Longrightarrow f(x) \leq f(y))$. By transfer we obtain $\forall^\alpha x \forall y\ (y {}^\alpha\!\approx x \Longrightarrow f(x) \leq f(y))$, each ${}^\alpha$standard point is a local maximum. By transfer this is true for any point.

Conversely. If any point of P is a local maximum, then f is upper bounded on P. Consider $\alpha \sqsubset \beta \sqsubset \gamma$. Let a be a ${}^\alpha$standard point of P, then, by hypothesis, $f(a)$ is a maximum of f on a neighborhood (${}^\alpha$standard by transfer) Q of a. Let ψ be a ${}^\gamma$upper test of f on P. It is easy to find one ${}^\gamma$upper test ψ_a of f on P such that $\psi_a(x) \leq f(a)$ for any $x {}^\alpha\!\approx a$ (take $\psi_a = f(a)\mathbb{I}_Q + \psi\mathbb{I}_{P-Q}$).

The general principle of choice, $C(\alpha, [\,])$ (see [15]), give a function $T : P \to \mathcal{E}(P)$ such that
1) $\forall^\alpha\ a \in P,\ [T(a) \in {}^\gamma(test)^*(f, P)$
2) $\forall^\alpha\ a \in P,\ \forall y \in P,\ y {}^\alpha\!\approx a \Longrightarrow T(a)(y) \leq f(a)]$.

The partial transfer give a function $T' : P \to \mathcal{E}(P)$ such that
1) $\forall^\beta\ a \in P,\ T'(a) \in {}^\gamma(test)^*(f, P)$
2) $\forall^\alpha\ a \in P,\ \forall y \in P,\ y {}^\alpha\!\approx a \Longrightarrow T'(a)(y) \leq f(a)$.

For the sake of simplicity we write $\psi_a = T'(a)$. Let E be a ${}^\beta$standard finite set which contains all ${}^\alpha$standard points of P. Consider the step function defined as follows: $\psi = \inf\{\psi_a, a \in E\}$. It is easy to show that $\psi \in {}^\alpha test_*(f) \cap {}^\gamma test^*(f)$.

By transfer, we conclude to the existence of $\psi \in {}^{\alpha}(test)_*(f,P) \cap {}^{\beta}(test)^*(f,P)$. The case of $\psi \in {}^{\beta}(test)_*(f,P) \cap {}^{\alpha}(test)^*(f,P)$ is similar. □

2. B- Extension to the case where P is non-compact

We first recall that x is said to be *almost* ${}^{\beta}$*standard* in P if and only if the ${}^{\beta}$shadow of x exists in P.

If P is non-compact, then we can define, using the same notation as in the compact case, the collections ${}^{\beta}test_*(f)$, ${}^{\beta}test^*(f)$ and ${}^{\beta}test(f)$ by letting:

$$\begin{aligned}
&\psi \in {}^{\beta}(test)^*(f,P) \iff \text{ for any } x \text{ almost } {}^{\beta}\text{standard in } P, \ [\psi(x) \geq f({}^{\beta}x)],\\
&\psi \in {}^{\beta}(test)_*(f,P) \iff \text{ for any} x \text{ almost } {}^{\beta}\text{standard in } P, \ [\psi(x) \leq f({}^{\beta}x)],\\
&{}^{\beta}(test)(f,P) = \{(\varphi,\psi) \in {}^{\beta}(test)_* \times {}^{\beta}(test)^*; \varphi \leq \psi\}.
\end{aligned}$$

However, we first prefer to consider the collections of upper and lower test functions, ${}^{\alpha,\beta}test_*(f)$, ${}^{\alpha,\beta}test^*(f)$ and also ${}^{\alpha,\beta}test(f)$ by letting:

$$\begin{aligned}
&\psi \in {}^{\alpha,\beta}(test)^*(f,P) \iff \forall x \in P, \ [{}^{\alpha}x \in P \implies f({}^{\beta}x) \leq \psi(x)],\\
&\psi \in {}^{\alpha,\beta}(test)_*(f,P) \iff \forall x \in P, \ [{}^{\alpha}x \in P \implies f({}^{\beta}x) \geq \varphi(x)],\\
&{}^{\alpha,\beta}(test)(f,P) = \{(\varphi,\psi) \in {}^{\alpha,\beta}(test)_* \times {}^{\alpha,\beta}(test)^*; \varphi \leq \psi\}.
\end{aligned}$$

The latter definitions of tests approximate the value of f at the ${}^{\beta}$ideal points so long as those are not too close to the boundary of P. This choice is intended to obtain a more general notion of integrability that the Lebesgue's one. We can remark that the hypothesis $\varphi \leq \psi$ is useless if P is compact.

We need some additional vocabulary

- A subinterval S of P is called a ${}^{\alpha}$*small-interval* if $\overline{S}$ is the cartesian product of N intervals $[a_i, b_i]$ such that for at least one i, $a_i \,{}^{\alpha}\!\approx b_i$ in $\overline{\mathbb{R}}$. We observe that, according to this definition, if a_i and b_i are jointly infinitely large, then S is a ${}^{\alpha}$small interval.
- We say that $\psi \in \mathcal{E}(P)$ has a ${}^{\alpha}$*Cauchy property*, and we denote it by ${}^{\alpha}\mathcal{C}(\psi,P)$ if and only if for all ${}^{\alpha}$small-interval S, $\int_S \psi \,{}^{\alpha}\!\approx 0$.
  If $(\varphi,\psi) \in {}^{\alpha,\beta}test(f)$ is ${}^{\alpha}$adapted, then, ${}^{\alpha}\mathcal{C}(\psi) \iff {}^{\alpha}\mathcal{C}(\varphi)$ and we say that (φ,ψ) is ${}^{\alpha}$*(Cauchy)*.

DEFINITION. If $(\varphi,\psi) \in {}^{\alpha,\beta}(test)(f,P)$ is ${}^{\alpha}$ adapted and ${}^{\alpha}\mathcal{C}(\psi,P)$, we say that (φ,ψ) is a ${}^{\alpha,\beta}$ *good-test* of f. We denote by ${}^{\alpha,\beta}$ *good-test*(f,P) the collection of the ${}^{\alpha,\beta}$ *good-tests* of f on P.

DEFINITION. A function is said to be *G-integrable* on P if and only if there exists $(\varphi,\psi) \in {}^{\alpha,\beta}$ *good-test*(f,P). We denote ${}_G\!\int_P f = {}^{\alpha}\!\int_P \psi = {}^{\alpha}\!\int_P \varphi$.

REMARK. The value ${}_G\int_P f$ seems to depend of the test (φ,ψ), but the Theorem 3 below proves the contrary. Indeed the Cauchy's type condition is essential for the uniqueness.

We proceed now to the demonstration of the fact that the G-integral unifies the Lebesgue integrability on compacts intervals and the existence of a kind of semi-convergent Lebesgue integral on non compacts intervals. Let us state the first lemma.

Lemma 1. *If ψ is any positive step function defined on an ${}^{\alpha}$standard compact interval then the condition ${}^{\alpha}\mathcal{C}(\psi)$ is equivalent to: $\forall^{\beta}\ \mathcal{P}\ [\mathcal{P}\ {}^{\alpha}$small-interval $] \Longrightarrow {}_G\int_{\mathcal{P}} \psi \,{}^{\alpha}\!\approx 0$.*

Proof. Necessary is obvious. Let us suppose that $\forall^{\beta}\ \mathcal{P}\ [\mathcal{P}\ {}^{\alpha}$small-interval $] \Longrightarrow {}_G\int_{\mathcal{P}} \psi \,{}^{\alpha}\!\approx 0$. We should like to remove the upper index β, but this is an illegal Borde's transfer because the constant ψ is not ${}^{\beta}$standard. Luckily, we can prove that any ${}^{\alpha}$small-interval is included in a ${}^{\beta}$standard ${}^{\alpha}$small-interval: Let $Q = \prod(a_i, b_i)$ be a ${}^{\alpha}$small-interval, and let $h > 0$ be a ${}^{\beta}$ standard ${}^{\alpha}$infinitesimal. The sub-interval of P, $S = P \cap \prod[{}^{\beta}a_i - h, {}^{\beta}b_i + h]$ is a ${}^{\alpha}$small-interval which is dominate by β and contains Q. As ψ is positive, $\int_Q \psi \,{}^{\alpha}\!\approx 0$. □

Theorem 3. *If P is compact, then f is L-integrable on P if and only if f is G-integrable on P. Moreover $\int_P f = {}_G\int_P f$.*

Proof. Let us suppose P compact. Then ${}^{\alpha,\beta}test(f,P) = {}^{\beta}test(f,P)$ because any element of P is ${}^{\alpha}$limited, so if f is G-integrable, it is obviously L-integrable.

Conversely, suppose that f is L-integrable. Let us first assume f positive. Let $S = [x,y]$ be a compact ${}^{\alpha}$small-interval. We can assume, due to the Lemma 1, that S is ${}^{\beta}$standard. The function f is L-integrable, so if we fix a γ such as $\beta \sqsubset \gamma$ there is $(\varphi,\psi) \in {}^{\gamma}test(f)$ such that (φ,ψ) is ${}^{\alpha}$adapted. As $\psi - \varphi$ is positive, $\int \psi - \varphi \,{}^{\alpha}\!\approx 0$ implies $\int_S \psi - \varphi \,{}^{\alpha}\!\approx 0$.

On the other hand, $(\varphi,\psi) \in {}^{\gamma}test(f,P)$ implies $(\varphi,\psi) \in {}^{\gamma}test(f,S)$ because the ${}^{\gamma}$shadow of an element of S stays in S. TheRefore, by Theorem 2, it suffices to prove that $\int_S f \,{}^{\alpha}\!\approx 0$ to get $\int_S \psi \,{}^{\alpha}\!\approx 0 \,{}^{\alpha}\!\approx \int_S \varphi$.

Let us suppose that $S = \prod_{i=1}^{N} I_i$ where $\overline{I_i} = [a_i, b_i]$ and $a = {}^{\alpha}a_1 = {}^{\alpha}b_1$. Put $S_n =]a - (1/n), a + (1/n)[\times \prod_{i=2}^{N} I_i$. Clearly $S \subset S_n$ for all n ${}^{\alpha}$standard. The ${}^{\alpha}$standard sequence $(h_n = f_{|S_n})_{n\in\mathbb{N}}$ decreases toward a function h such that $\int_P h = 0$. Therefore $\lim_{n\to\infty} \int_{S_n} f = 0$. Consequently, S being included in any ${}^{\alpha}$standard S_n $\int_S f \,{}^{\alpha}\!\approx 0$. So f is G-integrable.

If f is not positive, we decompose f in the classical form $f = f^{+} - f^{-}$. □

We proceed now to the cases where the interval of integration is non compact.

Then the G-integrability is equivalent to the existence of a kind of improper semi-convergent L-integral. The proof needs lemmas. The proof of Lemma 2 is in the proof of the Theorem 3.

Lemma 2. *If P is compact, if $(\varphi, \psi) \in {}^{\beta}(test)(f, P)$ and is ${}^{\alpha}$adapted, then $(\varphi, \psi) \in {}^{\alpha,\beta}good\text{-}test(f, P)$.*

Lemma 3. *For any interval $Q \subset P$. If $[Q] \sqsubseteq \alpha$, $(\varphi, \psi) \in {}^{\alpha,\beta}good\text{-}test(f, P)$, then $(\varphi_Q, \psi_Q) \in {}^{\alpha,\beta}good\text{-}test(f_Q, Q)$.*

Corollary. *If f is G-integrable, then for all intervals $Q \subset P$, f_Q is G-integrable.*

Proof of Lemma 3. Let Q be a sub-interval of P. We may suppose that $[Q] \sqsubseteq \alpha$. Let $(\varphi, \psi) \in {}^{\alpha,\beta}good\text{-}test(f)$. If φ_Q and ψ_Q denote the restrictions to Q of φ and ψ, then (φ_Q, ψ_Q) is obviously a ${}^{\alpha,\beta}$test of f which is ${}^{\alpha}$(Cauchy) and satisfies ${}^{\alpha}\int_Q \psi_Q = {}^{\alpha}\int_Q \varphi_Q$. It remains to prove that ${}^{\alpha}\int_Q \psi_Q \neq \pm\infty$. Let $[a, b]$ be the closure of P in $\overline{\mathbb{R}^N}$. The principle of choice (see [14]) ensure us the existence of the ${}^{\alpha}$standard function $Y : [a, b]^2 \to \overline{\mathbb{R}}$ such that $Y(a, a) = 0$ and $\forall^{\alpha}\ x, y \in [a, b]$, $Y(x, y) = {}^{\alpha}\int_{]x,y[} \psi_Q$.

Let $\Omega = \{(x, y) \in [a, b]^2;\ Y(x, y) = \pm\infty\}$. From the theorem of partial transfer (see [14]), we infer the existence of $\psi' \in \mathcal{E}(P)$, such that

$$\forall^{\beta}\ x, y \in [a, b],\ Y(x, y) = {}^{\beta}\int_{]x,y[} \psi'_Q \text{ and } {}^{\alpha}\mathcal{C}(\psi').$$

If x', y' (respectively $x"$, $y"$) are ${}^{\alpha}$standard (respectively ${}^{\beta}$standard) point of $[a, b]$, such that $x' \overset{\alpha}{\approx} x"$ and $y' \overset{\alpha}{\approx} y"$, then $\int_{]x",y"[} \psi' \overset{\alpha}{\approx} \int_{]x',y'[} \psi'$ since $]x", y"[-]x', y'[$ and $]x', y'[-]x", y"[$ are ${}^{\alpha}$finites unions of ${}^{\alpha}$small-intervals.

So, we have

$${}^{\beta}\int_{]x",y"[} \psi' = \pm\infty \Longleftrightarrow {}^{\beta}\int_{]x',y'[} \psi' = \pm\infty.$$

This implies that both Ω and $[a, b]^2 - \Omega$ are open in $[a, b]^2$. As $[a, b]^2$ is connected and $Y(a, b) \neq \pm\infty$, $\Omega = \phi$. This proves that (φ_Q, ψ_Q) is ${}^{\alpha}$adapted. □

DEFINITION. We denote by $\mathcal{K}(P)$ the set of all compacts intervals of P. A function f is said to *admit an improper L-integral I on P* if and only if
1) $\forall K \in \mathcal{K}(P)$, f_K is L-integrable,
2) for any interval Q of P, $\lim_{K \to Q, K \in \mathcal{K}(P)} \int f_K$ exists and is finite,
3) $\lim_{K \to P, K \in \mathcal{K}(P)} \int f_K = I$.

Then we have.

Theorem 4. *Let β be such that $\beta \sqsupset \alpha$. The next three conditions are equivalent:*

1) *f admits an improper L-integral I on P,*
2) *f is G-integrable on P and $I = {}_G\int_P f$,*
3) *$\exists(\varphi,\psi) \in {}^{\beta}(test)(f,P)^{\alpha}$adapted and ${}^{\alpha}\mathcal{C}(\psi)$.*

Proof. 1) $\Longrightarrow$ 2): If f admits an improper L-integral, the function $\mathcal{I}$ defined on $\mathcal{K}(P)$ by $\mathcal{I}(K) = \int_K f$, has a limit I as K tends to P. Consider K a ${}^{\beta}$standard sub-interval of P such that $K \overset{\alpha}{\hookrightarrow} P$ and any ${}^{\alpha}$standard x of P is in K. There exists $(\varphi,\psi) \in {}^{\gamma}(test)(f,K)$ ${}^{\beta}$adapted (where $\gamma \sqsupset \beta$). The hypothesis on K imply that if ${}^{\alpha}x \in P$, then ${}^{\alpha}x \in K$. If we extend φ and ψ by putting $\varphi(x) = \psi(x) = 0$ outsides K, then we have $(\varphi,\psi) \in {}^{\alpha,\gamma}(test)(f,P)$. Moreover, as $\mathcal{I}(K) \overset{\alpha}{\approx} I$ which is ${}^{\alpha}$limited, (φ,ψ) is ${}^{\alpha}$adapted. It remains to prove that (φ,ψ) is ${}^{\alpha}$Cauchy.

Let $\mathcal{P}$ be a ${}^{\alpha}$small-interval of P and $T = \mathcal{P} \cap K$.

As $\psi = 0$ outside of K, $\int_{\mathcal{P}} \psi = \int_T \psi$.

If there exists a ${}^{\alpha}$standard compact interval K_α such that $\mathcal{P} \subset K_\alpha \subset K$, then f_{K_α} is L-integrable and $(\varphi_{K_\alpha}, \psi_{K_\alpha}) \in {}^{\gamma}(test)(f_{K_\alpha}, K_\alpha)$. So, $\int_{\mathcal{P}} \psi = \int_{\mathcal{P}} \psi_{K_\alpha} \overset{\alpha}{\approx} 0$.

If $\mathcal{P}$ is ${}^{\alpha}$near of the boundary of K, then for the sake of simplicity we shall prove it for $N = 2$ (the general proof is only technically more difficult). An elementary reasoning gives that there is an ${}^{\alpha}$ideal sub-interval Q of P such that, T is entirely ${}^{\alpha}$infinitely close of a corner of Q (Fig. 2), or three sides of T are ${}^{\alpha}$infinitely close of the boundary of Q (Fig. 3) or, (Fig. 4) only two opposite sides of T are ${}^{\alpha}$infinitely close of the boundary of Q. By a rapid glance to Fig.4, we see that the third case reduces to the second. It can happen that T cuts the boundary of Q, but a further decomposition brings us back to the Figures 2 and 3.

Let us prove that $\int_T f \overset{\alpha}{\approx} 0$, in the cases 2 and 3. Thanks to the Borde's theorem of transfer, T can be supposed ${}^{\beta}$standard (the constants K and f are ${}^{\beta}$standard and α is strictly dominated by β).

In the situation of Fig. 2 we have $[a,b] \overset{\alpha}{\hookrightarrow} Q$ and $[a,b'] \overset{\alpha}{\hookrightarrow} Q$ so by the Definition 4, $\int_{[a,b]} f \overset{\alpha}{\approx} \int_{[a,b']} f$. This implies that $\int_{[c,b]} f \overset{\alpha}{\approx} 0$. Similarly, $\int_{[a',b]} f \overset{\alpha}{\approx} \int_{[a',b']} f$ implies that $\int_{[c',b]} f \overset{\alpha}{\approx} 0$. $\int_T f + \int_{[c',b]} f = \int_{[c,b]} f$, gives $\int_T f \overset{\alpha}{\approx} 0$. Now, T is a ${}^{\beta}$standard and compact sub-interval of K, (φ,ψ) being a ${}^{\beta}$adapted ${}^{\gamma}$test of f_K, the Lemma 3 gives that (φ_T,ψ_T) is a ${}^{\beta}$adapted ${}^{\gamma}$test of f_T, and by the definition of the L-integral $\int_T \psi \overset{\beta}{\approx} \int_T f$. Hence $\int_T \psi = \int \psi_T \overset{\alpha}{\approx} 0$.

The case of the Fig. 3 is similar: $\int_{[a,b]} f \overset{\alpha}{\approx} \int_{[a,b']} f$, $\int_T f + \int_{[a,b']} f = \int_{[a,b]} f$ and (φ_T,ψ_T) is a ${}^{\beta}$adapted ${}^{\gamma}$test of f_T.

2) $\Longrightarrow$ 1) Let us suppose that f is G-integrable, with an ${}^{\alpha}$standard G-integral I. Let K be a compact interval of P such that $K \overset{\alpha}{\hookrightarrow} P$. Consider β and γ such that $\beta \sqsupseteq [K]$ and $\alpha \sqsubseteq \beta \sqsubset \gamma$. Let (φ,ψ) be a ${}^{\beta,\gamma}$good-test of f on P. Throught the Theorem 3 and the Lemma 2 we deduce that f_K is L-integrable, $(\varphi_K,\psi_K) \in$

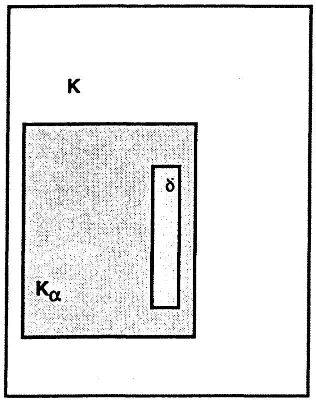

Fig. 1

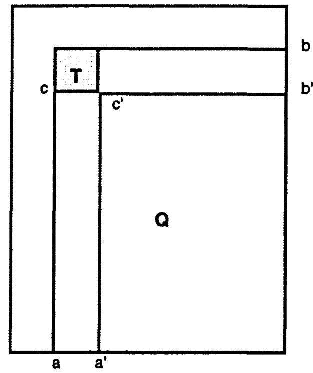

Fig. 2

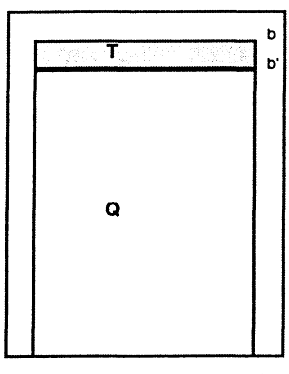

Fig. 3

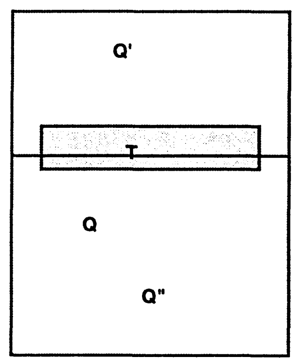

Fig. 4

${}^{\beta,\gamma}$*good-test*(f_K) and $\int_K f^{\beta} \approx \int \psi_K = \int_K \psi$. As $P - K$ is a ${}^{\alpha}$finite union of ${}^{\alpha}$small-intervals, $\int_K \psi^{\alpha} \approx \int_P \psi^{\beta} \approx I$. So $\int_K f^{\alpha} \approx I$. This prove that f admits an improper L-integral.

3) implies 2) is obvious.

The proof of 2) $\Longrightarrow$ 3), needs two lemma. □

Lemma 4. *If f is G-integrable, then for any ${}^{\alpha}$(small-interval) $\mathcal{P}$ such $\mathcal{P} \subset P$,* ${}_G\int_{\mathcal{P}} f^{\alpha} \approx 0$.

Proof of Lemma 4. Let $\mathcal{P}$ be a ${}^{\alpha}$small-interval of P; consider β, γ and δ such that $\alpha \sqsubseteq [\mathcal{P}] \sqsubseteq \gamma \sqsubset \beta \sqsubset \delta$. As f is G-integrable, there exists $(\varphi_0, \psi_0) \in {}^{\alpha,\beta}$*good-test*$(f)$. The partial transfer gives a ${}^{\gamma,\delta}$test of f on P, (φ, ψ), ${}^{\gamma}$adapted and ${}^{\alpha}\mathcal{C}(\psi)$.

As $\mathcal{P}$ is ${}^{\gamma}$standard, $(\varphi_{\mathcal{P}}, \psi_{\mathcal{P}}) \in {}^{\gamma,\delta}$*good-test*$(f, \mathcal{P})$. The definition of the G-integrability and the Cauchy's condition give $\int_{\mathcal{P}} f = {}^{\gamma} \int \psi_{\mathcal{P}} = \int_{\mathcal{P}} \psi \overset{\alpha}{\approx} 0$. □

Lemma 5. *Let β and f be such that $[f] = \alpha \sqsubseteq \beta$. Let K be a ${}^{\beta}$standard compact interval of P. If $(\varphi, \psi) \in {}^{\beta}test(f, K)$ is ${}^{\alpha}$adapted, then for any ${}^{\beta}$small-interval $\mathcal{P} \subset K$, $\int_{\mathcal{P}} \psi \overset{\alpha}{\approx} 0$ and $\int_{\mathcal{P}} \varphi \overset{\alpha}{\approx} 0$.*

Proof of Lemma 5. It is easy to show that $(\varphi^+, \psi^+) \in {}^{\beta}(test)(f^+, K)$ and $(\psi^-, \varphi^-) \in {}^{\beta}(test)(f^-, K)$. The real number $M = \max_{x \in K} \varphi^+(x)$ is clearly ${}^{\beta}$limited and this implies $\int_{\mathcal{P}} \varphi^+ \overset{\beta}{\approx} 0$. Similarly, we prove that $\int_{\mathcal{P}} \psi^- \overset{\beta}{\approx} 0$.

Since $\int_{\mathcal{P}} \psi^+ - \int_{\mathcal{P}} \psi^- - \int_{\mathcal{P}} \varphi^+ + \int_{\mathcal{P}} \varphi^- = \int_{\mathcal{P}} \psi - \varphi \overset{\alpha}{\approx} 0$, $\int_{\mathcal{P}} \psi^+ + \int_{\mathcal{P}} \varphi^- \overset{\alpha}{\approx} 0$. As these integrals are non negatives, we can conclude that both are ${}^{\alpha}$infinitesimales. □

Proof of 2) $\Longrightarrow$ 3). Let $(K_n)_{n \in \mathbb{N}}$ be a ${}^{\alpha}$standard increasing sequence of compact intervals of P which converges to P and such that for any n, $K_n \subset \mathring{K}_{n+1}$, the interior of K_{n+1}. Consider $\beta \sqsupset \alpha$ and $\varepsilon > 0$ ${}^{\beta}$standard, ${}^{\alpha}$infinitesimal. The principle of choice (see [15]) gives a sequence of step functions $(\varphi_n, \psi_n)_{n \in \mathbb{N}}$ such that

$$\forall^{\beta}\ n\ (\varphi_n, \psi_n) \in {}^{\beta}(test)(f, K_n) \text{ and } \int \psi_n - \varphi_n \leq \frac{\varepsilon}{2^n}.$$

If $\gamma \sqsupset \beta$, we obtain with partial transfer, a sequence of step functions also denoted by $(\varphi_n, \psi_n)_{n \in \mathbb{N}}$, such that

- $\forall^{\gamma}\ n\ (\varphi_n, \psi_n) \in {}^{\gamma}(test)(f, K_n)$
- $\forall^{\gamma}\ n \int \psi_n - \varphi_n \leq \varepsilon / 2^n$
- $\forall^{\beta}\ n,\ (\varphi_n, \psi_n) \in {}^{\beta}(test)(f, K_n)$.

Consider $(\varphi', \psi') \in {}^{\alpha,\beta}$*good-test*$(f, P)$ and denote by (φ'_n, ψ'_n) the restrictions of φ_n and ψ_n to $K_{n+1} - K_n$. Fix N a ${}^{\gamma}$standard ${}^{\beta}$unlimited, and put $\varphi = \varphi'_0 \vee \varphi'_1 \vee \ldots \vee \varphi'_N \vee \varphi'$ et $\psi = \psi'_0 \vee \psi'_1 \vee \ldots \vee \psi'_N \vee \psi'$ where $\vee$ is the symbol for the concantenation.

Any ${}^{\beta}$standard point of P is in a K_n with n ${}^{\beta}$standard; so, if ${}^{\beta}x \in P$, there exists a ${}^{\beta}$standard n such that $(\varphi({}^{\beta}x), \psi({}^{\beta}x)) = (\varphi_n({}^{\beta}x), \psi_n({}^{\beta}x))$. This implies that $(\varphi(x), \psi(x)) = (\varphi_n(x), \psi_n(x))$ or $(\varphi(x), \psi(x)) = (\varphi_{n+1}(x), \psi_{n+1}(x))$. So $\varphi(x) \leq f({}^{\beta}x) \leq \psi(x)$, $(\varphi, \psi) \in {}^{\beta}(test)(f, P)$. Moreover this test is ${}^{\alpha}$adapted since

$$\int_P \psi - \varphi = \sum_{n=0}^{N} \int_P \psi'_n - \varphi'_n \quad + \int_P \psi' - \varphi' \leq \varepsilon \sum_{n=0}^{N} \frac{1}{2^n} + \int_P \psi' - \varphi' \overset{\alpha}{\approx} 0.$$

Let us prove ${}^{\alpha}\mathcal{C}(\psi)$. Let $\mathcal{P}$ be a ${}^{\alpha}$small-interval of P, $\mathcal{P}_N = \mathcal{P} \cap K_N$ is a ${}^{\alpha}$small-interval of P too. Since K_N is a ${}^{\gamma}$standard compact interval, we can find a ${}^{\gamma}$standard ${}^{\alpha}$small-interval $\mathcal{P}_{\gamma}$ included in K_N and such that the symetric difference, $\mathcal{P}_N \triangle \mathcal{P}_{\gamma}$ be a ${}^{\alpha}$finite union of ${}^{\gamma}$small-interval. Lemma 5 implies $\int_{\mathcal{P}_N - \mathcal{P}_{\gamma}} \psi \overset{\alpha}{\approx} 0$ and

Fig 5

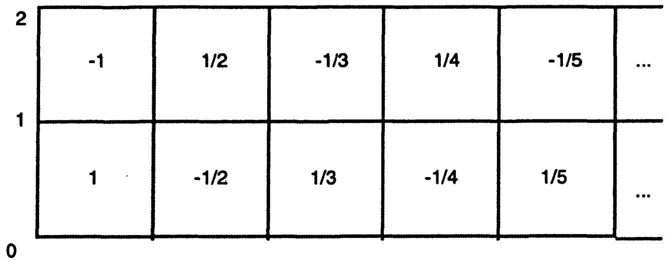

Fig. 5.

Fig 6

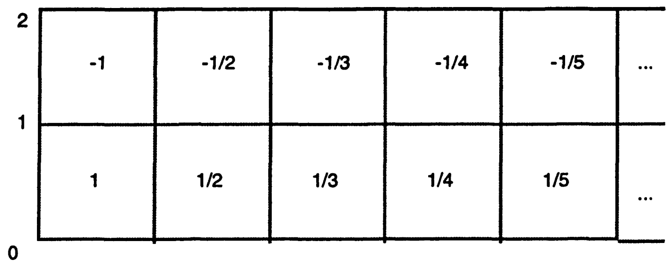

Fig. 6.

$\int_{\mathcal{P}_\gamma - \mathcal{P}_N} \psi \,{}^\alpha\!\approx 0$.

As $\mathcal{P}_\gamma$ is also a ${}^\alpha$small-interval of P, the Lemma 4 implies $\int_{\mathcal{P}_\gamma} f \,{}^\alpha\!\approx 0$ and, as $(\varphi, \psi) \in {}^\gamma test(f, \mathcal{P}_N)$, we have $(\varphi, \psi) \in {}^\gamma(test)(f, \mathcal{P}_\gamma)$. We deduce that $\int_{\mathcal{P}_\gamma} \psi \,{}^\alpha\!\approx 0$. Then $\int_{\mathcal{P}_N} \psi = \int_{\mathcal{P}_\gamma} \psi + \int_{\mathcal{P}_N - \mathcal{P}_\gamma} \psi - \int_{\mathcal{P}_\gamma - \mathcal{P}_N} \psi$ is ${}^\alpha$infinitesimal. Since $\psi = \psi'$ on $\mathcal{P} - \mathcal{P}_N$ we have $\int_{\mathcal{P} - \mathcal{P}_N} \psi \,{}^\alpha\!\approx 0$. So $\int_{\mathcal{P}} \psi \,{}^\alpha\!\approx 0$. □

As an obvious corollary, we have.

Theorem 5. *${}^\alpha$standard function f defined on a non compact and ${}^\alpha$standard interval I of $\mathbb{R}$ is G-integrable if and only if it admits an improper Lebesgue integral.*

REMARK. In the Definition 4, we make use of the term "improper L-integral" rather than that of "improper Lebesgue integral", because the last terminology has been used elsewhere with a different meaning when $N \geq 2$. In the classic approach, the class of functions which admit an improper Lebesgue integral on $\mathbb{R}^N$ ($N \geq 2$) contains only the Lebesgue integrable functions. It is, in our opinion, too small. Of course the class of integrability strongly depends on the shape allowed in definition of the ${}^\alpha$small domain, we have choosen intervals!!

EXAMPLE. The number in each square is the value of the function in its interior.

We can easily prove that this function is G-integrable and that $\int f = 0$. The function f hereunder is not G-integrable because the sequence $\int_{[0,n]\times[0,1]} f$ tends to $+\infty$ and $K_n = [0,n] \times [0,1]$ tends to $[0,+\infty[\times[0,1]$.

However, the sequence $(\int_{[0,n]\times[0,2]} f)$ tends to 0.

Now, we are going to characterize the Lebesgue-integrability, with the means of the ${}^{\beta}$tests and of the ${}^{\alpha,\beta}$tests. More precisely we can state

Theorem 6. *Consider $\beta \sqsupset \alpha$. The following conditions are equivalent.*

1) *$f$is Lebesgue-integrable on P*
2) *$|f|$est G-integrable on P,*
3) $\exists(\varphi,\psi) \in {}^{\beta}(test)(|f|,P)^{\alpha}$*adapted and* ${}^{\alpha}\mathcal{C}(\psi)$,
4) $\exists(\varphi,\psi) \in {}^{\beta}(test)(f,P)^{\alpha}$*adapted and* ${}^{\alpha}\mathcal{C}(|\psi|)$,
5) $\exists(\varphi,\psi) \in {}^{\beta}(test)(f,P)^{\alpha}$*adapted and* ${}^{\alpha}\int_P |\psi| < \infty$.

The easy proof of this theorem is left to the reader. The characterization (5) is equivalent to

$$\forall^{\alpha}\ \varepsilon > 0 \exists(\varphi,\psi) \in {}^{\alpha}test(f)[\int \psi - \varphi < \varepsilon \text{ and } {}^{\alpha}\int |\psi| < +\infty].$$

If f is standard it becomes: for any ${}^{\alpha}$standard $\varepsilon > 0$ there exists (φ,ψ) such that

$$\forall^{\text{almost standard}} x,\ \varphi(x) \le f({}^{\circ}x) \le \psi(x) \text{ and } \int \psi - \varphi < \varepsilon \text{ and } {}^{\alpha}\int |\psi| < +\infty.$$

This is the formulation in I.S.T. of the Loeb's one in [4] Theorem 1.14 p.170 and 3.4 p.190.

REMARK. We can prove all classical theorems of the classical theory of integration (th of Fubini, convergence theorems ...) in our theory without using the equivalence with the Lebesgue's definition of the integral.

3. Extensions of the G-integral and related problems

If a function f is G-integrable on two intervals P_1 and P_2, it maybe non-integrable on $P = P_1 \cup P_2$. A function f maybe G-integrable on an interval P but not integrable on the closure of P. For example, the function f defined by $f(x) = (1/x)\sin(1/x)$ if $x \neq 0$ and $f(0) = 0$, is G-integrable on $]0,1]$ but it is not integrable on $[0,1]$, (otherwise it would be Lebesgue integrable !).

Now, we can extend the class of integrables functions by saying that f is GG-integrable on P if there is a finite partition $\{P_i\}$ of P with intervals such that f be G integrable on each P_i. Then we let ${}_{GG}\int_P f = \sum_i {}_G\int_{P_i} f$. Of course we have to verify that the GG-integrability, as well as the value of the GG-integral do not depend on the partition $\{P_i\}$.

We don't know at the moment, if a modification of our definitions of integrability could give a process of integration which satisfies the same condition as Denjoy's concept does: if $F : [a,b] \to \mathbb{R}$ is continuous and derivable almost everywhere on $[a,b]$ then F' is integrable and $\int_{[a,x]} F' = F(x) - F(a)$.

Our choice of $\mathbb{R}^N$ as "space of integration" is motivated by physical considerations: whatever the space of states for a physical system, any standard state is represented in $\mathbb{R}^N$. However, the physical meaning of the integrability on $\mathbb{R}^N$, defined only as a formal extension of the integrability on $\mathbb{R}^k$ with standard k is not still heuristically clear for us today. We think that a deeper and rigorous reflexion on this subject still lies in the future. The affirmation that the infinitly large dimension can "replace" infinite dimension seems a too rapid one, and the current idea that a sort of S-integrability could completely replace the classic notions does not satisfy us either.

4. The language of the Relative Set Theory

It is built with two binary undefined predicates , the classic predicate $\ll . \in . \gg$ and another one, denoted by $\ll . \mathbf{st} . \gg$ which is a total preorder over the collection of sets. Two sets x and y such that $x \,\mathbf{st}\, y$ and $y \,\mathbf{st}\, x$ are called *equistandard.* A class $[a]$ of "equistandardness" is called a *level of standardness* or a *level of ideality* or shortly a *level.*

In the following, levels will be denoted by greek letters, $\sigma, \alpha, \beta \ldots$. Let $\beta = [b]$ and $\alpha = [a]$ be two levels. We say that β *dominates* α if $a \,\mathbf{st}\, b$. We denote this by $\alpha \sqsubseteq b$. We can also say that α is dominated by β, or that α is ${}^{\beta}$*standard.* If α does not dominate β, we say that β *strictly dominates* α. This will be denoted by $\alpha \sqsubset \beta$.

There exists a minimum level σ. It will be called the *standard level.* Any set defined in the classical ZFC set theory is standard.

Let $\mathcal{L}_{\mathrm{RST}}$ be the collection of the formulas of RST. If $F(x) \in \mathcal{L}_{\mathrm{RST}}$, we shall write

$$\begin{aligned}
&\forall^{\alpha}\ xF(x) \text{ for } \forall x[x \,\mathbf{st}\, a \Longrightarrow F(x)],\\
&\exists^{\alpha}\ xF(x) \text{ for } \exists x[x \,\mathbf{st}\, a \text{ and } F(x)],\\
&\forall^{[\ ]}xF(x) \text{ for } \forall xF(x),\\
&\exists^{[\ ]}xF(x) \text{ for } \exists x\ F(x).
\end{aligned}$$

A level of standardness β, or the empty bracket $[\ ]$, when it occurs as an upper index of a quantifier in a formula, is called a *level of quantification.* In order to extend the relations of domination to the levels of quantification, we set $[\ ] \sqsubseteq [\ ]$ and for any level of standardness α, $\alpha \sqsubset [\ ]$.

5. Some rules to handle the formulas in the Relative Set Theory

All the statements above concern formulas which belong to the sub-collection

$\mathcal{L}'_{\text{RST}}$ of $\mathcal{L}_{\text{RST}}$, of the formulas which are built from elementary formulas of the form $x \in y$, x and y being variables or constants, the logic connectors and the internal or external quantifiers $\forall^\gamma$, $\exists^\gamma$ where γ is a fixed level of quantification. The collection $\mathcal{L}'_{\text{RST}}$ is strictly included in $\mathcal{L}_{\text{RST}}$, for example, we can prove, that the formula of $\mathcal{L}_{\text{RST}}$, $\neg(b\,\mathbf{st}\,x)$, where b is a non-standard constant, is not equivalent to a formula of $\mathcal{L}'_{\text{RST}}$.

6. The homogeneity principle

It asserts roughly speaking that the universe of sets has the same structure at any level, more precisely it expresses that: in a formal discourse about sets of a level α involving levels of quantification dominating α, the height of the levels does not matter, only the relations of dominations between the levels are important. This can be more formally expressed by: if $F^{\alpha_1,\dots\alpha_n}$ is a formula of $\mathcal{L}'_{\text{RST}}$ with standard parameters and which involves the levels of quantifications $\alpha_1, \dots \alpha_n$, then for any levels of quantifications $\beta_1, \dots \beta_n$ such that , $\alpha_i \sqsubseteq \alpha_j \Longleftrightarrow \beta_i \sqsubseteq \beta_j$: $F^{\alpha_1,\dots\alpha_n} \Longleftrightarrow F^{\beta_1,\dots\beta_n}$.

As a direct consequence we have the principle of transfer. If $F(x)$ is a formula of the conventionnal mathematics with all its parameters ${}^\alpha$standard then for any level α: $[\forall^\alpha\ xF(x)] \Longleftrightarrow \forall x F(x)$.

7. The partial transfer theorem

It is so called because it involves an implication and not an equivalence (as the principle of transfer), and because all the indexes of quantification are not translated at the same level. It can be found and proved in [14].

If $F_1^{(\sigma,[\ ])}(x), F_2^{(\sigma,[\ ])}(x) \dots F_1^{(\sigma,[\ ])}(x)$ are formulas of $\mathcal{L}'_{\text{RST}}$ with all standard parameters and built only from the levels σ and $[\ ]$, then if $\sigma_2, \sigma_3, \dots, \sigma_n$ are levels such that $\sigma \sqsubset \sigma_2 \sqsubset \dots \sqsubset \sigma_n$

$$\begin{aligned} &\exists x\ [F_1^{(\sigma,[\ ])}(x) \wedge F_2^{(\sigma,[\ ])}(x) \dots \wedge F_n^{(\sigma,[\ ])}(x)] \\ &\quad \Longleftrightarrow \exists x[F_1^{(\sigma,[\ ])}(x) \wedge F_2^{(\sigma_2,[\ ])}(x) \dots \wedge F_n^{(\sigma_n,[\ ])}(x)]. \end{aligned}$$

The partial transfer theorem is often used in the cases where the classic approaches of non standard analysis would make use of the principles of permanence, but it is more general. An other usefull theorem is the particular one, due to P. Borde, a proof of which is given in [15].

8. Borde's transfer

Let F be a formula of $\mathcal{L}'_{\text{RST}}$. Let us suppose that its parameters have a level α which strictly dominates all the levels of quantifications, except $[\ ]$. Then,

$$\forall^\alpha\ xF(x) \Longleftrightarrow \forall x F(x).$$

9. Some rudiments of non standard analysis

Let α be a level. We say that a real number x is ${}^{\alpha}$*limited* if $\exists\lambda > 0; [|x| \leq \lambda]$. We say that two real numbers x and y are ${}^{\alpha}$*infinitesimaly close* and we write $x\,{}^{\alpha}\!\approx y$ if $\forall^{\alpha}\ \varepsilon > 0(|x-y| < e$. A number x is said ${}^{\alpha}$*infinitely large* if $\forall^{\alpha}\ \lambda(x > \lambda)$.

We shall frequently made use of the next properties and definitions:

- If $x \in \mathbb{R}$ is ${}^{\alpha}$limited, then there exists a real number b with $[b] \sqsubseteq \alpha$ such that $x\,{}^{\alpha}\!\approx b$. The number b is called the ${}^{\alpha}$*shadow of* x and we shall denote $b = {}^{\alpha}x$. We put ${}^{\alpha}x = +\infty$ if x is ${}^{\alpha}$infinitely large and ${}^{\alpha}x = -\infty$ if $-x$ is ${}^{\alpha}$infinitely large.
- If J is an open interval such that $[J] \sqsubseteq \alpha$ then

$$\forall^{\alpha}\ a[(a \in J \text{ and } x\,{}^{\alpha}\!\approx a) \Longrightarrow x \in J].$$

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